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Regularity Properties and Generalized Inverses of Delta-Related Operators

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Dedicated to the memory of Siegfried Prößdorf

Abstract. As a central topic certain relations between operator matrices are investigated which are called delta relations. The main aim of these relations is to reduce questions about classes of operators without invertibility symbol to those which admit an invertibility symbol. Particular attention is devoted to the generalized inversion of such operators. Different kinds of relations are introduced in order to analyze the "information" contained in the symbols of the related operators. Several examples are considered and the theory is also applied to singular integral operators with Carleman shift. Asymptotic solutions of equations characterized by those operators are presented. The approach simplifies several known results, makes the theory more rigorous from the operator theoretic point of view, and allows further conclusions in a very compact form.

Keywords: *Delta-related operators, factorization theory, generalized inverses, singular integral operators with Carleman shift*

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1. Basic definitions and examples

The goal of this paper is to present a relation that is suitable to connect bounded linear operators with different structures. It will guarantee the transfer of certain properties between different classes of operators. Because of the needs of applications, we are particularly interested in the study of regularity properties and the explicit representation of generalized inverses of the present operators.

Definition 1.1. Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces. We shall say that $\mathcal{T}$ is *Δ-related to $\mathcal{W}$* (briefly $\mathcal{T} \,\Delta\, \mathcal{W}$) if there is a bounded linear operator acting between Banach spaces $\mathcal{T}_\Delta : X_{1\Delta} \to X_{2\Delta}$ and two invertible bounded linear operators $E : Y_2 \to X_2 \oplus X_{2\Delta}$ and $F : X_1 \oplus X_{1\Delta} \to Y_1$ such that

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} = E\mathcal{W}F. \tag{1.1}$$

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Definition 1.2. Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces. We shall say that $\mathcal{T}$ is *Δ-related after extension to $\mathcal{W}$* (briefly $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$) if there is a bounded linear operator acting between Banach spaces $\mathcal{T}_\Delta : X_{1\Delta} \to X_{2\Delta}$ and invertible bounded linear operators

$$E : Y_2 \oplus Z \to X_2 \oplus X_{2\Delta}, \qquad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

$$F : X_1 \oplus X_{1\Delta} \to Y_1 \oplus Z, \qquad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} = E \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix} F, \tag{1.2}$$

where Z is a Banach space.

As an observation we point out that the denomination of Δ-relation was chosen because of the presence of three operators $\mathcal{T}$, $\mathcal{T}_\Delta$ and $\mathcal{W}$ in (1.1). As we shall see the properties of two operators $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$ may be highly dependent on $\mathcal{T}_\Delta$, too.

Example 1.1. It is evident that equivalent operators ($T = EWF$, where E and F are bijections) are Δ-related. Also, matricially coupled operators, i.e.

$$\begin{bmatrix} T & * \\ * & * \end{bmatrix}^{-1} = \begin{bmatrix} * & * \\ * & W \end{bmatrix}, \tag{1.3}$$

are Δ-related after extension, since we know from [2] that (1.3) holds if and only if T and W are equivalent after extension and this is a particular case of Definition 1.2 where $\mathcal{T}_\Delta = I$.

Example 1.2. Let α be a (non-identical) function which maps $\mathbb{R}$ into itself, has a Hölder continuous derivative and fulfills the Carleman condition $\alpha(\alpha(\xi)) = \xi$ (this yields $\alpha'(\xi) \neq 0$ on $\mathbb{R}$). This function allows the definition of a Carleman shift operator $(1 < p < +\infty)$

$$J : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n, \qquad (J\varphi)(\xi) = \widetilde{\varphi}(\xi) = \varphi(\alpha(\xi)) \tag{1.4}$$

that is supposed to be a bounded linear operator so that

$$\mathcal{K} = JS_{\mathbb{R}} + S_{\mathbb{R}}J$$

will be a compact operator, where $S_{\mathbb{R}}$ is the singular integral operator

$$S_{\mathbb{R}} : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n, \qquad (S_{\mathbb{R}}\varphi)(\xi) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(\tau)}{\tau - \xi} d\tau.$$

A concrete example of a function with the above conditions is $\alpha(\xi) = -\xi$. This particular function and the corresponding reflection operator (1.4) play a fundamental role in the study of several problems in mathematical physics as it is demonstrated in [18 - 20, 27].

We will study the singular integral operator with Carleman shift (see [10 - 14])

$$\mathcal{T} : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n, \qquad \mathcal{T} = A_{\mathcal{T}} P_{\mathbb{R}} + B_{\mathcal{T}} Q_{\mathbb{R}} \tag{1.5}$$

where

$$P_{\mathbb{R}} = \tfrac{1}{2}(I + S_{\mathbb{R}}), \quad Q_{\mathbb{R}} = \tfrac{1}{2}(I - S_{\mathbb{R}})$$
$$A_{\mathcal{T}} = aI + bJ, \quad B_{\mathcal{T}} = cI + dJ \qquad (a, b, c, d \in [L^\infty(\mathbb{R})]^{n\times n}).$$

The operator $\mathcal{T}$ defined in (1.5) is Δ-related to the paired operator [1)]

$$\mathcal{W} : [L^p(\mathbb{R})]^n \oplus [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n \oplus [L^p(\mathbb{R})]^n$$
$$\mathcal{W} = \begin{bmatrix} a & d \\ \tilde{b} & \tilde{c} \end{bmatrix} P_{\mathbb{R}} + \begin{bmatrix} c & b \\ \tilde{d} & \tilde{a} \end{bmatrix} Q_{\mathbb{R}}. \tag{1.6}$$

As a matter of fact, by the Gohberg-Krupnik-Litvinchuk identity (see [12] or [14], for instance) we have

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} = E\mathcal{W}E^{-1} \tag{1.7}$$

where

$$\mathcal{T}_\Delta : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n, \quad \mathcal{T}_\Delta = A_{\mathcal{T}_\Delta} P_{\mathbb{R}} + B_{\mathcal{T}_\Delta} Q_{\mathbb{R}}$$
$$A_{\mathcal{T}_\Delta} = aI - bJ, \quad B_{\mathcal{T}_\Delta} = cI - dJ$$
$$E : [L^p(\mathbb{R})]^n \oplus [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n \oplus [L^p(\mathbb{R})]^n, \quad E = \frac{1}{2}\begin{bmatrix} I_{[L^p(\mathbb{R})]^n} & J \\ I_{[L^p(\mathbb{R})]^n} & -J \end{bmatrix}.$$

Remark 1.3. The above example can be generalized to the case of a Carleman shift of order $m \geq 2$ by use of a generalization of the Gohberg-Krupnik-Litvinchuk identity that can be found in [11, 12].

Example 1.4. Let $A, B \in \mathcal{L}([L^p(\mathbb{R})]^n)$ and C be the operator of complex conjugation $(C\varphi)(\xi) = \overline{\varphi(\xi)}$. Then $\mathcal{T} = A + BC$ is Δ-related to

$$\mathcal{W} = \begin{bmatrix} A & B \\ CBC & CAC \end{bmatrix}$$

which is a linear and bounded operator acting on $[L^p(\mathbb{R})]^{2n}$. This is another consequence of the Gohberg-Krupnik-Litvinchuk identity, here represented in the form

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & -i\mathcal{T}i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I_{[L^p(\mathbb{R})]^n} & C \\ I_{[L^p(\mathbb{R})]^n} & -C \end{bmatrix} \mathcal{W} \begin{bmatrix} I_{[L^p(\mathbb{R})]^n} & I_{[L^p(\mathbb{R})]^n} \\ C & -C \end{bmatrix}.$$

Example 1.5. Let us consider Wiener-Hopf-Hankel operators on a finite interval $\Omega =]0,1[$

$$\mathcal{H} : [L^p(\Omega)]^n \to [L^p(\Omega)]^n, \qquad \mathcal{H} = P_\Omega \mathcal{F}^{-1}(\Phi_1 \cdot + \Phi_2 \cdot J)\mathcal{F}_{|P_\Omega X},$$

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1) Note that we use the same notation for the projectors $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$ defined on $[L^p(\mathbb{R})]^n$ or $[L^p(\mathbb{R})]^{2n}$.

where $X = [L^p(\mathbb{R}_+)]^n$, P_Ω is the projector that acts as the characteristic function on Ω, $(J\varphi)(\xi) = \varphi(-\xi)$, and Φ_1 and Φ_2 are elements of the L^p Fourier multiplier algebra. Several problems of mathematical physics can be reduced to equations characterized by this kind of operators. Consider for instance boundary/transmission problems for the Helmholtz equation involving a slit or a strip [9].

