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An Analysis of the Block Structure of Certain Subclasses of j_{qq}-Inner Functions

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Abstract. This paper is aimed at characterizing distinguished subclasses of j_{qq}-inner functions in terms of connected pairs of matrix-valued Hardy class functions. Moreover, an inverse problem for a given pair of matrix-valued Carathéodory functions is discussed.

Keywords: *j_{qq}-inner functions, connected pairs of matrix-valued Hardy class functions, Carathéodory functions generated by a j_{qq}-inner function*

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0. Introduction

The study of the class of j_{pq}-inner functions has been emphasized by several matricial generalizations of classical interpolation problems of Schur-Nevanlinna-Pick type. This is caused by the fact that the set of solutions of such interpolation problems can be parametrized with the aid of linear fractional transformations the generating matrix-valued functions of which are j_{pq}-inner functions appropriately constructed from the given data (see, e.g., [3, 10, 12] and the references cited). The converse question of constructing interpolation problems such that their solution set can be parametrized by a given j_{pq}-inner function in the way described above was studied in [3, 7]. Inverse problems for j_{pq}-inner functions with prescribed block information were treated in [2, 4 - 6].

In [15] we introduced the so-called ADD-parametrization of j_{qq}-inner functions which was initiated by former work of Arov [2] and Dewilde and Dym [8, 9]. The first goal of this paper is to characterize distinguished subclasses of j_{qq}-inner functions in terms of the ADD-parameters. It will turn out that the membership of a given j_{pq}-inner function to the Smirnov class (respectively, inverse Smirnov class) depends only on the connected pair for matrix-valued Hardy class functions, whereas the remaining part of the ADD-parametrization, namely the associated singular matrix-valued Carathéodory function, has no influence on this membership. Furthermore, we will study the following question: Let Ω_1 and Ω_2 be pseudocontinuable $q \times q$ Carathéodory functions of finite entropy. Does there exist a j_{pq}-inner function W belonging to the Smirnov class

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(respectively, inverse Smirnov class) such that Ω_1 and Ω_2 are the left and right $q \times q$ Carathéodory functions generated by W?

1. Some preliminaries and notations

Let us start with some notations. Throughout this paper, let p and q be positive integers. We will use $\mathbb{C}$ to denote the set of complex numbers. Further, let

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\},$$

$$\mathbb{C}_0 := \mathbb{C} \cup \{\infty\}, \qquad \mathbb{E} := \mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{T}).$$

The linear Lebesgue-Borel measure on the unit circle $\mathbb{T}$ will be denoted by $\underline{\lambda}$. For $t \in (0,\infty)$, let $\mathcal{L}^t(\mathbb{T})$ be the set of all Borel measurable functions $f : \mathbb{T} \longrightarrow \mathbb{C}$ for which $|f|^t$ is integrable with respect to $\underline{\lambda}$ on $\mathbb{T}$. If $\mathfrak{X}$ is a non-empty set, then $\mathfrak{X}^{p\times q}$ stands for the set of $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The null matrix which belongs to $\mathbb{C}^{p\times q}$ and the identity matrix that belongs to $\mathbb{C}^{q\times q}$ will be denoted by $0_{p\times q}$ and I_q, respectively. If the size of the identity matrix is clear, then we will omit the index. We will use the Löwner semi-ordering in the set $H_{q\times q}$ of all Hermitian $q \times q$ complex matrices, i.e., if $A \in H_{q\times q}$ and $B \in H_{q\times q}$ are such that $A - B$ is non-negative Hermitian, then we will write $A \geq B$ or $B \leq A$. If $A \in \mathbb{C}^{q\times q}$, then $\operatorname{Re} A$ stands for the real part of A.

We will work with several classes of holomorphic or meromorphic matrix-valued functions. If G is a simply connected domain of the extended complex plane $\mathbb{C}_0$, then $\mathcal{NM}(G)$ stands for the *Nevanlinna class* of all complex-valued functions which are meromorphic in G and which can be represented as quotient of two bounded holomorphic functions in G. If $g \in \mathcal{NM}(\mathbb{D})$ (respectively, $g \in \mathcal{NM}(\mathbb{E})$), then there exist a Borelian subset B_0 of $\mathbb{T}$ with $\underline{\lambda}(B_0) = 0$ and a Borel measurable function $\underline{g} : \mathbb{T} \longrightarrow \mathbb{C}$ such that

$$\lim_{r\to 1-0} g(rz) = \underline{g}(z) \qquad \left(\text{respectively, } \lim_{r\to 1+0} g(rz) = \underline{g}(z)\right)$$

for all $z \in \mathbb{T} \setminus B_0$. We will continue to use the notation $\underline{g}$ to designate a radial boundary function of a function g which belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times q}$ or $[\mathcal{NM}(\mathbb{E})]^{p\times q}$. If $g \in [\mathcal{NM}(\mathbb{D})]^{p\times q}$, then one says that g *admits a pseudocontinuation* (*into* $\mathbb{E}$) if there exists a function $g^\# \in [\mathcal{NM}(\mathbb{E})]^{p\times q}$ such that the radial boundary values $\underline{g}$ of g and $\underline{g^\#}$ of $g^\#$, respectively, coincide $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. If $\mathfrak{X}$ is a non-empty subset of $\mathbb{C}_0$, and if $f : \mathfrak{X} \longrightarrow \mathbb{C}^{p\times q}$ is given, then let the function $\widehat{f} : \widehat{\mathfrak{X}} \longrightarrow \mathbb{C}^{q\times p}$ be defined by $\widehat{\mathfrak{X}} := \{z \in \mathbb{C}_0 : 1/\overline{z} \in \mathfrak{X}\}$ and $\widehat{f}(z) := [f(1/\overline{z})]^*$. Observe that if f belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times q}$ (respectively, $[\mathcal{NM}(\mathbb{E})]^{p\times q}$), then $\widehat{f}$ belongs to $[\mathcal{NM}(\mathbb{E})]^{q\times p}$ (respectively, $[\mathcal{NM}(\mathbb{D})]^{q\times p}$), and $\underline{f}^*$ is a radial boundary function of $\widehat{f}$. We will work with a particular subclass of $\mathcal{NM}(\mathbb{D})$, namely with the so-called *Smirnov class* $\mathcal{N}_+(\mathbb{D})$. It consists of all holomorphic functions g which belong to $\mathcal{NM}(\mathbb{D})$ and for which the equality

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log^+ |\underline{g}(z)|\underline{\lambda}(dz) = \sup_{r\in[0,1)} \frac{1}{2\pi}\int_{\mathbb{T}} \log^+ |g(rz)|\underline{\lambda}(dz) \tag{1}$$

holds true, where $\log^+ x := \max(\log x, 0)$ for each $x \in [0,\infty)$. Note that the Hardy classes $H^t(\mathbb{D})$, $t \in (0,\infty]$, are subsets of the Smirnov class $\mathcal{N}_+(\mathbb{D})$. On the other hand, the class $\mathcal{S}_{p\times q}(\mathbb{D})$ of all $p \times q$ Schur functions is a subclass of $H^\infty(\mathbb{D})$. A function $f : \mathbb{D} \longrightarrow \mathbb{C}^{p\times q}$ is called $p\times q$ *Schur function* if f is holomorphic in $\mathbb{D}$ and if $I \geq f(z)[f(z)]^*$ for each $z \in \mathbb{D}$. If a $q \times q$ Schur function has unitary radial boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$, then it is said to be *inner*. A $p \times q$ Schur function f is called *strictly contractive* if $I_p - f(z)[f(z)]^*$ is positive Hermitian for each $z \in \mathbb{D}$. If f and g are inner $q \times q$ Schur functions, then f is called a *left* (respectively, *right*) *inner divisor of* g if $f^{-1}g$ (respectively, gf^{-1}) is an inner function.

Let $f \in [\mathcal{NM}(\mathbb{D})]^{p\times q}$. Then an inner function B that belongs to $\mathcal{S}_{p\times p}(\mathbb{D})$ (respectively, $\mathcal{S}_{q\times q}(\mathbb{D})$) is said to be a *left* (respectively, *right*) *denominator of* f if Bf (respectively, fB) belongs to $[\mathcal{N}_+(\mathbb{D})]^{p\times q}$. The concept of left and right denominators was created by Arov [1] during his investigations on Darlington synthesis. In particular, he developed a minimality concept. A left (respectively, right) denominator B_∇ of f is said to be *smallest* if, for every left (respectively, right) denominator B of f, there exists an inner function U which belongs to $\mathcal{S}_{p\times p}(\mathbb{D})$ (respectively, $\mathcal{S}_{q\times q}(\mathbb{D})$) such that $B = UB_\nabla$ (respectively, $B = B_\nabla U$). Arov [1: Lemma 5.1] proved that every function which belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times q}$ has smallest left as well as smallest right denominators.

