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Variational Integrals on Orlicz-Sobolev Spaces

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Abstract. We consider vector functions $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ minimizing variational integrals of the form $\int_\Omega G(\nabla u)\,dx$ with convex density G whose growth properties are described in terms of an N-function $A : [0,\infty) \to [0,\infty)$ with $\limsup_{t\to\infty} A(t)t^{-2} < \infty$. We then prove – under certain technical assumptions on G – full regularity of u provided that $n = 2$, and partial C^1-regularity in the case $n \geq 3$. The main feature of the paper is that we do not require any power growth of G.

Keywords: *Variational problems, minima, regularity theory, Orlicz-Sobolev spaces*
AMS subject classification: 49 N 60, 46 E 35

1. Introduction

Let Ω denote a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. For a given function $u_0 : \partial\Omega \to \mathbb{R}^N$ we consider the variational problem

$$\text{to find } u \in K \text{ such that } I(u) = \inf\nolimits_K I \tag{1.1}$$

where the class K consists of all vectorial functions agreing with u_0 on $\partial\Omega$ and the energy I is given by the expression

$$I(u) = \int_\Omega G(\nabla u)\,dx.$$

We impose the following conditions on the integrand $G : \mathbb{R}^{nN} \to \mathbb{R}$:

$$G \text{ is of class } C^2. \tag{1.2}$$

There exist constants $\lambda, \Lambda > 0$ and $\mu \geq 0$ such that, for all matrices $E, F \in \mathbb{R}^{nN}$,

$$\lambda(1 + |E|)^{-\mu}|F|^2 \leq D^2G(E)(F,F) \leq \Lambda|F|^2. \tag{1.3}$$

There is an N-function A and positive constants a and b such that

$$a \leq \frac{G(E)}{A(|E|)} \leq b \tag{1.4}$$

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is true for all matrices E whose norm is sufficiently large.

Our main concerns are the existence of solutions to problem (1.1) and their regularity properties. In standard variational calculus (see, e.g., [2, 4, 9, 10] and the references quoted therein) the answers to both questions are closely related to the growth behaviour of the integrand G which means that G is required to be of power growth for some exponent $p \geq 1$, i.e. (1.4) holds with $A(|E|)$ replaced by $|E|^p$ and the corresponding version of (1.3) reads

$$\lambda(1+|E|)^{p-2}|F|^2 \leq D^2G(E)(F,F) \leq \Lambda(1+|E|)^{p-2}|F|^2.$$

Of course, there are various ways of formulating a power growth condition for the integrand G but in all cases the density $G(E)$ is asymptoticaly close to some well-behaved integrand like $(1+|E|^2)^{p/2}$ $(E \in \mathbb{R}^{nN})$. As a consequence the natural space for problem (1.1) is the Sobolev class $W^1_p(\Omega, \mathbb{R}^N)$ and the existence of a minimizer u immediately follows with the help of the direct method. The p-growth condition is also used to investigate the regularity of the minimizer u via local comparison on balls $B \subset \Omega$ with the minimizer v of $\int_B (1+|\nabla v|^2)^{p/2}\,dx$ for boundary values u: assuming that some mean oscillation of ∇u on B is already small, it is possible to transfer some of the good regularity properties of v to the solution u by the way proving at least partial regularity with a possible set of singularities which is closed and of vanishing Lebesque measure.

In our case G is not of power growth for a fixed exponent $p \geq 1$: from (1.3) it only follows that G is of subquadratic growth, i.e. we have $\limsup_{|E|\to\infty} \frac{G(E)}{|E|^2} < \infty$, and, using the first inequality in (1.3), grows at least like $|E|^{2-\mu}$ as $|E| \to \infty$. Instead of power growth we require condition (1.4) which of course is much more flexible, for example we can consider the special case $G(E) = |E|\ln(1+|E|)$, i.e. $A(t) = t\ln(1+t)$, for which (1.3) holds with $\mu = 1$. Condition (1.4) also suggests the correct class in which we have to solve variational problem (1.1): K has to be chosen as a subclass of the Orlicz-Sobolev space $W^1_A(\Omega, \mathbb{R}^N)$ generated by the N-function A. When studying the smoothness of minimizers we can not rely on the local comparison with the smooth minimizer of some frozen functional. Using different methods we obtained (see Theorems 6.1 and 6.2 for a precise formulation) the following

Main Theorem. *Let conditions* (1.2) - (1.4) *hold and consider the solution* u *of problem* (1.1) *in* $K \cap W^1_A(\Omega, \mathbb{R}^N)$.

a) *If* $n = 2$ *and* $\mu \leq 1$, *then* $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ *for any* $0 < \alpha < 1$.

b) *Let* $n \geq 2$ *and suppose* $\mu < \frac{4}{n}$. *Then there is an open set* Ω_0 *in* Ω *such that* $|\Omega - \Omega_0| = 0$ *and* $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ *for all* $0 < \alpha < 1$ *where* $|\cdot|$ *denotes the Lebesgue measure.*

Regularity results under "general growth conditions" are mainly known for scalar problems ($N = 1$) and established with the help of the Hilbert-Haar method (see [13]) which uses convexity of the domain Ω and some smoothness properties of u_0 in a very essential way. So for convex domains Ω, for Lipschitz boundary data u_0 and under the assuption that $N = 1$ we have C^1-regularity of the minimizer u of the logarithmic energy $\int_\Omega |\nabla u|\ln(1+|\nabla u|)\,dx$ in any dimension n but we do not know if the same result is still valid without the hypothesis imposed on Ω and u_0. (In fact, our Main Theorem is true just for local I-minimizers from W^1_A.) The question of regularity is also adressed in the paper [12] of Marcellini still concentrating on the scalar case but dealing with local minimizers and general growth conditions. For example, he considers the integral $\int_\Omega |\nabla u|^p \ln(1+|\nabla u|)\,dx$ for $p>1$ but his arguments do not cover the limit case $p=1$.

In the vectorial setting $N>1$ there are even less references for (partial) regularity under general growth conditions: motivated by concrete examples from fluid mechanics and plasticity theory G. Seregin and the first author investigated the logarithmic model $\int_\Omega |\nabla u| \ln(1+|\nabla u|)\,dx$ for functions $u:\Omega\to\mathbb{R}^N$ (see [7, 8] and also [6]) and proved full regularity of local minimizers for the two-dimensional problem ($n=2$) and partial regularity up to dimension 4 (note that this is a slightly stronger result than statement b) of the Main Theorem). The present paper now describes the situation for integrands G satisfying conditions (1.2) - (1.4). In Section 2 we give a brief review of Orlicz-Sobolev spaces, in particular, we collect some technical lemmas which are used frequently. Next we show that our variational problem is well-posed on the corresponding class. Section 3 also introduces a regularisation of problem (1.1) and, as a byproduct, we obtain in Section 4 higher integrability of ∇u which together with certain Caccioppoli-type inequalities (see Section 5) forms the basis for the regularity proof. The proof of the Main Theorem proceeds in two steps: we first consider the case $n=2$ following a technique of Frehse and Seregin (see [6]), the general case $n\geq 3$ is based on a blow-up lemma for the squared mean oscillation of ∇u.

We wish to mention that a version of the Main Theorem for integrands with superquadratic growth is in preparation. As remarked earlier it would be of great interest to give optimal results for scalar problems: our arguments do not distinguish between vectorial and scalar functions and we conjecture that in the case $N=1$ the singular set should be empty for any dimension n.

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2. Preliminaries on Orlicz-Sobolev spaces

For a detailed account of the general setting we refer the reader to the book of Adams [1], we concentrate on some technical lemmas and recall some definitions. Following Adams we fix some N-function $A:[0,\infty)\to[0,\infty)$ having the following properties:

(N1) A is continuous, stricly increasing and convex.

(N2) $\lim_{t\downarrow 0}\frac{A(t)}{t}=0$ and $\lim_{t\to\infty}\frac{A(t)}{t}=\infty$.

(N3) There exist constants $k,t_0>0$ such that $A(2t)\leq kA(t)$ for all $t\geq t_0$.

In standard terminology (N3) means that A satisfies a Δ_2-condition near infinity. It is easy to see that (N3) implies the inequality

$$A(\lambda t)\leq A(\lambda t_0)+\left(1+k^{\frac{\ln\lambda}{\ln 2}+1}\right)A(t) \tag{2.1}$$

being valid for all $\lambda,t\geq 0$.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ denote a bounded domain. The Orlicz space $L_A(\Omega)$ is given by

$$\left\{ u : \Omega \to \mathbb{R} \text{ measurable} \,\middle|\, \int_\Omega A(|u|)\,dx < \infty \right\}.$$

$L_A(\Omega)$ together with the Luxemburg norm

$$\|u\|_{L_A(\Omega)} = \inf\left\{ l > 0 \,\middle|\, \int_\Omega A\left(\frac{1}{l}|u|\right) dx \leq 1 \right\}$$

carries the structure of a Banach space. For $u \neq 0$ the function $\varphi(\lambda) = \int_\Omega A(\lambda|u|)\,dx$ $(\lambda \geq 0)$ is strictly increasing with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Continuity of φ follows from Lebesgue's theorem on dominated convergence (an integrable majorant is given by (2.1)), hence there exist a unique number λ_* such that $\varphi(\lambda_*) = 1$. From the definition of $\|\cdot\|_{L_A}$ it follows that

$$\int_\Omega A\left(\frac{|u|}{\|u\|_{L_A}}\right) dx = 1 \tag{2.2}$$

and equation (2.2) is valid for any $u \in L_A(\Omega)$, $u \neq 0$.