We shall prove that $\mathcal{H}$ is Δ-related after extension to the Wiener-Hopf operator acting on the line

$$\mathcal{W} : P_{\mathbb{R}}(Y \oplus Y \oplus Y \oplus Y) \to P_{\mathbb{R}}(Y \oplus Y \oplus Y \oplus Y)$$

$$\mathcal{W} = P_{\mathbb{R}} \Phi_{\mathcal{W}} \cdot_{|P_{\mathbb{R}}(Y \oplus Y \oplus Y \oplus Y)}$$

with $Y = [L^p(\mathbb{R})]^n$ and

$$\Phi_{\mathcal{W}} = \begin{bmatrix} 0 & e^{-i\xi} & 0 & 0 \\ e^{i\xi} & \Phi_1 & -e^{-i\xi}\Phi_2 & 0 \\ 0 & e^{i\xi}\widetilde{\Phi_2} & -\widetilde{\Phi_1} & e^{i\xi} \\ 0 & 0 & e^{-i\xi} & 0 \end{bmatrix}.$$

In fact,

$$\begin{bmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{H}_\Delta \end{bmatrix} = E \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_{Q_{\mathbb{R}}(Y^2 \oplus Y^2)} \end{bmatrix} F \tag{1.8}$$

where E and F are linear invertible bounded operators defined by

$$E \ : \ Y^2 \oplus (Y \oplus Y) \to [(P_\Omega X \oplus P_{]1,+\infty[} X \oplus X) \oplus Q_{\mathbb{R}} Y^2] \oplus Y^2$$

$$E = \frac{1}{2} \left[\begin{array}{cc|c} E_1 & 0 & 0 \\ 0 & I_{Q_{\mathbb{R}}Y^2} & 0 \\ \hline 0 & 0 & I_{Y^2} \end{array}\right] \left[\begin{array}{c|cc} I_{Y^2} & 0 & e^{-i\xi}J \\ & e^{i\xi}J & \Phi_2 + \Phi_1 J \\ \hline I_{Y^2} & 0 & -e^{-i\xi}J \\ & -e^{i\xi}J & \Phi_2 - \Phi_1 J \end{array}\right]$$

$$E_1 = \left[\begin{array}{cc|c} I_{P_\Omega X} & -P_\Omega \mathcal{F}^{-1}(\Phi_1 \cdot + \Phi_2 \cdot J)\mathcal{F}_{|P_{]1,+\infty[}X} & 0 \\ 0 & I_{P_{]1,+\infty[}X} & 0 \\ \hline 0 & 0 & I_X \end{array}\right] \left[\begin{array}{c|c} W_1 & I_X - W_1 W_{-1} \\ \hline -I_X & W_{-1} \end{array}\right] \mathcal{F}^{-1}$$

$$(W_a\varphi)(\xi) = P_{\mathbb{R}_+}\varphi(\xi - a) = P_+\varphi(\xi - a)$$

$$F \ : \ [((P_\Omega X \oplus P_{]1,+\infty[} X) \oplus X) \oplus Q_{\mathbb{R}} Y^2] \oplus Y^2 \to P_{\mathbb{R}}(Y^2 \oplus Y^2) \oplus Q_{\mathbb{R}}(Y^2 \oplus Y^2)$$

$$F = \left[\begin{array}{c|c} I_{P_{\mathbb{R}}(Y^2 \oplus Y^2)} & 0 \\ \hline Q_{\mathbb{R}}(\Phi_{\mathcal{W}})_{|P_{\mathbb{R}}(Y^2 \oplus Y^2)} & I_{Q_{\mathbb{R}}(Y^2 \oplus Y^2)} \end{array}\right] \left[\begin{array}{cc|c} P_{\mathbb{R}}F_1 & 0 & I_{Y^2} \\ -Q_{\mathbb{R}}\Phi_3 \cdot P_{\mathbb{R}}F_1 & I_{Q_{\mathbb{R}}Y^2} & \\ \hline JP_{\mathbb{R}}F_1 & 0 & -J_{|Y^2} \\ -JQ_{\mathbb{R}}\Phi_3 \cdot P_{\mathbb{R}}F_1 & J_{|Q_{\mathbb{R}}Y^2} & \end{array}\right]$$

$$F_1 = \begin{bmatrix} -\mathcal{F}W_{-1}\left(P_+\mathcal{F}^{-1}(\Phi_1 \cdot + \Phi_2 \cdot J)\mathcal{F}_{|X} - I_X\right) & \mathcal{F}I_X \\ \mathcal{F}I_X & 0 \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & \Phi_1 + \Phi_2 J \end{bmatrix}.$$

Additionally, the operator $\mathcal{H}_\Delta$ is given by

$$\mathcal{H}_\Delta : P_{]1,+\infty[}X \oplus X \oplus Q_{\mathbb{R}}Y^2 \oplus Y^2 \to P_{]1,+\infty[}X \oplus X \oplus Q_{\mathbb{R}}Y^2 \oplus Y^2$$

$$\mathcal{H}_\Delta = \begin{bmatrix} I_{P_{]1,+\infty[}X} & 0 & 0 & 0 \\ 0 & I_X & 0 & 0 \\ 0 & 0 & I_{Q_{\mathbb{R}}Y^2} & 0 \\ 0 & 0 & 0 & \begin{bmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & \Phi_1 - \Phi_2 J \end{bmatrix} P_{\mathbb{R}} + Q_{\mathbb{R}} \end{bmatrix}.$$

Relation (1.8) was derived with the help of several space decompositions. We avoided to calculate the factors in a more compact form, because one can see here directly that E and F are invertible operators.

Example 1.6. Δ-relations after extension are transitive, i.e.

$$\left.\begin{array}{c} A \overset{\bullet}{\Delta} B \\ B \overset{\bullet}{\Delta} C \end{array}\right\} \implies A \overset{\bullet}{\Delta} C$$

and therefore it is possible to relate questions for compositions of operators which are Δ-related (after extension). This is very useful if we are working with higher order operators where certain iteration processes are applied in order to reduce the complexity of such operators (see [6, 8]).

Example 1.7. Another situation where the transitivity property is useful appears when we want to change the spaces in which the operators are defined. Consider, for instance, Wiener-Hopf-Hankel operators defined between Bessel potential spaces (that is the usual setting in various applications [18 - 20, 27])

$$\mathcal{H} : \widetilde{H}^{u,p}(\mathbb{R}_+) \to H^{v,p}(\mathbb{R}_+), \qquad \mathcal{H} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}(\Phi_1 \cdot + \Phi_2 \cdot J)\mathcal{F}_{|\widetilde{H}^{u,p}(\mathbb{R}_+)} \tag{1.9}$$

where, considering u and v as multi-indices of n components and $1 < p < +\infty$, $\widetilde{H}^{u,p}(\mathbb{R}_+)$ is the Bessel potential space of distributions over $\mathbb{R}$ with support contained in $\overline{\mathbb{R}_+}$ and $H^{v,p}(\mathbb{R}_+)$ denotes the space of generalized functions on $\mathbb{R}_+$ which have extensions into $\mathbb{R}$ that belong to the correspondent Bessel potential space over the full line, $H^{v,p}(\mathbb{R})$. By $r_{\mathbb{R}\to\mathbb{R}_+}$ we denote the operator of restriction from $H^{v,p}(\mathbb{R})$ to $H^{v,p}(\mathbb{R}_+)$. In addition, we suppose that $\left((\xi - i)^v\Phi_1(\xi)(\xi + i)^{-u}\right)^{\pm 1}$ and $(\xi - i)^v\Phi_2(\xi)(\xi + i)^{-u}$ are L^p Fourier multipliers and $(J\varphi)(\xi) = \varphi(-\xi)$.

Using Bessel potential operators, we can guarantee an equivalence relation between $\mathcal{H}$ and the lifted operator

$$\mathcal{H}_0 : [L^p(\mathbb{R}_+)]^n \to [L^p(\mathbb{R}_+)]^n$$
$$\mathcal{H}_0 = P_+\mathcal{F}^{-1}\left((\xi - i)^v\Phi_1(\xi + i)^{-u} \cdot + (\xi - i)^v\Phi_2(-\xi + i)^{-u} \cdot J\right)\mathcal{F}_{|[L^p(\mathbb{R}_+)]^n}$$

with $P_+ = P_{\mathbb{R}_+}$ (see [4, 15, 26] where classes of this kind of operators are analyzed). On the other hand, one can extend and "double" $\mathcal{H}_0$ in a way that we construct the operator

$$\mathcal{W} : [L^p(\mathbb{R}_+)]^n \oplus [L^p(\mathbb{R}_+)]^n \to [L^p(\mathbb{R}_+)]^n \oplus [L^p(\mathbb{R}_+)]^n$$

$$\mathcal{W} = P_+\mathcal{F}^{-1} \times \begin{bmatrix} (\xi - i)^v(\Phi_1 - \Phi_2(\widetilde{\Phi_1})^{-1}\widetilde{\Phi_2})(\xi + i)^{-u} & -(\xi - i)^v\Phi_2(\widetilde{\Phi_1})^{-1}(-\xi - i)^{-v} \\ (-\xi + i)^u(\widetilde{\Phi_1})^{-1}\widetilde{\Phi_2}(\xi + i)^{-u} & (-\xi + i)^u(\widetilde{\Phi_1})^{-1}(-\xi - i)^{-v} \end{bmatrix} \times \mathcal{F}_{|[L^p(\mathbb{R}_+)]^{2n}}$$

so that $\mathcal{H}_0 \overset{*}{\Delta} \mathcal{W}$ (see [9] in a similar case). Therefore, using the transitivity property we find that $\mathcal{H}$ defined in (1.9) is Δ-related after extension to $\mathcal{W}$.

2. General aspects of the transfer of properties by delta relations

Suppose that we have a Δ-relation after extension $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$. In the following table we characterize various regularity properties of $\mathcal{T}$ depending on certain properties of $\mathcal{W}$.

Properties of $\mathcal{T}$	$\dim\ker \mathcal{W} = 0$	$\dim\ker \mathcal{W} < \infty$	$\ker \mathcal{W}$ is complemented	$\ker \mathcal{W}$ is closed
codim im $\mathcal{W}$ $= 0$	invertible	right invertible and Fredholm	right invertible	surjective
codim im $\mathcal{W}$ $< \infty$	left invertible and Fredholm	Fredholm	right regularizable	codim im $\mathcal{T} < \infty$ $\ker \mathcal{T}$ is closed
im $\mathcal{W}$ is complemented	left invertible	left regularizable	generalized invertible	n. n.
im $\mathcal{W}$ is closed	injective	$\dim\ker \mathcal{T} < \infty$ im $\mathcal{T}$ is closed	n. n.	normally solvable

Tab. 1: Regularity classes of an operator $\mathcal{T}$ in a Δ-relation after extension $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$

For more details of this classification see [16, 24].