Note that the set $\mathcal{C}_q(\mathbb{D})$ of all $q \times q$ *Carathéodory functions*, i.e., the set of all functions $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{D}$ and which satisfy $\operatorname{Re}\Omega(z) \geq 0_{q\times q}$ for all $z \in \mathbb{D}$, is a subset of $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ (see [13: Corollary 2]). If $g : \mathbb{D} \longrightarrow \mathbb{C}$ admits a representation

$$g(w) = \alpha \cdot \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{z+w}{z-w} \log k(z)\underline{\lambda}(\mathrm{d}z)\right\} \tag{2}$$

for all $w \in \mathbb{D}$ where α is some number which belongs to $\mathbb{T}$ and where $k : \mathbb{T} \longrightarrow [0,\infty)$ is some Borel measurable function which satisfies

$$\frac{1}{2\pi}\int_{\mathbb{T}} |\log k| \mathrm{d}\underline{\lambda} < \infty,$$

then g necessary belongs to $\mathcal{N}_+(\mathbb{D})$. Such functions g are called *outer*. A function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is said to be *outer* (in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$) if $\det\Phi$ is an outer function. An outer function Φ in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is called *normalized* if $\Phi(0)$ is non-negative Hermitian. If both Φ and Ψ are outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, then it can be easily seen that the product $\Phi\Psi$ is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

It seems to be useful to state some further properties of outer functions, which were proved by Arov [1] and which we will use in the following.

Lemma 1.

(a) *If Φ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, then* $\det\Phi$ *does not vanish in* $\mathbb{D}$ *and Φ^{-1} is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ as well.*

(b) *Let $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ be such that* $\det\Phi$ *does not vanish in* $\mathbb{D}$. *If Φ^{-1} also belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, then both functions Φ and Φ^{-1} are necessarily outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*

2. On some interrelations between outer functions in the Smirnov class and functions of the Potapov class

Throughout this section, let m be a positive integer, and let J be an $m \times m$ signature matrix, i.e., J belongs to $\mathbb{C}^{m\times m}$ and satisfies as well $J = J^*$ as $J^2 = I$. A matrix $A \in \mathbb{C}^{m\times m}$ is called *J-contractive* if $J \geq A^*JA$. If $A \in \mathbb{C}^{m\times m}$ even satisfies $A^*JA = J$, then A is said to be *J-unitary.* We will work with the *Potapov class* $\mathfrak{P}_J(\mathbb{D})$, i.e., the set of all $m \times m$ matrix-valued functions W which satisfy the following three conditions:

(i) W is meromorphic in $\mathbb{D}$.

(ii) The function $\det W$ does not vanish identically in $\mathbb{D}$.

(iii) For each z which belongs to the set $\mathbb{H}_W$ of all points of analyticity of W, the matrix $W(z)$ is J-contractive.

The Potapov class $\mathfrak{P}_J(\mathbb{D})$ is a subclass of $[\mathcal{NM}(\mathbb{D})]^{m\times m}$ (see, e.g., [12: Corollary 2]). In particular, every function W which belongs to $\mathfrak{P}_J(\mathbb{D})$ has radial boundary values $\underline{W}$ $\underline{\lambda}$-a.e. on $\mathbb{T}$. If $W \in \mathfrak{P}_J(\mathbb{D})$ satisfies $\underline{W}^*J\underline{W} = J \quad \underline{\lambda}$ – a.e. on $\mathbb{T}$, then W is said to be a *J-inner function.*

In the following, let us mainly concentrate on the $(p+q)\times(p+q)$ signature matrix

$$j_{pq} := \operatorname{diag}(I_p, -I_q)\ . \tag{3}$$

If we will consider a function W which belongs to the Potapov class $\mathfrak{P}_{j_{pq}}(\mathbb{D})$, then let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{4}$$

be the block partition of W where W_{11} is a $p \times p$ block. Observe that every J-inner function W admits a pseudocontinuation $W^{\#}$ (into $\mathbb{E}$). For each $z \in \mathbb{E}$ which fulfills $1/\overline{z} \in \mathbb{H}_W$ and $\det W(1/\overline{z}) \neq 0$, this pseudocontinuation $W^{\#}$ admits the representation $W^{\#}(z) = J[W(1/\overline{z})]^{-*}J$. In particular, if W is a j_{pq}-inner function, then all the matrix-valued functions W_{11}, W_{12}, W_{21} and W_{22} admit pseudocontinuations (into $\mathbb{E}$).

The following lemma, which can be traced back to [8, 9, 16], will be useful for our further considerations.

Lemma 2. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then* $\det W_{22}$ *does not vanish identically, and*

$$S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21} \qquad S_{12} := W_{12}W_{22}^{-1}$$
$$S_{21} := -W_{22}^{-1}W_{21} \qquad S_{22} := W_{22}^{-1}$$

are matrix-valued Schur functions, whereby S_{12} and S_{21} are even strictly contractive. If W is even a j_{pq}-inner function, then $\det W_{11}$ *does not vanish identically and*

$$S_{12} = (\widehat{W_{11}^{\#}})^{-1}\widehat{W_{21}^{\#}}, \qquad S_{21} = -\widehat{W_{12}^{\#}}(\widehat{W_{11}^{\#}})^{-1}, \qquad S_{11} = (\widehat{W_{11}^{\#}})^{-1}. \tag{5}$$

Now we want to discuss certain matrix-valued functions built from a function which belongs to the Potapov class $\mathfrak{P}_{j_{pq}}(\mathbb{D})$. First we state a characterization of the subclass of all outer functions which belong to $[\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$.

Theorem 1. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then the following statements are equivalent:*

(i) *W is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$.*

(ii) *$S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21}$ is an outer function in $\mathcal{S}_{p\times p}(\mathbb{D})$, and W_{22} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*

A proof of Theorem 1 can be easily obtained combining [5: Theorem 2] and Lemma 1. We omit the details.

Theorem 2. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then:*

(a) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $g := \frac{1}{2}(I + W_{22}^{-1}W_{21}f)$ is an outer function which belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$.*

(b) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $\det(W_{22}+W_{21}f)$ does not vanish identically and $\Omega := (W_{22}+W_{21}f)^{-1}(W_{22}-W_{21}f)$ belongs to $\mathcal{C}_q(\mathbb{D})$.*

(c) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $(W_{22}+W_{21}f)^{-1}$ belongs to $[H^2(\mathbb{D})]^{q\times q}$.*

Proof. Lemma 2 shows that $\det W_{22}$ does not vanish identically and that as well $S_{22} := W_{22}^{-1}$ as $S_{21} := -W_{22}^{-1}W_{21}$ are matrix-valued Schur functions where S_{21} is strictly contractive. Let $f \in \mathcal{S}_{p\times q}(\mathbb{D})$. Then $h := S_{21}f$ is a strictly contractive $q \times q$ Schur function. Hence $\det(I_q + h)$ and $\det(I_q - h)$ nowhere vanish in $\mathbb{D}$. Applying a result due to Arov [1] (see also [13: Corollary 1]), we then obtain that $\frac{1}{2}(I + h)$ and $\frac{1}{2}(I - h)$ are outer functions which belong to $\mathcal{S}_{q\times q}(\mathbb{D})$. In particular, part (a) is verified. Since h is strictly contractive the identities

$$W_{22} + W_{21}f = W_{22}(I - h) \qquad \text{and} \qquad W_{22} - W_{21}f = W_{22}(I + h) \tag{6}$$

imply that $\det(W_{22} + W_{21}f)$ does not vanish identically and that the representation

$$\Omega = (I - h)^{-1}(I + h) \tag{7}$$

holds, which shows that Ω belongs to $\mathcal{C}_q(\mathbb{D})$ (see, e.g., [10: Proposition 2.1.2]). Moreover, the first identity in (6) implies that $G := (W_{22} + W_{21}f)^{-1}$ satisfies $G = (I - h)^{-1}S_{22}$. Hence we see that, as product of the outer function $(I - h)^{-1}$ and the $q \times q$ Schur function S_{22}, the function G belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Since W has j_{pq}-contractive values we get from a result due to Potapov [18] (see also [10: Theorem 1.3.3]) that W^* has j_{pq}-contractive values as well. Thus $W_{22}W_{22}^* - W_{21}W_{21}^* \geq I$ and therefore

$$I - S_{21}S_{21}^* = I - W_{22}^{-1}W_{21}W_{21}^*W_{22}^{-*} \geq W_{22}^{-1}W_{22}^{-*}.$$

Since the $p\times q$ Schur function f has contractive values, we obtain $S_{21}S_{21}^* \geq S_{21}ff^*S_{21}^* = hh^*$. Using (7) and the first identity in (6), then it follows

$$\begin{aligned}
\operatorname{Re}\Omega &= \operatorname{Re}[(I - h)^{-1}(I + h)] \\
&= \frac{1}{2}(I - h)^{-1}[(I + h)(I - h)^* + (I - h)(I + h)^*](I - h)^{-*} \\
&= (I - h)^{-1}(I - hh^*)(I - h)^{-*}
\end{aligned}$$

$$\begin{aligned}&\geq (I-h)^{-1}(I-S_{21}S_{21}^*)(I-h)^{-*}\\&\geq (W_{22}+W_{21}f)^{-1}(W_{22}+W_{21}f)^{-*}\\&= GG^*.\end{aligned}$$

Hence the corresponding radial boundary values fulfill the inequality

$$\operatorname{Re}\underline{\Omega} \;\geq\; \underline{G}\,\underline{G}^* \tag{9}$$

λ-a.e. on $\mathbb{T}$. Since Ω belongs to $\mathcal{C}_q(\mathbb{D})$ we know from [13: Lemma 4] that $\operatorname{Re}\underline{\Omega}$ belongs to $[\mathcal{L}^1(\mathbb{T})]^{q\times q}$. Thus the inequality (9) implies that $\underline{G}$ belongs to $[\mathcal{L}^2(\mathbb{T})]^{q\times q}$. The application of the maximum modulus principle for the Smirnov class (see, e.g., [11: Theorem 2.11]) then shows that G belongs to $[H^2(\mathbb{D})]^{q\times q}$ ∎

In view of Lemma 2, the following theorem can be verified analogously to the proof of Theorem 2. We will omit the details.

