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Oscillation Theorems for Certain Class of Nonlinear Difference Equations

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Abstract. Some new oscillation results for certain class of forced nonlinear difference equations of the form

$$\Delta(a_n\,\Delta(x_n + p_n x_{n-k})) + q_n\,f(x_{n+1-l}) = e_n \qquad (n \in \mathbb{N}_0)$$

are established. Examples which dwell upon the importance of the results are also given.

Keywords: *Nonlinear difference equations, oscillation*

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1. Introduction

In recent years there has been an increasing interest in the study of the oscillatory and asymptotic behaviour of solutions of difference equations (see, e.g., [1 - 6] and the references therein). Numerous results exist for homogeneous difference equations with or without delay, however, the work on forced equations is scanty. Therefore, the purpose of the present paper is to investigate the oscillatory behaviour of solutions of a class of forced nonlinear difference equations of the form

$$\Delta\big(a_n\,\Delta(x_n + p_n x_{n-k})\big) + q_n\,f(x_{n+1-l}) = e_n \qquad (n \in \mathbb{N}_0) \tag{1}$$

where

$\Delta x_n = x_{n+1} - x_n$

$k, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{n_0} = \{n \in \mathbb{N}_0 : n \geq n_0\}$

$\{a_n\}$ $(a_n > 0), \{p_n\}, \{q_n\}$ $(q_n \geq 0)$ and $\{e_n\}$ are real sequences

$f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $u\,f(u) > 0$ for $u \neq 0$

$\Delta(a_n\,\Delta E_n) = e_n$ for some real sequence $\{E_n\}$.

Let $M = \max\{k, l\}$ and $N_0 \in \mathbb{N}_0$. By a *solution* of the difference equation (1) we mean a real sequence $\{x_n\}_{n \geq N_0 - M}$ which satisfies (1) for $n \geq N_0$. A solution $\{x_n\}_{n \geq N_0 - M}$ is said to be *non-oscillatory* if all terms x_n are eventually of fixed sign. Otherwise it is said to be *oscillatory*.

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Now for each result pertinent to the difference equation (1) that we shall prove, we require some of the following conditions:

(c_1) $0 \le p_n \le A < 1$ for a constant A.

(c_2) $p_n p_{n-k} \ge 0$ and $-1 \le -B \le p_n \le A \le 1$ for positive constants A and B.

(c_3) There are constants $\delta, \eta > 0$ such that $|u| > \delta$ implies $|f(u)| > \eta$.

(c_4) $\sum_{n=n_0}^{\infty}(n+1)q_n = \infty$.

(c_5) $\sum_{n=n_0}^{\infty}(n+1)^{\alpha}q_n = \infty$ where $\alpha < 1$.

(c_6) $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$.

(c_7) $\sum_{n=n_0}^{\infty} \frac{1}{a_{n-l}} (\sum_{s=n}^{\infty} q_s) = \infty$.

(c_8) f is non-decreasing and superlinear, i.e. $\int_c^{\infty} \frac{du}{f(u)} < \infty$ and $\int_{-c}^{-\infty} \frac{du}{f(u)} < \infty$ for every $c > 0$.

(c_9) f is sublinear, i.e. $\int_0^{c} \frac{du}{f(u)} < \infty$ and $\int_0^{-c} \frac{du}{f(u)} < \infty$ for every $c > 0$.

(c_{10}) $f(uv) \ge f(u)f(v)$ for all $u > 0$ and large v, and $\sum_{n=n_0}^{\infty} q_s f(R_{n+1-l}) = \infty$ where $R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}$.

(c_{11}) $\lim_{n\to\infty} E_n = 0$.

(c_{12}) $\{E_n\}$ is k-periodic.

We may note that $p_n p_{n-k} \ge 0$ if $\{p_n\}$ is k-periodic.

2. Oscillation of equation (1) when $lim_{n\to\infty} E_n = 0$

We begin with the following lemma which is needed in the sequel.

