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Spectral Asymptotics for Variational Fractals

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Abstract. In this paper a generalization of a classic result of H. Weyl concerning the asymptotics of the spectrum of the Laplace operator is proved for variational fractals. Physically we are studying the density of states for the diffusion trough a fractal media. A variational fractal is a couple $(K,\mathcal{E})$ where K is a self-similar fractal and $\mathcal{E}$ is an energy form with some similarity properties connected with those of K. In this class we can find some of the most widely studied families of fractals as nested fractals, p.c.f. fractals, the Sierpinski carpet etc., as well as some regular self-similar Euclidean domains. We will see that if $r(x)$ is the number of eigenvalues associated with $\mathcal{E}$ smaller than x, then $r(x) \sim x^{\nu/2}$, where ν is the intrinsic dimension of $(K,\mathcal{E})$.

Keywords: *Fractals, Brownian motion, spectral dimension, spectral asymptotics*

AMS subject classification: Primary 47 B 06, secondary 60 J 65, 31 C 25

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following classic eigenvalues problem:

$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Δ is the usual Laplace operator in $\mathbb{R}^n$. It is well known that the eigenvalues, i.e. the real numbers λ such that problem (1.1) has a non-trivial solution, are a positive infinite discrete set, and have finite multiplicity.

A question that naturally occurs in the study of some physical phenomena, such as waves or diffusions problems is how the sequence of the eigenvalues goes to infinity.

An answer to this question has been given by H. Weyl in [25]. Define the density of states $r^0(x)$ as the number, with multiplicity, of eigenvalues of problem (1.1) smaller than x. Then Weyl's celebrated formula states that

$$r^0(x) = \frac{1}{(2\pi)^n} B_n |\Omega| x^{\frac{n}{2}} + o(x^{\frac{n}{2}}), \tag{1.2}$$

as $x \to +\infty$. Here B_n is the measure of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ is the measure of Ω.

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In this paper a similar result is proved for a large class of fractal sets, the class of variational fractals. The main difficulty in considering the analogue of problem (1.1) on a fractal, is the definition of the Laplace operator on the fractal. Because non-trivial fractals are typically not differentiable structures, the question is what is the Laplace operator on a fractal? A probabilistic way to answer this question is to say that the Laplace operator is the generator of the Brownian motion on the fractal. However in this way we have to say what is the Brownian motion on a fractal. This approach to the problem has been carried out by several authors (see, for example, [3, 10, 14] where a Brownian motion is constructed and studied on the Sierpinski gasket. Later Lindstrøm in [16] constructed a Brownian motion on a big class of fractal objects, the class of nested fractals. The procedure used by these authors is to obtain the Brownian motion as a renormalized limit of random walks. Other examples of construction of Brownian motions on different fractals are [2, 12]. So we know that it is possible to consider a "Laplace operator" on some fractals. The analogue of formula (1.2) should be the same formula with the Hausdorff dimension d_f instead of n and the Hausdorff measure instead of the Lebesgue measure.

Unfortunately, this is wrong and it is wrong for a rather simple fact. The fact is that the Hausdorff dimension, which is a purely static object can not be the intrinsic dimension of the fractal. The notion of intrinsic dimension for an object is strictly linked to the notion of intrinsic metric on it, which is linked again to the existence of an energy (i.e. an intrinsic Laplace operator) on the object. This is true for Riemannian geometry, where the metric tensor and the Laplace-Beltrami operator are strictly correlated, and it holds also for the fractals [5, 20 - 22].

The result we will prove in the present paper, namely Theorem 2.2, states that

$$0 < \liminf_{x \to +\infty} r^0(x) x^{-\frac{\nu}{2}} \leq \limsup_{x \to +\infty} r(x) x^{-\frac{\nu}{2}} < +\infty, \tag{1.3}$$

where ν should be interpreted as the intrinsic dimension of the fractal structure. Because we use the correct intrinsic dimension of the fractal, our proof works also for some regular domains of $\mathbb{R}^n$, as cubes, which are some trivial examples of variational fractals.

Before ending the present section we have to do some remarks. The first is that in (1.3) we have lost the constants that appeared in (1.2). This is inevitable if we want a result which holds in general for variational fractals. In fact, it is proved, for example, that no equality holds in (1.3) in the case of the Sierpinski gasket [9].

Second, similar theorems have been proved by other authors for different classes of fractals. We refer particularly to [7, 13]. We discuss more extensively the relation of our result with other results in Section 3. However, we want to stress some facts about our result.