Theorem 3. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then:*

(a) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $h := \frac{1}{2}(I + gW_{12}W_{22}^{-1})$ is an outer function which belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$.*

(b) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, the function $\det(W_{22}+gW_{12})$ does not vanish identically and $\Omega := (W_{22}-gW_{12})(W_{22}+gW_{12})^{-1}$ belongs to $\mathcal{C}_q(\mathbb{D})$.*

(c) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, the function $(W_{22}+gW_{12})^{-1}$ belongs to $[H^2(\mathbb{D})]^{q\times q}$.*

In the case that the given function W is even a j_{pq}-inner function, the pseudocontinuability of W yields the possibility to verify two further results comparable with Theorems 2 and 3. The corresponding proofs can be obtained similarly to the proof of Theorem 2.

Theorem 4. *Let W be a j_{pq}-inner function. Then:*

(a) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $g := \frac{1}{2}(I + f\widehat{W_{12}^{\#}}(\widehat{W_{11}^{\#}})^{-1})$ is an outer function which belongs to $\mathcal{S}_{p\times p}(\mathbb{D})$.*

(b) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $\det(\widehat{W_{11}^{\#}}+f\widehat{W_{12}^{\#}})$ does not vanish identically, and $\Omega := (\widehat{W_{11}^{\#}}-f\widehat{W_{12}^{\#}})(\widehat{W_{11}^{\#}}+f\widehat{W_{12}^{\#}})^{-1}$ belongs to $\mathcal{C}_p(\mathbb{D})$.*

(c) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $(\widehat{W_{11}^{\#}}+f\widehat{W_{12}^{\#}})^{-1}$ belongs to $[H^2(\mathbb{D})]^{p\times p}$.*

Theorem 5. *Let W be a j_{pq}-inner function. Then:*

(a) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $h := \frac{1}{2}(I + (\widehat{W_{11}^{\#}})^{-1}\widehat{W_{21}^{\#}}g)$ is an outer function which belongs to $\mathcal{S}_{p\times p}(\mathbb{D})$.*

(b) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, the function $\det(\widehat{W_{11}^{\#}}+\widehat{W_{21}^{\#}}g)$ does not vanish identically, and $\Omega := (\widehat{W_{11}^{\#}}+\widehat{W_{21}^{\#}}g)^{-1}(\widehat{W_{11}^{\#}}-\widehat{W_{21}^{\#}}g)$ belongs to $\mathcal{C}_p(\mathbb{D})$.*

(c) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, the function $(\widehat{W_{11}^{\#}}+\widehat{W_{21}^{\#}}g)^{-1}$ belongs to $[H^2(\mathbb{D})]^{p\times p}$.*

3. Particular subclasses of j_{pq}-inner functions

If J is an $m \times m$ signature matrix, then a function W which belongs to $\mathfrak{P}_J(\mathbb{D})$ is called to be of *Smirnov type* (respectively, *inverse Smirnov type*) if W (respectively, W^{-1}) belongs to $[\mathcal{N}_+(\mathbb{D})]^{m\times m}$. A J-inner function which is as well of Smirnov type as of inverse Smirnov type is said to be *Arov-singular*. The aim of this section is to give necessary and sufficient conditions for the situation that a given function which belongs to the Potapov class $\mathfrak{P}_{j_{pq}}(\mathbb{D})$ is of Smirnov type (respectively, inverse Smirnov type), where j_{pq} is the $(p+q)\times(p+q)$ signature matrix defined by (3). In particular, we will focus our attention to j_{pq}-inner functions. First we are going to study characterizations via the block structure. Hereby, we continue to use the block partition (4) of W where W_{11} is a $p \times p$ block. A j_{pq}-inner function W is Arov-singular if and only if both functions W and W^{-1} belong to $[\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$. By virtue of Theorem 1 and Lemma 1, we thus get that a given j_{pq}-inner function W is Arov-singular if and only if $S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21}$ is an outer function in $\mathcal{S}_{p\times p}(\mathbb{D})$ and W_{22} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

Now we will characterize the subclass of all functions which belong to $\mathfrak{P}_{j_{pq}}(\mathbb{D})$ and which are of Smirnov type.

Theorem 6. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then the following statements are equivalent:*

- **(i)** *W is of Smirnov type.*
- **(ii)** *W_{22} is outer in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*
- **(iii)** *W_{22}^{-1} is outer in $\mathcal{S}_{q\times q}(\mathbb{D})$.*
- **(iv)** *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $W_{22} + W_{21}f$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*
- **(v)** *For some $f_0 \in \mathcal{S}_{p\times q}(\mathbb{D})$, $W_{22} + W_{21}f_0$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*
- **(vi)** *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $(W_{22} + W_{21}f)^{-1}$ is outer in $[H^2(\mathbb{D})]^{q\times q}$.*
- **(vii)** *For some $f_0 \in \mathcal{S}_{p\times q}(\mathbb{D})$, $(W_{22} + W_{21}f_0)^{-1}$ is outer in $[H^2(\mathbb{D})]^{q\times q}$.*
- **(viii)** *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $W_{22} + gW_{12}$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*
- **(ix)** *For some $g_0 \in \mathcal{S}_{q\times p}(\mathbb{D})$, $W_{22} + g_0W_{12}$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*
- **(x)** *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $(W_{22} + gW_{12})^{-1}$ is outer in $[H^2(\mathbb{D})]^{q\times q}$.*
- **(xi)** *For some $g_0 \in \mathcal{S}_{q\times p}(\mathbb{D})$, $(W_{22} + g_0W_{12})^{-1}$ is outer in $[H^2(\mathbb{D})]^{q\times q}$.*

Proof. Our proof is organized as follows:

$$
\begin{array}{ccccccc}
(x) & \Longleftrightarrow & (viii) & \Longrightarrow & (ix) & \Longleftrightarrow & (xi) \\
 & & \Uparrow & & \Downarrow & & \\
(i) & \Longleftrightarrow & & (ii) & & \Longleftrightarrow & (iii) \\
 & & \Uparrow & & \Downarrow & & \\
(vii) & \Longleftrightarrow & (v) & \Longleftarrow & (iv) & \Longleftrightarrow & (vi)
\end{array}
$$

(i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii): These equivalences are proved in [5: Theorem 2].

(iv) $\Leftrightarrow$ (vi) and (v) $\Leftrightarrow$ (vii): Apply part (c) of Theorem 2 and part (a) of Lemma 1.

(ii) $\Rightarrow$ (iv): Let $f \in \mathcal{S}_{p\times q}(\mathbb{D})$. From Theorem 2 we know that $\phi := I + W_{22}^{-1}W_{21}f$ is an outer function which belongs to $[H^\infty(\mathbb{D})]^{q\times q}$. Since the product of two outer functions which belong to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is outer as well we see from $W_{22}+W_{21}f = W_{22}\phi$ that (ii) implies (iv).

(iv) $\Rightarrow$ (v) and (viii) $\Rightarrow$ (ix) are trivial.

(v) $\Rightarrow$ (ii): Suppose (v). According to part (a) of Theorem 2, $\psi := I + W_{22}^{-1}W_{21}f_0$ is an outer function that belongs to $[H^\infty(\mathbb{D})]^{q\times q}$. Thus Lemma 1 shows that ψ^{-1} is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Then the identity $W_{22} = (W_{22}+W_{21}f_0)\psi^{-1}$ yields (ii).

(viii) $\Leftrightarrow$ (x) and (ix) $\Leftrightarrow$ (xi): Apply part (c) of Theorem 3 and part (a) of Lemma 1.