Lemma 1. *Let condition (c_2) hold, and let $\{E_n\}$ be a real bounded and $\{x_n\}$ a real eventually positive sequence. Then the sequence $\{z_n\}$, $z_n = x_n + p_n x_{n-k} - E_n$, is bounded if and only if $\{x_n\}$ is bounded. Further, if $\{z_n\}$ is of one sign for large n, then $\{x_n\}$ is unbounded implies that $\{z_n\}$ is unbounded and $z_n > 0$ for large n.*

Proof. Let $x_n > 0$ and $x_{n-k} > 0$ for $n \ge n_1 > n_0 \in \mathbb{N}_0$. Clearly, $\{x_n\}$ is bounded implies that $\{z_n\}$ is bounded. Next suppose that $\{z_n\}$ is bounded. Assume $\{x_n\}$ is unbounded. So there exists a sequence $\{n_j\}$ of natural numbers such that

$$lim_{j\to\infty} n_j = \infty, \qquad \lim_{j\to\infty} x_{n_j} = \infty, \qquad x_{n_j} = \max\{x_n : n_1 \le n \le n_j\}.$$

Since $n - k \le n$,

$$x_{n_j - k} \le \max\{x_n : n_1 \le n \le n_{j-k}\} \le \max\{x_n : n_1 \le n \le n_j\} = x_{n_j}.$$

Thus

$$z_{n_j} + E_{n_j} \ge x_{n_j} - Bx_{n_j - k} \ge (1-B)x_{n_j}$$

leads to a contradiction as $j \to \infty$. Hence $\{x_n\}$ is bounded.

Next suppose that $z_n > 0$ or $z_n < 0$ for $n > n_1^* > n_1$. Clearly, $\{x_n\}$ is unbounded implies that $\{z_n\}$ is unbounded. If $z_n < 0$ for $n \ge n_1^*$, then arguing as above, we obtain $E_{n_j} > z_{n_j} + E_{n_j} \ge (1-B)x_{n_j}$ which leads to a contradiction as $j \to \infty$. Thus $z_n > 0$ for $n \ge n_1^*$. This completes the proof of the lemma ∎

Theorem 2. *Let the conditions* $(c_2),(c_3),(c_4),(c_6)$ *and* (c_{11}) *hold. In addition, assume that the sequence* $\{a_n\}$ *is bounded. Then all bounded solutions of equation* (1) *are either oscillatory or tend to zero as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution of equation (1) for $n \in \mathbb{N}_{n_0}$. So there exists an $n_1 \in \mathbb{N}_{n_0}$ such that $x_n > 0$ or $x_n < 0$ for $n \geq n_1$. Let $x_n > 0$ for $n \geq n_1$. Hence there exists an $n_2 > n_1$ such that $x_{n-k} > 0$ and $x_{n-l} > 0$ for $n \geq n_2$. Letting $z_n = x_n + p_n x_{n-k} - E_n$ for $n \geq n_2$, we obtain from equation (1)

$$\Delta(a_n \Delta z_n) = -q_n f(x_{n+1-l}) \leq 0. \tag{2}$$

Then $\{z_n\}$ is bounded and $z_n > 0$ or $z_n < 0$ for $n \geq n_3 \geq n_2$. Let $z_n > 0$ for $n \geq n_3$. From (2) and condition (c_6) it follows that $a_n \Delta z_n > 0$ for $n \geq n_4 > n_3$ and hence $\Delta z_n > 0$ for $n \geq n_4$. Let $\lim_{n\to\infty} z_n = \lambda$, $0 < \lambda < \infty$. Clearly, there exists an integer $n_5 > n_4$ such that $n - k > n_2$ for $n \geq n_5$ and hence, for $n \geq n_5$,

$$(1-A) z_n < x_n + |E_n| + |E_{n-k}|. \tag{3}$$

For $0 < \varepsilon < (1-A)\lambda$ there exists $n_6 > n_5$ such that $(1-A)z_n < x_n + \varepsilon$ for $n \geq n_6$. Thus, $\liminf_{n\to\infty} x_n > 0$. On the other hand, multiplying (2) by $(n+1)$ and summing the resulting equality we obtain

$$\sum_{s=n_5}^{n-1} (s+1) q_s f(x_{s+1-l}) \leq n_5 a_{n_5} \Delta z_{n_5} + K z_n < \infty$$

because $a_n \leq K$ $(n \geq n_5)$ for some constant K. This in turn implies, in view of condition (c_3) and $\liminf_{n\to\infty} x_n > 0$, a contradiction to condition (c_4). Thus $z_n < 0$ for $n \geq n_3$. Consequently, from the definition of z_n we get $0 \leq (1-B)\limsup_{n\to\infty} x_n \leq 0$. Hence $\lim_{n\to\infty} x_n = 0$. The case $x_n < 0$ for $n \geq n_1$ may be proved similarly. This completes the proof of the theorem ∎