Lemma 2.1. *Consider a sequence $\{u_m\}$ in $L_A(\Omega)$ such that $\int_\Omega A(|u_m|)\,dx \to 0$ as $m \to \infty$. Then we have:*

a) *$\int_\Omega A(\lambda|u_m|)\,dx \to 0$ as $m \to \infty$ for any $\lambda \geq 0$.*

b) $\lim_{m\to\infty} \|u_m\|_{L_A} = 0$.

Proof. The statements can be deduced from [1: Section 8.13] or [11: Lemma 3.10.4]. For the readers convenience we sketch the simple proof.

a) It is enough to consider the case $\lambda > 1$. Passing to a subsequence if necessary we may assume that $A(|u_m(x)|) \to 0$ for a.a. $x \in \Omega$. Let us fix $\delta > 0$. Then, by Egoroff's theorem, there exists a measurable subset E_δ of Ω such that $|E_\delta| \leq \delta$ and $A(|u_m|) \to 0$ uniformly on $\Omega \setminus E_\delta$. Property (N1) implies that also $|u_m| \to 0$ uniformly on $\Omega \setminus E_\delta$. Recalling (2.1) we see that

$$\begin{aligned}
&\int_\Omega A(\lambda|u_m|)\,dx \\
&\quad \leq \int_{\Omega\setminus E_\delta} A(\lambda|u_m|)\,dx + \int_{E_\delta} \left[A(\lambda t_0) + \left(k^{\frac{\ln\lambda}{\ln 2}+1}\right)A(|u_m|)\right]dx \\
&\quad \leq \int_{\Omega\setminus E_\delta} A(\lambda|u_m|)\,dx + A(\lambda t_0)\delta + \left(1 + k^{\frac{\ln\lambda}{\ln 2}+1}\right)\int_\Omega A(|u_m|)\,dx \\
&\quad \to A(\lambda t_0)\delta
\end{aligned}$$

as $m \to \infty$, hence

$$0 \leq \liminf_{m\to\infty} \int_\Omega A(\lambda|u_m|)\,dx \leq \limsup_{m\to\infty} \int_\Omega A(|u_m|)\,dx \leq A(\lambda t_0)\delta$$

and statement a) of Lemma 2.1 follows.

b) We argue by contradiction and consider a subsequence $\{u_m\}$ such that $\|u_m\|_{L_A(\Omega)} \geq \delta$ for some $\delta > 0$. But from (2.2) and statement a) of the lemma we obtain

$$1 = \int_\Omega A\left(\frac{|u_m|}{\|u_m\|_{L_A}}\right) dx \leq \int_\Omega A\left(\frac{1}{\delta}|u_m|\right) dx \to 0$$

as $m \to \infty$ ∎

For a function $v \in L^1(\mathbb{R}^n)$ we introduce its mollification

$$v_\rho(x) = \int_{\mathbb{R}^n} \omega_\rho(x-y)v(y)\,dy \qquad (\rho > 0)$$

where ω is a smooth function supported in the unit ball such that $\omega \geq 0$ and $\int_{\mathbb{R}^n} \omega\,dx = 1$. We let $\omega_\rho(x) = \rho^{-n}\omega(\frac{x}{\rho})$. The next lemma then follows from [1: Theorem 8.20 and Subsection 8.14].

Lemma 2.2. *Let the measurable function $u : \mathbb{R}^n \to \mathbb{R}$ vanish outside the domain Ω and assume in addition that $u \in L_A(\Omega)$. Then $u_\rho \to u$ in $L_A(\Omega)$ as $\rho \downarrow 0$.*

Proof. From Jensen's inequality we infer

$$\begin{aligned} A(|u_\rho(x)|) &\leq A\left(\int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|\,dy\right) \\ &\leq \int_{\mathbb{R}^n} \omega_\rho(y-x)\,A(|u(y)|)\,dy \\ &= [A(|u|)]_\rho(x). \end{aligned} \tag{2.3}$$

If the statement of the lemma is wrong, then we can find a sequence $\rho_m \downarrow 0$ and a number $\varepsilon > 0$ such that $\int_\Omega A(|u_{\rho_m} - u|)\,dx \geq \varepsilon$ for all $m \in \mathbb{N}$. From $\|u_{\rho_m} - u\|_{L^1(\Omega)} \to 0$ we get $u_{\rho_m}(x) \to u(x)$ a.e. on Ω at least for a subsequence. Let us fix $\delta > 0$. Then, for a suitable measurable set E_δ with $|E_\delta| < \delta$, we have $u_{\rho_m} \to u$ uniformly on $\Omega \setminus E_\delta$. Next we use monotonicity and convexity of A to see that

$$\begin{aligned} \varepsilon &\leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \int_{E_\delta} A(|u_{\rho_m} - u|)\,dx \\ &\leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \int_{E_\delta} A(\tfrac{1}{2}\,2|u_{\rho_m}| + \tfrac{1}{2}\,2|u|)\,dx \\ &\leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \frac{1}{2}\int_{E_\delta} A(2|u_{\rho_m}|)\,dx + \frac{1}{2}\int_{E_\delta} A(2|u|)\,dx \end{aligned}$$

and according to (2.3) we get the inequality

$$\begin{aligned} \varepsilon &\leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \frac{1}{2}\int_{E_\delta} [A(2|u|)]_{\rho_m}\,dx + \frac{1}{2}\int_{E_\delta} A(2|u|)\,dx \\ &\leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \frac{1}{2}\int_\Omega \big|[A(2|u|)]_{\rho_m} - A(2|u|)\big|\,dx + \int_{E_\delta} A(2|u|)\,dx. \end{aligned}$$

Since $A(2|u|) \in L^1(\Omega)$, the last integral is bounded by $\frac{\varepsilon}{2}$ provided δ is sufficiently small. In this case we obtain the contradiction

$$\frac{\varepsilon}{2} \leq \int_{\Omega\setminus E_\delta} A(|u_{\rho_m} - u|)\,dx + \frac{1}{2}\int_\Omega \big|[A(2|u|)]_{\rho_m} - A(2|u|)\big|\,dx \to 0$$

as $m \to \infty$ ∎

With similar arguments (compare also [11: Theorem 3.15.4]) we deduce the following

Lemma 2.3. *Consider $u \in L_A(\Omega)$ and extend u to $\mathbb{R}^n$ by letting $u = 0$ outside Ω. Then*

$$\|u(x+h) - u(x)\|_{L_A(\Omega)} \to 0 \qquad as \quad h \to 0, h \in \mathbb{R}^n.$$

Proof. If the statement of the lemma is wrong, then we can arrange

$$\int_\Omega A\big(|u(x+h_m) - u(x)|\big)\, dx \geq \varepsilon$$

for a sequence $h_m \to 0$ and some $\varepsilon > 0$ (compare Lemma 2.1). Clearly, $\|u(x+h_m) - u(x)\|_{L^1(\Omega)} \to 0$ and as before we can arrange $|u(x+h_m) - u(x)| \to 0$ uniformly on $\Omega \setminus E_\delta$, $E_\delta \subset \Omega$ with $|E_\delta| < \delta$. The properties of A imply

$$\begin{aligned}
&\int_\Omega A\big(|u(x+h_m) - u(x)|\big)\, dx \\
&\quad \leq \int_{\Omega\setminus E_\delta} A\big(|u(x+h_m) - u(x)|\big)\, dx + \frac{1}{2}\int_{E_\delta} A(2|u(x+h_m)|)\, dx + \frac{1}{2}\int_{E_\delta} A(2|u|)\, dx \\
&\quad = \int_{\Omega\setminus E_\delta} A\big(|u(x+h_m) - u(x)|\big)\, dx + \frac{1}{2}\int_{E_\delta} A(2|u|)\, dx + \frac{1}{2}\int_{(E_\delta\setminus h_m)\cap\Omega} A(2|u|)\, dx.
\end{aligned}$$

Recalling $A(2|u|) \in L^1(\Omega)$ we can fix δ in such a way that the last two integrals are bounded by $\frac{\varepsilon}{4}$. This gives

$$\frac{\varepsilon}{2} \leq \int_{\Omega\setminus E_\delta} A\big(|u(x+h_m) - u(x)|\big)\, dx \to 0$$

as $m \to \infty$, and the lemma is established ∎

After these preparations we introduce the Orlicz-Sobolev space

$$W_A^1(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} \Big| \, u, |\nabla u| \in L_A(\Omega) \Big\}$$

which together with the norm

$$\|u\|_{W_A^1(\Omega)} = \|u\|_{L_A(\Omega)} + \|\nabla u\|_{L_A(\Omega)}$$

is a Banach space. (Of course, A is still required to satisfy conditions (N1) - (N3)). We further let

$$\overset{\circ}{W}{}_A^1(\Omega) = \text{closure of } C_0^\infty(\Omega) \text{ in } W_A^1(\Omega) \text{ with respect to } \|\cdot\|_{W_A^1(\Omega)}.$$