(ii) $\Rightarrow$ (viii): Let $g \in \mathcal{S}_{q\times p}(\mathbb{D})$. From Theorem 3 we know that $\delta := I + gW_{12}W_{22}^{-1}$ is an outer function which belongs to $[H^\infty(\mathbb{D})]^{q\times q}$. In view of $W_{22}+gW_{12} = \delta W_{22}$ thus we obtain that (viii) is necessary for (ii).

(ix) $\Rightarrow$ (ii): Suppose (ix). Part (a) of Theorem 3 yields that $\theta := I + g_0W_{12}W_{22}^{-1}$ is an outer function which belongs to $[H^\infty(\mathbb{D})]^{q\times q}$. Thus part (a) of Lemma 1 shows that θ^{-1} is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Hence we get from $W_{22} = \theta^{-1}(W_{22}+g_0W_{12})$ that (ii) follows from (ix) ∎

Now we turn our attention to the subclass of all functions which belong to $\mathfrak{P}_{j_{pq}}(\mathbb{D})$ and which are of inverse Smirnov type. We recall a first characterization of these functions.

Theorem 7. *Let $W \in \mathfrak{P}_{j_{pq}}(\mathbb{D})$. Then the following statements are equivalent:*

(i) *W is of inverse Smirnov type.*

(ii) *$W_{11} - W_{12}W_{22}^{-1}W_{21}$ is outer in $\mathcal{S}_{p\times p}(\mathbb{D})$.*

(iii) *$(W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1}$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$.*

A proof of Theorem 7 is given in [5: Theorem 2]. If we study j_{pq}-inner functions W, then we obtain further equivalent conditions.

Theorem 8. *Let W be a j_{pq}-inner function. Then the following statements are equivalent:*

(i) *W is of inverse Smirnov type.*

(iv) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $\widehat{W_{11}^{\#}} + f\widehat{W_{12}^{\#}}$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$.*

(v) *For some $f_0 \in \mathcal{S}_{p\times q}(\mathbb{D})$, $\widehat{W_{11}^{\#}} + f_0\widehat{W_{12}^{\#}}$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$.*

(vi) *For each $f \in \mathcal{S}_{p\times q}(\mathbb{D})$, $(\widehat{W_{11}^{\#}} + f\widehat{W_{12}^{\#}})^{-1}$ is outer in $[H^2(\mathbb{D})]^{p\times p}$.*

(vii) *For some $f_0 \in \mathcal{S}_{p\times q}(\mathbb{D})$, $(\widehat{W_{11}^{\#}} + f_0\widehat{W_{12}^{\#}})^{-1}$ is outer in $[H^2(\mathbb{D})]^{p\times p}$.*

(viii) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}}g$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$.*

(ix) *For some $g_0 \in \mathcal{S}_{q\times p}(\mathbb{D})$, $\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}}g_0$ is outer in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$.*

(x) *For each $g \in \mathcal{S}_{q\times p}(\mathbb{D})$, $(\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}}g)^{-1}$ is outer in $[H^2(\mathbb{D})]^{p\times p}$.*

(xi) *For some $g_0 \in \mathcal{S}_{q\times p}(\mathbb{D})$, $(\widehat{W_{11}^{\#}} + \widehat{W_{21}^{\#}}g_0)^{-1}$ is outer in $[H^2(\mathbb{D})]^{p\times p}$.*

Proof. Taking into account Theorem 7, our proof is organized as follows:

$$\begin{array}{ccccccccc} & & & & (ii) & & & & \\ (vii) & \Longleftrightarrow & (v) & \Longrightarrow & \Updownarrow & \Longrightarrow & (viii) & \Longleftrightarrow & (x) \\ & & \Uparrow & & (i) & & \Downarrow & & \\ (vi) & \Longleftrightarrow & (iv) & \Longleftarrow & \Updownarrow & \Longleftarrow & (ix) & \Longleftrightarrow & (xi) \\ & & & & (iii) & & & & \end{array}$$

(iv) $\Leftrightarrow$(vi) and (v) $\Leftrightarrow$(vii): Apply part (c) of Theorem 4 and part (a) of Lemma 1.

(ii) $\Rightarrow$ (iv): Let $f \in \mathcal{S}_{p\times q}(\mathbb{D})$. From Theorem 4 we know that $\phi := I + f\widehat{W_{12}^{\#}}(\widehat{W_{11}^{\#}})^{-1}$ is an outer function which belongs to $[H^\infty(\mathbb{D})]^{p\times p}$. From Lemma 2 we see

$$\widehat{W_{11}^{\#}} + f\widehat{W_{12}^{\#}} = \phi\widehat{W_{11}^{\#}} = \phi(W_{11} - W_{12}W_{22}^{-1}W_{21})$$

and hence, in view of (ii), that $\widehat{W_{11}^{\#}} + f\widehat{W_{12}^{\#}}$ is the product of two outer functions which belong to $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$. Thus (iv) holds true.

(iv) $\Rightarrow$ (v) and (viii) $\Rightarrow$ (ix): These implications are obvious.

(v) $\Rightarrow$ (ii): Suppose that (v) is valid. By virtue of part (a) of Theorem 4, the function $\psi := I + f_0\widehat{W_{12}^{\#}}(\widehat{W_{11}^{\#}})^{-1}$ is an outer function that belongs to $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$. Using (v), Lemma 1, the identity

$$W_{11} - W_{12}W_{22}^{-1}W_{21} = (\widehat{W_{11}^{\#}})^{-1} = (\widehat{W_{11}^{\#}} + f_0\widehat{W_{12}^{\#}})^{-1}\phi$$

and Lemma 2, we obtain then (ii).

(viii) $\Leftrightarrow$ (x) and (ix) $\Leftrightarrow$ (xi): Apply part (c) of Theorem 5 and part (a) of Lemma 1.

(iii) $\Rightarrow$ (viii): Let $g \in \mathcal{S}_{q\times p}(\mathbb{D})$. From Theorem 5 we know that $\delta := I + (\widehat{W_{11}^{\#}})^{-1}\widehat{W_{21}^{\#}}g$ is an outer function which belongs to $[H^2(\mathbb{D})]^{p\times p}$. In view of Lemma 2, we have

$$\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}}g = \widehat{W_{11}^{\#}}\delta = (W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1}\delta\ .$$

Thus (viii) is necessary for (iii).

(ix) $\Rightarrow$ (iii): Let (ix) be satisfied. Since we know from part (a) of Theorem 5 that $\Theta := I + (\widehat{W_{11}^{\#}})^{-1}\widehat{W_{21}^{\#}}g_0$ is an outer function which belongs to $[H^\infty(\mathbb{D})]^{p\times p}$. Thus Lemma 1 shows that Θ^{-1} is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{p\times p}$. Hence (iii) follows from $(W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1} = \widehat{W_{11}^{\#}} = (\widehat{W_{11}^{\#}} + \widehat{W_{12}^{\#}}g_0)\Theta^{-1}$ ∎

In our further considerations we are going to study an other type of characterization of the particular subclasses of j_{qq}-inner functions which is closely related to the so-called ADD-parametrization of j_{qq}-inner functions (see [15]). For our aims in this paper, we need only parts of the ADD-parametrization. For this reason, we will not recall a complete description of the ADD-parametrization, but we will restrict our attention to the following notion which was introduced in [15]. An ordered pair $[\Phi, \Psi]$ of functions which belong to $[H^2(\mathbb{D})]^{q\times q}$ is called *left* (respectively, *right*) *connected pair of* $[H^2(\mathbb{D})]^{q\times q}$*-functions* if there is an inner $q\times q$ Schur function V such that $\underline{\Psi} = \underline{V}\ \underline{\Phi}^*$ (respectively, $\underline{\Psi} = \underline{\Phi}^*\underline{V}$) holds true $\underline{\lambda}$-a.e. on $\mathbb{T}$. Every such function V is said to be an *inner function which realizes this left* (respectively, *right*) *connection of* $[\Phi, \Psi]$.

A useful property of left and right connected pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions is their pseudocontinuability:

Proposition 1. *If $[\Phi, \Psi]$ is a left (respectively, right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions, and if V is an inner $q \times q$ Schur function which realizes this left (respectively, right) connection, then Φ and Ψ admit pseudocontinuations $\Phi^\#$ and $\Psi^\#$, respectively, which satisfy*

$$\Psi = V\widehat{\Phi^\#} \quad \text{and} \quad \Phi = \widehat{\Psi^\#}V \qquad \left(\text{respectively, } \Psi = \widehat{\Phi^\#}V \text{ and } \Phi = V\widehat{\Psi^\#}\right)$$

A proof of Proposition 2 is given in [15: Proposition 2.3]. Note that the function V is unique if the function $\det\Phi$ (or the function $\det\Psi$) does not vanish identically in $\mathbb{D}$ (see [15: Lemma 2.5]). In order to prepare further characterizations of the subclasses of j_{qq}-inner functions considered above we introduce convenient subclasses of connected pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions.,

Definition 1. Let $[\Phi, \Psi]$ be a left connected or a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. The pair $[\Phi, \Psi]$ is called to be *of Smirnov type* (respectively, *of inverse Smirnov type*) if Φ (respectively, Ψ) is an outer function (in $[H^2(\mathbb{D})]^{q\times q}$).

