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On the Dirac Operator with an Electromagnetic Potential

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Abstract. A new approach based on the construction of some special biquaternionic projection operators is proposed for analysis and solution of the Dirac equation with electromagnetic potential. There is given an example of the application of this technique which allows us to find the solutions for some class of potentials.

Keywords: *Dirac's operator, biquaternions, eikonal equation, exact solutions*

AMS subject classification: 81 Q 05

1. Introduction

In this work we consider the Dirac equation with an electromagnetic potential using its biquaternionic form:

$$(D + f(x)I + M^{\alpha})u(x) = 0. \tag{1}$$

Here D is the *Moisil-Theodoresco operator* (also sometimes called *Dirac operator*), $D = \sum_{k=1}^{3} i_k \partial_k$, i_k are standard basic quaternions, $\partial_k = \frac{\partial}{\partial x_k}$, f and u are biquaternion-valued functions, I is the identity operator, and M^{α} is the operator of multiplication from the right-hand side by the biquaternion α. Equation (1) may be obtained from the classic Dirac equation

$$\left(i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + im + ie\left(\gamma_0 \phi(x) + \sum_{k=1}^{3} \gamma_k A_k(x) \right) \right) \Phi(x) = 0, \tag{2}$$

by a simple matrix transform introduced in [3] (see also [4] and [5: Section 12]). Equations (1) and (2) are equivalent, any solution of (1) with the aid of the matrix transform may be converted to a solution of (2) and vice versa.

The simplest case, when the scalar part of the potential f is zero and the vector part is the gradient of an abritrary scalar function, was completely studied in [4, 9] due to the possible factorization

$$D - \frac{\operatorname{grad}\eta}{\eta} I + M^{\alpha} = \eta(D + M^{\alpha})\eta^{-1} I, \tag{3}$$

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and due to the fact that the integral representations as well as the solutions to some boundary value problems corresponding to the operator $D + M^\alpha$ were obtained in [5]. Note that the case $\mathrm{Vec}(f) \equiv 0$ seems to be at least of the same level of difficulty as the gradient case but up to now it is not clear how to solve it.

The principal idea of the article is to reduce in some sense equation (1) to the gradient case. For this purpose we use some specially constructed projection operators based on the algebraic properties of biquaternionic zero divisors. As one of the possible applications we show how this technique allows us to obtain the solutions of (1) in some special cases. In order to simplify the exposition we consider the operator $D + fI$ without M^α because the last constant term as a rule does not represent any considerable difficulty.

2. Preliminaries

Let us denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (=biquaternions). Each element q of $\mathbb{H}(\mathbb{C})$ is represented in the form $q = \sum_{k=0}^{3} q_k i_k$, where $\{q_k\} \subset \mathbb{C}$, i_0 is the unit and i_k $(k = \overline{1,3})$ are standard quaternionic imaginary units. We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with i_k $(k = \overline{0,3})$. We will use also the vector representation of $q \in \mathbb{H}(\mathbb{C})$: $q = \mathrm{Sc}(q) + \mathrm{Vec}(q)$, where $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = \vec{q} = \sum_{k=1}^{3} q_k i_k$. A complex quaternion of the form $q = \vec{q}$ will be called *purely vectorial*. We identify them with vectors from $\mathbb{C}^3$. The quaternion $\overline{q} = q_0 - \vec{q}$ is called *conjugated* to q.

Let us denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. For different equivalent descriptions of $\mathfrak{S}$ see, e.g., [5: p. 28]. We will use two of them:

$$a \in \mathfrak{S} \quad \Longleftrightarrow \quad \begin{array}{l} 1.\ a\overline{a} = 0 \\ 2.\ a^2 = 2a_0 a. \end{array} \tag{4}$$

As usual, zero is not included to $\mathfrak{S}$.