The main result of this section is the following

Theorem 2.1. *Let Ω denote a bounded Lipschitz domain. Then*

$$\mathring{W}_A^1(\Omega) = W_A^1(\Omega) \cap \mathring{W}_1^1(\Omega).$$

Proof. We choose functions $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ $(1 \le j \le M)$ according to $\varphi_j \ge 0$ and $\sum_{j=1}^M \varphi_j \equiv 1$ on $\bar{\Omega}$. We may further assume that in the case $\partial'\Omega = \partial\Omega \cap \operatorname{supp}\varphi_j \ne \emptyset$ the set $\partial'\Omega$, after rotation and translation, is the graph of a Lipschitz function f, more precisely, we assume that

$$\operatorname{supp}\varphi_j \subset B_{r/2}(0) \quad \text{and} \quad \Omega_r = \Omega \cap B_r(0) = \{y \in \mathbb{R}^n \,|\, |y| < r, y_n > f(y')\}$$

with f satisfying $|f(y_1') - f(y_2')| \le L|y_1' - y_2'|$. Here we write $y = (y', y_n)$ for points y in $\mathbb{R}^n$. Then, for $y \in \Omega_{\frac{r}{2}}$ and $h > 0$ sufficiently small, we get $z = (y_1, \dots, y_{n-1}, f(y') + h) \in \Omega_r$ and

$$\operatorname{dist}(z, \partial\Omega \cap B_r(0)) \ge \left(1 - \frac{L^2}{1+L^2}\right)^{\frac{1}{2}} h = \frac{1}{\sqrt{1+L^2}} h. \tag{2.4}$$

For proving (2.4) let $z_1 \in \partial\Omega \cap B_r(0)$, i.e. $z_1 = (y_1', f(y_1'))$. Then

$$\begin{aligned} |z - z_1|^2 &= |y' - y_1'|^2 + (f(y') + h - f(y_1'))^2 \\ &= |y' - y_1'|^2 + (f(y') - f(y_1'))^2 + 2\sqrt{\varepsilon}\, h \frac{1}{\sqrt{\varepsilon}} (f(y') - f(y_1')) + h^2 \\ &\ge (1-\varepsilon)h^2 + |y' - y_1'|^2 \left\{1 - \frac{1}{\varepsilon} L^2 + L^2\right\}, \end{aligned}$$

and with $\varepsilon = \frac{L^2}{1+L^2}$ the claim follows ∎

Let us now consider $u \in W_A^1(\Omega) \cap \mathring{W}_1^1(\Omega)$. We want to show that there exists a sequence $\{u_m\}$ in $C_0^\infty(\Omega)$ such that

$$\|u - u_m\|_{W_A^1(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

To this purpose we write $u = \sum_{j=1}^M u^j$, $u^j = \varphi_j u \in W_A^1(\Omega)$. Then it is sufficient to prove that for each $j = 1, \dots, M$ there is a sequence $\{u_m^j\}$ in $C_0^\infty(\Omega)$ with the property

$$\|u^j - u_m^j\|_{W_A^1(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

In this case $u_m = \sum_{j=1}^M u_m^j$ has the desired property.

If $\operatorname{supp} u^j \cap \partial\Omega = \emptyset$, we choose a sequence $\rho_m \downarrow 0$ and define $u_m^j = (u^j)_{\rho_m} \in C_0^\infty(\Omega)$ (at least for large enough m). The desired convergence follows from Lemma 2.2. Assume now $\operatorname{supp}\varphi_j \cap \partial\Omega \ne \emptyset$ and let us use the notation introduced before. We further let $u^j = 0$ outside Ω and define

$$u_h^j(x) = u^j(x - he_n) \quad \text{where } e_n = (0, \dots, 0, 1).$$

For $h > 0$ sufficiently small we have $u_h^j \in W_A^1(\Omega) \cap \mathring{W}_1^1(\Omega)$ and according to Lemma 2.3 we know $u_h^j \to u^j$ in $W_A^1(\Omega)$ as $h \downarrow 0$. Thus there exists a sequence $h_m \downarrow 0$ such that $\|u^j_{h_m} - u^j\|_{W^1_A(\Omega)} \le \frac{1}{2m}$. Let $0 < \rho < \frac{1}{1+L^2}h_m^2$. Then (see (2.4)) $(u^j_{h_m})_\rho \in C_0^\infty(\Omega)$ and (compare Lemma 2.2) $(u^j_{h_m})_\rho \to u^j_{h_m}$ in $W^1_A(\Omega)$ as $\rho \downarrow 0$. Hence we can select a sequence $p_m \downarrow 0$ with the property

$$\|(u^j_{h_m})_{\rho_m} - u^j_{h_m}\|_{W^1_A(\Omega)} \le \frac{1}{2m}$$

and we may therefore define $u^j_m = (u^j_{h_m})_{\rho_m}$. This proves the inclusion

$$W^1_A(\Omega) \cap \mathring{W}^1_1(\Omega) \subset \mathring{W}^1_A(\Omega).$$

The opposite inclusion is a direct consequence of property (N2): first observe that $\|u_m - u\|_{L_A(\Omega)} \to 0$ implies convergence in the sense that $\int_\Omega A(|u_m - u|)\,dx \to 0$ which follows from convexity of A together with the definition of $\|\cdot\|_{L_A(\Omega)}$. But property (N2) then shows that $\int_\Omega |u_m - u|\,dx \to 0$. So, if $u_m \in C_0^\infty(\Omega)$ converges to $u \in W^1_A(\Omega)$ in the norm of this space, then the above argument shows $\|u_m - u\|_{W^1_1(\Omega)} \to 0$, i.e. $u \in \mathring{W}^1_1(\Omega) \cap W^1_A(\Omega)$ ∎

We finish this section with the following inequality of Poincaré type.

Lemma 2.4. *For $u \in \mathring{W}^1_A(\Omega)$ we have the inequality*

$$\|u\|_{L_A(\Omega)} \le d\|\nabla u\|_{L_A(\Omega)},$$

d denoting the diameter of Ω.

Proof. We just consider the case $u \in C_0^\infty(\Omega)$ and assume that Ω is a subdomain of the cube $[0,d]^n$. Then

$$u(x', x_n) = \int_0^{x_n} \frac{\partial}{\partial x_n} u(x',t)\,dt, \qquad \text{hence } \frac{|u(x)|}{ld} \le \int_0^d \frac{1}{l}|\nabla u(x',t)|\,dt \quad (l > 0).$$

Jensen's inequality implies

$$A\left(\frac{|u(x)|}{ld}\right) \le \frac{1}{d}\int_0^d A\left(\frac{|\nabla u|}{l}\right)\,dx_n$$

and therefore (choose $l = \|\nabla u\|_{L_A(\Omega)}$)

$$\int_\Omega A\left(\frac{|u|}{ld}\right)\,dx \le \int_\Omega A\left(\frac{|\nabla u|}{l}\right)\,dx = 1$$

which can be seen by integrating the foregoing inequality over $[0,d]^{n-1}$ with respect to the variable x' followed by an integration over $[0,d]$ with respect to x_n. From the last inequality we get $\|u\|_{L_A(\Omega)} \le ld = d\|\nabla u\|_{L_A(\Omega)}$ ∎

As a consequence of Lemma 2.4 we see that $\|v\|_{\mathring{W}_A^1(\Omega)} = \|\nabla v\|_{L_A(\Omega)}$ is equivalent to the original norm on $\mathring{W}_A^1(\Omega)$.

Remark 2.1. For later purposes we note that for any function A satisfying (N1) and (N2) there exists a complementary function A^* satisfying (N1) and (N2) too, such that

$$\int_\Omega |uv|\,dx \le 2\|u\|_{L_A(\Omega)}\|v\|_{L_{A^*}(\Omega)} \tag{2.5}$$

for all $u \in L_A(\Omega)$ and $v \in L_{A^*}(\Omega)$ (compare [1]).

Remark 2.2. The embedding $W_A^1(\Omega) \to W_1^1(\Omega)$ is continuous.

Remark 2.3. It is easy to see that all results of this section extend to vector functions $u : \Omega \to \mathbb{R}^N$.

3. An existence theorem for convex variational problems on Orlicz-Sobolev spaces

Let $G : \mathbb{R}^{nN} \to \mathbb{R}$ be a function with the following properties:

$$G \text{ is convex on the whole space } \mathbb{R}^{nN}. \tag{3.1}$$

There exist positive constants C_1, C_2 and a function A satisfying (N1) - (N3) such that, for all $E \in \mathbb{R}^{nN}$,

$$C_1(A(|E|) - 1) \le G(E) \le C_2(A(|E|) + 1). \tag{3.2}$$

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a given function $u_0 \in W_A^1(\Omega, \mathbb{R}^N)$ we introduce the convex class

$$K = u_0 + \mathring{W}_A^1(\Omega, \mathbb{R}^N) \tag{3.3}$$

on which the functional

$$I(u) = \int_\Omega G(\nabla u)\,dx$$

is well defined.

Theorem 3.1. *Under the assumptions* (3.1) - (3.3) *there exists at least one solution to the problem*

$$\text{to find } u \in K \text{ such that } \; I(u) = \inf_K I. \tag{3.4}$$

If G is strictly convex, then the solution is unique.