Remark 1. From [15: Remark 2.2] it is clear that $[\Phi, \Psi]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions of Smirnov type (respectively, of inverse Smirnov type) if and only if $[\Psi, \Phi]$ is a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions of inverse Smirnov type (respectively, of Smirnov type).

Remark 2. If $[\Phi, \Psi]$ is a left (or right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions of Smirnov type or inverse Smirnov type, then one can easily see that the functions $\det\Phi$ and $\det\Psi$ do not vanish identically.

If W is a j_{qq}-inner function, then the pair $[\Phi_l, \Psi_l]$ given by

$$\Phi_l := (W_{22} + W_{21})^{-1} \quad \text{and} \quad \Psi_l := (\widehat{W_{11}^\#} + \widehat{W_{12}^\#})^{-1}$$

turns out to be a (well-defined) left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions (see [15: Section 5]). It is said to be the *left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W*. Similarly, the pair $[\Phi_r, \Psi_r]$ constructed from an arbitrary j_{qq}-inner function W via

$$\Phi_r := (W_{22} + W_{12})^{-1} \quad \text{and} \quad \Psi_r := (\widehat{W_{11}^\#} + \widehat{W_{21}^\#})^{-1}$$

is a (well-defined) right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions (see [15: Section 5]) which is called the *right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W*.

The following theorem shows how one can use these two pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions in order to describe the situation that a given j_{qq}-inner function is of Smirnov type and of inverse Smirnov type, respectively.

Theorem 9. *Let W be a j_{qq}-inner function. Further, let $[\Phi_l, \Psi_l]$ and $[\Phi_r, \Psi_r]$ be the left connected pair and the right connected pair, respectively, of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W. Then the following statements are equivalent:*

(i) *W is of Smirnov type (respectively, of inverse Smirnov type).*

(ii) *$[\Phi_l, \Psi_l]$ is of Smirnov type (respectively, of inverse Smirnov type).*

(iii) *$[\Phi_r, \Psi_r]$ is of Smirnov type (respectively, of inverse Smirnov type).*

Proof. The constant function $f : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ with value I_q belongs to $S_{q\times q}(\mathbb{D})$. Thus the application of Theorem 6 yields the asserted characterizations of j_{qq}-inner functions of Smirnov type. Analogously, Theorem 7 provides the other equivalence ∎

Observe that Theorem 9 shows that the singular $q \times q$ Carathéodory function which occurs in the ADD-parametrization of a given j_{qq}-inner function W has no influence whether it is of Smirnov type or of inverse Smirnov type. Furthermore, it is obvious from Theorem 9, how any j_{qq}-inner function W can be characterized to be Arov-singular by the left (respectively, right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions generated by W.

4. A modification of an inner-outer factorization theorem

The goal of this section is to work out a Smirnov class version of the inner-outer factorization theorem for the class $[H^2(\mathbb{D})]^{q\times q}$ (see, e.g., Masani [17]). For this purpose we need the following result which goes back to Arov [3].

Proposition 2. *For each $\Xi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$, there are a function $\nabla \in [H^\infty(\mathbb{D})]^{q\times q}$ and an outer function $\psi \in H^\infty(\mathbb{D})$ such that $\Xi = \frac{1}{\psi}\nabla$, where Ξ is an outer function (in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$) if and only if ∇ is an outer function in $[H^\infty(\mathbb{D})]^{q\times q}$.*

To treat the uniqueness aspect of the desired factorization we will use the following

Lemma 3. Let V_1 and V_2 be inner $q \times q$ Schur functions, and let Φ_1 and Φ_2 be outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

(a) If $V_1\Phi_1 = V_2\Phi_2$, then there is a unitary $q \times q$ complex matrix A such that $V_1 = V_2A^*$ and $\Phi_1 = A\Phi_2$.

(b) If $\Phi_1V_1 = \Phi_2V_2$, then there is a unitary $q \times q$ complex matrix B such that $V_1 = B^*V_2$ and $\Phi_1 = \Phi_2B$.

Proof. Using Proposition 2 we can choose outer functions Δ_1 and Δ_2 in $[H^\infty(\mathbb{D})]^{q\times q}$ and outer functions φ_1 and φ_2 in $H^\infty(\mathbb{D})$ such that $\Phi_1 = \frac{1}{\varphi_1}\Delta_1$ and $\Phi_2 = \frac{1}{\varphi_2}\Delta_2$. The function $\Theta := V_1(\varphi_2\Delta_1)$ obviously belongs to $[H^2(\mathbb{D})]^{q\times q}$. Assume $V_1\Phi_1 = V_2\Phi_2$. Then Θ admits the representation $\Theta := V_2(\varphi_1\Delta_2)$, and we have $(\underline{\varphi_2}\,\underline{\Delta_1})^*(\underline{\varphi_2}\,\underline{\Delta_1}) = (\underline{\varphi_1}\,\underline{\Delta_2})^*(\underline{\varphi_1}\,\underline{\Delta_2})$ $\underline{\lambda}$-a.e. on $\mathbb{T}$. Using part (b) of Theorem 10, we see that there is a unitary $q \times q$ complex matrix A such that $\varphi_2\Delta_1 = A\varphi_1\Delta_2$ and $V_1 = V_2A^*$. Thus $\Phi_1 = A\Phi_2$. Part (a) is proved. Part (b) can be verified analogously ∎

Now we are able to prove the announced inner-outer factorization in the Smirnov class.

Theorem 10. *Let $\Theta \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ be such that the function* $\det\Theta$ *does not vanish identically in $\mathbb{D}$. Then:*

(a) *There is a unique inner $q \times q$ Schur function V and a unique normalized outer function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = V\Phi$.*

(b) *There is a unique inner $q \times q$ Schur function U and a unique normalized outer function $\Psi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = \Psi U$.*

Proof. According to Proposition 2 there exist a normalized outer function φ in $H^\infty(\mathbb{D})$ and a function $\Sigma \in [H^\infty(\mathbb{D})]^{q\times q}$ such that $\Theta = \frac{1}{\varphi}\Sigma$. From the inner-outer factorization theorem for the Hardy class $[H^2(\mathbb{D})]^{q\times q}$ (see, e.g., [17]) we see then that there are a unique inner $q \times q$ Schur function V and a unique normalized outer function

$\Delta \in [H^2(\mathbb{D})]^{q\times q}$ such that $\Sigma = V\Delta$. Consequently, $\Theta = V\frac{1}{\varphi}\Delta$. Using again Proposition 2 we get that $\Phi := \frac{1}{\varphi}\Delta$ is a normalized outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ which satisfies $\Theta = V\Phi$. The uniqueness of this representation follows immediately from Lemma 3. Thus part (a) is verified. Part (b) can be proved analogously ∎

Corollary 1. *Let $\Theta \in [\mathcal{NM}(\mathbb{D})]^{q\times q}$ be such that* $\det\Theta$ *does not vanish identically.*

(a) *Let L be a smallest left denominator of Θ. Then there are a unique inner $q\times q$ Schur function V and a unique normalized outer function Φ in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = L^{-1}V\Phi$. In particular, L and V have no common non-constant inner left divisor.*

(b) *Let $\widetilde{L}$ and $\widetilde{V}$ be inner $q\times q$ Schur functions, and let $\widetilde{\Phi}$ be an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = \widetilde{L}^{-1}\widetilde{V}\widetilde{\Phi}$. Then $\widetilde{L}$ is a left denominator of Θ. The functions $\widetilde{L}$ and $\widetilde{V}$ have no common non-constant inner left divisor if and only if $\widetilde{L}$ is a smallest left denominator of Θ.*

Proof. Since L is a left denominator of Θ, the function $G := L\Theta$ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Then we know from Theorem 10 that there are a unique inner function $V \in \mathcal{S}_{q\times q}(\mathbb{D})$ and a unique normalized outer function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $L\Theta = V\Phi$. Hence the assertion stated in (a) follows. From $\Theta = \widetilde{L}^{-1}\widetilde{V}\widetilde{\Phi}$ we get $\widetilde{L}\Theta = \widetilde{V}\widetilde{\Phi} \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$, i.e., $\widetilde{L}$ is a left denominator of Θ. Consequently, we have $\widetilde{L} = XL$ with some inner $q\times q$ Schur function X. Thus we obtain

$$L^{-1}V\Phi = \Theta = \widetilde{L}^{-1}\widetilde{V}\widetilde{\Phi} = L^{-1}X^{-1}\widetilde{V}\widetilde{\Phi}$$

and therefore $XV\Phi = \widetilde{V}\widetilde{\Phi}$. Then the application of Lemma 3 shows that there is a unitary $q\times q$ complex matrix A such that $XV = \widetilde{V}A$, i.e., X is a left inner divisor of V. If L and V have no common non-constant left inner divisor, then we get from $\widetilde{L} = XL$ and $XV = \widetilde{V}A$ that X is a constant function (with unitary value), i.e., $\widetilde{L}$ is a smallest left denominator of Θ. Conversely, if $\widetilde{L}$ is a smallest left denominator of Θ, then the application of part (a) completes the proof ∎

Analogously to Corollary 1 the following result can be checked.