Theorem 3. *Let the conditions* $(c_1),(c_2),(c_4),(c_6),(c_8)$ *and* (c_{11}) *hold. In addition, assume that the sequence* $\{a_n\}$ *is bounded. Then every solution of equation* (1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof. In view of Theorem 2, it is enough to prove that no non-oscillatory solution of equation (1) is unbounded. Let $\{x_n\}$ be an unbounded non-oscillatory solution of equation (1). Proceeding as in the proof of Theorem 2 and setting $y_n = (1-A)z_n - \varepsilon$, we obtain $0 < y_n < x_n$, $\Delta y_n > 0$, $\Delta(a_n \Delta y_n) \leq 0$ and

$$\Delta(a_n \Delta y_n) + (1-A) q_n f(y_{n+1-l}) \leq 0. \tag{4}$$

The rest of the proof is similar to that of Theorem 2 and hence the details are omitted ∎

Theorem 4. *Let the conditions* $(c_2),(c_6),(c_7)$ *and* (c_{11}) *hold. In addition, suppose that the function* f *is monotonically increasing and*

$$\lim_{|u|\to\infty} \inf \frac{|f(u)|}{|u|^{1+\beta}} > 0 \qquad (\beta > 0).$$

Then every solution of equation (1) *is oscillatory or tends to zero as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a non-oscillatory solution of equation (1) such that $x_n > 0$ for $n \geq n_1 \in \mathbb{N}_{n_0}$. Proceeding as in Theorem 2, we obtain $z_n > 0$ and $\lim_{n\to\infty} z_n = \infty$ if $\{x_n\}$ is unbounded.

Letting y_n as in Theorem 3, we obtain (4) and $y_n \to \infty$ as $n \to \infty$. Summing (4) twice and using $|f(u)| > K|u|^{1+\beta}$ for $|u|$ large and $y_n \to \infty$ as $n \to \infty$, we have

$$(1-A)K \sum_{j=n_5}^{n-1} \frac{1}{a_{j-l}} \left(\sum_{s=j}^{\infty} q_s \right) < \int_{y_{n_5-l}}^{y_{n-l}} \frac{dt}{t^{1+\beta}} < \frac{1}{\beta y_{n_5-l}^{\beta}}$$

which is a contradiction to condition (c_7). Hence the sequence $\{x_n\}$ is bounded. The rest of the proof is similar to that of Theorem 2 and hence the details are omitted ∎

In the following we obtain a result which holds for sublinear functions f.

Theorem 5. *Let the conditions* $(c_2), (c_6), (c_9), (c_{10})$ *and* (c_{11}) *hold. If* $l \geq 1$ *and the function* f *is monotonically increasing, then every solution of equation* (1) *is oscillatory or tends to zero as* $n \to \infty$.

Proof. Proceeding as in Theorem 4 and defining y_n as in Theorem 3 we have

$$y_n > \sum_{s=n_4}^{n-1} \Delta y_s > a_n \Delta y_n R_n, \quad \text{where} \quad R_n = \sum_{s=n_5}^{n-1} \frac{1}{a_s} \quad (n \geq n_5 > n_4).$$

Now summing from n_6 to $n-1$, we obtain

$$(1-A) \sum_{s=n_6}^{n-1} q_s f(R_{s+1-l}) \leq -\sum_{s=n_6}^{n-1} \frac{\delta(a_s \Delta y_s)}{f(a_s \Delta y_s)} \leq \int_{a_n \Delta y_n}^{a_{n_6} \Delta y_{n_6}} \frac{dt}{f(t)} \leq \int_{0}^{a_{n_6} \Delta y_{n_6}} \frac{dt}{f(t)}.$$

This in view of condition (c_9) contradicts condition (c_{10}). Thus the sequence $\{x_n\}$ is bounded. The rest of the proof is similar to that of Theorem 2 and hence the details are omitted ∎

Finally, in this section we obtain a result subject to the condition $a_n = 1$ for all $n \in \mathbb{N}_{n_0}$.