Proof. Consider a sequence $\{u_m\}$ in K such that $I(u_m) \to \gamma = \inf_K I \ge -C_1$ as $m \to \infty$. According to (3.2) we have

$$\int_\Omega A(|\nabla u_m|)\,dx \le \beta < \infty \tag{3.5}$$

for some constant β independent of m. Assuming $\beta \ge 1$ and using convexity of A we get

$$A\left(\frac{1}{\beta}|\nabla u_m|\right) \le \frac{1}{\beta}A(|\nabla u_m|)$$

which together with (3.5) implies $\|\nabla u_m\|_{L_A(\Omega)} \le \beta$. Applying Lemma 2.4 to $u_m - u_0$ we arrive at

$$\|u_m\|_{W^1_A(\Omega)} \le (\operatorname{diam}\Omega + 1)\big[\beta + \|u_0\|_{W^1_A(\Omega)}\big]. \tag{3.6}$$

Using compactness of the embedding $W^1_1(\Omega,\mathbb{R}^N) \to L^1(\Omega,\mathbb{R}^N)$ together with Remark 2.2 and (3.6) we find $\hat{u}$ in $L^1(\Omega,\mathbb{R}^N)$ such that $u_m \to \hat{u}$ in $L^1(\Omega,\mathbb{R}^N)$ (for a subsequence). On the other hand, property (N2) of the function A together with (3.5) gives the existence of $V \in L^1(\Omega,\mathbb{R}^{nN})$ such that $\nabla u_m \rightharpoonup V$ in $L^1(\Omega,\mathbb{R}^{nN})$ (for a subsequence). Clearly, $V = \nabla\hat{u}$, hence $u_m \rightharpoonup \hat{u}$ in the space $W^1_1(\Omega,\mathbb{R}^N)$. Since $u_m - u_0 \in \mathring{W}^1_1(\Omega,\mathbb{R}^N)$, we get $\hat{u} - u_0 \in \mathring{W}^1_1(\Omega,\mathbb{R}^N)$. According to Mazur's lemma we can arrange

$$v_m = \sum_{j=m}^{N(m)} c_j^m u_j \to \hat{u} \qquad \text{in } W^1_1(\Omega,\mathbb{R}^N)$$

for suitable sequences $N(m) \in \mathbb{N}$ and $c_j^m \ge 0$ wit $\sum_{j=m}^{N(m)} c_j^m = 1$, and for some subsequence we may also assume $\nabla v_m \to \nabla\hat{u}$ a.e. on Ω. Let $\varepsilon > 0$ and select m_0 such that $I(u_m) < \gamma + \varepsilon$ for all $m \ge m_0$. This implies $(m \ge m_0)$

$$I(v_m) = \int_\Omega G\left(\sum_{j=m}^{N(m)} c_j^m \nabla u_j\right) dx \le \sum_{j=m}^{N(m)} c_j^m \int_\Omega G(\nabla u_j)\,dx < \gamma + \varepsilon,$$

i.e. $\int_\Omega G(\nabla v_m)\,dx < \gamma + \varepsilon$ for $m \ge m_0$.

Applying Fatou's lemma (recall that according to (3.2) G is bounded from below) we deduce $I(\hat{u}) \le \gamma$, which shows that $\nabla\hat{u}$ is of class $L_A(\Omega,\mathbb{R}^{nN})$. Recalling (3.6) we have $\sup_m \|u_m\|_{L_A(\Omega)} < \infty$. Passing to a further subsequence we may assume $u_m \to \hat{u}$ a.e. on Ω and by quoting Fatou's lemma one more time, we get $\int_\Omega A(|\hat{u}|)\,dx < \infty$ so that $\hat{u} \in W^1_A(\Omega,\mathbb{R}^N)$. Finally, we observe $\hat{u} - u_0 \in W^1_A \cap \mathring{W}^1_1(\Omega,\mathbb{R}^N)$ and Theorem 2.1 implies $\hat{u} - u_0 \in \mathring{W}^1_A(\Omega,\mathbb{R}^N)$, i.e. $\hat{u} \in K$. Thus $\hat{u}$ is a solution to (3.4). Uniqueness for strictly convex G is immediate ∎

Next we introduce a suitable regularisation of problem (3.4). To this purpose we assume from now on $u_0 \in W^1_2(\Omega,\mathbb{R}^N)$ (see (3.3)) and define for $\delta > 0$

$$G_\delta(E) = \frac{\delta}{2}|E|^2 + G(E) \qquad (E \in \mathbb{R}^{nN}).$$

In addition we require that G is subquadratic, i.e. there exists a constant C_3 such that

$$G(E) \le C_3(|E|^2 + 1) \qquad \text{for all } E \in \mathbb{R}^{nN}. \tag{3.7}$$

We then discuss the variational problem

$$I_\delta(w) = \int_\Omega G_\delta(\nabla w)\,dx \to \min \qquad \text{in } K' = u_0 + \mathring{W}^1_2(\Omega,\mathbb{R}^N). \tag{3.8}$$

Lemma 3.1. *Assume that G is strictly convex and let (3.2) and (3.7) hold. Let $u_\delta \in K'$ denote the unique solution of problem (3.8). Then, as $\delta \downarrow 0$, we have*

$$u_\delta \to \hat{u} \quad \textit{in} \quad W_1^1(\Omega, \mathbb{R}^N), \qquad \delta \int_\Omega |\nabla u_\delta|^2\, dx \to 0, \qquad I_\delta(u_\delta) \to I(\hat{u}),$$

$\hat{u} \in K$ denoting the unique solution of problem (3.4).

Proof. From (3.2) and (3.7) we get the growth estimates

$$\frac{\delta}{2}|E|^2 - C_1 \le G_\delta(E) \le \left(\frac{\delta}{2} + C_3\right)|E|^2 + C_3,$$

hence (3.8) is well-posed and due to the strict convexity of G_δ has a unique solution u_δ. From

$$I_\delta(u_\delta) \le I_\delta(u_0) \le I_1(u_0) \qquad \text{together with} \qquad A(|E|) \le \frac{1}{C_1}G(E) + 1$$

we deduce

$$\int_\Omega A(|\nabla u_\delta|)\, dx \le M < \infty \qquad \text{for all } \ \delta \in (0,1],$$

and as in the proof of Theorem 3.1 we find a sequence $\delta_m \downarrow 0$ and a function $\tilde{u} \in W_1^1(\Omega, \mathbb{R}^N)$ such that ($u_m = u_{\delta_m}$)

$$u_m \to \tilde{u} \ \text{ in } L^1(\Omega, \mathbb{R}^N), \quad \nabla u_m \to \nabla \tilde{u} \ \text{ in } L^1(\Omega, \mathbb{R}^{nN}), \quad \tilde{u} - u_0 \in \mathring{W}_1^1(\Omega, \mathbb{R}^N).$$

Moreover, $I(\tilde{u}) \le \liminf_{m\to\infty} I(u_m)$, i.e. $\nabla\tilde{u} \in L_A(\Omega, \mathbb{R}^{nN})$, and by repeating the arguments from the proof of Theorem 3.1 we find that $\tilde{u} \in K$. We further have the chain of inequalities

$$I(\tilde{u}) \le \liminf_{m\to\infty} I(u_m) \le \liminf_{m\to\infty} I_{\delta_m}(u_m) = \tau_1 \le \tau_2 = \limsup_{m\to\infty} I_{\delta_m}(u_m) \le I(w)$$

being valid for any $w \in K'$. Consider $v \in K$. Then, by Theorem 2.1, there exists a sequence $\varphi_m \in C_0^\infty(\Omega, \mathbb{R}^N)$ such that $\varphi_m \to v - u_0$ in $W_A^1(\Omega, \mathbb{R}^N)$. Let $w_m = u_0 + \varphi_m$. Then $w_m \to v$ in $W_A^1(\Omega, \mathbb{R}^N)$. The functional I is convex and locally bounded from above on the space $W_A^1(\Omega, \mathbb{R}^N)$, hence continuous, so that $I(w_m) \to I(v)$. Therefore, $\tilde{u}$ is I-minimizing in K which means $\tilde{u} = \hat{u}$. Moreover, $u_\delta \to \hat{u}$ in $W_1^1(\Omega, \mathbb{R}^N)$ as $\delta \downarrow 0$ not only for a subsequence, and $I(\hat{u}) = \tau_1 = \tau_2$, i.e. $I(\hat{u}) = \lim_{\delta\downarrow 0} I_\delta(u_\delta)$. Using $I(\hat{u}) \le I(u_\delta) \le I_\delta(u_\delta)$ we see $I(u_\delta) \to I(\hat{u})$ and $\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2\, dx \to 0$ as $\delta \downarrow 0$, and the proof is complete ∎

4. Higher integrability of the gradient

Assume that (3.2) and (3.7) hold with A satisfying (N1) - (N3). Let A^* denote the N-function complementary to A. In this section we require the following structural conditions for the integrand G:

$$G \text{ is of class } C^2(\mathbb{R}^{nN}). \tag{4.1}$$

There is a constant $C_4 > 0$ such that

$$|Q|^2 |D^2G(Q)| \leq C_4(G(Q)+1) \qquad \text{for all } Q \in \mathbb{R}^{nN}. \tag{4.2}$$

With suitable constants $\lambda > 0$ and $\mu \geq 0$ we have

$$D^2G(Q)(E,E) \geq \lambda(1+|Q|)^{-\mu}|E|^2 \qquad \text{for all } E, Q \in \mathbb{R}^{nN}. \tag{4.3}$$

There is a constant $C_5 > 0$ with property

$$A^*(|DG(Q)|) \leq C_5(A(|Q|)+1) \qquad \text{for all } Q \in \mathbb{R}^{nN}. \tag{4.4}$$

Note that (4.2) together with (3.7) implies boundedness of D^2G.