Theorem covers partially and sometimes completely, analog theorems existing in the literature. We postpone to Section 3 the discussion on the relationship between the present paper and similar earlier papers. Furthermore, where for other authors the exponent ν in (1.3) is the spectral dimension of the fractal, which means a number for which (1.3) holds, we have the geometric interpretation of ν as intrinsic dimension of the fractal.

Acknowledgments. I am in strong debt with U. Mosco who introduced me to the study of Dirichlet forms on fractals, and helped me, in every phase of the preparation of this paper, with many constructive discussions.

2. Notation and results

In this section we will give the basic notation and state our main result.

2.1 Notation. We recall some notations from [20] about homogeneous fractal spaces.

2.1.1 Homogeneous spaces. A *homogeneous space* is a pair (K,d), where K is a topological space and d is a pseudo-distance on K, such that the following assumption are satisfied:

(i) The pseudo-balls $B(x,r)$ form a basis of open neighborhoods of K.

(ii) There exists two positive constants ν and c, such that for every $r > 0$ and $\varepsilon \in (0,1)$, the ball $B(x,r)$ contains at most $c\varepsilon^{-\nu}$ points whose mutual distance is greater than εr.

We recall that, by definition, a pseudo-distance satisfies the assumptions of a distance, with possibly a multiplicative constant $c_T \geq 1$ in the triangle inequality. A sufficient condition to satisfy (ii) is the following:

(ii)$'$ There exists a Radon measure μ on K and a positive constant c_0 such that:

$$0 < c_0\, \mu(B(x,R)) \left(\frac{r}{R}\right)^{\nu} \leq \mu(B(x,r)),$$

for every $x \in K$ and $0 < r \leq R$. If the opposite inequality also holds for some positive constant c', then we say that ν is the *homogeneous dimension* of (K,d).

2.1.2 Energy forms. Let K be a separable Hausdorff topological space and μ a Radon measure on it. A strongly local regular Dirichlet form $(\mathcal{G},\mathcal{D})$ with domain $\mathcal{D} \subset L^2(K,\mu)$ will be called an *energy form* on K (cf. [8, 19]). It is well known [8: Corollary 1.3.1] that it is possible to associate to every Dirichlet form $(\mathcal{G},\mathcal{D})$ a non-positive self-adjoint Markov generator G on $L^2(K,\mu)$. The correspondence is determined by

$$\begin{aligned} &\mathrm{Dom}(G) \subset \mathcal{D} \\ &\mathcal{G}(u,v) = -\langle Gu, v\rangle_{L^2(K,\mu)} \quad (u \in \mathrm{Dom}(G),\ v \in \mathcal{F}). \end{aligned}$$

In this way when we speak about eigenvalues, eigenfunctions, resolvent of the form $(\mathcal{G},\mathcal{D})$ we mean these objects for the associated Markov generator. For example, an eigenvalue of the form $(\mathcal{G},\mathcal{D})$ is a real number λ for which there exists $u \in \mathcal{D}$, called the *eigenfunction* associated with λ, such that $\mathcal{G}(u,v) = \lambda \int_K uv\, d\mu$ for every $v \in \mathcal{D}$.

If the eigenvalues of the generator associated with $(\mathcal{G},\mathcal{D})$ have finite multiplicities and form a discrete set, then the following variational characterization of the eigenvalues holds:

Theorem 2.1 (Min-Max). *Let $(\mathcal{G}, \mathcal{D})$ be a Dirichlet form such that its eigenvalues have finite multiplicities and form a discrete set. Let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ be the ordered sequence of its eigenvalues. Then*

$$\lambda_k = \max_{v_1,\ldots,v_{k-1} \in L^2(K,\mu)} \min_{\substack{u \perp v_1,\ldots,v_{k-1} \\ u \in \mathcal{D}}} \frac{\mathcal{G}[u]}{\|u\|^2} \tag{2.1}$$

for $k \geq 1$.

For the proof see [24: Theorem XIII.2]. For further informations about Dirichlet forms we refer to [8, 19].

2.1.3 Self-similar fractals. We introduce the notion of self-similar fractals in the sense of [11]. Let $\mathbb{R}^D$ be the D-dimensional Euclidean space, $D \geq 1$, $|x - y|$ the Euclidean distance and $B_e(x, r)$ the Euclidean ball centered at $x \in \mathbb{R}^D$ of radius $r \geq 0$. Let $\Psi = \{\psi_1, \ldots, \psi_N\}$ be N given maps $\psi_i : \mathbb{R}^D \to \mathbb{R}^D$ such that:

(i) There exists real numbers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N > 1$ such that

$$|\psi_i(x) - \psi_i(y)| = \alpha_i^{-1}|x - y|$$

for every $x, y \in \mathbb{R}^D$ and $i = 1, \ldots, N$.