Corollary 2. *Let $\Theta \in [\mathcal{NM}(\mathbb{D})]^{q\times q}$ be such that* $\det\Theta$ *does not vanish identically.*

(a) *Let R be a smallest right denominator of Θ. Then there are a unique inner $q\times q$ Schur function U and a unique normalized outer function Ψ in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = \Psi U R^{-1}$. In particular, R and U have no common non-constant inner right divisor.*

(b) *Let $\widetilde{R}$ and $\widetilde{U}$ be inner $q\times q$ Schur functions, and let $\widetilde{\Psi}$ be an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ such that $\Theta = \widetilde{\Psi}\widetilde{U}\widetilde{R}^{-1}$. Then $\widetilde{R}$ is a right denominator of Θ. The functions $\widetilde{R}$ and $\widetilde{U}$ have no common non-constant inner right divisor if and only if $\widetilde{R}$ is a smallest right denominator of Θ.*

Now we turn our attention to such denominators of $[\mathcal{NM}(\mathbb{D})]^{q\times q}$-functions which produce outer matrix-valued functions.

Lemma 4. *Let* $\Theta \in [\mathcal{NM}(\mathbb{D})]^{q\times q}$.

(a) *Let* V *be a left (respectively, right) denominator of* Θ *such that* $\Psi := V\Theta$ *(respectively,* $\Phi := \Theta V$*) is an outer function in* $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. *Then* V *is a smallest left (respectively, smallest right) denominator of* Θ.

(b) *Let* V *be a smallest left (respectively, smallest right) denominator of* Θ. *Then* $\Psi := V\Theta$ *(respectively,* $\Phi := \Theta V$*) is an outer function in* $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ *if and only if* $\det\Theta$ *does not vanish identically and* Θ^{-1} *belongs to* $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

Proof. (a): The function Ψ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Assume that V_∇ is a smallest left denominator of Θ. Then $\Psi_\nabla := V_\nabla\Theta$ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ as well. Since V and V_∇ are inner $q\times q$ Schur functions we obtain

$$\underline{\Psi_\nabla}^*\underline{\Psi_\nabla} = \underline{\Psi}^*\underline{\Psi} \qquad \underline{\lambda}\text{-a.e. on } \mathbb{T}\ . \tag{10}$$

Since Ψ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, we see from Theorem 10 the existence of an inner $q\times q$ Schur function U such that $\Psi_\nabla = U\Psi$. Thus it follows

$$\Psi = U^{-1}\Psi_\nabla\ . \tag{11}$$

On the other hand, we have

$$\Psi = V\Theta = VV_\nabla^{-1}\Psi_\nabla\ . \tag{12}$$

Since Ψ is an outer function the identity (10) shows that $\det\Psi_\nabla$ does not vanish identically in $\mathbb{D}$. Hence comparing (11) and (12) we infer that

$$V = U^{-1}V_\nabla\ . \tag{13}$$

Because V_∇ is a smallest left denominator of Θ it follows that U^{-1} is an inner $q\times q$ Schur function. Since U is also an inner $q\times q$ Schur function we can conclude that U is a constant function with unitary value. The minimality of V_∇ and (13) then imply that V is a smallest left denominator of Θ. The case that V is a right denominator of Θ such that Φ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ can be treated analogously.

(b): First assume that Ψ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Then the functions $\det\Psi$ and $\det\Theta$ do not vanish identically in $\mathbb{D}$, and Ψ^{-1} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Since $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$ we obtain from the representation $\Theta^{-1} = \Psi^{-1}V$ that Θ^{-1} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

Conversely, now assume that $\det\Theta$ does not vanish identically in $\mathbb{D}$ and that Θ^{-1} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Using Theorem 10 we see that there are an outer function Ξ in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and an inner $q\times q$ Schur function V_Δ such that $\Theta^{-1} = \Xi V_\Delta$. Since $\det\Xi$ does not vanish in $\mathbb{D}$, it follows

$$\Xi^{-1} = V_\Delta\Theta\ . \tag{14}$$

Because Ξ (and therefore Ξ^{-1}) is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, we thus see that there is a left denominator of Θ, namely V_Δ, such that $V_\Delta\Theta$ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. According to part (a) this left denominator V_Δ is necessarily a smallest one. Since V is a smallest left denominator of Θ, the function $W := VV_\Delta^{-1}$ is a constant function with unitary value. Moreover, we get from (14) that $\Psi = V\Theta = W\Xi^{-1}$. Since Ξ^{-1} is an outer function thus it is clear that Ψ is an outer function. Hence the asserted equivalence is verified. The other case can be proved analogously ∎

5. Aspects of left and right connected pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions

Our first goal of our further considerations is a characterization of the situation that a left or right connected pair of Smirnov type (respectively, of inverse Smirnov type) $[\Phi, \Psi]$ exists if the $[H^2(\mathbb{D})]^{q\times q}$-function Φ is given. Recall that we already know that in this case the matrix-valued functions Φ and Ψ necessarily admit pseudocontinuations (see Proposition 2).

Proposition 3. *Let $\Phi \in [H^2(\mathbb{D})]^{q\times q}$ be such that $\det \Phi$ does not vanish identically in $\mathbb{D}$. Suppose that Φ admits a pseudocontinuation $\Phi^\#$. Then the following statements are equivalent:*

(i) *There is a function $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ such that $[\Phi, \Psi]$ is a left connected pair of inverse Smirnov type.*

(ii) *There is a function $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ such that $[\Phi, \Psi]$ is a right connected pair of inverse Smirnov type.*

(iii) *There is a function $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ such that $[\Psi, \Phi]$ is a left connected pair of Smirnov type.*

(iv) *There is a function $\Psi \in [H^2(\mathbb{D})]^{q\times q}$ such that $[\Psi, \Phi]$ is a right connected pair of Smirnov type.*

(v) $(\widehat{\Phi\#})^{-1} \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

Proof. (i) $\Rightarrow$ (v) (respectively, (ii) $\Rightarrow$ (v)): From (i) (respectively, (ii)) and Remark 2 we see that Ψ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Hence Ψ^{-1} belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ as well. Lemma 2.5 in [15] shows that $V := \Psi(\widehat{\Phi\#})^{-1}$ (respectively, $V := (\widehat{\Phi\#})^{-1}\Psi$) is an inner $q \times q$ Schur function. Since $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$, the identity $(\widehat{\Phi\#})^{-1} = \Psi^{-1}V$ (respectively, $(\widehat{\Phi\#})^{-1} = V\Psi^{-1}$) provides (v).

(v) $\Rightarrow$ (i) (respectively, (v) $\Rightarrow$ (ii)): According to Theorem 10 we get from (v) that there is an outer function $\Sigma \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and an inner $q \times q$ Schur function V such that $(\widehat{\Phi\#})^{-1} = \Sigma V$ (respectively, $(\widehat{\Phi\#})^{-1} = V\Sigma$). Then $\Psi := \Sigma^{-1}$ is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$ which satisfies $\Psi = V\widehat{\Phi\#}$ (respectively, $\Psi = \widehat{\Phi\#}V$). Thus we can conclude that V is a left (respectively, right) denominator of $\widehat{\Phi\#}$. Consequently, Proposition 2 in [15] yields (i) (respectively, (ii)).

(i) $\Leftrightarrow$ (iv) and (ii) $\Leftrightarrow$ (iii): Use Remark 1 ∎

In view of Lemma 2.5 in [15] and Proposition 1 we introduce the following notions.

Definition 2. Let $[\Phi, \Psi]$ be a left (respectively, right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. Suppose that the function $\det \Phi$ does not vanish identically in $\mathbb{D}$. Let V be the (unique) inner $q \times q$ Schur function which realizes the left (respectively, right) connection of $[\Phi, \Psi]$.

(a) The pair $[\Phi, \Psi]$ is called *minimal* if V is a smallest right (respectively, left) denominator of $\widehat{\Psi\#}$.

(b) The pair $[\Phi, \Psi]$ is said to be *cominimal* if V is a smallest left (respectively, right) denominator of $\widehat{\Phi\#}$.

From Remark 2 in [15] we see immediately the following

Remark 3. The pair $[\Phi, \Psi]$ is a minimal left (respectively, minimal right) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions if and only if $[\Psi, \Phi]$ is a cominimal right (respectively, cominimal left) connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions.