Theorem 2.1. *Table 1 holds true, i.e. if $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$ and $\mathcal{W}$ has the property (i, j) $(i, j = 1, \ldots, 4)$ in Table 1, then $\mathcal{T}$ has the same property.*

Proof. Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces such that $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$ (see Definition 1.2). Then by (1.2), as E and F are bounded invertible linear operators, we obtain:

(i) $\ker\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix}\right) \simeq \ker\mathcal{W}$.

(ii) $\mathrm{im}\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix}\right)$ is closed if and only if $\mathrm{im}\mathcal{W}$ is closed. In this case, $(X_2 \oplus X_{2\Delta})/\mathrm{im}\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix}\right) \simeq Y_2/\mathrm{im}\mathcal{W}$.

From (i) and (ii) we derive Table 1 ∎

Corollary 2.2. *Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces such that $\mathcal{T} \overset{\bullet}{\Delta} \mathcal{W}$, and let $\mathcal{T}_\Delta$ an operator which satisfies* (1.2). *Then $\mathcal{W}$ is a Fredholm operator if and only if $\mathcal{T}$ and $\mathcal{T}_\Delta$ are Fredholm operators.*

Proof. This is a direct consequence of (i) and (ii) in the proof of Theorem 2.1 ∎

Remark 2.3. In general Table 1 is not valid in the converse direction, i.e., we can not change the roles of $\mathcal{T}$ and $\mathcal{W}$ in Table 1 if we still have $\mathcal{T} \overset{\bullet}{\Delta} \mathcal{W}$. This is due to the non-symmetric character of the Δ-relation (after extension). For instance, $\mathcal{T}$ may be left invertible, $\mathcal{T}_\Delta$ right invertible, and $\mathcal{W}$ generalized but not one-sided invertible. Evidently, one can only say: If $\mathcal{T}$ belongs to the regularity class (i, j) and $\mathcal{T}_\Delta$ to the class (i', j'), then $\mathcal{W}$ belongs to the class $(\max\{i, i'\}, \max\{j, j'\})$.

Theorem 2.4. *Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators acting between Banach spaces. In addition, suppose that $\mathcal{T}$ and $\mathcal{W}$ are generalized invertible operators. If there is a generalized invertible operator $U_\Delta : Y_{1\Delta} \to Y_{2\Delta}$ such that*

$$\ker\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right) \simeq \ker\mathcal{W} \tag{2.1}$$

$$(X_2 \oplus Y_{2\Delta})\big/\mathrm{im}\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right) \simeq Y_2/\mathrm{im}\mathcal{W}, \tag{2.2}$$

then $\mathcal{T} \overset{\bullet}{\Delta} \mathcal{W}$.

Proof. Let us denote by $\mathcal{T}^-$, $\mathcal{W}^-$ and U_Δ^- some generalized inverses of $\mathcal{T}$, $\mathcal{W}$ and U_Δ, respectively. These generalized inverses allow the following decompositions:

$$X_1 = \mathrm{im}(\mathcal{T}^-\mathcal{T}) \oplus \ker\mathcal{T} \qquad X_2 = \mathrm{im}\mathcal{T} \oplus \ker(\mathcal{T}\mathcal{T}^-) \tag{2.3}$$

$$Y_1 = \ker\mathcal{W} \oplus \mathrm{im}(\mathcal{W}^-\mathcal{W}) \qquad Y_2 = \ker(\mathcal{W}\mathcal{W}^-) \oplus \mathrm{im}\mathcal{W} \tag{2.4}$$

$$Y_{1\Delta} = \mathrm{im}(U_\Delta^- U_\Delta) \oplus \ker U_\Delta \qquad Y_{2\Delta} = \mathrm{im}U_\Delta \oplus \ker(U_\Delta U_\Delta^-). \tag{2.5}$$

By hypothesis (2.1) there is an invertible bounded linear operator

$$J_1 : \ker\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right) \to \ker\mathcal{W}.$$

On the other hand, from the right-side identities of (2.3) and (2.5) we obtain

$$(X_2 \oplus Y_{2\Delta})\big/\mathrm{im}\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right) \simeq \ker\left(\begin{bmatrix} \mathcal{T}\mathcal{T}^- & 0 \\ 0 & U_\Delta U_\Delta^- \end{bmatrix}\right)$$

and from (2.4)

$$Y_2/\mathrm{im}\mathcal{W} \simeq \ker(\mathcal{W}\mathcal{W}^-).$$

Therefore, attending to (2.2), there is an invertible bounded linear operator

$$J_2 : \ker\left(\begin{bmatrix} \mathcal{T}\mathcal{T}^- & 0 \\ 0 & U_\Delta U_\Delta^- \end{bmatrix}\right) \to \ker(\mathcal{W}\mathcal{W}^-).$$

With those invertible operators we construct operators E and F, that are represented in the sense of decompositions (2.4) and

$$X_1 \oplus Y_{1\Delta} = \mathrm{im}\left(\begin{bmatrix} \mathcal{T}^-\mathcal{T} & 0 \\ 0 & U_\Delta^- U_\Delta \end{bmatrix}\right) \oplus \ker\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right)$$

$$X_2 \oplus Y_{2\Delta} = \mathrm{im}\left(\begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\right) \oplus \ker\left(\begin{bmatrix} \mathcal{T}\mathcal{T}^- & 0 \\ 0 & U_\Delta U_\Delta^- \end{bmatrix}\right)$$

$$E : Y_2 \oplus (X_2 \oplus Y_{2\Delta}) \to (X_2 \oplus Y_{2\Delta}) \oplus Y_2$$

$$E = \left[\begin{array}{cc|cc} 0 & 0 & I_{\mathrm{im}\,(\mathrm{diag}[\mathcal{T}, U_\Delta])} & 0 \\ -J_2^{-1} & 0 & 0 & 0 \\ \hline I_{\ker(\mathcal{W}\mathcal{W}^-)} & 0 & 0 & J_2 \\ 0 & I_{\mathrm{im}\,\mathcal{W}} & 0 & 0 \end{array}\right]$$

$$F : (X_1 \oplus Y_{1\Delta}) \oplus Y_2 \to Y_1 \oplus (X_2 \oplus Y_{2\Delta})$$

$$F = \left[\begin{array}{cc|cc} 0 & J_1 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{W}^{-1}_{|\mathrm{im}\,(\mathcal{W}^-\mathcal{W})} \\ \hline \mathrm{diag}[\mathcal{T}, U_\Delta]_{|\mathrm{im}\,(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta])} & 0 & 0 & 0 \\ 0 & 0 & J_2^{-1} & 0 \end{array}\right]$$

where

$$\mathcal{W}_{|\mathrm{im}(\mathcal{W}^-\mathcal{W})} : \mathrm{im}(\mathcal{W}^-\mathcal{W}) \to \mathrm{im}\mathcal{W}, \quad \mathcal{W}_{|\mathrm{im}(\mathcal{W}^-\mathcal{W})}\varphi = \mathcal{W}\varphi$$

$$\mathrm{diag}[\mathcal{T}, U_\Delta]_{|\mathrm{im}(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta])} : \mathrm{im}(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta]) \to \mathrm{im}(\mathrm{diag}[\mathcal{T}, U_\Delta])$$

$$\mathrm{diag}[\mathcal{T}, U_\Delta]_{|\mathrm{im}(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta])}\varphi = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & U_\Delta \end{bmatrix}\varphi$$

are invertible operators, due to (2.3)-(2.5), with inverses denoted by $\mathcal{W}^{-1}_{|\mathrm{im}(\mathcal{W}^-\mathcal{W})}$ and $\mathrm{diag}[\mathcal{T}, U_\Delta]^{-1}_{|\mathrm{im}(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta])}$, respectively.

Moreover, note that E and F are invertible bounded linear operators with inverses

$$E^{-1} : (X_2 \oplus Y_{2\Delta}) \oplus Y_2 \to Y_2 \oplus (X_2 \oplus Y_{2\Delta})$$

$$E^{-1} = \left[\begin{array}{cc|cc} 0 & -J_2 & 0 & 0 \\ 0 & 0 & 0 & I_{\mathrm{im}\,\mathcal{W}} \\ \hline I_{\mathrm{im}\,(\mathrm{diag}[\mathcal{T}, U_\Delta])} & 0 & 0 & 0 \\ 0 & I_{\ker(\mathrm{diag}[\mathcal{T}\mathcal{T}^-, U_\Delta U_\Delta^-])} & J_2^{-1} & 0 \end{array}\right]$$

$$F^{-1} : Y_1 \oplus (X_2 \oplus Y_{2\Delta}) \to (X_1 \oplus Y_{1\Delta}) \oplus Y_2$$

$$F^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & \mathrm{diag}[\mathcal{T}, U_\Delta]^{-1}_{|\mathrm{im}\,(\mathrm{diag}[\mathcal{T}^-\mathcal{T}, U_\Delta^- U_\Delta])} & 0 \\ J_1^{-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & J_2 \\ 0 & \mathcal{W}_{|\mathrm{im}\,(\mathcal{W}^-\mathcal{W})} & 0 & 0 \end{array}\right].$$

With the operator U_Δ given by hypothesis and the invertible bounded linear operators

E and F we obtain

$$\begin{bmatrix} \mathcal{T} & 0 & 0 \\ 0 & U_\Delta & 0 \\ 0 & 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix} F,$$

where $Z_1 = Y_2$ and $Z = X_2 \oplus Y_{2\Delta}$. Therefore, we have a Δ-relation after extension (1.2) with

$$\mathcal{T}_\Delta : Y_{1\Delta} \oplus Y_2 \to Y_{2\Delta} \oplus Y_2, \qquad \mathcal{T}_\Delta = \begin{bmatrix} U_\Delta & 0 \\ 0 & I_{Y_2} \end{bmatrix}$$

and E and F given above ∎

For bounded linear operators acting between Banach spaces

$$V : X_1 \oplus X_2 \to Y_1 \oplus Y_2, \qquad V = [V_{ij}]_{\substack{i=1,2 \\ j=1,2}}$$

we shall denote by $R_{11}(V)$ the restricted operator of V to the first component spaces,

$$R_{11}(V) : X_1 \to Y_1, \qquad R_{11}(V) = V_{11}.$$

Theorem 2.5. *Let $\mathcal{T} : X_1 \to X_2$ and $\mathcal{W} : Y_1 \to Y_2$ be bounded linear operators such that $\mathcal{T}$ is Δ-related after extension to $\mathcal{W}$ in the sense of Definition 1.2. In this case:*

(i) *If $\mathcal{W}$ is a generalized invertible operator and $\mathcal{W}^- : Y_2 \to Y_1$ is a generalized inverse of it, then $\mathcal{T}$ is also generalized invertible and*

$$\mathcal{T}^- : X_2 \to X_1, \qquad \mathcal{T}^- = R_{11}\left(F^{-1} \begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1}\right) \tag{2.6}$$

is a generalized inverse of $\mathcal{T}$.

