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The Stokes System in Domains with Outlets of Bounded and Connected Cross-Sections

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Abstract. The Stokes system with prescribed fluxes is investigated. By smoothness assumptions on the boundary and by the boundedness of the diameters of the outlets it is ensured that the divergence equation in each bounded subdomain is solvable, the Poincaré inequality is valid and the constants in all the corresponding estimates are bounded *independently of the location.* We derive existence, uniqueness and regularity results in two different frameworks: On one hand we use weighted function spaces generated by L^q-norms, $1 < q < \infty$, where the weight is of exponential type and apply a technique of Maz'ya and Plamenevskii. On the other hand we use local spaces, since in order to solve the problem with non-zero flux it seems to us that to formulate results in local spaces is more adequate and physical senseful.

Keywords: *Stokes systems, non-compact boundaries, weighted spaces, local spaces*

AMS subject classification: 35 Q 30

1. Introduction

An interesting task and fairly open field in the theory of hydrodynamics is the investigation of flows in unbounded domains Ω with non-compact boundary $\partial\Omega$. In our paper we investigate the system

$$\text{(S)} \quad \begin{cases} -\Delta \mathbf{v}(x) + \nabla p(x) = \mathbf{f}(x) & (x \in \Omega) \\ \operatorname{div} \mathbf{v}(x) = 0 & (x \in \Omega) \\ \mathbf{v}(x) = 0 & (x \in \partial\Omega). \end{cases}$$

Here Ω is a domain which has several outlets connected by a smooth bounded domain. Each outlet has a bounded and connected cross-section. Such domains are of special interest as model problems for practical applications. They reflect systems of channels or pipes. In addition to problem (S) we prescribe the fluxes for the outlets to determine the velocity field $\mathbf{v}(x) = (v_1(x), \dots, v_n(x))$ and the corresponding pressure $p(x)$ uniquely (uniqueness means unique up to an additive constant in the pressure).

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This problem was extensively studied in [6] by Ladyzhenskaya and Solonnikov in the Hilbert space setting. In this context it turned out that for such domains it is not senseful to restrict the considerations to vector fields with finite Dirichlet integral if the flux is different from zero (see also Amick [2, 3] for straight cylinders).

In [15, 16] Pileckas derived solvability properties of problem (S) with prescribed fluxes or pressure drop in a whole class of unbounded domains in the framework of weighted Sobolev spaces generated by the L^q-norm. Interesting for our purposes is the fact that he applied these general results to the special case of cylindrical outlets with circular cross-section. His paper motivated our attempt to generalize the results of [6] to L^q-spaces $(1 < q < \infty)$ using techniques of Maz'ya and Plamenevskii [7] and Solonnikov [18].

We will formulate solvability properties of problem (S) with prescribed fluxes in two different frameworks: On one hand we will use weighted function spaces, where the weight is of exponential type. On the other hand we will use local spaces (for the exact definitions see Sections 2 and 5).

In particular, the straight infinite cylinder as a domain with two cylindrical exits (having a common axis) is a special example for domains where our results are valid.

The usage of weights of exponential type seems rather natural from the following observations (for details we refer to [10: Sections 2.2 and 3.1].

Assume for the moment that we want to solve problem (S) with zero fluxes through all outlets. We will denote this problem by $(\mathrm{S})_0$. Moreover, we restrict ourselves to a straight infinite cylinder and the data are smooth and have compact support. Performing the complex Fourier transformation $\mathbf{F}_{x_3\to\lambda}$ with respect to the variable x_3 along the cylindrical axis leads to a family of problems (S_λ) in the cross-section. Suppose for $\beta \in \mathbb{R}$ that the line $\mathbb{R} + i\beta$ is free of eigenvalues of this family which means problem (S_λ) has a unique solution $\mathbf{v}_\lambda$ for each $\lambda \in \mathbb{R} + i\beta$. Then the inverse Fourier transformation applied to the family $\mathbf{v}_\lambda$ yields the solution $\mathbf{v}$ for the problem in the cylinder. Parseval's identity then provides $\int_{\mathbb{R}} e^{2\beta t}|v(t)|^2\,dt = \int_{\mathbb{R}+i\beta} |\hat{v}(\lambda)|^2\,d\lambda$. This relation is the motivation to solve problem $(\mathrm{S})_0$ in spaces of functions with exponential weights in case of domains with cylindrical outlets.