(ii) There exists a non-empty bounded open set $V \subset \mathbb{R}^D$ such that

$$\bigcup_{i=1}^{N} \psi_i(V) \subset V$$

and $\psi_i(V) \cap \psi_j(V) = \emptyset$ if $i \neq j$.

Under these assumption we have [11: Section 5.3/Theorem (1)]

(i) There exists a unique non-empty compact set $K \subset \mathbb{R}^D$, the *invariant set of* Ψ, such that

$$K = \bigcup_{i=1}^{N} \psi_i(K).$$

(ii) There exists a unique probability measure μ on K, the *invariant measure of* Ψ, such that

$$\mu = \sum_{i=1}^{N} \alpha_i^{-d_f} (\psi_i)_{\#}(\mu) \tag{2.2}$$

where $(\psi_i)_{\#}(\mu)(\cdot) \equiv \mu(\psi_i^{-1}(\cdot))$, and d_f, the *fractal dimension* of K, is defined as the unique solution of

$$\sum_{i=1}^{N} \alpha_i^{-d_f} = 1. \tag{2.3}$$

It is possible to prove that d_f is the Hausdorff dimension of K, while μ is the d_f-Hausdorff measure on $\mathbb{R}^D$ restricted to K and normalized.

For arbitrary n-ples of indices $i_1, \ldots, i_n \in \{1, \ldots, N\}$ we define

$$\psi_{i_1 \cdots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n} \qquad \text{and} \qquad K_{i_1 \cdots i_n} = \psi_{i_1 \cdots i_n}(K).$$

We call $K_{i_1 \cdots i_n}$ an *n-complex*. The *boundary* of K [20] is defined as

$$\Gamma = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{N} \psi_i^{-1}(K_i \cap K_j).$$

This is a closed subset of K, $\mu(K_i \cap K_j) = 0$ for $i \neq j$, which yields $\mu(\Gamma) = 0$ (cf. [18, 20]). Notice that if we use (2.2) to calculate $\mu(K_i)$, we obtain $\mu(K_i) = \alpha_i^{-d_f}$ $(i = 1, \ldots, N)$ so that (2.2) can be rewritten as

$$\mu = \sum_{i=1}^{N} \mu(K_i)(\psi_i)_{\#}(\mu). \tag{2.4}$$

An equivalent formulation of this formula is

$$\int_K f \, d\mu = \sum_{i=1}^{N} \mu(K_i) \int_K (f \circ \psi_i) \, d\mu$$

which holds for every $f \in L^1(K, \mu)$. If we take $f = g 1_{K_i}$ for $g \in L^1(K, \mu)$ and $i = 1, \ldots, N$, we obtain

$$\int_K (g \circ \psi_i) \, d\mu = \frac{1}{\mu(K_i)} \int_{K_i} g \, d\mu$$

which gives a "change of variables formula" for the measure μ.

2.1.4 Variational fractals. Suppose that together with the self-similar fractal K, the invariant set of a given family $\Psi \equiv \{\psi_1, \ldots, \psi_N\}$ and its invariant measure μ, also an energy form $(\mathcal{E}, \mathcal{F})$ on K is given. We will further assume the following self-similar property for the energy:

There exists $\sigma < 1$ such that

$$\mathcal{E}[u] = \sum_{i=1}^{N} \mu(K_i)^{\sigma} \mathcal{E}[u \circ \psi_i] \tag{2.6}$$

for every $u \in \mathcal{F}$.

Then the pair $(K, \mathcal{E})$ is called a *variational fractal* [20]. The scaling factors α_i, N and σ for length mass and energy are the basic "physical" constants for a variational fractal.

2.1.5 Homogeneous fractal spaces. Given a variational fractal $(K, \mathcal{E})$, define the pseudo-distance $d(x, y) = |x - y|^{\delta}$ on K by requiring that d^2 scales like the energy:

$$d^2(x, y) = \sum_{i=1}^{N} \mu(K_i)^{\sigma} d^2(\psi_i(x), \psi_i(y))$$

for $x, y \in K$. This determines δ as the unique solution of

$$\sum_{i=1}^{N} \mu(K_i)^{\sigma} \alpha_i^{-2\delta} = 1.$$

One of the fundamental results of [20] says that in these notation the pair (K, d) is a homogeneous space of homogeneous dimension $\nu = \delta^{-1} d_f$. The scaling factor of the energy is linked to the homogeneous dimension of the fractal by the identity

$$\sigma = \frac{\nu - 2}{\nu}. \tag{2.7}$$

The homogeneous dimension ν is the *intrinsic dimension* of the variational fractal, while d, defined above, is the *intrinsic (pseudo-)metric* on the fractal. These concepts have been introduced by U. Mosco in [20]. We refer to this paper and to [17, 21] for a complete explanation of these topics both from a mathematical and physical point of view.