(ii) *If $\mathcal{T}$ and $\mathcal{T}_\Delta$ (cf. Definition 1.2) are generalized invertible operators, and $\mathcal{T}^-$ and $\mathcal{T}_\Delta^-$ are generalized inverses of $\mathcal{T}$ and $\mathcal{T}_\Delta$, respectively, then $\mathcal{W}$ is also a generalized invertible operator and*

$$\mathcal{W}^- : Y_2 \to Y_1, \qquad \mathcal{W}^- = R_{11}\left(F \begin{bmatrix} \mathcal{T}^- & 0 \\ 0 & \mathcal{T}_\Delta^- \end{bmatrix} E\right)$$

is a generalized inverse of $\mathcal{W}$.

Proof. (i) If $\mathcal{W}^- : Y_2 \to Y_1$ is an operator such that $\mathcal{W}\mathcal{W}^-\mathcal{W} = \mathcal{W}$, then

$$\begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix} \begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix} = \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_Z \end{bmatrix}.$$

Thus, by (1.2), we have

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} F^{-1} \begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1} \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix} = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}_\Delta \end{bmatrix}. \tag{2.7}$$

If we write

$$F^{-1}\begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1} : X_2 \oplus X_{2\Delta} \to X_1 \oplus X_{1\Delta}$$

in the form

$$F^{-1}\begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1} = \begin{bmatrix} A & * \\ * & * \end{bmatrix} \tag{2.8}$$

(where the right term was abbreviated) and substitute (2.8) into (2.7), then a direct computation shows that

$$A : X_2 \to X_1, \qquad A = R_{11}\left(F^{-1}\begin{bmatrix} \mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1}\right)$$

is a generalized inverse of $\mathcal{T}$.

Proposition (ii) can be proved in a similar way and therefore is omitted here ∎

According to the proofs of the above theorems, we can derive similar results for the special case $\mathcal{T} \,\Delta\, \mathcal{W}$ instead of $\mathcal{T} \,\overset{\bullet}{\Delta}\, \mathcal{W}$. For instance, about the generalized invertibility of a Δ-related operator, we have the following result.

Corollary 2.6. *Let $\mathcal{T} : X_1 \to X_2$ be Δ-related to $\mathcal{W} : Y_1 \to Y_2$ in the notation of Definition 1.1. If $\mathcal{W}$ is a generalized invertible operator and $\mathcal{W}^- : Y_2 \to Y_1$ is a generalized inverse of $\mathcal{W}$, then $\mathcal{T}$ is also a generalized invertible operator and*

$$\mathcal{T}^- : X_2 \to X_1, \qquad \mathcal{T}^- = R_{11}(F^{-1}\mathcal{W}^- E^{-1})$$

is a generalized inverse of $\mathcal{T}$.

Theorem 2.7. *Suppose that $\mathcal{T} : X_1 \to X_2$ is Δ-related after extension to $\mathcal{W} : Y_1 \to Y_2$. If $\mathcal{W}^-$ is a generalized inverse of $\mathcal{W}$, we obtain projectors onto the image and the kernel of $\mathcal{T}$, respectively, in the form*

$$P_{\ker \mathcal{T}} = R_{11}(F^{-1})\,(I_{Y_1} - \mathcal{W}^-\mathcal{W})\,R_{11}(F)$$
$$P_{\operatorname{im} \mathcal{T}} = R_{11}\left(E\begin{bmatrix} \mathcal{W}\mathcal{W}^- & 0 \\ 0 & I_Z \end{bmatrix} E^{-1}\right).$$

Proof. If $\mathcal{W}^-$ is a generalized inverse of $\mathcal{W}$, then by Theorem 2.5 we know already that $\mathcal{T}$ is also generalized invertible and additionally we can present a generalized inverse $\mathcal{T}^-$ of $\mathcal{T}$. In this sense, the first part can be proved by a computation of $I_{X_1} - \mathcal{T}^-\mathcal{T}$, with the help of (1.2) and (2.6),

$$\begin{aligned} I_{X_1} - \mathcal{T}^-\mathcal{T} &= I_{X_1} - R_{11}\left(F^{-1}\begin{bmatrix}\mathcal{W}^- & 0\\ 0 & I_Z\end{bmatrix}E^{-1}\right) R_{11}\left(E\begin{bmatrix}\mathcal{W} & 0\\ 0 & I_Z\end{bmatrix}F\right)\\ &= I_{X_1} - R_{11}\left(\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\right) R_{11}\left(\begin{bmatrix}\mathcal{T} & 0\\ 0 & \mathcal{T}_\Delta\end{bmatrix}\right)\\ &= I_{X_1} - R_{11}\left(\begin{bmatrix}A_{11}\mathcal{T} & A_{12}\mathcal{T}\\ A_{21}\mathcal{T}_\Delta & A_{22}\mathcal{T}_\Delta\end{bmatrix}\right)\\ &= R_{11}(F^{-1}F) - R_{11}\left(F^{-1}\begin{bmatrix}\mathcal{W}^- & 0\\ 0 & I_Z\end{bmatrix}E^{-1}\,E\begin{bmatrix}\mathcal{W} & 0\\ 0 & I_Z\end{bmatrix}F\right)\\ &= R_{11}\left(F^{-1}\left(\begin{bmatrix}I_{Y_1} & 0\\ 0 & I_Z\end{bmatrix} - \begin{bmatrix}\mathcal{W}^-\mathcal{W} & 0\\ 0 & I_Z\end{bmatrix}\right)F\right)\\ &= R_{11}\left(F^{-1}\begin{bmatrix}I_{Y_1} - \mathcal{W}^-\mathcal{W} & 0\\ 0 & 0\end{bmatrix}F\right). \end{aligned}$$

Therefore,

$$P_{\ker\mathcal{T}} = R_{11}(F^{-1})(I_{Y_1} - \mathcal{W}^-\mathcal{W})R_{11}(F).$$

The formula for $P_{\operatorname{im}\mathcal{T}}$ is proved in an analogous way ∎

Finally, we present an identification of the structure of Δ-related after extension operators.

Theorem 2.8. *Consider a Banach space X and two bounded linear operators $\mathcal{T} : X \to X$ and $\mathcal{W} : X \to X$. If $\mathcal{T} \overset{*}{\Delta} \mathcal{W}$, then there are bounded linear operators A, B, C and D between Banach spaces such that $\mathcal{T}$ and $\mathcal{W}$ admit the representations*

$$\mathcal{T} = R_{11}(A - BD^{-1}C) \tag{2.9}$$

$$\mathcal{W} = R_{11}(D - CA^{-1}B) \tag{2.10}$$

where A and D are invertible operators with inverses A^{-1} and D^{-1}, respectively.

Proof. From the hypotheses there are Banach spaces Z, $X_{1\Delta}$ and $X_{2\Delta}$, a bounded linear operator $\mathcal{T}_\Delta : X_{1\Delta} \to X_{2\Delta}$, and invertible bounded linear operators $E : X \oplus Z \to X \oplus X_{2\Delta}$ and $F : X \oplus X_{1\Delta} \to X \oplus Z$ so that

$$\begin{bmatrix}\mathcal{T} & 0\\ 0 & \mathcal{T}_\Delta\end{bmatrix} = E\begin{bmatrix}\mathcal{W} & 0\\ 0 & I_Z\end{bmatrix}F.$$

Therefore,

$$\begin{aligned} &A : X \oplus X_{1\Delta} \to X \oplus X_{2\Delta}, && A = EF\\ &B : X \oplus Z \to X \oplus X_{2\Delta}, && B = E\begin{bmatrix}I_X - \mathcal{W} & 0\\ 0 & 0\end{bmatrix}\\ &C : X \oplus X_{1\Delta} \to X \oplus Z, && C = F\\ &D : X \oplus Z \to X \oplus Z, && D = \begin{bmatrix}I_X & 0\\ 0 & I_Z\end{bmatrix} \end{aligned}$$

are bounded linear operators that fulfill (2.9) and (2.10) with $A^{-1} = F^{-1}E^{-1}$ and $D^{-1} = D$ ∎

Remark 2.9. The above result is deeply related with systems theory. In particular, it is well-known that if, for some bounded linear operator U, $A = A(x) = xI_X - U$ (where X is a complex Banach space), then

$$W(x) = D - CA^{-1}B \tag{2.11}$$

is a realization [1] of $W(x)$.