If we allow varying cross-sections we are not too far away from the situation of cylindrical outlets if we presume that the diameters are bounded from above and below and that the domain has the *uniform C^l-regularity property* as defined by Adams [1: IV.4.6]). The point is that these assumptions yield the solvability of the divergence equation in each bounded subdomain of Ω with estimates where the constant is bounded *independently of the location* and the validity of Poincaré's inequality.

Already in [6] it is stated that problem $(\mathrm{S})_0$ has a unique solution decaying exponentially to zero if the force does not increase too fast in direction of the axis (not faster than e^{cx_3} for some specified constant c, see the end of Section 3 there). In [6] this is a kind of byproduct and the main point is, of course, to investigate the influence of the non-zero flux on the behaviour of the Dirichlet norm of the velocity field.

Their result is one more motivation to consider forces which increase or decrease exponentially with a certain (bounded by some constant) rate. In this frame we prove that there exists a unique solution to problem $(\mathrm{S})_0$ such that it together with all derivatives increases or decreases with the same rate as the force (with the usual regularity shift).

Comparing this setting with the results of Ladyzhenskaya and Solonnikov we generalize [6] not only allowing $q \neq 2$ but also finding the precise relation between the asymptotic behaviour of the data and the solution and considering higher order regularity.

For straight cylindrical exits one is used to think about the decay of the solution of problem (S) to the corresponding Poiseuille flows in the outlets (which are determined by the shape of the cross-section and the flux). For domains where the cross-section may vary it is not adequate to speak about an asymptotic decay in this sense. It is meaningless as long as there is no serious restriction [1)] on the behaviour of the boundary.

While studying spatial stability we use weighted spaces, whose topology is stronger than that induced by local spaces. Nevertheless, in order to solve the problem with non-zero flux, we have to choose $\beta < 0$ but then the topology becomes weaker and hence less suitable. For that it seems to us that to formulate results in local spaces is more adequate and physical senseful in this situation.

This paper is structured as follows. In the next Section 2 we define the domain Ω and state the precise assumptions on the shape of the domain which are necessary for our approach. Moreover, we define the weighted function spaces which we use. To prepare the derivation of existence and uniqueness results for problem $(S)_0$ in weighted spaces we provide two auxiliary lemmata. The first concerns the case $q = 2$ and force with compact support and the second is a local estimate of Cattabriga type. In Section 3 we deal with problem $(S)_0$. Theorem 1 characterizes the solvability properties in weighted spaces generated by the L^q-norm, $1 < q < \infty$. Sections 4 and 5 show how to derive L^q-results for non-zero fluxes. Defining a *flux carrier* similar to that used in [17, 18] this question is answered from different points of view: in Section 4 in weighted spaces and in Section 5 in local spaces.

Everything is formulated for space dimension $n = 3$, but it is easy to see that the same arguments are applicable for $n = 2$ if the embeddings are changed appropriately.

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2. Definitions and auxiliary results

The precise definition of the domain $\Omega \subset \mathbb{R}^3$ needs some preparation. We suppose that the boundary $\partial\Omega$ is a submanifold without boundary. To prove existence of weak solutions it is sufficient that Ω has the *uniform* C^1-property. For more regularity of the solution more regularity of the boundary is necessary. This is formulated precisely in Theorem 1. If we refer to the whole domain Ω, then Cartesian coordinates $x = (x_1, x_2, x_3)$ are used. Let J be the fixed number of outlets. Then