2.2 Main result. Let $(K, \mathcal{E})$ be a variational fractal, with energy form $(\mathcal{E}, \mathcal{F})$. Define $\mathcal{F}^0$ as the closure in the norm $\sqrt{\mathcal{E}[\cdot] + \|\cdot\|_{L^2(K,\mu)}}$ of $\mathcal{F} \cap C_0(K)$, where $C_0(K)$ is the space of continuous functions f on K such that $\mathrm{Supp}(f) \cap \Gamma = \emptyset$.

We define the *density of states* $r(x)$ with Neumann (or reflecting) boundary conditions as the number of eigenvalues of $(\mathcal{E}, \mathcal{F})$, with their multiplicity, lesser than equal to x.

Conversely, we define the densities of states $r^0(x)$ with Dirichlet (or absorbent) boundary conditions as the number of eigenvalues of $(\mathcal{E}, \mathcal{F}^0)$, with their multiplicity, lesser than equal to x.

With these definition the densities of states are increasing possibly infinite real functions. We will assume that the domain $\mathcal{F}$ of the form is compactely embedded in $L^2(K, \mu)$. This implies by [24:Theorem XIII.64] that the eigenvalues of $(\mathcal{E}, \mathcal{F})$ have finite multiplicities and form a discrete set, so that both the densities are finite functions.

Now we are in a position to state our main result.

Theorem 2.2. *Let $(K, \mathcal{E})$ be a variational fractal with energy form $(\mathcal{E}, \mathcal{F})$ and homogeneous dimension ν. Suppose $\mathcal{F}$ is compactely embedded in $L^2(K, \mu)$. Then there exist three strictly positive constants c_1, c_2 and $\bar{x}$ such that*

$$c_1 x^{\nu/2} \leq r^0(x) \leq r(x) \leq c_2 x^{\nu/2} \tag{2.8}$$

for every $x \geq \bar{x}$.

This theorem, in some sense, generalizes the classic result of H. Weyl (1.2). However, as was pointed out at the end of Section 1 in Theorem 2.2, we have lost the constants that appeared in (1.2).

3. Applications

In this section we will give some example of variational fractals.

3.1 Euclidean Domains. There is a large collections of regular domain in $\mathbb{R}^D$ which are self-similar and variational fractals but are not fractals in a strict sense, i.e. their Euclidean dimension, Hausdorff dimension and intrinsic (homogeneous) dimension are the same. The simplest example of this class is the cube in $\mathbb{R}^D$.

Let $D \geq 1$ and define $Q = [0,1]^D \subset \mathbb{R}^D$. Fix an integer $\alpha \geq 2$, and define $N = \alpha^D$ ($\alpha_i = \alpha$ for $i = 1, \ldots, N$). Then Q is a self-similar fractal, in the sense of Subsection 2.1.3, if we think to Q as the union of N small cubes of size α^{-1}. Notice that in this case, by (2.3), we have $d_f = D$. Furthermore, Q becomes a variational fractal, in the sense of Subsection 2.1.4 , if we equip it with the energy

$$\mathcal{E}[u] = \int_Q |\nabla u(x)|^2 dx$$

defined for every u in the Sobolev space $H^1(Q^0)$. In this case $\sigma = \frac{D-2}{2}$, $\delta = 1$ and $\nu = D$. Theorem 2.2 is contained in [6: Chapter VI.4].

Other regular domain which are variational fractals are for example tetrahedrons.

3.2 Nested fractals. Nested fractals have been introduced from Lindstrøm in [16]. These are self-similar fractals, in the sense of Subsection 2.1.3, with the following extra requirement, called *nesting*:

- There exists a finite set $F \subset K$ such that

$$K_{i_1,\ldots,i_n} \cap K_{j_1,\ldots,j_n} = F_{i_1,\ldots,i_n} \cap F_{j_1,\ldots,j_n}$$

for every $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)$.

In the family of the nested fractals we find the Sierpinski gasket, the Koch curve, the Lindstrøm snowflake etc. (cf. [16]). The nesting hypothesis implies that nested fractals are finitely ramified fractals, which roughly speaking means that it is possible to disconnect them by removing a finite number of points of the fractal.

For nested fractals we have $\alpha_i \equiv \alpha > 1$ for $i = 1, \ldots, N$, so that, by using (2.3), $d_f = \frac{\log N}{\log \alpha}$.