Theorem 6. *Let the conditions* $(c_2), (c_5)$ *and* (c_{11}) *hold. If the function* f *is monotonically increasing and* $\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0$, *then every solution of equation* (1) *is oscillatory or tends to zero as* $n \to \infty$.

Proof. Proceeding as in Theorem 4 and setting $v_n = \frac{\Delta y_n}{y_{n-l}}$, we have

$$\Delta \nu_n + \nu_n^2 + (1-A) q_n \frac{f(y_{n+1-l})}{y_{n+1-l}} \leq 0.$$

Multiplying the above inequality by $(n+1)^\alpha$ and then summing from n_5 to $n-1$, we obtain

$$0 > -n_5^\alpha \nu_{n_5} - \alpha \sum_{s=n_5}^{n-1} \xi^{\alpha-1}\nu_s + \sum_{s=n_5}^{n-1} (s+1)^\alpha v_s^2 + K(1-A)\sum_{s=n_5}^{n-1}(s+1)^\alpha q_s$$

for $s < \xi < s+1$ because, for large $|u|, f(u) > Ku$ and $y_n \to \infty$ as $n \to \infty$. By the method of completing the squares we obtain

$$K(1-A)\sum_{s=n_5}^{n-1}(s+1)^\alpha q_s < n_5^\alpha \nu_{n_5} + \frac{\alpha^2(n_5+1)^{\alpha-1}}{4(1-\alpha)}$$

which is a contradiction to the condition (c_5). Hence $\{x_n\}$ is bounded. The rest of the proof is similar to that of Theorem 2 and hence the details are omitted ∎

Remark. We may note that in Theorems 2 - 6 we obtain $\lim_{n\to\infty} x_n = 0$ for a non-oscillatory solution $\{x_n\}$ of equation (1) in the case $z_n < 0$ for large n. If $0 < p_n \le A < 1$ and $\{E_n\}$ is oscillatory or $\{E_n\} \equiv 0$, $z_n < 0$ implies that $0 < x_n + \delta_n + p_n x_{n-k} < E_n$ which is a contradiction. Hence in Theorems 1 - 6 (Theorem 2) the conclusion would read as "*Every (every bounded) solution is oscillatory*" if the condition (c_2) is replaced by the condition (c_1) and $\{E_n\}$ is either oscillatory or $\{E_n\} \equiv 0$.

In the following we give some examples to illustrate the above results.

Example 1. Consider the difference equation

$$\begin{aligned}\Delta^2\left(x_n + \frac{2(-1)^n+1}{4}x_{n-2}\right) &+ \frac{12(n-1)^2}{(n-2)n(n+1)(n+2)}x_{n-1}^3 \\ &= \frac{2}{n(n+1)(n+2)} + \frac{2(-1)^n(2n^2-4n+1)+1}{2n(n-1)(n-2)} \\ &+ \frac{12}{(n-2)(n-1)n(n+1)(n+2)} \quad (n \ge 3).\end{aligned} \tag{5}$$

Here $a_n = 1$,

$$E_n = \frac{1}{n} + \frac{2(-1)^n+1}{4(n-2)} + \frac{1}{n(n-1)(n-2)} \quad \text{and} \quad -1 < -\frac{1}{4} \le p_n \le \frac{3}{4} < 1.$$

From Theorem 2, it follows that all bounded solutions of equation (5) are either oscillatory or tend to zero as $n \to \infty$. In particular, $\{x_n\} = \left\{\frac{1}{n}\right\}$ is a solution of equation (5) tending to zero.