Proposition 4. *Let $[\Phi, \Psi]$ be a left or right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. Suppose that* $\det \Phi$ *does not vanish identically in $\mathbb{D}$. Let V be the inner $q \times q$ Schur function which realizes this left or right connection, respectively. Further, let Φ_o and Ψ_o be outer functions that belong to $[H^2(\mathbb{D})]^{q\times q}$, and let Φ_i and Ψ_i be inner $q \times q$ Schur functions such that $\Phi = \Phi_o\Phi_i$ and $\Psi = \Psi_i\Psi_o$ (respectively, $\Phi = \Phi_i\Phi_o$ and $\Psi = \Psi_o\Psi_i$). Then the following statements hold true:*

(a) *$[\Phi, \Psi]$ is a minimal left or minimal right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions if and only if V and Φ_i have no common non-constant inner right or left divisor, respectively.*

(b) *$[\Phi, \Psi]$ is a cominimal left or cominimal right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions if and only if V and Ψ_i have no common non-constant inner left or right divisor, respectively.*

Proof. (a): We will consider that case that $[\Phi, \Psi]$ is a minimal left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions. Using Proposition 1 we have $\widehat{\Psi\#} = \Phi V^{-1} = \Phi_o\Phi_i V^{-1}$. From part (b) of Corollary 2 we get immediately the asserted equivalence. The case that $[\Phi, \Psi]$ is a minimal right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions can be proved similarly.

(b): Use part (a) and Remark 3 ∎

Lemma 5.

(a) *Every left or right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions of Smirnov type is necessarily minimal.*

(b) *Every left or right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions of inverse Smirnov type is necessarily cominimal.*

Proof. (a): Let $[\Phi, \Psi]$ be a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions, and let V be the inner $q \times q$ Schur function which realizes this connection. According to Proposition 1, we have $\Phi = \widehat{\Psi\#}V$. Assume that $[\Phi, \Psi]$ is a pair of Smirnov type, i. e., that Φ is an outer function in $[H^2(\mathbb{D})]^{q\times q}$. Then Remark 2 and Lemma 4 show that V is a minimal right denominator of $\widehat{\Psi\#}$, i. e., the pair $[\Phi, \Psi]$ is minimal. The case that $[\Phi, \Psi]$ is a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions can be treated analogously.

(b): Use part (a) and Remarks 1 and 3 ∎

Proposition 5. *Let $[\Phi, \Psi]$ be a left or right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions such that* $\det \Phi$ *does not vanish identically.*

(a) *Let the pair $[\Phi, \Psi]$ be minimal. Then $[\Phi, \Psi]$ is of Smirnov type if and only if $(\widehat{\Psi\#})^{-1}$ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*

(b) *Let the pair $[\Phi, \Psi]$ be cominimal. Then $[\Phi, \Psi]$ is of inverse Smirnov type if and only if $(\widehat{\Phi\#})^{-1}$ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.*

Proof. (a): Assume that $[\Phi, \Psi]$ is a left connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions which is minimal. Let V be the (unique) inner $q \times q$ Schur function which realizes this left connection. Then V is a minimal right denominator of $\widehat{\Psi\#}$. In view of Proposition 1, we have $\Phi = \widehat{\Psi\#}V$. Thus we obtain from Lemma 4 that $[\Phi,\Psi]$ is of Smirnov type if and only if $(\widehat{\Psi\#})^{-1}$ belongs to $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. The case that $[\Phi,\Psi]$ is a right connected pair of $[H^2(\mathbb{D})]^{q\times q}$-functions can be treated similarly.

(b): Use part (a) and Remarks 1 and 3 ∎

We have not characterized all left (respectively, right) connected pairs which are minimal as well as cominimal. What can be said about the existence of cominimal pairs which are not minimal? The following example gives an affirmative answer to the last question.

Example 1. Let Φ_o be an outer function of $H^2(\mathbb{D}) \setminus \{0\}$ which admits a pseudocontinuation $\Phi_o^\#$ such that there exists a non-constant smallest (left) denominator L of $\widehat{\Phi_o^\#}$. Then $\Phi := \Phi_o L$ is a pseudocontinuable function which belongs to $H^2(\mathbb{D})\setminus\{0\}$. Let V be a smallest (left) denominator of $\widehat{\Phi\#}$, and let $\Psi := V\widehat{\Phi\#}$. Then $[\Phi,\Psi]$ is a left connected pair of $H^2(\mathbb{D})$-functions which is cominimal and not minimal.

Indeed, the definition of Φ and Ψ implies that $[\Phi,\Psi]$ is a left connected pair of $H^2(\mathbb{D})$-functions and V is the unique inner function which realizes this left connection. The cominimality follows directly from the fact that V is a smallest left denominator of $\widehat{\Phi\#}$. From the definition of Φ, we get $\widehat{\Phi\#} = L^{-1}\widehat{\Phi_o^\#}$. Then it follows $L\widehat{\Phi\#} = \widehat{\Phi_o^\#} = L^{-1}\Lambda_i\Lambda_o$, i.e., L^2 is a left denominator of $\widehat{\Phi\#}$. In view of the fact that V is a smallest left denominator of $\widehat{\Phi\#}$, there is an inner function $X \in \mathcal{S}_{1\times 1}(\mathbb{D})$ such that $L^2 = XV$. This implies that L und V have common non-constant inner divisors. From Proposition 1, we get $\widehat{\Psi\#}V = \Phi$ and finally $\widehat{\Psi\#} = \Phi_o L V^{-1}$. By virtue of part (b) of Corollary 2, we get that V is not a smallest right denominator of $\widehat{\Psi\#}$, i. e., the pair $[\Phi,\Psi]$ is not minimal.

6. Carathéodory functions generated by a j_{qq}-inner function

If W is a j_{qq}-inner function, then both functions

$$\Omega_l := (W_{22}+W_{21})^{-1}(W_{22}-W_{21}) \quad \text{and} \quad \Omega_r := (W_{22}-W_{12})(W_{22}+W_{12})^{-1}$$

are (well-defined) $q\times q$ Carathéodory functions (see [15: Propositions 5.1 and 5.2]). These functions Ω_l and Ω_r, which are called the *left $q\times q$ Carathéodory function* and the *right $q\times q$ Carathéodory function*, respectively, *generated by* W, play an essential role in the context of ADD-parametrization of j_{qq}-inner functions (see [15]). From Propositions 5.4 and 5.5 in [15] we know that these $q\times q$ Carathéodory functions Ω_l and Ω_r necessarily admit pseudocontinuations and that the identities

$$\Omega_l + \widehat{\Omega_l^\#} = 2\Phi_l\widehat{\Phi_l^\#} = 2\widehat{\Psi_l^\#}\Psi_l \quad \text{and} \quad \Omega_r + \widehat{\Omega_r^\#} = 2\widehat{\Phi_r^\#}\Phi_r = 2\Psi_r\widehat{\Psi_r^\#}$$

are satisfied where $[\Phi_l,\Psi_l]$ and $[\Phi_r,\Psi_r]$ are the left and right connected pairs of $[H^2(\mathbb{D})]^{q\times q}$-functions, respectively, generated by the given j_{qq}-inner function W. Since the functions $\det \Phi_l$ and $\det \Phi_r$ do not vanish identically in $\mathbb{D}$, the functions $\det(\Omega_l+\widehat{\Omega_l^{\#}})$ and $\det(\Omega_r+\widehat{\Omega_r^{\#}})$ do not vanish identically in $\mathbb{D}$ as well. Thus, in view of Theorems 2.1 and 2.2 in [11], it is readily checked that Ω_l and Ω_r are $q \times q$ Carathéodory functions of finite entropy. A $q \times q$ Carathéodory function Ω is said to be *of finite entropy* if

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log[\det(\mathrm{Re}\ \underline{\Omega})]\mathrm{d}\underline{\lambda} > -\infty .$$

If Ω_1 and Ω_2 are arbitrary pseudocontinuable $q \times q$ Carathéodory functions of finite entropy, then Theorem 6.8 in [15] gives necessary and sufficient conditions for the case that the set $\mathcal{I}_{\Omega_1,\Omega_2}$ of all j_{qq}-inner functions W such that Ω_1 and Ω_2 are the left and right $q\times q$ Carathéodory functions, respectively, generated by W is non-empty. The goal of our following considerations is to study the problem of the existence of a function W which belongs to particular subclasses of j_{qq}-inner functions and which satisfies $\Omega_l = \Omega_1$ and $\Omega_r = \Omega_2$. Hereby, note that a $q \times q$ Schur function f is said to be *of finite entropy* if

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log[\det(I - \underline{f}\ \underline{f}^*)]\mathrm{d}\underline{\lambda} > -\infty .$$

The formulation of Theorem 6.8 in [15], already mentioned above, contains the description of the case $\mathcal{I}_{\Omega_1,\Omega_2} \neq \emptyset$ in terms of the existence of inner $q\times q$ Schur functions which satisfy some identities. However, its proof additionally yields an explicit interrelation between functions which belong to $\mathcal{I}_{\Omega_1,\Omega_2}$ and these inner $q \times q$ Schur functions, which is the following

Theorem 11. *Let Ω_1 and Ω_2 be $q \times q$ Carathéodory functions of finite entropy. Suppose that both functions Ω_1 and Ω_2 admit pseudocontinuations. Then the functions $f_1 := (I - \Omega_1)(I + \Omega_1)^{-1}$ and $f_2 := (I - \Omega_2)(I + \Omega_2)^{-1}$ are pseudocontinuable $q \times q$ Schur functions of finite entropy. In particular,*

$$\rho_1 := I - f_1\widehat{f_1^{\#}}, \quad \rho_2 := I - f_2\widehat{f_2^{\#}}, \quad \sigma_1 := I - \widehat{f_1^{\#}} f_1, \quad \sigma_2 := I - \widehat{f_2^{\#}} f_2$$

are functions whose determinants do not vanish identically. Let ϕ_1 and ϕ_2 be the unique normalized outer left spectral factors of $\underline{\rho_1}$ and $\underline{\rho_2}$, respectively, and let ψ_1 and ψ_2 be the unique normalized outer right spectral factors of $\underline{\sigma_1}$ and $\underline{\sigma_2}$, respectively.