An energy form on a nested fractal is introduced by considering the Dirichlet form associated with the Markov generator associated with a standard diffusion on K (cf. [16]). With this energy the fractal becomes a variational fractal, in the sense of Subsection 2.1.4, with $\sigma < 0$, $\delta = \frac{d_f(1-\sigma)}{2}$ and $\nu = \frac{2}{1-\sigma} < 2$. The intrinsic dimension ν is equal to the spectral dimension d_s found by Fukushima in [7]. Our Theorem 2.2 is, in the case of nested fractals, equivalent to [7: Theorem 4.1].

However, we have a clear geometric interpretation of the spectral dimension d_s as the intrinsic (homogeneous) dimension of the fractal. This fact was already pointed out in [5, 22].

3.3 P.C.F. fractals. P.C.F. fractals have been introduced by J. Kigami in [12]. They are described in an abstract way as quotient spaces of of the free semigroup on a finite alphabet. They have an extra requirement of finitely ramified type called post critical finiteness. We do not discuss here the P.C.F. hypothesis, for which we refer to [12, 13], neither we make any example, but we recall that the class of nested fractals is contained in the class of P.C.F. fractals as a particular case and that P.C.F. fractals are finitely ramified.

A P.C.F. fractal K reduces to a self-similar fractals, in the sense of Subsection 2.1.3, if we assume the following:

- $K \subset \mathbb{R}^D$.
- The self-similar maps F_i (cf. [13: Definition 1.1]) satisfy the requirements i) and ii) for the ψ_i given in Subsection 2.1.3.
- The Bernoulli measure considered on it is the invariant measure μ introduced in Subsection 2.1.3.

Energy forms on P.C.F. fractals have been introduced by J. Kigami and M. L. Lapidus in [13] by means of regular harmonic structures.

There is not a general proof of the existence of a regular harmonic structure for a P.C.F. fractal (neither there is a general construction for a P.C.F. fractal structure), but there are many examples of regular harmonic structures in several concrete cases [13]. If a regular harmonic structure exists on a P.C.F. fractal K, then it is possible to introduce an energy form on K by using it.

The relevant fact is that the energy form associated to a regular harmonic structure is subject to a scaling law equivalent to (2.6) [13: Lemma 6.1 and (A.4)]. This reduces a P.C.F. fractal, with the restriction stated above, to a variational fractal, for which Theorem 2.2 holds.

There could be more than one harmonic structure on a P.C.F. fractal, but if we consider the harmonic structure with the scaling law (2.6) (cf. [13: Equation (A.4)]) we obtain again Theorem 2.2.

In fact [13: Lemma 6.1 and (A.4)] state that

$$\mathcal{E}[u] = \sum_{i=1}^{N} \mu_i^{-1/S} \mathcal{E}[u \circ F_i]$$

where $\mu_i = \mu(K_i)$. This relation is equivalent to our (2.6) and yields $S = -\frac{1}{\sigma}$. Then spectral dimension is defined by [13: Equation (A.5)] as

$$d_S^* = \frac{2S}{S+1}.$$

Recalling our definition (2.7) of σ and that $S = -\frac{1}{\sigma}$, we obtain $\nu = d_S^*$. So our intrinsic dimension ν is the spectral dimension founded by J. Kigami and M. L. Lapidus [13:

Theorem A.2].

[13]	the present paper
N	N
F_i	ψ_i
$\rho_{(0)}(x)$	$r^{(0)}(x)$
λ/r_i	$\mu(K_i)^\sigma$
S	$-1/\sigma$
$d_S^* = \frac{2S}{S+1}$	ν

Table 1: Glossary

Table 1 is a small glossary to compare terms used in the present paper with those used in 13].

3.4 The Sierpinski carpet. The Brownian motion on the Sierpinski carpet and the spectral properties of its generator have been widely studied by several authors. See, for example, [2, 4] and references in these papers. Here we want to stress only that the Sierpinski carpet is a variational fractal, in the sense of Subsection 2.1.4 , and the hypothesis of Theorem 2.2 are fullfilled.

Following the notation of Subsection 2.1.3 take $D = 2$, $N = 8$, $\alpha_i = 3$ for $i = 1, \dots, 8$ and define the similarity maps

$$\psi_i(x) = \frac{1}{3}(x - x_i) + x_i \qquad (i = 1, \dots, 8)$$

where

$$x_1 = (0,0), \quad x_2 = (0, \frac{1}{2}), \quad x_3 = (0,1), \quad x_4 = (\frac{1}{2}, 1)$$
$$x_5 = (1,1), \quad x_6 = (1, \frac{1}{2}), \quad x_7 = (1,0), \quad x_8 = (\frac{1}{2}, 0).$$

The associated invariant set K is the Sierpinski carpet.