$$\Omega \equiv \Omega^0 \cup \Omega^1 \cup \cdots \cup \Omega^J$$

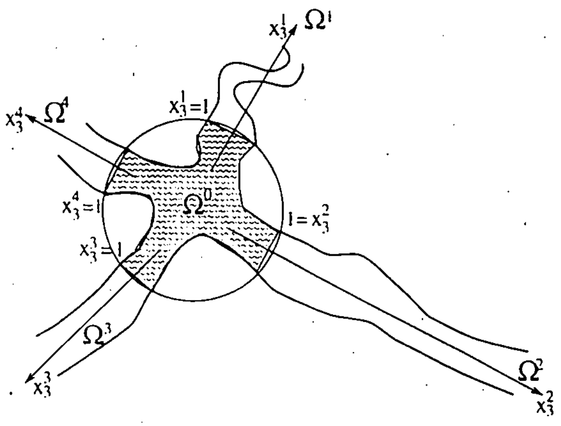
[1)] For example, if the cross-section varies *periodically* (see [9]).

with $\Omega^j \cap \Omega^k = \emptyset$ if $j \neq k$ and $j, k \neq 0$. We assume that there exists a number $R \in \mathbb{R}$ such that $\Omega^0 \subset \Omega \cap B_R$ where B_R is the ball with center 0 and radius R. Without loss of generality we set $R = 1$. For each outlet Ω^j we may find Cartesian coordinates x^j such that in these coordinates

$$\Omega^j = \left\{ x^j \in \mathbb{R}^3 : (x_1^j, x_2^j) \in \Sigma^j(x_3^j) \text{ and } x_3^j > 1 \right\}$$

where $\Sigma^j(x_3^j) \subset \mathbb{R}^2$ is a bounded domain for all $x_3^j > 1$. We assume that the diameters of the outlets are bounded from above and below as follows:

$$d \leq \operatorname{diam} \Sigma^j(x_3^j) \leq D \qquad \text{where } 0 < d \leq D < \infty.$$



The flux through the j^{th} outlet is denoted by $\mathcal{F}^j \equiv \int_{\Sigma^j(x_3^j)} v_3(x^j)\, dx_1^j dx_2^j$. A necessary condition for the solvability of problem (S) is $\sum_{j=1}^J \mathcal{F}^j = 0$. This is clear from the physical meaning of the conditions $\operatorname{div} \mathbf{v} = 0$ in Ω and $\mathbf{v} = 0$ on $\partial\Omega$. Moreover, under these conditions the fluxes are constant in each outlet.

We consider problem (S) in a class of functions with exponential weights:

Take $\beta \in \mathbb{R}$ and let ρ_β be a smooth positive function on $\overline{\Omega}$ with $\rho_\beta(x) = e^{\beta x_3^j}$ on Ω^j for $j = 1, \ldots, J$. With the help of the usual Sobolev space $W^{l,q}(\Omega)$ $(l \in \mathbb{N}_0 \cup \{-1\}, 1 < q < \infty)$ we define for $\mathbf{v} \in C_0^\infty(\overline{\Omega})$ the norm

$$\|\mathbf{v}; W_\beta^{l,q}(\Omega)\| \equiv \|\rho_\beta \mathbf{v}; W^{l,q}(\Omega)\|. \tag{1}$$

$W_\beta^{l,q}(\Omega)$ is the closure of $C_0^\infty(\overline{\Omega})$ and $\mathring{W}_\beta^{l,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ under $\|\cdot; W_\beta^{l,q}(\Omega)\|$. Note that $W_{\beta_1}^{s,q}(\Omega) \subset W_{\beta_2}^{l,q}(\Omega)$ if $\beta_1 \geq \beta_2$ and $s \geq l$. It is easy to see that

$$W_\beta^{l,q}(\Omega) = \left\{ \mathbf{v} \in W_{loc}^{l,q}(\overline{\Omega}) : \|\mathbf{v}; W_\beta^{l,q}(\Omega)\| < \infty \right\}$$

using standard cut-off arguments. In the case $l = 0$ we write $W^{0,q}_{\beta}(\Omega) = L^q_{\beta}(\Omega)$. If no confusion may arise we will use the same notation for vector fields with components in these spaces. Analogously to the usual Sobolev spaces (see [1: Theorem III.3.8]) for $\mathbf{f} \in W^{-1,q}_{\beta}(\Omega)$ we dispose of the representation