Example 2. Consider the difference equation

$$\begin{aligned}\Delta\left[\frac{2n+2}{2n+1}\Delta\left(x_n + \frac{(-1)^n}{n(n-2)}x_{n-2}\right)\right] &+ \frac{2(2n+3)}{(n-1)^3}x_{n-1}^3 \\ &= \frac{2}{n(2n+1)} - \frac{2}{(n+1)(2n+3)}\end{aligned} \quad (n \ge 4). \tag{6}$$

Here

$$a_n = \frac{2n+2}{2n+1}, \qquad E_n = \frac{1}{n}, \qquad -1 < -\frac{1}{15} \le p_n \le \frac{1}{8} < 1.$$

From Theorem 3 it follows that all solutions of equation (6) are oscillating or tend to zero as $n \to \infty$. In particular, $\{x_n\} = \{(-1)^n n\}$ is an oscillatory solution of equation (6).

3. Oscillation of equation (1) when $\{E_n\}$ is periodic

In this section we obtain conditions for the oscillation of all solutions of equation (1) when the sequence $\{E_n\}$ is k-periodic.

Theorem 7. *Let the conditions* $(c_1), (c_3), (c_4), (c_6)$ *and* (c_{12}) *holds. If the sequence* $\{a_n\}$ *is bounded, then every bounded solution of equation* (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution of equation (1) for $n \ge n_1 \in \mathbb{N}_{n_0}$. Proceeding as in Theorem 2 we obtain $(1-A)(z_n + E_n) \le x_n$ for $n \ge n_2$. Since $\{E_n\}$ is k-periodic, there exist real constants b_1 and b_2 and sequences $\{n'_j\}$ and $\{n''_j\}$ of natural numbers such that

$$\lim_{j\to\infty} n'_j = \lim_{j\to\infty} n''_j = \infty, \quad E_{n'_j} = b_1, \qquad E_{n''_j} = b_2, \qquad b_1 \le E_n \le b_2.$$

For $n > n_3 > \max\{n_2, n'_j\}$ where $n'_j > n_1$ we have

$$0 < (1-A)(z_{n_j} + E_{n_j}) \le (1-A)(z_n + b_1) \le (1-A)(z_n + E_n) \le x_n.$$

Setting $y_n = (1-A)(z_n + b_1)$, we obtain $0 < y_n \le x_n$, $\{y_n\}$ is bounded, $\Delta y_n > 0$ and $\Delta(a_n \Delta y_n) \le 0$ for $n \ge n_3$. If $\lim_{n\to\infty} y_n = \lambda$ $(0 < \lambda < \infty)$, then for $0 < \varepsilon < \lambda$ there exists an integer $n_4 > n_3$ such that $0 < \lambda - \varepsilon < y_{n+1-l} \le x_{n+1-l}$ for $n \ge n_4$. Hence $f(x_{n+1-l}) > \lambda^* > 0$ for $n \ge n_4$. From (2) we get

$$0 \ge \Delta(a_n \Delta y_n) + (1-A) q_n f(x_{n+1-l}) \ge \Delta(a_n \Delta y_n) + (1-A)\lambda^* q_n$$

for $n \ge n_4$. The rest of the proof is similar to that of Theorem 2 and hence the details are omitted ∎

Theorem 8. *Let the conditions* $(c_1), (c_4), (c_6), (c_8)$ *and* (c_{12}) *hold. If the sequence* $\{a_n\}$ *is bounded, then every solution of equation* (1) *is oscillatory.*

Proof. The proof is similar to that of Theorems 3 and 7. Therefore the details are omitted ∎

Remark. One can obtain further results similar to Theorems 4 - 6. Details are omitted.

Example 3. Consider the difference equation

$$\Delta^2\left(x_n + \frac{2}{3}x_{n-4}\right) + x_{n-1}^3 = -\frac{10}{3}\cos\frac{n\pi}{2} \qquad (n \ge 4). \tag{7}$$

Here the sequence $\{E_n\}$,

$$E_n = \frac{5}{3}\sin\frac{n\pi}{2} + \frac{1}{2}\sin^3\frac{n\pi}{2},$$

is 4-periodic. From Theorem 8 it follows that all solutions of equation (7) are oscillatory. In particular, $\{x_n\} = \left\{\sin\frac{n\pi}{2}\right\}$ is an oscillatory solution of equation (7).

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