In order to give to K the structure of variational fractal we equip it with a energy form $\mathcal{E}$ satisfying (2.6). The existence of such a form has been proved S. Kusuoka and Z. X. Yin in [15: Section 8/Example 1]. The couple $(K, \mathcal{E})$ becomes a variational fractal, and so Theorem 2.2 holds for the Sierpinski carpet.

4. Proof of main result

In this section we will prove our main result, Theorem 2.2. The proof we are going to give follows closely the classical proof of the original Weyl's Theorem given in [6]. In particular, only intrinsic physics characteristics of the fractal, and their scaling laws, enter in the proof. As in the classical proof (cf. [6: Chapter VI.4]), only the scaling properties of mass and energy are used to prove (2.8).

For $i = 1, \dots, N$ define the Dirichlet forms $(\mathcal{E}_i, \mathcal{F}_i)$ as

$$\mathcal{E}_i[u] = \mu(K_i)^\sigma \mathcal{E}[u \circ \psi_i] \tag{4.1}$$

for every $u \in \mathcal{F}_i = \left\{u \in L^2(K_i, \mu_{|K_i}) : u \circ \psi_i \in \mathcal{F}\right\}$. Notice that $(K_i, \mathcal{E}_i)$ for $i = 1, \ldots, N$ are variational fractals. With this notation (2.6) becomes

$$\mathcal{E}[u] = \sum_{i=1}^{N} \mathcal{E}_i[u_{|K_i}]$$

for every $u \in \mathcal{F}$.

The following elementary result gives the scaling rates of the eigenvalues of the energy form in term of the intrinsic dimension of the fractal K.

Lemma 4.1. *A real number λ is an eigenvalue for $(\mathcal{E}, \mathcal{F}^{(0)})$ if and only if $\lambda\mu(K_i)^{-\frac{2}{\nu}}$ is an eigenvalue for $(\mathcal{E}_i, \mathcal{F}_i^{(0)})$ for every $i = 1, \ldots, N$. Furthermore, the multiplicity is preserved.*

Proof. We will prove the lemma only for the form $(\mathcal{E}, \mathcal{F})$, being the case $(\mathcal{E}, \mathcal{F}^0)$ identical. We start proving the "only if part". Let $u \in \mathcal{F}$ be an eigenfunction of $(\mathcal{E}, \mathcal{F})$ associated with the eigenvalue λ. For a fixed $i = 1, \ldots, N$ define the surjective map

$$\mathcal{F} \ni v \mapsto \hat{v}_i = v \circ \psi_i^{-1} \in \mathcal{F}_i. \tag{4.2}$$

For $v \in \mathcal{F}$, by definition (4.1) of the form $(\mathcal{E}_i, \mathcal{F}_i)$, we obtain

$$\mathcal{E}_i(\hat{u}_i, \hat{v}_i) = \mu(K_i)^\sigma \mathcal{E}(\hat{u}_i \circ \psi_i, \hat{v}_i \circ \psi_i) = \mu(K_i)^\sigma \mathcal{E}(u, v).$$

Now because u is an eigenfunction of the form $(\mathcal{E}, \mathcal{F})$, the previous equation yields

$$\begin{aligned}
\mathcal{E}_i(\hat{u}_i, \hat{v}_i) &= \lambda\mu(K_i)^\sigma \int_K uv \, d\mu \\
&= \lambda\mu(K_i)^\sigma \int_K (\hat{u}_i \circ \psi_i)(\hat{v}_i \circ \psi_i) \, d\mu \\
&= \lambda\mu(K_i)^{\sigma-1} \int_{K_i} \hat{u}_i \hat{v}_i d\mu \\
&= \lambda\mu(K_i)^{-2/\nu} \int_{K_i} \hat{u}_i \hat{v}_i d\mu
\end{aligned}$$

where we used (2.5). Because the map (4.2) is surjective the previous relation implies that $\lambda\mu(K_i)^{-2/\nu}$ is an eigenvalue of $(\mathcal{E}_i, \mathcal{F}_i)$ associated with $\hat{u}_i$. This concludes the proof of the "only if part" of the lemma.