We will consider $\mathbb{H}(\mathbb{C})$-valued functions given in a domain $\Omega \subset \mathbb{R}^3$. On the set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the well-known Moisil-Theodoresco operator is defined by the formula $D = \sum_{k=1}^{3} i_k \partial_k$, which was introduced for the first time in [6, 7]. Let us introduce the integral operators

$$(Tf)(x) = \int_\Omega \mathcal{K}(x-y) f(y)\, d\Omega_y \quad (x \in \mathbb{R}^3) \tag{5}$$

$$(Kf)(x) = -\int_\Gamma \mathcal{K}(x-y)\vec{n}(y) f(y)\, d\Gamma_y \quad (x \in \mathbb{R}^3 \setminus \Gamma) \tag{6}$$

which are the analogs corresponding to D of the complex T-operator and Cauchy-type operator, respectively. Here $\Gamma = \partial\Omega$ is a Liapunov surface, $\vec{n} = \sum_{k=1}^{3} n_k i_k$ is the outward unit normal to Γ, and $\mathcal{K}(x) = -\frac{\vec{x}}{4\pi |x|^3}$.

We will need the following properties of the introduced integral operators (see, e.g., [2: Chapter 1]).

Theorem 1.

1) (Borel-Pompeiu formula): *Let* $f \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$. *Then*

$$(Kf)(x) + (TDf)(x) = f(x) \quad (x \in \Omega).$$

2) (Cauchy integral formula): *Let* $f \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega}) \cap \mathrm{Ker}D(\Omega)$. *Then*

$$f = Kf \quad in \ \Omega. \tag{7}$$

3) *Let* $f \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$. *Then*

$$DTf = f \quad in \ \Omega. \tag{8}$$

3. Projection operators for the disturbed Moisil-Theodoresco equation

Let us consider the equation

$$(D + f)u = 0, \tag{9}$$

where $D = \sum_{k=1}^3 i_k \partial_k$, f is a given $\mathbb{H}(\mathbb{C})$-valued function, and u is also an $\mathbb{H}(\mathbb{C})$-valued function. First, we will consider the case, when the values of f and of $\vec{f}$ in all points are not zero divisors. Then let us introduce a complex-valued function $\gamma = \sqrt{\vec{f}^2}$, where it is not important which of the roots to select. Then the function $\gamma + \vec{f}$ determines a zero divisor in all the domain of definition of the function f, we have

$$(\gamma + \vec{f})(\gamma - \vec{f}) = \gamma^2 - \vec{f}^2 = 0.$$

Using the corresponding idempotents $\frac{1}{2\gamma}(\gamma \pm \vec{f})$ we define the two operators

$$P^+ = \frac{1}{2\gamma}(\gamma + \vec{f})I \qquad \text{and} \qquad P^- = \frac{1}{2\gamma}(\gamma - \vec{f})I,$$

where I is the identity operator. These operators are mutually complementary, orthogonal projection operators on the set of $\mathbb{H}(\mathbb{C})$-valued functions.

We ahve the following

Proposition 1. *The operator* $D + fI$ *can be rewritten in the form*

$$D + fI = P^+ D_{\xi_+} + P^- D_{\xi_-} = D_{\xi_+} P^+ + D_{\xi_-} P^-, \tag{10}$$

where the complex-valued functions ξ_+ *and* ξ_- *are defined as* $\xi_+ = f_0 + \gamma$, $\xi_- = f_0 - \gamma$ *and* $D_{\xi_\pm} = D + \xi_\pm I$.

Recall that $\vec{f}^2 = \vec{f} \cdot \vec{f} = -\langle \vec{f}, \vec{f} \rangle$, where the last term is a scalar product of two vectors.

Let us verify (10):

$$\begin{aligned} P^+ D_{\xi_+} + P^- D_{\xi_-} &= \frac{1}{2\gamma}(\gamma + \vec{f})(D + \xi_+ I) + \frac{1}{2\gamma}(\gamma - \vec{f})(D + \xi_- I) \\ &= D + \frac{1}{2\gamma}\{(\gamma + \vec{f})(f_0 + \gamma) + (\gamma - \vec{f})(f_0 - \gamma)\} I \\ &= D + \frac{1}{2\gamma}\{2\gamma f_0 + 2\gamma \vec{f}\} I \\ &= D + fI. \end{aligned}$$