Lemma 4.1. *Let the conditions* (3.2), (3.7), (4.1), (4.2), (4.3) *with* $\mu < 2$ *and* (4.4) *hold and consider the solution* $\hat{u}$ *of problem* (3.4) *with* $u_0 \in W_2^1(\Omega, \mathbb{R}^N)$. *Then we have* $h = (1+|\nabla \hat{u}|)^{1-\mu/2} \in W^1_{2,\mathrm{loc}}(\Omega)$.

Proof. For $\delta \leq 1$ consider the regularized problem (3.8) whose solution u_δ is of class $W^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ which can be seen by applying the standard difference quotient technique (compare [9]). From the weak form of the Euler equation we infer after integration by parts

$$\int_\Omega D^2G_\delta(\nabla u_\delta)(\nabla \partial_\alpha u_\delta, \nabla \Phi)\,dx = 0$$

being valid for $\alpha = 1, \ldots, n$ and any $\Phi \in W_2^1(\Omega, \mathbb{R}^N)$ with compact support in Ω. We insert $\Phi = \eta^2 \partial_\alpha u_\delta$ $(0 \leq \eta \leq 1, \eta \in C_0^1(\Omega))$, and get with standard calculations (from now on summation with respect to α)

$$\int_\Omega \eta^2 D^2G_\delta(\nabla u_\delta)(\nabla\partial_\alpha u_\delta, \nabla\partial_\alpha u_\delta)\,dx = -2\int_\Omega \eta D^2G_\delta(\nabla u_\delta)(\nabla\partial_\alpha u_\delta, \nabla\eta \otimes \partial_\alpha u_\delta)\,dx.$$

Using

$$|D^2G_\delta(X)(Y,Z)| \leq \left(D^2G_\delta(X)(Y,Y)\right)^{1/2}\left(D^2G_\delta(X)(Z,Z)\right)^{1/2}$$

and (4.2) we deduce

$$\begin{aligned}\int_\Omega \eta^2 D^2G_\delta(\nabla u_\delta)(\nabla\partial_\alpha u_\delta, \nabla\partial_\alpha u_\delta)\,dx &\leq c_1(\eta)\int_\Omega |D^2G_\delta(\nabla u_\delta)||\nabla u_\delta|^2\,dx \\ &\leq c_2(\eta)\left\{\int_\Omega \delta|\nabla u_\delta|^2\,dx + I(u_\delta) + 1\right\}.\end{aligned}$$

By Lemma 3.1 this implies the bound

$$\int_{\Omega^*} D^2G_\delta(\nabla u_\delta)(\nabla\partial_\alpha u_\delta, \nabla\partial_\alpha u_\delta)\, dx \le c_3(\Omega^*) < \infty$$

for any subdomain $\Omega^* \subset\subset \Omega$ with $c_3(\Omega^*)$ independent of δ.

Using (4.3) we arrive at

$$\int_{\Omega^*} (1 + |\nabla u_\delta|)^{-\mu} |\nabla^2 u_\delta|^2\, dx \le c_4(\Omega^*).$$

Let $h_\delta = (1 + |\nabla u_\delta|)^{1-\mu/2}$. Since $\mu < 2$ we have $|\nabla h_\delta| \le (1 - \frac{\mu}{2})(1 + |\nabla u_\delta|)^{-\mu/2} |\nabla^2 u_\delta|$, hence

$$\int_{\Omega^*} |\nabla h_\delta|^2\, dx \le c_5(\Omega^*). \tag{4.5}$$

By Lemma 3.1 the sequence $\{h_\delta\}$ is bounded in $L^1(\Omega)$ which together with (4.5) implies local boundedness of $\{h_\delta\}$ in $W^1_{2,\mathrm{loc}}(\Omega)$. Thus there exists $h \in W^1_{2,\mathrm{loc}}(\Omega)$ such that $h_\delta \rightharpoonup h$ in $W^1_{2,\mathrm{loc}}(\Omega)$ as $\delta \downarrow 0$ at least for sequence $\delta \downarrow 0$. We want to show that $h = (1 + |\nabla \hat{u}|)^{1-\mu/2}$. To this purpose we write

$$\begin{aligned} &I_\delta(u_\delta) - I(\hat{u}) \\ &= \frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2\, dx + I(u_\delta) - I(\hat{u}) \\ &= \frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2\, dx + \int_\Omega DG(\nabla \hat{u}) : (\nabla u_\delta - \nabla \hat{u})\, dx \\ &\quad + \int_\Omega \int_0^1 D^2G((1-t)\nabla \hat{u} + t\nabla u_\delta)(\nabla u_\delta - \nabla \hat{u}, \nabla u_\delta - \nabla \hat{u})(1-t)\, dt dx. \end{aligned}$$

From (4.4) we get $|DG(\nabla \hat{u})| \in L_{A^*}(\Omega)$ and recalling Remark 2.1 we see that

$$\int_\Omega DG(\nabla \hat{u}) : \nabla\varphi\, dx = 0 \qquad \text{for all } \varphi \in \mathring{W}^1_A(\Omega, \mathbb{R}^N). \tag{4.6}$$

Inserting (4.6) into the formula for $I_\delta(u_\delta) - I(\hat{u})$ and using Lemma 3.1 we obtain

$$\int_\Omega \int_0^1 D^2G((1-t)\nabla \hat{u} + t\nabla u_\delta)(\nabla u_\delta - \nabla \hat{u}, \nabla u_\delta - \nabla \hat{u})(1-t)\, dt dx \to 0$$

as $\delta \downarrow 0$. But then (see (4.3))

$$\int_\Omega (1 + |\nabla u_\delta| + |\nabla \hat{u}|)^{-\mu} |\nabla u_\delta - \nabla \hat{u}|^2\, dx \to 0 \qquad \text{as } \delta \downarrow 0,$$

and therefore

$$(1 + |\nabla u_\delta| + |\nabla \hat{u}|)^{-\mu} |\nabla u_\delta - \nabla \hat{u}|^2 \to 0 \qquad \text{a.e.} \tag{4.7}$$

at least for a subsequence. We may assume that also $h_\delta \to h$ a.e. on Ω which means that $\nabla u_\delta(x)$ has a finite limit for almost all x in Ω. But then (4.7) implies $\nabla u_\delta \to \nabla \hat{u}$ a.e. and we get the desired claim $h = (1 + |\nabla \hat{u}|)^{1-\mu/2} \in W^1_{2,\mathrm{loc}}(\Omega)$ ∎

The embedding theorem implies the following

Lemma 4.2. *Under the assumptions of Lemma* 4.1 *we have* $\nabla\hat{u} \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ *for any* $p < \infty$, *if* $n = 2$, *and for* $p = 2$, *if* $n \geq 3$, $\mu \leq \frac{4}{n}$. *In the case of* $n \geq 3$ *we can take* $p = (2-\mu)\frac{n}{n-2}$.

5. Some inequalities of Caccioppoli type

In this section we give appropriate versions of Caccioppoli type inequalities for our variational problem (3.4).

Lemma 5.1. *Let conditions* (3.2), (3.7), (4.1), (4.2), (4.3) *with* $\mu < \frac{4}{n}$ *and* (4.4) *hold. Let* $\hat{u}$ *denote the unique solution of problem* (3.4) *with* $u_0 \in W^1_2(\Omega, \mathbb{R}^N)$. *Suppose that the ball* $B_R(x_0)$ *is compactly contained in* Ω. *Then, for any* $0 < t < 1$ *and any* $Q \in \mathbb{R}^{nN}$, *we have the estimate*

$$\int_{B_{tR}(x_0)} |\nabla(1+|\nabla\hat{u}|)^{1-\mu/2}|^2\,dx \;\leq\; c(1-t)^{-2}R^{-2}\int_{B_R(x_0)} |D^2G(\nabla\hat{u})|\,|\nabla\hat{u}-Q|^2\,dx$$

with c *independent of* t *and* R.