(a) *If $\mathcal{I}_{\Omega_1,\Omega_2}$ is non-empty and W belongs to $\mathcal{I}_{\Omega_1,\Omega_2}$, then $c_2 := W_{22}^{-1}\psi_2^{-1}$ and $b_2 := \phi_2^{-1}(W_{11} - W_{12}W_{22}^{-1}W_{21})$ are inner $q\times q$ Schur functions which satisfy the identities*

$$f_1 = c_2\psi_2\widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 \tag{15}$$

and

$$W = \begin{bmatrix} I & f_2 \\ \widehat{f_2^{\#}} & I \end{bmatrix} \cdot \mathrm{diag}(\rho_2^{-1}\phi_2, \psi_2^{-1}) \cdot \mathrm{diag}(b_2, c_2^{-1}) . \tag{16}$$

(b) *If $\mathcal{I}_{\Omega_1,\Omega_2}$ is non-empty and W belongs to $\mathcal{I}_{\Omega_1,\Omega_2}$, then $b_1 := \phi_1^{-1}W_{22}^{-1}$ and $c_1 := (W_{11} - W_{12}W_{22}^{-1}W_{21})\psi_1^{-1}$ are inner $q \times q$ Schur functions which fulfill*

$$f_2 = c_1\psi_1\widehat{f_1^{\#}}\rho_1^{-1}\phi_1 b_1 \tag{17}$$

and

$$W = \operatorname{diag}(c_1, b_1^{-1}) \cdot \operatorname{diag}(\psi_1\sigma_1^{-1}, \phi_1^{-1}) \cdot \begin{bmatrix} I & \widehat{f_1^{\#}} \\ f_1 & I \end{bmatrix} . \tag{18}$$

(c) *If c_2 and b_2 are inner $q \times q$ Schur functions which satisfy* (15)*, then W defined by* (16) *belongs to $\mathcal{I}_{\Omega_1,\Omega_2}$, and the representations $c_2 = W_{22}^{-1}\psi_2^{-1}$ and $b_2 = \phi_2^{-1}(W_{11} - W_{12}W_{22}^{-1}W_{21})$ of c_2 and b_2 hold true.*

(d) *If c_1 and b_1 are inner $q \times q$ Schur functions which satisfy* (17)*, then W defined by* (18) *belongs to $\mathcal{I}_{\Omega_1,\Omega_2}$, and the representations $c_1 = (W_{11} - W_{12}W_{22}^{-1}W_{21})\psi_1^{-1}$ and $b_1 = \phi_1^{-1}W_{22}^{-1}$ of c_1 and b_1 hold true.*

Proof. The representation (16) of W follows from the equations

$$(c_2\psi_2)^{-1} f_1 = \widehat{f_2^{\#}}\rho_2^{-1}\phi_2 b_2 \qquad \text{and} \qquad \phi_2 b_2 + f_2(c_2\psi_2)^{-1} f_1 = \rho_2^{-1}\phi_2 b_2$$

which are an immediate consequences of identity (15). The rest of the assertion stated in parts (a) and (c) was already verified in the proof of Theorem 6.8 in [15]. Applying Proposition 8 in [14], parts (b) and (d) can be proved analogously ∎

Using Theorem 11 we will be able to characterize the situation that the set $\mathcal{I}_{\Omega_1,\Omega_2}$ contains j_{qq}-inner functions which are of Smirnov type and of inverse Smirnov type, respectively.

Corollary 3. *Let the assumptions of Theorem* 11 *be satisfied. Suppose that $\mathcal{I}_{\Omega_1,\Omega_2}$ is non-empty, and let W belong to $\mathcal{I}_{\Omega_1,\Omega_2}$. Then:*

(a) *The following statements are equivalent:*

(i) *W is a j_{qq}-inner function of Smirnov type.*

(ii) *$c_2 := W_{22}^{-1}\psi_2^{-1}$ is a constant $q \times q$ Schur function with unitary value.*

(iii) *$b_1 := \phi_1^{-1}W_{22}^{-1}$ is a constant $q \times q$ Schur function with unitary value.*

(b) *The following statements are equivalent:*

(iv) *W is a j_{qq}-inner function of inverse Smirnov type.*

(v) *$b_2 := \phi_2^{-1}(W_{11} - W_{12}W_{22}^{-1}W_{21})$ is a constant $q \times q$ Schur function with unitary value.*

(vi) *$c_1 := (W_{11} - W_{12}W_{22}^{-1}W_{21})\psi_1^{-1}$ is a constant $q \times q$ Schur function with unitary value.*

Proof. If ϕ (respectively, ψ) is an outer function in $[H^\infty(\mathbb{D})]^{q\times q}$ and if b (respectively, c) is an inner $q \times q$ Schur function, then $F := \phi b$ (respectively, $G := c\psi$) is also an outer function in $[H^\infty(\mathbb{D})]^{q\times q}$ if and only if b (respectively, c) is a constant $q \times q$ Schur function (with unitary value) (see, e.g., [17]). Thus the combination of Theorem 2 in [5] and Theorem 11 yields the asserted equivalences ∎

Obviously, Corollary 3 can be used to characterize Arov-singular j_{qq}-inner functions W which belong to $\mathcal{I}_{\Omega_1,\Omega_2}$. Moreover, in view of Theorem 11 and Corollary 3, we get immediately necessary and sufficient conditions for the existence of functions which belong to the considered subclasses of j_{qq}-inner functions and which have prescribed associated left and right $q \times q$ Carathéodory functions:

Theorem 12. Let the assumptions of Theorem 11 be satisfied. Then:

(a) The following statements are equivalent:

(i) There is a j_{qq}-inner function W of Smirnov type such that Ω_1 and Ω_2 are the left $q \times q$ Carathéodory function and the right $q \times q$ Carathéodory function, respectively, generated by W.

(ii) There are an inner $q \times q$ Schur function b_2 and a unitary $q \times q$ complex matrix u_2 such that $f_1 = u_2 \widehat{\psi_2 f_2^\# \rho_2^{-1} \phi_2} b_2$.

(iii) There are an inner $q \times q$ Schur function c_1 and a unitary $q \times q$ complex matrix u_1 such that $f_2 = c_1 \widehat{\psi_1 f_1^\# \rho_1^{-1} \phi_1} u_1$.

(b) The following statements are equivalent:

(iv) There is a j_{qq}-inner function W of inverse Smirnov type such that Ω_1 and Ω_2 are the left $q \times q$ Carathéodory function and the right $q \times q$ Carathéodory function, respectively, generated by W.

(v) There are an inner $q \times q$ Schur function c_2 and a unitary $q \times q$ complex matrix v_2 such that $f_1 = c_2 \widehat{\psi_2 f_2^\# \rho_2^{-1} \phi_2} v_2$.

(vi) There are an inner $q \times q$ Schur function b_1 and a unitary $q \times q$ complex matrix v_1 such that $f_2 = v_1 \widehat{\psi_1 f_1^\# \rho_1^{-1} \phi_1} b_1$.

(c) The following statements are equivalent:

(vii) There is an Arov-singular j_{qq}-inner function W such that Ω_1 and Ω_2 are the left $q \times q$ Carathéodory function and the right $q \times q$ Carathéodory function, respectively, generated by W.

(viii) $f_1 = u_2 \widehat{\psi_2 f_2^\# \rho_2^{-1} \phi_2} v_2$, for some unitary $q \times q$ complex matrices u_2 and v_2.

(xi) $f_2 = v_1 \widehat{\psi_1 f_1^\# \rho_1^{-1} \phi_1} u_1$, for some unitary $q \times q$ complex matrices u_1 and v_1.

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