At the same time

$$\begin{aligned} D_{\xi_+} P_+ + D_{\xi_-} P_- &= (D + \xi_+)\frac{1}{2\gamma}(\gamma + \vec{f}) I + (D + \xi_-)\frac{1}{2\gamma}(\gamma - \vec{f}) I \\ &= D + \frac{1}{2\gamma}\{\xi_+(\gamma + \vec{f}) + \xi_-(\gamma - \vec{f})\} I \\ &= D + \frac{1}{2\gamma}\{\gamma(\xi_+ + \xi_-) + \vec{f}(\xi_+ - \xi_-)\} I \\ &= D + \frac{1}{2\gamma}\{2\gamma f_0 + 2\gamma \vec{f}\} I \\ &= D + fI. \end{aligned}$$

Of course, (10) does not signify that $D_{\xi_\pm}$ commute with $P^\pm$. They "commute" only "simultaneously". Moreover,

$$D_{\xi_+} P^+ u = (D + \xi_+)\frac{1}{2\gamma}(\gamma + \vec{f}) u = \frac{1}{2}\left(Du + D\left[\frac{\vec{f}}{\gamma} u\right] + \xi_+ u + \xi_+ \frac{\vec{f}}{\gamma} u\right),$$

and if $u \in \operatorname{Ker}(D + fI)$, then

$$\begin{aligned} D_{\xi_+} P^+ u &= \frac{1}{2}\left(Du + (\gamma + f_0) u + (\gamma + f_0)\frac{\vec{f}}{\gamma} u + D\left[\frac{\vec{f}}{\gamma} u\right]\right) \\ &= \frac{1}{2}\left(\gamma u + \frac{f_0 \vec{f}}{\gamma} u + D\left[\frac{\vec{f}}{\gamma} u\right]\right). \end{aligned}$$

For $D_{\xi_-} P^-$ from (1) we obtain the same but with the sign minus.

Similar arguments lead to the relations

$$D + \bar{f} I = P^+ D_{\xi_-} + P^- D_{\xi_+} = D_{\xi_-} P^+ + D_{\xi_+} P^-, \tag{11}$$

where $\bar{f} = f_0 - \vec{f}$. Then from (10) and (11) we obtain the inverse relations

$$\begin{aligned} D_{\xi_+} &= P^+(D + fI) + P^-(D + \bar{f} I) \\ D_{\xi_-} &= P^+(D + \bar{f} I) + P^-(D + fI). \end{aligned}$$

Now let us consider the equation

$$\vec{g}_+^2 = \xi_+^2, \tag{12}$$

where $\vec{g}_+ = \operatorname{grad}\mu_+$, and, in fact, it is an equation for μ_+. Equation (12) is very well known and is called *eikonal equation* (for its solution see, e.g., [1: Section 2.3] and [8: Section 3.1]). Such a vector $\vec{g}_+$ exists for any scalar function ξ_+. The same for the function ξ_-:

$$\vec{g}_-^2 = \xi_-^2, \tag{13}$$

$\vec{g}_- = \operatorname{grad}\mu_-$. Then let us introduce pairs of projection operators

$$Q^\pm = \frac{1}{2\xi_+}(\xi_+ \pm \vec{g}_+)I \qquad \text{and} \qquad R^\pm = \frac{1}{2\xi_-}(\xi_- \pm \vec{g}_-)I$$

corresponding to ξ_+ and ξ_-, respectively. For the operators D_{ξ_+} and D_{ξ_-} we have

$$\begin{aligned} D_{\xi_+} &= Q^+ D_{\vec{g}_+} + Q^- D_{-\vec{g}_+} = D_{\vec{g}_+} Q^+ + D_{-\vec{g}_+} Q^- \\ D_{\xi_-} &= R^+ D_{\vec{g}_-} + R^- D_{-\vec{g}_-} = D_{\vec{g}_-} R^+ + D_{-\vec{g}_-} R^-, \end{aligned}$$

where $D_{\vec{g}_\pm} = D + \vec{g}_\pm I$. For the vectors $\vec{g}_\pm$ we have another representation:

$$\vec{g}_\pm = -\frac{\operatorname{grad}\eta_\pm}{\eta_\pm},$$

where $\eta_\pm(x) = C_\pm e^{-\mu_\pm(x)}$ with C_+ and C_- complex constants. Then

$$D_{\vec{g}_\pm} = D + \operatorname{grad}\mu_\pm I = D - \frac{\operatorname{grad}\eta_\pm}{\eta_\pm} I.$$