Proof. As before we make use of the δ-approximation introduced in (3.8) and observe the equation (summation with respect to $\alpha = 1, \dots, n$)

$$\int_{B_R(x_0)} D^2G_\delta(\nabla u_\delta)\big(\nabla\partial_\alpha u_\delta, \nabla(\eta^2[\partial_\alpha u_\delta - Q_\alpha])\big)\,dx \;=\; 0 \tag{5.1}$$

valid for $\eta \in C^1_0(B_R(x_0))$, $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{tR}(x_0)$, $|\nabla\eta| \leq c_1(1-t)^{-1}R^{-1}$. Similarly to the proof of Lemma 4.1 we deduce from (5.1) the estimate

$$\begin{aligned}&\int_{B_R(x_0)} \eta^2 D^2G_\delta(\nabla u_\delta)(\nabla\partial_\alpha u_\delta, \nabla\partial_\alpha u_\delta)\,dx\\ &\qquad\leq c_2\|\nabla\eta\|^2_{L^\infty}\int_{B_R(x_0)} \{\delta|\nabla u_\delta - Q|^2 + |D^2G(\nabla u_\delta)|\,|\nabla u_\delta - Q|^2\}dx.\end{aligned}$$

Let h_δ and h denote the functions introduced in the proof of Lemma 4.1. Then the foregoing inequality implies

$$\int_{B_{tR}(x_0)} |\nabla h_\delta|^2\,dx \;\leq\; c_3\|\nabla\eta\|^2_{L^\infty}\int_{B_R(x_0)} \Big\{\delta|\nabla u_\delta - Q|^2 + |D^2G(\nabla u_\delta)|\,|\nabla u_\delta - Q|^2\Big\}dx$$

so that (recall Lemma 3.1 and $h_\delta \rightharpoonup h$ in $W^1_{2,\mathrm{loc}}(\Omega)$)

$$\int_{B_{tR}(x_0)} |\nabla h|^2\,dx \;\leq\; c_3\|\nabla\eta\|^2_{L^\infty}\liminf_{\delta\downarrow 0}\int_{B_R(x_0)} |D^2G(\nabla u_\delta)|\,|\nabla u_\delta - Q|^2\,dx.$$

From (4.5) and the following considerations together with our assumption $\mu < \frac{4}{n}$ we find some $p > 2$ such that $\int_{\Omega^*} |\nabla u_\delta|^p \, dx \le c_4(\Omega^*) < \infty$ for any subdomain $\Omega^* \subset\subset \Omega$. Hence we deduce $|\nabla u_\delta - Q|^2 \rightharpoonup \Theta$ in $L^{1+\varepsilon}_{\text{loc}}(\Omega)$ for some $\varepsilon > 0$ and a function Θ in this space. But $\nabla u_\delta \to \nabla \hat{u}$ a.e. as $\delta \downarrow 0$ implies $\Theta = |\nabla \hat{u} - Q|^2$. The sequence $\{|D^2G(\nabla u_\delta)|\}$ is bounded independently of δ (combine (3.7) with (4.2)) and $|D^2G(\nabla u_\delta)| \to |D^2G(\nabla \hat{u})|$ a.e. on Ω as $\delta \downarrow 0$. From this it is now immediate that

$$\lim_{\delta \downarrow 0} \int_{B_R(x_0)} |D^2G(\nabla u_\delta)| \, |\nabla u_\delta - Q|^2 \, dx = \int_{B_R(x_0)} |D^2G(\nabla \hat{u})| \, |\nabla \hat{u} - Q|^2 \, dx$$

and we arrive at

$$\int_{B_{tR}(x_0)} |\nabla h|^2 \, dx \le c_5 \left(\frac{1}{R} \frac{1}{1-t} \right)^2 \int_{B_R(x_0)} |D^2G(\nabla \hat{u})| \, |\nabla \hat{u} - Q|^2 \, dx$$

and the proof is finished ∎

Lemma 5.2. *Let $n = 2$. Assume that* (3.2), (3.7), (4.1), (4.2), (4.3) *with $\mu \le 1$ are satisfied. Let u_δ denote the solution of problem* (3.8) *and define the functions*

$$h_\delta = (1 + |\nabla u_\delta|)^{1-\mu/2}$$
$$H_\delta = \left(D^2G_\delta(\nabla u_\delta)(\nabla \partial_\alpha u_\delta, \nabla \partial_\alpha u_\delta)\right)^{1/2}.$$

Then, for any $0 < \delta \le 1$ and any disc $B_{2R}(x_0)$ compactly ontained in Ω, we have the inequality $(T_{2R}(x_0) = B_{2R}(x_0) - \overline{B_R(x_0)})$

$$\int_{B_R(x_0)} H_\delta^2 \, dx \le cR^{-1} \left(\int_{T_{2R}(x_0)} H_\delta^2 \, dx \right)^{1/2} \int_{T_{2R}(x_0)} H_\delta h_\delta \, dx.$$

Note, that Lemma 5.2 does not require condition (4.4).

Proof. We use equation (5.1) with $B_R(x_0)$ replaced by $B_{2R}(x_0)$ and with $\eta \in C^1_0(B_{2R}(x_0))$, $0 \le \eta \le 1$, $\eta = 1$ on $B_R(x_0)$ and $|\nabla \eta| \le \frac{c_1}{R}$. Since $\nabla \eta$ is supported on the closure of the ring $T_{2R}(x_0)$, we obtain (Q_δ denoting the mean value of ∇u_δ with respect to $T_{2R}(x_0)$)

$$\int_{B_{2R}(x_0)} \eta^2 D^2G_\delta(\nabla u_\delta)(\nabla \partial_\alpha u_\delta, \nabla \partial_\alpha u_\delta) \, dx$$
$$\le c_2 \left(\int_{T_{2R}(x_0)} D^2G_\delta(\nabla u_\delta)(\nabla \partial_\alpha u_\delta, \nabla \partial_\alpha u_\delta) \, dx \right)^{1/2}$$
$$\times \left(\int_{T_{2R}(x_0)} D^2G_\delta(\nabla u_\delta)\Big(\nabla \eta \otimes [\partial_\alpha u_\delta - (Q_\delta)_\alpha], \nabla \eta \otimes [\partial_\alpha u_\delta - (Q_\delta)_\alpha]\Big) dx \right)^{1/2},$$

hence

$$\int_{B_R(x_0)} H_\delta^2\,dx \le c_3\frac{1}{R}\left(\int_{T_{2R}(x_0)} H_\delta^2\,dx\right)^{1/2}\left(\int_{T_{2R}(x_0)} |D^2G_\delta(\nabla u_\delta)|\,|\nabla u_\delta - Q_\delta|^2\,dx\right)^{1/2}. \quad (5.2)$$

Using the boundedness of $|D^2G_\delta(\nabla u_\delta)|$ and Poincare's inequality (recall $n = 2$) we get

$$\left(\int_{T_{2R}(x_0)} |D^2G_\delta(\nabla u_\delta)|\,|\nabla u_\delta - Q_\delta|^2\,dx\right)^{1/2} \le c_4 \int_{T_{2R}(x_0)} |\nabla^2 u_\delta|\,dx. \quad (5.3)$$

Finally, we observe (see (4.3)) $h_\delta H_\delta \ge \sqrt{\lambda}(1+|\nabla u_\delta|)^{1-\mu}|\nabla^2 u_\delta|$ and $\mu \le 1$ shows that $h_\delta H_\delta \ge \sqrt{\lambda}\,|\nabla^2 u_\delta|$. Inserting this into (5.3) and using (5.2), the lemma is established ∎

6. Proof of the regularity results

We first give a precise version of the Main Theorem from Section 1.

Theorem 6.1. *Suppose that $n \ge 2$ and let Ω denote a bounded Lipschitz domain in $\mathbb{R}^n$. Let (3.2), (3.7), (4.1), (4.2), (4.3) with $\mu < \frac{4}{n}$ and (4.4) hold. Let $\hat{u}$ denote the unique solution of problem (3.4) with $u_0 \in W_2^1(\Omega, \mathbb{R}^N)$. Then there is an open subset Ω_0 of Ω such that $|\Omega - \Omega_0| = 0$ and $\hat{u} \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for any $0 < \alpha < 1$.*

Theorem 6.2. *Under the same conditions as in Theorem 6.1 let $n = 2$ and suppose $\mu \le 1$ in (4.3). Then $\hat{u} \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ for any $0 < \alpha < 1$.*

Remark 6.1. In the two-dimensional case we have partial regularity for $\mu < 2$, whereas $\mu \le 1$ implies full regularity. We would like to know if $\mu < 2$ is sufficient for $\Omega = \Omega_0$.

Remark 6.2. With minor changes (see [7]) all results extend to local I-minimizers from the space $W_A^1(\Omega, \mathbb{R}^N)$.

We first consider the two-dimensional case, i.e we start with the

Proof of Theorem 6.2. We follow [6]. As demonstrated there we can deduce from Lemma 5.2 that for any $q \ge 1$ and any compact subdomain $\Omega^* \subset \Omega$ there is a constant $K = K(q, \Omega^*) \in (0, \infty)$ such that, for any $0 < \delta \le 1$, the inequality

$$\int_{B_R(x_0)} H_\delta^2\,dx \le K|\ln R|^{-q} \quad (6.1)$$

is valid for all discs $B_R(x_0) \subset \Omega^*$. Clearly, $|\nabla\{DG_\delta(\nabla u_\delta)\}|^2 \le c_1 H_\delta^2$. If we choose $q > 2$ in (6.1), then the variant of Morrey's lemma given in [5] shows that $DG_\delta(\nabla u_\delta)$ is continuous on Ω^* with modulus of continuity independent of δ. Since $\nabla u_\delta \to \nabla \hat{u}$ a.e. on Ω we deduce continuity of $DG(\nabla \hat{u})$. We have

$$\frac{1}{|Q|} DG(Q) : Q \geq \frac{1}{|Q|} \left(G(Q) - G(0)\right) \geq \frac{1}{2} \frac{1}{|Q|} G(Q)$$

at least for sufficiently large Q which follows from (3.2) and (N2), hence $\frac{1}{|Q|} DG(Q) : Q \to \infty, |Q| \to \infty$, and therefore DG is a homeomorphism. This proves continuity of $\nabla \hat{u}$. The complete claim, i.e. $\nabla \hat{u} \in C^{0,\alpha}(\Omega, \mathbb{R}^{nN})$, follows from the next lemma and Remark 6.3, a direct proof (not using partial regularity theory) is given in [6] ∎

Lemma 6.1. *Assume that the hypothesis of Theorem* 6.1 *are satisfied. Fix some* $L > 0$ *and define* C_0 *as before formula* (6.3). *Then, for all* $\tau \in (0,1)$, *we find a number* $\varepsilon = \varepsilon(\tau, L) > 0$ *such that*

$$|(\nabla \hat{u})_{x_0,R}| < L \qquad \textit{and} \qquad \fint_{B_R(x_0)} |\nabla \hat{u} - (\nabla \hat{u})_{x_0,R}|^2 \, dx < \varepsilon^2$$

imply

$$\fint_{B_{\tau R}(x_0)} |\nabla \hat{u} - (\nabla \hat{u})_{x_0,\tau R}|^2 dx \leq C_0 \tau^2 \fint_{B_R(x_0)} |\nabla \hat{u} - (\nabla \hat{u})_{x_0,R}|^2 dx$$

for any ball $B_R(x_0) \subset \Omega$. *Here and in what follows we write* $\fint_{B_r(y)} f \, dx$ *or* $(f)_{y,r}$ *to denote the mean value of a function* f *with respect to a ball* $B_r(y)$.