In the sense that (2.11) can transform some input data to an output, (2.11) is called a characteristic operator function or a transfer function. Therefore Theorem 2.8 implies that Δ-related operators after extension (acting between the same Banach space) can be interpreted as restrictions to the first component spaces of some characteristic operators.

On the other hand, the operators $A - BD^{-1}C$ and $D - CA^{-1}B$ are said to be Schur coupled [3]. In the finite-dimensional case this notion allows special relations between matrices (see [3] where a characterization of Schur coupling in terms of Hermitian matrices is presented). Therefore (2.9) and (2.10) tell us that the Δ-related after extension operators $\mathcal{T}$ and $\mathcal{W}$ can be seen as restrictions to the first components of Schur coupled operators.

3. On singular integral operators with Carleman shift

In this section we like to present two concrete applications, first the explicit generalized inversion of certain composed operators and second the asymptotic representation of solutions of the corresponding equations under sufficient smoothness assumptions.

Let us consider once again the operator $\mathcal{T}$ with Carleman shift in (1.5). Suppose now that $\det A_{\mathcal{W}} \neq 0$ and $\det B_{\mathcal{W}} \neq 0$, where

$$A_{\mathcal{W}}(\xi) = \begin{bmatrix} a(\xi) & d(\xi) \\ b(\alpha(\xi)) & c(\alpha(\xi)) \end{bmatrix} \in [L^\infty(\mathbb{R})]^{2n\times 2n}$$

$$B_{\mathcal{W}}(\xi) = \begin{bmatrix} c(\xi) & b(\xi) \\ d(\alpha(\xi)) & a(\alpha(\xi)) \end{bmatrix} \in [L^\infty(\mathbb{R})]^{2n\times 2n}.$$

We say that

$$G = B_{\mathcal{W}}^{-1} A_{\mathcal{W}} \tag{3.1}$$

admits a so-called (see [21: Chapter V, §5]) *generalized factorization in* $[L^p(\mathbb{R})]^{2n}$ $(1 < p < +\infty)$, if

$$G(\xi) = G_-(\xi)D(\xi)G_+(\xi) \qquad (\xi \in \mathbb{R}) \tag{3.2}$$

where

$$D(\xi) = \operatorname{diag}\left[\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\kappa_1}, \ldots, \left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\kappa_{2n}}\right] \qquad (\kappa_j \in \mathbb{Z},\ \lambda_\pm(\xi) = \xi \pm i)$$

$$\lambda_-^{-1}G_- \in Q_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}, \quad \lambda_+^{-1}G_+ \in P_{\mathbb{R}}[L^q(\mathbb{R})]^{2n\times 2n}$$

$$\lambda_+^{-1}G_+^{-1} \in P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}, \quad \lambda_-^{-1}G_-^{-1} \in Q_{\mathbb{R}}[L^q(\mathbb{R})]^{2n\times 2n} \qquad \left(q = \tfrac{p}{p-1}\right)$$

$$G_- P_{\mathbb{R}} G_-^{-1} \in \mathcal{L}([L^p(\mathbb{R})]^{2n}).$$

We will use the notation

$$G_{\pm} = \begin{bmatrix} g_{11\pm} & g_{12\pm} \\ g_{21\pm} & g_{22\pm} \end{bmatrix} \qquad \text{and} \qquad G_{\pm}^{-1} = \begin{bmatrix} g_{11\pm}^{(-1)} & g_{12\pm}^{(-1)} \\ g_{21\pm}^{(-1)} & g_{22\pm}^{(-1)} \end{bmatrix}$$

where $g_{kl\pm}, g_{kl\pm}^{(-1)}$ $(k,l = 1,2)$ are $n \times n$ blocks. When η is a $2n$ multi-index, $\eta = (\eta_1, \ldots, \eta_{2n})$, let

$$\zeta^{\eta_\bullet} = \operatorname{diag}[\zeta^{\eta_1}, \ldots, \zeta^{\eta_n}] \qquad \text{and} \qquad \zeta^{\eta_{\bullet\bullet}} = \operatorname{diag}[\zeta^{\eta_{n+1}}, \ldots, \zeta^{\eta_{2n}}].$$

We also recall that $\tilde{\varphi}(\xi) = \varphi(\alpha(\xi))$.

Proposition 3.1. *If the matrix function G (see (3.1)) has a generalized factorization in $[L^p(\mathbb{R})]^{2n}$, then the operator with Carleman shift $\mathcal{T}$, defined in (1.5), is generalized invertible, and a generalized inverse of $\mathcal{T}$ is given by*

$$\begin{aligned}
\mathcal{T}^- = {} & \\
\frac{1}{2}\Big[& \big(g_{11+}^{(-1)} P_{\mathbb{R}} + g_{11-} Q_{\mathbb{R}}\big)\big((\lambda_-/\lambda_+)^{-\kappa_\bullet} P_{\mathbb{R}} + Q_{\mathbb{R}}\big)(c\tilde{a} - \tilde{d}b)^{-1}\big(g_{11-}^{(-1)}\tilde{a} - g_{12-}^{(-1)}\tilde{d}\big) \\
& + \big(g_{12+}^{(-1)} P_{\mathbb{R}} + g_{12-} Q_{\mathbb{R}}\big)\big((\lambda_-/\lambda_+)^{-\kappa_{\bullet\bullet}} P_{\mathbb{R}} + Q_{\mathbb{R}}\big)(c\tilde{a} - \tilde{d}b)^{-1}\big(g_{21-}^{(-1)}\tilde{a} - g_{22-}^{(-1)}\tilde{d}\big) \\
& + \big(\widetilde{g_{21-}}(I - Q_{\mathbb{R}} - \mathcal{K}) + \widetilde{g_{21+}^{(-1)}}(Q_{\mathbb{R}} + \mathcal{K})\big)\big(I - Q_{\mathbb{R}} - \mathcal{K} + (\widetilde{\lambda_-}/\widetilde{\lambda_+})^{-\kappa_\bullet}(Q_{\mathbb{R}} + \mathcal{K})\big) \\
& \times \big(\widetilde{c\tilde{a} - \tilde{d}b}\big)^{-1}\big(\widetilde{g_{11-}^{(-1)}}a - \widetilde{g_{12-}^{(-1)}}d\big)J + \big(\widetilde{g_{22-}}(I - Q_{\mathbb{R}} - \mathcal{K}) + \widetilde{g_{22+}^{(-1)}}(Q_{\mathbb{R}} + \mathcal{K})\big) \\
& \times \big(I - Q_{\mathbb{R}} - \mathcal{K} + \widetilde{(\widetilde{\lambda_-}/\lambda_+)}^{-\kappa_{\bullet\bullet}}(Q_{\mathbb{R}} + \mathcal{K})\big)\big(\widetilde{c\tilde{a} - \tilde{d}b}\big)^{-1}\big(\widetilde{g_{21-}^{(-1)}}a - \widetilde{g_{22-}^{(-1)}}d\big)J \qquad (3.3) \\
& + \big(g_{11+}^{(-1)} P_{\mathbb{R}} + g_{11-} Q_{\mathbb{R}}\big)\big((\lambda_-/\lambda_+)^{-\kappa_\bullet} P_{\mathbb{R}} + Q_{\mathbb{R}}\big)(c\tilde{a} - \tilde{d}b)^{-1}\big(g_{12-}^{(-1)}c - g_{11-}^{(-1)}b\big)J \\
& + \big(g_{12+}^{(-1)} P_{\mathbb{R}} + g_{12-} Q_{\mathbb{R}}\big)\big((\lambda_-/\lambda_+)^{-\kappa_{\bullet\bullet}} P_{\mathbb{R}} + Q_{\mathbb{R}}\big)(c\tilde{a} - \tilde{d}b)^{-1}\big(g_{22-}^{(-1)}c - g_{21-}^{(-1)}b\big)J \\
& + \big(\widetilde{g_{21-}}(I - Q_{\mathbb{R}} - \mathcal{K}) + \widetilde{g_{21+}^{(-1)}}(Q_{\mathbb{R}} + \mathcal{K})\big)\big(I - Q_{\mathbb{R}} - \mathcal{K} + (\widetilde{\lambda_-}/\widetilde{\lambda_+})^{-\kappa_\bullet}(Q_{\mathbb{R}} + \mathcal{K})\big) \\
& \times \big(\widetilde{c\tilde{a} - \tilde{d}b}\big)^{-1}\big(\widetilde{g_{12-}^{(-1)}}\tilde{c} - \widetilde{g_{11-}^{(-1)}}\tilde{b}\big) + \big(\widetilde{g_{22-}}(I - Q_{\mathbb{R}} - \mathcal{K}) + \widetilde{g_{22+}^{(-1)}}(Q_{\mathbb{R}} + \mathcal{K})\big) \\
& \times \big(I - Q_{\mathbb{R}} - \mathcal{K} + (\widetilde{\lambda_-}/\widetilde{\lambda_+})^{-\kappa_{\bullet\bullet}}(Q_{\mathbb{R}} + \mathcal{K})\big)\big(\widetilde{c\tilde{a} - \tilde{d}b}\big)^{-1}\big(\widetilde{g_{22-}^{(-1)}}\tilde{c} - \widetilde{g_{21-}^{(-1)}}\tilde{b}\big)\Big]
\end{aligned}$$

where

$$\lambda_\pm(\xi) = \xi \pm i \qquad (\xi \in \mathbb{R}) \tag{3.4}$$

and $\mathcal{K}$ is the compact operator

$$\mathcal{K} = \frac{1}{2}(JS_{\mathbb{R}}J + S_{\mathbb{R}}) \tag{3.5}$$

Proof. The generalized factorization (3.2) of G implies (see [21: Chapter V, §5]) that $\mathcal{W}$ is a Fredholm operator. Therefore $\mathcal{W}$ is a generalized invertible operator and