In order to prove the "if part" let $\lambda\mu(K_i)^{-2/\nu}$ be an eigenvalue of the form $(\mathcal{E}_i, \mathcal{F}_i)$ associated with the eigenfunction $u \in \mathcal{F}_i$. This means that

$$\begin{aligned}
\mu(K_i)^\sigma \mathcal{E}(u \circ \psi_i, v \circ \psi_i) &= \mathcal{E}_i(u, v) \\
&= \lambda\mu(K_i)^{-2/\nu} \int_{K_i} uv \, d\mu_{|K_i} \\
&= \lambda\mu(K_i)^{1-2/\nu} \int_K (u \circ \psi_i)(v \circ \psi_i) \, d\mu \\
&= \lambda\mu(K_i)^\sigma \int_K (u \circ \psi_i)(v \circ \psi_i) \, d\mu
\end{aligned}$$

for every $v \in \mathcal{F}_i$. Because $\mathcal{F}_i \ni v \mapsto v \circ \psi_i \in \mathcal{F}$ is surjective the previous relation implies that λ is an eigenvalue of $(\mathcal{E}, \mathcal{F})$ associated with $u \circ \psi_i$. This concludes the proof of the "if part" of the lemma. It is easy to check that that multiplicities are preserved ∎

The next lemma is the key step in proving Theorem 2.2. It gives a recursive estimate of the densities of states.

Lemma 4.2. *Let $x \geq 0$. Then*

$$\sum_{i=1}^{N} r^0(x\mu(K_i)^{2/\nu}) \leq r^0(x) \leq r(x) \leq \sum_{i=1}^{N} r(x\mu(K_i)^{2/\nu}). \tag{4.3}$$

Before proving this result we will show how Theorem 2.2 follows from it.

Proof of Theorem 2.2. We start by proving the last inequality in (2.8). Define $\mu_i = \mu(K_i)$ for $i = 1, \ldots, N$. We can assume that

$$0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_N < 1.$$

Define $x_0 = 1$ and $x_1 = x_0\mu_1^{-2/\nu} > x_0$. Because $r(x)$ is bounded on finite intervals, there exists $c > 0$ such that

$$r(x) \leq cx^{\nu/2} \tag{4.4}$$

for every $x \in [x_0, x_1]$. Now define $x_2 = x_1\mu_N^{-2/\nu} > x_1$. Then for every $x \in [x_1, x_2]$ we have

$$x_0 \leq x_0 \left(\frac{\mu_i}{\mu_1}\right)^{2/\nu} \leq x_1\mu_i^{2/\nu} \leq x\mu_i^{2/\nu} \leq x_1 \left(\frac{\mu_i}{\mu_1}\right)^{2/\nu}$$

for every $i = 1, \ldots, N$, i.e. $x\mu_i^{2/\nu} \in [x_0, x_1]$ for every $i = 1, \ldots, N$. Lemma 4.2 and (4.4) yields

$$r(x) \leq \sum_{i=1}^{N} r(x\mu_i^{2/\nu}) \leq cx^{\nu/2} \sum_{i=1}^{N} \mu_i = cx^{\nu/2} \tag{4.5}$$

for every $x \in [x_1, x_2]$.

Now we proceed inductively defining $x_{n+1} = x_n\mu_N^{-2/\nu}$, for every integer $n \geq 2$. Suppose that

$$r(x) \leq cx^{\nu/2} \tag{4.6}$$

for every $x \in [x_1, x_n]$. It is elementary to check that if $x \in [x_1, x_{n+1}]$, then $x\mu_i^{2/\nu} \in [x_1, x_n]$. Lemma 4.2 and the inductive hypothesis (4.6) imply that (4.5) holds for every $x \in [x_1, x_{n+1}]$. Obviously, $\lim_{n\to+\infty}[x_1, x_n] = [x_1, +\infty)$, this means that $r(x) \leq cx^{\nu/2}$ for every $x \geq x_1$.

The proof of the first inequality in (2.8) is identical, while the proof of the second one is contained in Lemma 4.2. ∎

Proof of Lemma 4.2. We start proving the last inequality in (4.3). Define the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ as

$$\begin{aligned}\tilde{\mathcal{F}} &= \left\{f \in L^2(K,\mu) : f_{|K_i} \in \mathcal{F}_i \text{ for all } i = 1, \ldots, N\right\} \\ &= \left\{f \in L^2(K,\mu) : f_{|K_i} \circ \psi_i \in \mathcal{F} \text{ for all } i = 1, \ldots, N\right\}\end{aligned}$$

and

$$\tilde{\mathcal{E}}[u] = \sum_{i=1}^{N} \mu(K_i)^{\sigma} \mathcal{E}[u \circ \psi_i] = \sum_{i=1}^{N} \mathcal{E}_i[u_{|K_i}].$$

It is clear that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is an extension of $(\mathcal{E}, \mathcal{F})$. This form represents the fractal media K where the subdomains $K_1, \ldots, K_N$ have been artificially separated.