Remark 6.3. It is standard to show (see [9]) that $\hat{u}$ is of class $C^{1,\alpha}$ for any $0 < \alpha < 1$ in an open neighborhood of some point $x_0 \in \Omega$ if and only if

$$\sup_{r>0} |(\nabla \hat{u})_{x_0,r}| < \infty \quad \text{and} \quad \fint_{B_r(x_0} |\nabla \hat{u} - (\nabla \hat{u})_{x_0,r}|^2 \, dx \to 0 \ \ (r \downarrow 0). \qquad (6.2)$$

Let Ω_0 denote the collection of points satisfying both conditions. Then Ω_0 is open, $|\Omega - \Omega_0| = 0$ and $\hat{u} \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$. In the case $n = 2$ together with $\mu \leq 1$ we already know $\hat{u} \in C^1(\Omega, \mathbb{R}^N)$ but then (6.2) holds everywhere and $\hat{u} \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ follows which completes the proof of Theorem 6.2.

Proof of Lemma 6.1. First note that $\nabla \hat{u} \in L^2_{\text{loc}}(\Omega, \mathbb{R}^N)$ follows from Lemma 4.2. As usual we argue by contradiction assuming that there exists a sequence of balls $B_{R_m}(x_m) \subset \Omega$ such that

$$|A_m| \leq L, \quad A_m = (\nabla \hat{u})_{x_m,R_m}, \quad \fint_{B_{R_m}(x_m)} |\nabla \hat{u} - (\nabla \hat{u})_{x_m,R_m}|^2 \, dx = \varepsilon_m^2 \to 0$$

and

$$\fint_{B_{\tau R_m}(x_m)} |\nabla \hat{u} - (\nabla \hat{u})_{x_m,\tau R_m}|^2 \, dx > C_0 \tau^2 \varepsilon_m^2.$$

Let $a_m = (\hat{u})_{x_m, R_m}$ and

$$v_m(z) = \frac{1}{\varepsilon_m R_m}\left(\hat{u}(x_m + R_m z) - a_m - R_m A_m z\right) \qquad (z \in B_1).$$

After passing to subsequences we can arrange (by writing our assumptions in terms of the scaled functions)

$$\begin{aligned}
&A_m \to A, \quad v_m \to v \ \text{ strongly in } \ L^2(B_1, \mathbb{R}^N)\\
&\nabla v_m \rightharpoondown \nabla v \quad \text{in} \ \ L^2(B_1, \mathbb{R}^{nN})\\
&\varepsilon \nabla v_m \to 0 \quad \text{in} \ \ L^2(B_1, \mathbb{R}^{nN}) \ \text{ and a.e. on } \ \Omega.
\end{aligned}$$

Here A denotes a suitable matrix in $\mathbb{R}^{nN}$ and v is some function from the space $W_2^1(B_1, \mathbb{R}^N)$. It is then easy to show that v is a solution of the following elliptic system with constant coefficients (see [7] for details):

$$\int_{B_1} D^2G(A)(\nabla v, \nabla \varphi)\, dz = 0 \qquad \text{for all } \ \varphi \in C_0^1(B_1, \mathbb{R}^N),$$

hence v is of class $C^\infty(B_1, \mathbb{R}^N)$ and satisfies the Campanato estimate

$$\fint_{B_r} |\nabla v - (\nabla v)_r|^2\, dz \le K r^2 \fint_{B_1} |\nabla v|^2\, dz$$

with K depending on n, N and on the modulus of ellipticity of $D^2G(A)$. Let us set $C_0 = 2K$. Then the proof of the lemma will be complete as soon as we can show

$$\nabla v_m \to \nabla v \qquad \text{strongly in } \ L^2_{\text{loc}}(B_1, \mathbb{R}^{nN}). \tag{6.3}$$

To this purpose we first observe the identity

$$\begin{aligned}
\varepsilon_m^2 \iint\limits_{B_1\,0}^{\quad 1} &\varphi D^2G\big(A_m + \varepsilon_m \nabla v + s\varepsilon_m(\nabla v_m - \nabla v)\big)\\
&\times (\nabla v_m - \nabla v, \nabla v_m - \nabla v)(1-s)\, ds dz\\
&= \int_{B_1} \varphi\big\{G(A_m + \varepsilon_m \nabla v_m) - G(A_m + \varepsilon_m \nabla v)\big\} dz\\
&\quad - \int_{B_1} \varepsilon_m \varphi DG(A_m + \varepsilon_m \nabla v) : (\nabla v_m - \nabla v)\, dz,
\end{aligned} \tag{6.4}$$

$\varphi \geq 0$ denoting some function in $C_0^1(B_1)$. The right-hand side of (6.4) equals

$$\left.\begin{aligned}
&\int_{B_1} G(A_m + \varepsilon_m \nabla v_m)\,dz \\
&\qquad - \int_{B_1} \Big\{(1-\varphi)G(A_m + \varepsilon_m \nabla v_m) + \varphi G(A_m + \varepsilon_m \nabla v)\Big\}\,dz \\
&\qquad - \int_{B_1} \varepsilon_m \varphi DG(A_m + \varepsilon_m \nabla v) : (\nabla v_m - \nabla v)\,dz \\
&\quad \leq \int_{B_1} G\big(A_m + \varepsilon_m \nabla[v_m + \varphi(v - v_m)]\big)\,dz \\
&\qquad - \int_{B_1} G\big(A_m + \varepsilon_m[\varphi \nabla v + (1-\varphi)\nabla v_m]\big)\,dz \\
&\qquad - \int_{B_1} \varepsilon_m \varphi DG(A_m + \varepsilon_m \nabla v) : (\nabla v_m - \nabla v)\,dz.
\end{aligned}\right\} \tag{6.5}$$

For the estimate in (6.5) we used the minimality of $\hat{u}$ together with convexity of G. Let $X_m = A_m + \varepsilon_m\big((1-\varphi)\nabla v_m + \varphi \nabla v\big)$. Then we may write

$$\begin{aligned}
&\int_{B_1} G\big(A_m + \varepsilon_m \nabla[v_m + \varphi(v - v_m)]\big)\,dz - \int_{B_1} G(X_m)\,dz \\
&\quad = \int_{B_1} G\big(X_m + \varepsilon_m \nabla\varphi \otimes (v - v_m)\big)\,dz - \int_{B_1} G(X_m)\,dz \\
&\quad = \varepsilon_m \int_{B_1} DG(X_m) : \big(\nabla\varphi \otimes (v - v_m)\big)\,dz \\
&\qquad + \varepsilon_m^2 \int_{B_1}\int_0^1 (1-s) D^2G\big(X_m + s\varepsilon_m \nabla\varphi \otimes (v - v_m)\big) \\
&\qquad \times \big(\nabla\varphi \otimes (v - v_m), \nabla\varphi \otimes (v - v_m)\big)\,ds\,dz,
\end{aligned}$$

the last integral being bounded by the quantity (recall $D^2G \in L^\infty$)

$$c_1 \varepsilon_m^2 \int_{B_1} |\nabla\varphi|^2 |v - v_m|^2\,dz.$$

Inserting this into (6.5) and returning to (6.4) we deduce

$$\begin{aligned}
&\text{l.h.s. of (6.4)} \\
&\quad \leq \varepsilon_m^2 c_1 \int_{B_1} |\nabla\varphi|^2 |v - v_m|^2\,dz + \varepsilon_m \int_{B_1} DG(X_m) : \big(\nabla\varphi \otimes (v - v_m)\big)\,dz \\
&\quad - \varepsilon_m \int_{B_1} \varphi DG(A_m + \varepsilon_m \nabla v) : (\nabla v_m - \nabla v)\,dz.
\end{aligned} \tag{6.6}$$