$$\mathcal{W}^- = (G_+^{-1} P_{\mathbb{R}} + G_- Q_{\mathbb{R}})(D^{-1} P_{\mathbb{R}} + Q_{\mathbb{R}}) G_-^{-1} B_{\mathcal{W}}^{-1} \tag{3.6}$$

is a generalized inverse of $\mathcal{W}$ (see [21: Chapter V]). Consequently, from (1.7) and Corollary 2.6 we obtain a generalized inverse of $\mathcal{T}$,

$$\mathcal{T}^- = R_{11}(E\mathcal{W}^-E^{-1}) = \frac{1}{2}(\mathcal{W}_{11}^{(-)} + J\mathcal{W}_{21}^{(-)} + \mathcal{W}_{12}^{(-)}J + J\mathcal{W}_{22}^{(-)}J), \tag{3.7}$$

where we use the notation

$$\mathcal{W}^- = \begin{bmatrix} \mathcal{W}_{11}^{(-)} & \mathcal{W}_{12}^{(-)} \\ \mathcal{W}_{21}^{(-)} & \mathcal{W}_{22}^{(-)} \end{bmatrix}$$

with $\mathcal{W}_{kl}^{(-)} : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n$. Finally, the formula of a generalized inverse of $\mathcal{T}$ presented in the theorem is obtained by a combination of (3.6), (3.7) and

$$JP_{\mathbb{R}} = (Q_{\mathbb{R}} + \mathcal{K})J \qquad \text{and} \qquad JQ_{\mathbb{R}} = (P_{\mathbb{R}} - \mathcal{K})J,$$

where $\mathcal{K}$ is defined in (3.5) ∎

Remark 3.2. Other generalized inverses of $\mathcal{T}$, for instance a reflexive one or the Moore-Penrose inverse, can be obtained if we replace (3.6) by different kinds of generalized inverses of $\mathcal{W}$ that can be constructed following the ideas of [25].

Now let us introduce some additional assumptions. We consider the case where G is an invertible Hölder continuous matrix function on $\mathbb{R}$ (cf. condition (i) in the next theorem), with a diagonalizable jump at infinity (see condition (ii) in the following result). Actually, the last assumption can be replaced by the Jordan form case. This will imply only a reorganization of the elements in the final representation of a generalized inverse of $\mathcal{T}$ presented below.

Theorem 3.3. *Suppose that G (see (3.1)) has the following properties:*

(i) $G \in \mathcal{G}[C^\beta(\ddot{\mathbb{R}})]^{2n\times 2n}$ *for some* $\beta \in]0,1]$.

(ii) *There are constant $(n \times n)$-matrices V_{kl} $(k,l = 1,2)$ and complex numbers η_j $(j = 1,\dots,2n)$ such that*

$$\frac{1}{p} - 1 < \operatorname{Re}\eta_j < \frac{1}{p} \qquad (p \in]1,+\infty[)$$

$$G^{-1}(+\infty)G(-\infty) = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \operatorname{diag}\left[e^{2\pi i\eta_1},\dots,e^{2\pi i\eta_{2n}}\right] \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}.$$

(iii) $\beta > \frac{1}{p} - \min_j \operatorname{Re}\eta_j$.

(iv) $\mathcal{W} : [L^p(\mathbb{R})]^{2n} \to [L^p(\mathbb{R})]^{2n}$ *is a Fredholm operator.*

Then the singular integral operator with Carleman shift $\mathcal{T}$ (see (1.5)) is generalized *invertible, and a generalized inverse $\mathcal{T}^-$ of $\mathcal{T}$ is given by* (3.3) *with*

$$\begin{aligned}
g_{11+}^{(-1)} &= V_{11}\lambda_+^{-\eta_\bullet-\kappa_\bullet} + (H_+)_{11}\lambda_+^{-\kappa_\bullet}, \quad g_{12+}^{(-1)} = V_{12}\lambda_+^{-\eta_{\bullet\bullet}-\kappa_{\bullet\bullet}} + (H_+)_{12}\lambda_+^{-\kappa_{\bullet\bullet}} \\
g_{21+}^{(-1)} &= V_{21}\lambda_+^{-\eta_\bullet-\kappa_\bullet} + (H_+)_{21}\lambda_+^{-\kappa_\bullet}, \quad g_{22+}^{(-1)} = V_{22}\lambda_+^{-\eta_{\bullet\bullet}-\kappa_{\bullet\bullet}} + (H_+)_{22}\lambda_+^{-\kappa_{\bullet\bullet}} \\
g_{11-} &= \left[\left(G_{11}(+\infty)V_{11} + G_{12}(+\infty)V_{21}\right)\lambda_-^{-\eta_\bullet} + (H_-)_{11}\right]\lambda_-^{-\kappa_\bullet} \\
g_{12-} &= \left[\left(G_{11}(+\infty)V_{12} + G_{12}(+\infty)V_{22}\right)\lambda_-^{-\eta_{\bullet\bullet}} + (H_-)_{12}\right]\lambda_-^{-\kappa_{\bullet\bullet}} \\
g_{21-} &= \left[\left(G_{21}(+\infty)V_{11} + G_{22}(+\infty)V_{21}\right)\lambda_-^{-\eta_\bullet} + (H_-)_{21}\right]\lambda_-^{-\kappa_\bullet} \\
g_{22-} &= \left[\left(G_{21}(+\infty)V_{12} + G_{22}(+\infty)V_{22}\right)\lambda_-^{-\eta_{\bullet\bullet}} + (H_-)_{22}\right]\lambda_-^{-\kappa_{\bullet\bullet}} \\
g_{11-}^{(-1)} &= g_{22-}/(g_{11-}g_{22-} - g_{12-}g_{21-}), \quad g_{12-}^{(-1)} = -g_{12-}/(g_{11-}g_{22-} - g_{12-}g_{21-}) \\
g_{21-}^{(-1)} &= -g_{21-}/(g_{11-}g_{22-} - g_{12-}g_{21-}), \quad g_{22-}^{(-1)} = g_{11-}/(g_{11-}g_{22-} - g_{12-}g_{21-})
\end{aligned}$$

where $\kappa_1, \ldots, \kappa_{2n}$ *are the partial indices of a generalized factorization of*

$$G = B_{\mathcal{W}}^{-1} A_{\mathcal{W}} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

and $(H_\pm)_{kl}$ *are* $n \times n$ *elements of*

$$H_- = \begin{bmatrix} (H_-)_{11} & (H_-)_{12} \\ (H_-)_{21} & (H_-)_{22} \end{bmatrix} \in Q_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}$$

$$H_+ = \begin{bmatrix} (H_+)_{11} & (H_+)_{12} \\ (H_+)_{21} & (H_+)_{22} \end{bmatrix} \in P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}$$

such that

$$\left.\begin{aligned} G(+\infty)V + H_-\mathrm{diag}[\lambda_-^{\eta_1}, \ldots, \lambda_-^{\eta_{2n}}] \\ V + H_+\mathrm{diag}[\lambda_+^{\eta_1}, \ldots, \lambda_+^{\eta_{2n}}] \end{aligned}\right\} \in \mathcal{G}[L^\infty(\mathbb{R})]^{2n\times 2n} \tag{3.8}$$

for a constant matrix, with $n \times n$ *elements* V_{kl},

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Proof. From the equality $P_{\mathbb{R}}GP_{\mathbb{R}} + Q_{\mathbb{R}} = W(I - Q_{\mathbb{R}}GP_{\mathbb{R}})$, where $I - Q_{\mathbb{R}}GP_{\mathbb{R}}$ is an invertible operator with inverse given by $I + Q_{\mathbb{R}}GP_{\mathbb{R}}$, it follows that

$$\mathcal{W} = B_{\mathcal{W}}W : [L^p(\mathbb{R})]^{2n} \to [L^p(\mathbb{R})]^{2n}$$

is a Fredholm operator if and only if

$$P_{\mathbb{R}}G_{|P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n}} : P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n} \to P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n}$$

is a Fredholm operator as well. Therefore, from [7: Theorem 2.2] (see also [23] for some particular cases) we obtain that G admits a generalized factorization (see (3.2))

$$G(\xi) = G_-(\xi)\mathrm{diag}\left[\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\kappa_1}, \ldots, \left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\kappa_{2n}}\right] G_+(\xi)$$

with

$$\begin{aligned} G_- &= \big(G(+\infty)V + H_-\mathrm{diag}[\lambda_-^{\eta_1}, \ldots, \lambda_-^{\eta_{2n}}]\big)\mathrm{diag}[\lambda_-^{-\kappa_1-\eta_1}, \ldots, \lambda_-^{-\kappa_{2n}-\eta_{2n}}] \\ G_+ &= \mathrm{diag}[\lambda_+^{\eta_1+\kappa_1}, \ldots, \lambda_+^{\eta_{2n}+\kappa_{2n}}]\big(V + H_+\mathrm{diag}[\lambda_+^{\eta_1}, \ldots, \lambda_+^{\eta_{2n}}]\big)^{-1} \end{aligned}$$

where V is a constant matrix, and $H_+ \in P_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}$ and $H_- \in Q_{\mathbb{R}}[L^p(\mathbb{R})]^{2n\times 2n}$ are such that (3.8) is fulfilled. Now the properties of Δ-related operators, and in particular an application of Proposition 3.1, imply the statement ∎

Corollary 3.4. *Let us substitute conditions* (ii) *and* (iv) *in Theorem* 3.3 *by the following ones:*