We claim that the totality of eigenvalues and eigenfunctions of the form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ consists of the eigenvalues and eigenfunctions of the forms $(\mathcal{E}_i, \mathcal{F}_i)$, for $i = 1, \ldots, N$, where each eigenfunction $u \in \mathcal{F}_i$ is extended to the whole K defining $u(x) = 0$ for $x \notin K_i$ (this fact is physically evident and reflects the fact that several separated vibrating systems performs vibrations without interacting with each other, cf. [6: Chapter VI.4]).

In order to prove mathematically this statement consider an eigenfunction u of $(\mathcal{E}_i, \mathcal{F}_i)$ associated to the eigenvalue λ. Then we can define $\tilde{u} \in \tilde{\mathcal{F}}$ by $\tilde{u}(x) = u(x)$ if $x \in K_i$ and $\tilde{u}(x) = 0$ if $x \notin K_i$. It is clear that such a function belong to $\tilde{\mathcal{F}}$, furthermore

$$\tilde{\mathcal{E}}(\tilde{u}, v) = \mathcal{E}_i(u, v_{|K_i}) = \lambda \int_{K_i} u v_{|K_i} \, d\mu = \lambda \int_K \tilde{u} v \, d\mu$$

for every $v \in \tilde{\mathcal{F}}$. This means that every eigenfunction of $(\mathcal{E}_i, \mathcal{F}_i)$ for $i = 1, \ldots, N$ gives an eigenfunction of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$, and it is clear that the multiplicity is preserved.

Conversely, if u is an eigenfunction of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ associated with the eigenvalue λ, then $u = \sum_{i=1}^{N} u 1_{K_i}$. It easy to check that $u 1_{K_i}$ for every $i = 1, \ldots, N$ is an eigenfunctions of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ associated to λ which vanish outside K_i. This means that we may consider the totality of the eigenfunctions of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ as constituted of functions supported in only one K_i for $i = 1, \ldots, N$.

It is clear that if u is an eigenfunction of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ associated to λ and supported in K_i, then

$$\mathcal{E}_i(u_{|K_i}, v) = \tilde{\mathcal{E}}(u, \tilde{v}) = \lambda \int_K u \tilde{v} d\mu = \lambda \int_{K_i} u_{|K_i} v \, d\mu,$$

i.e. $u_{|K_i}$ is an eigenfunction of $(\mathcal{E}_i, \mathcal{F}_i)$ associated with the same eigenvalue.

Define the density of the states $\tilde{r}(x)$ of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ in the obvious way (cf. beginning of Subsection 2.2). Then the property we have proved can be summarized in

$$\tilde{r}(x) = \sum_{i=1}^{N} r_i(x). \tag{4.7}$$

Now, because $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is an extension of $(\mathcal{E}, \mathcal{F})$,

$$\max_{v_1, \ldots, v_{k-1} \in L^2(K)} \min_{u \perp v_1, \ldots, v_{k-1} \atop u \in \tilde{\mathcal{F}}} \frac{\tilde{\mathcal{E}}[u]}{\|u\|^2} \leq \max_{v_1, \ldots, v_{k-1} \in L^2(K)} \min_{u \perp v_1, \ldots, v_{k-1} \atop u \in \mathcal{F}} \frac{\mathcal{E}[u]}{\|u\|^2},$$

thus by the Min-Max principle, Theorem 2.1 we obtain $r(x) \leq \tilde{r}(x)$. This relation together with (4.7) yields

$$r(x) \leq \sum_{i=1}^{N} r_i(x).$$

To complete the proof of the last inequality in (4.3) it suffices to notice that by Lemma 4.1 it follows that $r_i(x) = r(x\mu(K_i)^{2/\nu})$. The second inequality in (4.3) is an immediate consequence of the Min-Max principle, Theorem 2.1 , because $\mathcal{F}^0 \subset \mathcal{F}$. The first inequality in (4.3) can be proved in the same manner of the last one. Define the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}^0)$ where

$$\begin{aligned}\tilde{\mathcal{F}}^0 &= \left\{ f \in L^2(K,\mu) : f_{|K_i} \in \mathcal{F}_i^0 \text{ for all } i = 1,\dots,N \right\} \\ &= \left\{ f \in L^2(K,\mu) : f_{|K_i} \circ \psi_i \in \mathcal{F}^0 \text{ for all } i = 1,\dots,N \right\}\end{aligned}$$

and notice that $\tilde{\mathcal{F}}^0 \subset \mathcal{F}^0$. The proof is identical to that of the last inequality in (4.3), replacing $\tilde{\mathcal{F}}$ with $\tilde{\mathcal{F}}^0$. This repetition is omitted ∎

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