Next we observe

$$
\begin{aligned}
&\varepsilon_m \int_{B_1} DG(X_m) : (\nabla\varphi \otimes (v - v_m))\, dz \\
&\qquad - \varepsilon_m \int_{B_1} \varphi DG(A_m + \varepsilon_m \nabla v) : (\nabla v_m - \nabla v)\, dz \\
&\quad = \varepsilon_m \int_{B_1} \big[DG(X_m) - DG(A_m + \varepsilon_m \nabla v)\big] : (\nabla\varphi \otimes (v - v_m))\, dz \\
&\qquad - \varepsilon_m \int_{B_1} DG(A_m + \varepsilon_m \nabla v) : \nabla(\varphi[v_m - v])\, dz \\
&\quad = \varepsilon_m I_m^1 - \varepsilon_m I_m^2
\end{aligned}
$$

with

$$
\begin{aligned}
I_m^2 &= \int_{B_1} \big[DG(A_m + \varepsilon_m \nabla v) - DG(A_m)\big] : \nabla(\varphi[v_m - v])\, dz \\
&= \varepsilon_m \int_{B_1}\int_0^1 D^2G(A_m + s\varepsilon_m \nabla v)(\nabla v, \nabla(\varphi[v_m - v]))\, ds dz
\end{aligned}
$$

and

$$
\begin{aligned}
I_m^1 &= \varepsilon_m \int_{B_1}\int_0^1 D^2G\big(A_m + \varepsilon_m \nabla v + s\varepsilon_m(1-\varphi)(\nabla v_m - \nabla v)\big) \\
&\qquad \times \big(\nabla v_m - \nabla v, \nabla\varphi \otimes (v - v_m)\big)(1-\varphi)\, ds dz \\
&\le c_2 \varepsilon_m \int_{B_1} |\nabla v_m - \nabla v|\, |\nabla\varphi|\, |v - v_m|\, dz.
\end{aligned}
$$

We get from (6.6)

$$
\begin{aligned}
&\int_{B_1}\int_0^1 \varphi D^2G\big(A_m + \varepsilon_m \nabla v + s\varepsilon_m(\nabla v_m - \nabla v)\big)(\nabla v_m - \nabla v, \nabla v_m - \nabla v)(1-s)\, ds dz \\
&\qquad \le c_1 \int_{B_1} |\nabla\varphi|^2 |v - v_m|^2\, dz + c_2 \int_{B_1} |\nabla v_m - \nabla v|\, |\nabla\varphi|\, |v - v_m|\, dz \\
&\qquad\quad - \int_{B_1}\int_0^1 D^2G(A_m + s\varepsilon_m \nabla v)\big(\nabla v, \nabla(\varphi[v_m - v])\big)\, ds dz.
\end{aligned}
$$

The convergence properties of the sequence $\{v_m\}$ imply that the first two integrals on the right-hand side vanish as $m \to \infty$. For the third integral this follows from

$$
\int_0^1 D^2G(A_m + s\varepsilon_m \nabla v)(\nabla v, \cdot)\, ds \to D^2G(A)(\nabla v, \cdot)
$$

in $L^2_{\text{loc}}(B_1, \mathbb{R}^{nN})$ together with $\nabla(\varphi[v_m - v]) \to 0$ in $L^2(B_1, \mathbb{R}^{nN})$. Finally, we make use of the ellipticity condition (4.3) and get

$$0 = \lim_{m \to \infty} \int_{B_1} \varphi\big(1 + |A_m| + \varepsilon_m|\nabla v| + \varepsilon_m|\nabla v_m - \nabla v|\big)^{-\mu}|\nabla v_m - \nabla v|^2\, dz.$$

In order to prove (6.3) we fix a radius $0 < r < 1$. Using the local boundedness of ∇v we deduce after appropriate choice of φ

$$\int\limits_{B_r \cap [\varepsilon_m|\nabla v_m| \le M]} |\nabla v_m - \nabla v|^2\, dz \to 0 \qquad \text{as } m \to \infty \tag{6.7}$$

for any number $M > 0$. We introduce the auxiliary function

$$\varphi_m(z) = \frac{1}{\varepsilon_m}\Big\{(1 + |\varepsilon_m \nabla v_m + A_m|)^{1-\mu/2} - (1 + |A_m|)^{1-\mu/2}\Big\}.$$

By Lemma 4.1 φ_m is of class $W^1_{2,\text{loc}}(B_1)$, we further have $|\varphi_m| \le c_3|\nabla v_m|$ and Lemma 5.1 implies after scaling

$$\begin{aligned}\int_{B_r} |\nabla\varphi_m|^2\, dz &\le c_4(r) \int_{B_1} |D^2G(A_m + \varepsilon_m \nabla v_m)|\,|\nabla v_m|^2\, dz \\ &\le c_5(r) \int_{B_1} |\nabla v_m|^2\, dz \\ &= c_5(r),\end{aligned}$$

hence $\|\varphi_m\|_{W^1_2(B_r)} \le c_6(r)$. For M sufficiently large (depending on L but not on m) we have, for a.a. points in $B_r \cap [\varepsilon_m|\nabla v_m| > M]$,

$$\varphi_m \ge \frac{1}{2}\varepsilon_m^{-1}\varepsilon_m^{1-\mu/2}|\nabla v_m|^{1-\mu/2}, \qquad \text{hence} \qquad |\nabla v_m|^2 \le 2^{\frac{4}{2-\mu}}\varepsilon_m^{\frac{\mu}{2}\frac{4}{2-\mu}}\varphi_m^{\frac{4}{2-\mu}}.$$

The assumption $\mu < \frac{4}{n}$ implies $\frac{4}{2-\mu} < \frac{2n}{n-2}$ ($= \infty$, if $n = 2$). The latter inequality together with the uniform local W^1_2-bound for φ_m then shows that

$$\int\limits_{B_r \cap [\varepsilon_m|\nabla v_m| > M]} |\nabla v_m|^2\, dz \to 0, \qquad \text{hence} \qquad \int\limits_{B_r \cap [\varepsilon_m|\nabla v_m| > M]} |\nabla v_m - \nabla v|^2\, dz \to 0$$

as $m \to \infty$. Combining this with (6.7) claim (6.3) is established ∎

7. Some examples and applications to problems from fluid mechanics

In this section we briefly discuss some examples to which Theorems 6.1 and 6.2 apply.

Example 1. For $1 \le p < 2$ let $A(t) = t^p \ln(1+t)$ $(t \ge 0)$ and define

$$G(E) = \begin{cases} A(|E|) & \text{if } |E| \ge 1 \\ g_0(|E|) & \text{if } |E| \le 1 \end{cases} \qquad (E \in \mathbb{R}^{nN})$$

where g_0 is a quadratic polynomial on $\mathbb{R}$ chosen in such a way that G is of class C^2.

Lemma 7.1.

a) *For $s > 0$ sufficiently large we have $A^*(A'(s)) \le 2A(s)$.*

b) *Condition (4.3) is satisfied with $\mu = 2 - p$.*

Proof. Statement b) is immediate; for statement a) we just observe $A^*(A'(s)) = sA'(s) - A(s)$ ∎

Corollary 7.1. *Let u denote the minimizer to the integrand G defined above. Then we have full regularity for $n = 2$ and partial regularity if $p > 2 - \frac{4}{n}$, in particular, partial regularity holds, if $n = 3$.*

Proof. Clearly, statement a) of Lemma 7.1 implies (4.4), all other conditions required in Theorems 6.1 and 6.2 are obvious ∎

Example 2. For $t \ge 0$ let $A(t) = t \ln\big(1 + \ln(1+t)\big)$ and consider $G(E) = A(|E|)$ $(E \in \mathbb{R}^{nN})$.

Lemma 7.2.

a) *We have the inequality $A^*(A'(s)) \le 2A(s)$ for large enough s.*

b) *Condition (4.3) holds for any $\mu > 1$.*

Corollary 7.2. *Let u denote the associated minimizer. Then ∇u is partially Hölder continuous in dimensions 2 and 3.*

Example 3. For $0 < \alpha \le 1$ and $t \ge 0$ let

$$A(t) = \int_0^t s^{1-\alpha}(ar \sinh s)^\alpha ds = \int_0^t s^{1-\alpha} \ln(s + \sqrt{s^2+1})^\alpha ds.$$

In the case $n = N$ the associated energy density $G(E) = A(|E|)$ occurs as model for a certain generalized Newtonian fluid: let v denote the velocity field of an incompressible fluid and assume that v is time independent and also small. Then, in the case of the Sutterby fluid model (see [3]), v is the minimizer of $\int_\Omega G(\mathcal{E}(v))dx$ (+ potential terms) subject to the constraint $\operatorname{div} v = 0$ and for appropriate boundary conditions. Here $\mathcal{E}(v)$ is the symmetric gradient $\frac{1}{2}(\nabla v + \nabla v^T)$. For $\alpha = 1$ we obtain the Prandtl-Eyring model discussed in [8], hence we assume $\alpha < 1$.

Lemma 7.3.

a) *We have* $A^*(A'(s)) \leq 2A(s)$ *provided* s *is large enough.*

b) *Condition* (4.3) *is true for* $\mu = \alpha$.

Corollary 7.3. *Let* u *denote the minimizer from Theorem* 6.1 *or* 6.2 *with* G *defined above. Then we have regularity in the two-dimensional case and partial regularity if* $n = 3$. *Partial regularity holds for any* n *such that* $\alpha < \frac{4}{n}$.

Remark 7.1. With minor changes the above result extends to the fluid model where u has to satisfy $\operatorname{div} u = 0$ and is a minimizer of $\int_\Omega \int_0^{|\mathcal{E}(u)|} s^{1-\alpha} \ln(s + \sqrt{s^2+1})^\alpha ds$.

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