(ii) *There are* $2n$ *complex numbers* η_j $(j = 1, \ldots, 2n)$ *such that*

$$\frac{1}{p} - 1 < \mathrm{Re}\eta_j < \frac{1}{p} \qquad (p \in]1, \infty[)$$

$$\lim_{\xi\to\pm\infty} \mathrm{diag}\left[\left(\frac{\lambda_+(\xi)}{\lambda_-(\xi)}\right)^{\eta_1}, \ldots, \left(\frac{\lambda_+(\xi)}{\lambda_-(\xi)}\right)^{\eta_{2n}}\right] = G(\pm\infty).$$

(iv) $\mathcal{W} : [L^p(\mathbb{R})]^{2n} \to [L^p(\mathbb{R})]^{2n}$ *is an invertible operator.*

Then, with the help of the methods exposed in [7] *and* [23], *we find an inverse of* $\mathcal{T}$. *Due to the large size of the formulas we do not present* $\mathcal{T}^{-1}$ *explicitly.*

Now we will study the asymptotic behavior of the solution $\varphi \in [L^p(\mathbb{R})]^n$ $(1 < p < \infty)$ of the equation

$$\mathcal{T}\varphi = \psi \in [L^p(\mathbb{R})]^n \tag{3.9}$$

where

$$\mathcal{T} = A_{\mathcal{T}}P_{\mathbb{R}} + B_{\mathcal{T}}Q_{\mathbb{R}} \qquad \text{with} \quad \begin{cases} A_{\mathcal{T}} = a(\xi)I + b(\xi)J \\ B_{\mathcal{T}} = c(\xi)I + d(\xi)J \end{cases}$$

and a, b, c, d are continuous $n \times n$ elements such that

$$G(\xi) = \frac{1}{c(\xi)a(\alpha(\xi)) - d(\alpha(\xi))b(\xi)} \times \begin{bmatrix} a(\alpha(\xi))a(\xi) - b(\alpha(\xi))b(\xi) & a(\alpha(\xi))d(\xi) - b(\xi)c(\alpha(\xi)) \\ c(\xi)b(\alpha(\xi)) - a(\xi)d(\alpha(\xi)) & c(\alpha(\xi))c(\xi) - d(\alpha(\xi))d(\xi) \end{bmatrix}$$

exists and represents a Hölder-continuous $(2n \times 2n)$-matrix function on $\mathbb{R}$, of order ν, admitting (different) limits at $\pm\infty$. In the study of equation (3.9) we will consider the case where ψ is such that there exists

$$f(\infty) = \lim_{\xi\to\infty}\left((\xi - i)^2 \begin{bmatrix} \left(c(\xi)I + d(\xi)J\right)^{-1}\psi(\xi) \\ \left(b(\xi)I + a(\xi)J\right)^{-1}\psi(\xi) \end{bmatrix} + (\xi - i)^3 \begin{bmatrix} \left(\left(c(\xi)I + d(\xi)J\right)^{-1}\psi(\xi)\right)' \\ \left(\left(b(\xi)I + a(\xi)J\right)^{-1}\psi(\xi)\right)' \end{bmatrix}\right)$$

and

$$g(\infty) = \lim_{\xi\to\infty}\frac{\xi - i}{2i}\begin{bmatrix} \left(c(\xi)I + d(\xi)J\right)^{-1}\psi(\xi) \\ \left(b(\xi)I + a(\xi)J\right)^{-1}\psi(\xi) \end{bmatrix} =: \begin{bmatrix} g_1(\infty) \\ g_2(\infty) \end{bmatrix}.$$

Let us introduce the matrix functions

$$Y_{\eta_*}(\xi) = \operatorname{diag}\left[(\xi - i)^{-\eta_1 - 1}, \dots, (\xi - i)^{-\eta_n - 1}\right]$$
$$Z_{\eta_*}(\xi) = \operatorname{diag}\left[(\xi + i)^{-\eta_1}(\xi - i)^{-1}, \dots, (\xi + i)^{-\eta_n}(\xi - i)^{-1}\right].$$

By $[Y_{\eta_*}(\xi)]_\pm$ and $[Z_{\eta_*}(\xi)]_\pm$ we denote the boundary values of the corresponding matrices $Y_{\eta_*}(\xi)$ and $Z_{\eta_*}(\xi)$ on $\mathbb{R}$, respectively. Similar notation is used for η_{**} instead of η_*.

Theorem 3.5. *Consider the above assumptions, and additionally suppose the following:*

(i) *There is a constant C such that*

$$|G(\xi_1) - G(\xi_2)| \le C\frac{|\xi_1 - \xi_2|^\nu}{(1 + |\xi_1|)^\nu(1 + |\xi_2|)^\nu}$$

for $\xi_1, \xi_2 < -1$ and $\xi_1, \xi_2 > 1$, separately (see [22]).

(ii) *The matrix $G(+\infty)G^{-1}(-\infty)$ has $2n$ distinct* [2] *eigenvalues with logarithmic values $2\pi\eta_j$ $(j = 1, \dots, 2n)$ such that*

$$\max_{1\le j,k\le 2n}\operatorname{Re}(\eta_j - \eta_k) < \nu \qquad \text{and} \qquad \frac{1}{p} - 1 < \operatorname{Re}(\eta_j) < \frac{1}{p}.$$

(iii) *There are four constant $(n \times n)$-matrices T_{kl} $(k, l = 1, 2)$ such that $T = [T_{kl}]_{k,l=1}^2$ is an invertible matrix which fulfills*

$$G(+\infty)G^{-1}(-\infty) = T\operatorname{diag}\left[e^{2\pi i\eta_1}, \dots, e^{2\pi i\eta_{2n}}\right]T^{-1}.$$

We will use the notation

$$w = \min\left(\nu - \max_{1\le j,k\le 2n}\operatorname{Re}(\eta_j - \eta_k), 1 - \max_{1\le j\le 2n}\operatorname{Re}(\eta_j)\right) - \varepsilon$$

---

[2] This assumption can be dropped on the price of rather complicated formulas, cf. Theorem 3.3.

for an arbitrary small $\varepsilon > 0$. *Let* $\psi \in [L^p(\mathbb{R})]^n$. *If there is a solution in* $[L^p(\mathbb{R})]^{2n}$ *of the equation*

$$\mathcal{W}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}(c(cI+dJ)^{-1}+b(bI+aJ)^{-1})\psi\\ (\tilde{d}(cI+dJ)^{-1}+\tilde{a}(bI+aJ)^{-1})\psi\end{bmatrix}, \tag{3.10}$$

where $\mathcal{W}$ *is defined in* (1.6), *then, for* $\xi \to \pm\infty$,

$$\begin{aligned}\varphi(\xi) = \frac{1}{2}\Big[& g_1(\infty) + (A_1T_{11}+A_2T_{21})(\lambda_-/\lambda_+)^{\eta_\bullet}(-\infty)[Y_{\eta_\bullet}(\xi)]_+(c_1+\mathcal{O}(|\xi|^{-w}))\\ & + g_2(\infty) + (A_1T_{12}+A_2T_{22})(\lambda_-/\lambda_+)^{\eta_{\bullet\bullet}}(-\infty)[Y_{\eta_{\bullet\bullet}}(\xi)]_+(c_2+\mathcal{O}(|\xi|^{-w}))\\ & + T_{11}[Z_{\eta_\bullet}(\xi)]_-(c_3+\mathcal{O}(|\xi|^{-w})) + T_{12}[Z_{\eta_{\bullet\bullet}}(\xi)]_-(c_4+\mathcal{O}(|\xi|^{-w}))\\ & + (A_3T_{11}+A_4T_{21})(\lambda_-/\lambda_+)^{\eta_\bullet}(-\infty)[Y_{\eta_\bullet}(\alpha(\xi))]_+(c_1+\mathcal{O}(|\alpha(\xi)|^{-w}))\\ & + (A_3T_{12}+A_4T_{22})(\lambda_-/\lambda_+)^{\eta_{\bullet\bullet}}(-\infty)[Y_{\eta_{\bullet\bullet}}(\alpha(\xi))]_+(c_2+\mathcal{O}(|\alpha(\xi)|^{-w}))\\ & + T_{21}[Z_{\eta_\bullet}(\alpha(\xi))]_-(c_3+\mathcal{O}(|\alpha(\xi)|^{-w})) + T_{22}[Z_{\eta_{\bullet\bullet}}(\alpha(\xi))]_-(c_4+\mathcal{O}(|\alpha(\xi)|^{-w}))\Big]\end{aligned}$$

is a representation of a solution of equation (3.9), *where* c_1, c_2, c_3 *and* c_4 *are constant* $(n \times 1)$*-matrices which depend on the coefficients of* G *and on* ψ *and*

$$\begin{aligned} A_1 &= \frac{c(\alpha(-\infty))c(-\infty)-d(-\infty)d(\alpha(-\infty))}{a(-\infty)c(\alpha(-\infty))-d(-\infty)b(\alpha(-\infty))}\\ A_2 &= \frac{c(\alpha(-\infty))b(-\infty)-d(-\infty)a(\alpha(-\infty))}{a(-\infty)c(\alpha(-\infty))-d(-\infty)b(\alpha(-\infty))}\\ A_3 &= \frac{a(-\infty)d(\alpha(-\infty))-b(\alpha(-\infty))c(-\infty)}{a(-\infty)c(\alpha(-\infty))-d(-\infty)b(\alpha(-\infty))}\\ A_4 &= \frac{a(-\infty)a(\alpha(-\infty))-b(\alpha(-\infty))b(-\infty)}{a(-\infty)c(\alpha(-\infty))-d(-\infty)b(\alpha(-\infty))}. \end{aligned}$$