In [9] this operator was studied using the factorization

$$D - \frac{\operatorname{grad}\eta_\pm}{\eta_\pm} I = \eta_\pm D \eta_\pm^{-1} I.$$

A similar representation we obtain for $D_{-\vec{g}_\pm}$: $D_{-\vec{g}_\pm} = \eta_\pm^{-1} D \eta_\pm I$. Thus, for the operators D_{ξ_+} and D_{ξ_-} we have

$$\begin{aligned} D_{\xi_+} &= Q^+ \eta_+ D \eta_+^{-1} I + Q^- \eta_+^{-1} D \eta_+ I = \eta_+ D \eta_+^{-1} Q^+ + \eta_+^{-1} D \eta_+ Q^- \\ D_{\xi_-} &= R^+ \eta_- D \eta_-^{-1} I + R^- \eta_-^{-1} D \eta_- I = \eta_- D \eta_-^{-1} R^+ + \eta_-^{-1} D \eta_- R^-. \end{aligned}$$

Finally, for the operator $D + fI$ we obtain the representation

$$\begin{aligned} D + fI &= P^+ Q^+ \eta_+ D \eta_+^{-1} I + P^+ Q^- \eta_+^{-1} D \eta_+ I \\ &\quad + P^- R^+ \eta_- D \eta_-^{-1} I + P^- R^- \eta_-^{-1} D \eta_- I \\ &= P^+ \eta_+ D \eta_+^{-1} Q^+ + P^+ \eta_+^{-1} D \eta_+ Q^- \\ &\quad + P^- \eta_- D \eta_-^{-1} R^+ + P^- \eta_-^{-1} D \eta_- R^-. \end{aligned} \tag{14}$$

Now let us concentrate on the case $D+\xi$, where $\xi = \mathrm{Sc}(\xi)$ and

$$\vec{g}^{\,2} = \xi^2,$$

$\vec{g} = \operatorname{grad}\mu = -\frac{\operatorname{grad}\eta}{\eta}$ and $\eta = Ce^{-\mu}$. Define $Q^{\pm} = \frac{1}{2\xi}(\xi \pm \vec{g}\,)I$. Then

$$D + \xi I = Q^+ D_{\vec{g}} + Q^- D_{-\vec{g}}.$$

We obtain that

$$u \in \operatorname{Ker}(D+\xi) \quad \Longleftrightarrow \quad \begin{cases} Q^+\eta D\eta^{-1}u = 0 \\ Q^-\eta^{-1}D\eta u = 0. \end{cases}$$

In another form:

$$\left.\begin{aligned} Q^+Dv &= 0 \\ Q^-Dw &= 0 \end{aligned}\right\}$$

where $v = \frac{u}{\eta}$ and $w = \eta u$. It is not difficult to find the general solution of both equations. Using Theorem 1 we obtain

$$\begin{aligned} v &= K\phi_1 + TQ^-\psi_1 \\ w &= K\phi_2 + TQ^+\psi_2 \end{aligned}$$

where ϕ_k and ψ_k are arbitrary biquaternion-valued functions. The main problem is to describe the intersection $\operatorname{Ker}(Q^+\eta D\eta^{-1}I) \cap \operatorname{Ker}(Q^-\eta^{-1}D\eta I)$ whiuch is exactly $\operatorname{Ker}(D+\xi I)$.

In the following section we show how all this machinery may give the result in some special case which includes an ample class of potential functions ξ.

4. Example of the application of the projectors' technique

Let us consider the following example. Assume that the function ξ satisfies two conditions:

1) $\Delta\xi = 0$
2) $(\operatorname{grad}\xi)^2 = C^2$

where C is an arbitrary complex number. The simplest example of such a function ξ is the linear function $\xi = ax_1 + bx_2 + cx_3 + d$, where a, b, c, d are complex constants. One more example is the function $\xi = az^n + cx_3 + d$, where $z := x_1 + ix_2$. We will construct a class of particular solutions for the equation

$$(D+\xi)u = 0. \tag{15}$$

Due to condition 2) we can immediately construct the vector $\vec{g}$ as

$$\vec{g} = \frac{1}{C}\xi \operatorname{grad}\xi.$$

Then

$$\vec{g} = \operatorname{grad}\mu = -\frac{\operatorname{grad}\eta}{\eta},$$

where $\mu = \frac{\xi^2}{2C}$ and $\eta = e^{-\frac{\xi^2}{2C}}$. The projection operators in that case have the form

$$Q^{\pm} := \frac{1}{2\xi}(\xi \pm \operatorname{grad}\mu) = \frac{1}{2}\left(1 \pm \frac{1}{C}\operatorname{grad}\xi\right).$$

Let us consider the function $f = Q^+e^{-\mu} + Q^-e^{\mu}$ and apply the operator D to it. Consider first

$$\begin{aligned} D[Q^+e^{-\mu}] &= D\left[\left(\frac{1}{2} + \frac{\operatorname{grad}\mu}{2\xi}\right)e^{-\mu}\right] \\ &= -\frac{1}{2}\operatorname{grad}\mu\cdot e^{-\mu} - \frac{\operatorname{grad}\xi}{2\xi^2}\operatorname{grad}\mu\cdot e^{-\mu} - \frac{1}{2\xi}(\operatorname{grad}\mu)^2e^{-\mu} - \frac{1}{2\xi}e^{-\mu}\Delta\mu. \end{aligned}$$

Using here the definition of the operator Q^+ and the fact that $(\operatorname{grad}\mu)^2 = \xi^2$ we obtain

$$D[Q^+e^{-\mu}] = -\xi Q^+e^{-\mu} + \frac{1}{2\xi}e^{-\mu}\left(D - \frac{\operatorname{grad}\xi}{\xi}\right)\operatorname{grad}\mu.$$

Reasoning along similar lines we obtain an analogous result for the function Q^-e^{μ}. Namely,

$$D[Q^-e^{\mu}] = -\xi Q^-e^{\mu} - \frac{1}{2\xi}e^{\mu}\left(D - \frac{\operatorname{grad}\xi}{\xi}\right)\operatorname{grad}\mu.$$

Thus,

$$D[Q^+e^{-\mu} + Q^-e^{\mu}] = -\xi(Q^+e^{-\mu} + Q^-e^{\mu}) + \frac{1}{2\xi}(e^{-\mu} - e^{\mu})\xi D\xi^{-1}\operatorname{grad}\mu.$$

Consider

$$\xi D\xi^{-1}\operatorname{grad}\mu = \xi D\xi^{-1}\frac{1}{C}\xi\operatorname{grad}\xi = \frac{1}{C}\xi D\operatorname{grad}\xi = -\frac{1}{C}\xi\Delta\xi = 0.$$

We obtain that

$$D[Q^+e^{-\mu} + Q^-e^{\mu}] + \xi(Q^+e^{-\mu} + Q^-e^{\mu}) = 0.$$

That is, $Q^+e^{-\mu} + Q^-e^{\mu} \in \operatorname{Ker}(D + \xi I)$. Moreover, $f := Q^+e^{-\mu}C_1 + Q^-e^{\mu}C_2 \in \operatorname{Ker}(D + \xi I)$, where C_1 and C_2 are arbitrary constant complex quaternions. Thus, the following proposition is true.

Proposition 2. *Let $\Delta\xi = 0$ and $(\operatorname{grad}\xi)^2 = C^2$ in some domain $\Omega \subset \mathbb{R}^3$ which may coincide with the whole $\mathbb{R}^3$, C be an arbitrary complex constant different from zero. Then the function*

$$f = \left(1 + \frac{1}{C}\operatorname{grad}\xi\right)e^{-\frac{\xi^2}{2C}}C_1 + \left(1 - \frac{1}{C}\operatorname{grad}\xi\right)e^{\frac{\xi^2}{2C}}C_2,$$

where C_1 and C_2 are arbitrary constant complex quaternions, is a solution of equation (15) *in the domain Ω.*

As can be seen immediately, we constructed also a particular solution to the Schrödinger equation with a quaternionic potential $(\Delta + (\operatorname{grad}\xi - \xi^2))f = 0$, where f is an $\mathbb{H}(\mathbb{C})$-valued function.

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