Proof. From the equivalence relation

$$B_{\mathcal{W}}^{-1}\mathcal{W} = W \qquad (= GP_{\mathbb{R}} + Q_{\mathbb{R}}) \tag{3.11}$$

and the hypotheses we know that there is a solution of the equation

$$W\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}(c(\xi)+d(\xi)J)^{-1}\psi(\xi)\\ (b(\xi)+a(\xi)J)^{-1}\psi(\xi)\end{bmatrix} \tag{3.12}$$

and this solution is the same as the solution of equation (3.10). Under the assumptions of the theorem we can apply [22: Theorem 3.1] to equation (3.12) and, consequently, obtain the following representation of a solution of equation (3.12), for $\xi \to \pm\infty$:

$$\begin{bmatrix} \varphi_1(\xi) \\ \varphi_2(\xi) \end{bmatrix} = \begin{bmatrix} g_1(\infty) \\ g_2(\infty) \end{bmatrix} + G^{-1}(-\infty) \begin{bmatrix} T_{11}\,(\lambda_-/\lambda_+)^{\eta_\bullet}(-\infty) & T_{12}\,(\lambda_-/\lambda_+)^{\eta_{\bullet\bullet}}(-\infty) \\ T_{21}\,(\lambda_-/\lambda_+)^{\eta_\bullet}(-\infty) & T_{22}\,(\lambda_-/\lambda_+)^{\eta_{\bullet\bullet}}(-\infty) \end{bmatrix} \times \begin{bmatrix} Y_{\eta_\bullet}(\xi) & 0 \\ 0 & Y_{\eta_{\bullet\bullet}}(\xi) \end{bmatrix}_+ \begin{bmatrix} c_1 + \mathcal{O}(|\xi|^{-w}) \\ c_2 + \mathcal{O}(|\xi|^{-w}) \end{bmatrix} + \begin{bmatrix} T_{11} Z_{\eta_\bullet}(\xi) & T_{12} Z_{\eta_{\bullet\bullet}}(\xi) \\ T_{21} Z_{\eta_\bullet}(\xi) & T_{22} Z_{\eta_{\bullet\bullet}}(\xi) \end{bmatrix}_- \begin{bmatrix} c_3 + \mathcal{O}(|\xi|^{-w}) \\ c_4 + \mathcal{O}(|\xi|^{-w}) \end{bmatrix}, \tag{3.13}$$

where c_1, c_2, c_3 and c_4 are constant vectors and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is a constant invertible matrix so that

$$G(+\infty)G^{-1}(-\infty) = T\mathrm{diag}\big[e^{2\pi i \eta_1}, \dots, e^{2\pi i \eta_{2n}}\big]T^{-1}.$$

Further, from the Δ-relation (1.7) and from (3.11) we obtain that equation (3.12) is equivalent to

$$\begin{bmatrix} \mathcal{T}(\frac{1}{2}\varphi_1 + \frac{1}{2}J\varphi_2) \\ \mathcal{T}_\Delta(\frac{1}{2}\varphi_1 - \frac{1}{2}J\varphi_2) \end{bmatrix} = \begin{bmatrix} \psi \\ \frac{1}{2}\big((c-dJ)(c+dJ)^{-1} + (b-aJ)(b+aJ)^{-1}\big)\psi \end{bmatrix}.$$

Therefore, $\frac{1}{2}\varphi_1 + \frac{1}{2}J\varphi_2$ is a solution of the equation $\mathcal{T}\varphi = \psi$ and from (3.13) we arrive at the desired representation ∎

4. Further aspects about the type of the relation

In brief we present some additional cases for the discussion of properties of operators $\mathcal{T}$ which are related to operator matrices $\mathcal{W}$ with a "simpler structure".

Example 4.1. Let $\mathcal{C} \subset \mathcal{L}(X)$ be a class of bounded linear operators on a Banach space X and $\mathrm{alg}\,\mathcal{C}$ the subalgebra of $\mathcal{L}(X)$ that is finitely generated by operators from $\mathcal{C}$. Then every $\mathcal{T} \in \mathrm{alg}\,\mathcal{C}$ is equivalent after extension to a pure $\mathcal{C}$-operator matrix, more precisely

$$\begin{bmatrix} \mathcal{T} & 0 \\ 0 & I_{X^{m-1}} \end{bmatrix} = E\mathcal{W}F$$

where $m \in \mathbb{N}$, $I_{X^{m-1}} \in \mathcal{L}(X^{m-1})$ is the unit operator, E and F are upper/lower triangular operator matrices whose diagonal elements equal the unity $I \in \mathcal{L}(X)$, and $\mathcal{W} \in \mathcal{C}^{m\times m}$ (see [8]). For instance,

$$\begin{bmatrix} T_1 + T_2T_3 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -T_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ -T_3 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T_3 & I \end{bmatrix}. \tag{4.1}$$

Consequently, a generalized inverse $\mathcal{W}^-$ of $\mathcal{W}$ implies a generalized inverse $\mathcal{T}^- = R_{11}(\mathcal{W}^-)$ of $\mathcal{T}$ where R_{11} denotes the restriction on the first element of a matrix.

Let us consider the particular case where $X = [L^2(\mathbb{R}_+)]^n$ and $\mathcal{C}$ denotes a class of Wiener-Hopf operators with Fourier symbols in $[L^\infty(\mathbb{R})]^{n\times n}$, which is a large class for the applications [17]. We like to study the singularities of the solution of an equation $\mathcal{T}\varphi = \psi$ for $\mathcal{T} \in \operatorname{alg}\mathcal{C}$ without smoothness assumptions on the Fourier symbols of the operators if ψ is smooth on $\overline{\mathbb{R}_+}$ and rapidly decreasing (for somplicity). Note that, in the half-line case, the (Hilbert) Sobolev space setting can always be translated in terms of the present situation by the use of the lifting procedure [28].

Theorem 4.2. *Under the above assumptions and*

$$\mathcal{W} = P_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} \in \mathcal{G}\,\mathcal{L}([L^2(\mathbb{R}_+)]^{nm})$$

the unique solution $\varphi = [\varphi_j]_{j=1,\dots,n}$ of the equation $\mathcal{T}\varphi = \psi$ satisfies

$$\varphi_j(x) = \mathcal{O}(|x|^{-\gamma_j - 1/2}) \qquad as\ \ |x| \to 0$$

where γ_j are the numbers provided by a bounded operator (or strong) factorization (see [5, 24]) of $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ with respect to the intermediate space [7]

$$Z = H^{\gamma_1,2}(\mathbb{R}) \oplus \cdots \oplus H^{\gamma_{nm},2}(\mathbb{R}) \qquad (|\gamma_j| < \tfrac{1}{2}).$$

Proof. The solution is representable by the first n components of the nm-vector

$$\begin{bmatrix} \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{-1} & 0 \\ 0 & I_{[L^2(\mathbb{R}_+)]^{n(m-1)}} \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} I_{[L^2(\mathbb{R}_+)]^n} & 0 \\ * & I_{[L^2(\mathbb{R}_+)]^{n(m-1)}} \end{bmatrix} \mathcal{W}_+^{-1}\mathcal{W}_-^{-1} \begin{bmatrix} \psi \\ 0 \end{bmatrix},$$

where $\mathcal{W}_\pm^{-1} = P_+A_\pm^{-1}$ are operators on $[L^2(\mathbb{R}_+)]^{nm}$ generated by a strong factorization of A,

$$A = A_-A_+,$$

with respect to the intermediate space Z so that $A_\pm$ are invertible operators having the factor properties

$$A_+[L^2(\mathbb{R}_+)]^{nm} = \widetilde{H}^{\gamma_1,2}(\mathbb{R}_+) \oplus \cdots \oplus \widetilde{H}^{\gamma_{nm},2}(\mathbb{R}_+)$$
$$A_-\left[\widetilde{H}^{\gamma_1,2}(\mathbb{R}_-) \oplus \cdots \oplus \widetilde{H}^{\gamma_{nm},2}(\mathbb{R}_-)\right] = [L^2(\mathbb{R}_-)]^{nm}.$$

Thus a theorem of Abel type yields the same result for φ as for the first n components of the solution of the corresponding Wiener-Hopf system [23] ∎

Remark 4.3. Other properties which are transferred are, for instance, regularity properties of the operators; if $\Phi \in \mathcal{G}[C(\ddot{\mathbb{R}})]^{nm\times nm}$, then $\mathcal{T}$ is Fredholm if and only if

$$\det(\mu\Phi(+\infty) + (1-\mu)\Phi(-\infty)) \neq 0 \qquad (\mu \in]0,1[)$$

otherwise $\mathcal{T}$ is not normally solvable; if Φ is continuous but not invertible on $\ddot{\mathbb{R}}$, then $\mathcal{T}$ is generalized invertible (with infinite defect numbers) if and only if $\operatorname{rank}\Phi(\xi) = \text{const}$ otherwise $\mathcal{T}$ is not normally solvable.

The spectra and essential spectra of $\mathcal{T}$ and $\mathcal{W}$ are not so directly related. We can modify, e.g., formula (4.1) and obtain

$$\begin{bmatrix} T_1 + T_2T_3 - \lambda I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -T_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 - \lambda I & T_2 \\ -T_3 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ T_3 & I \end{bmatrix}$$

i.e. $\mathcal{T} - \lambda I$ and the middle factor $\mathcal{W}_\lambda$ satisfy the same relation as $\mathcal{T}$ and $\mathcal{W}$ do. So at least we obtain explicitly $\operatorname{ess\,sp}\mathcal{T} = \{\lambda : \Phi_\lambda \text{ is 2-singular}\}$ where Φ_λ denotes the Fourier symbol of $\mathcal{W}_\lambda$.

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