$$\begin{aligned} \mathbf{f} &= \mathbf{f}_0 + (\operatorname{div} \mathbf{f}_1, \operatorname{div} \mathbf{f}_2, \operatorname{div} \mathbf{f}_3) \qquad (\mathbf{f}_i \in L^q_{\beta}(\Omega), i = 0, \dots, 3) \\ \|\mathbf{f}; W^{-1,q}_{\beta}(\Omega)\| &= \sum_{i=0}^{3} \|\mathbf{f}_i; L^q_{\beta}(\Omega)\|. \end{aligned} \tag{2}$$

The application of a functional $\mathbf{f} \in W^{-1,q}_{\beta}(\Omega)$ is indicated by $\langle \mathbf{f}, \cdot \rangle_{\Omega}$.

Since the domain under consideration is uniformly of C^1-type (if $q = 2$ even uniformly Lipschitz condition is enough), all outlets have bounded cross-sections and $\mathbf{v}$ is zero on the boundary, the *Poincaré inequality* holds: $\|\mathbf{v}; L^q(\Omega)\| \le c_p \|\nabla \mathbf{v}; L^q(\Omega)\|$.

As far as standard L^2-theory is concerned the following is well-known: Let $\mathbf{f} \in W^{-1,2}(\Omega)$. Then there exists a uniquely determined $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$, such that

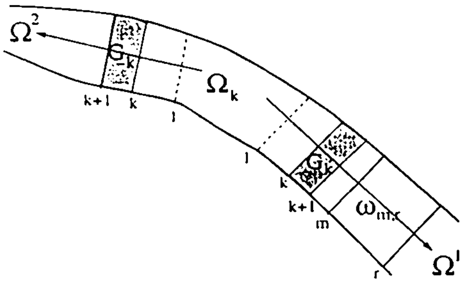
$$\textbf{(WF)} \quad \begin{cases} \displaystyle\int_{\Omega} \nabla \mathbf{v} : \nabla \Psi \, dx = \langle \mathbf{f}, \Psi \rangle_{\Omega} & \text{for all } \Psi \in \mathring{W}^{1,2}(\Omega) \text{ with } \operatorname{div} \Psi = 0 \\ \|\nabla \mathbf{v}; L^2(\Omega)\| \le c \, \|\mathbf{f}; W^{-1,2}(\Omega)\|. \end{cases}$$

We call $\mathbf{v}$ *weak* solution of problem $(\mathrm{S})_0$.

If $\mathbf{f} \in W^{-1,2}_{\beta}(\Omega)$, then we expect a similar estimate in terms of $L^2_{\beta}(\Omega)$ at least for a certain range of β near $\beta = 0$. But this cannot be shown simply using the Riesz representation theorem. This is due to the fact that if we endow $W^{1,2}_{\beta}(\Omega)$ with a scalar product, then in this scalar product gradients and solenoidal functions are no longer perpendicular and (WF) cannot be generalized to this case. To overcome this difficulty we will first consider right-hand sides with compact support. The aim is to prove an auxiliary lemma for $q = 2$ and $\mathbf{f}$ with compact support which is in $W^{-1,2}_{\beta}(\Omega)$ for some β with $|\beta| < \beta^*$. For such force a weak solution $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$ exists since due to its compact support $\mathbf{f}$ is also in $W^{-1,2}(\Omega)$. We consider portions of the outlets and investigate how the weighted norm of this solution depends on the distance between the part in which $\mathbf{v}$ is considered and the part where $\mathbf{f}$ has its support. It is shown that the influence of such force vanishes exponentially (see Lemma 1).

To avoid unnecessary indices we will perform all proofs in Sections 2 and 3 for a domain with two exits Ω^1 and Ω^2. For the different kinds of bounded domains we set $G_0 \equiv \Omega^0$, $\Omega_1 = \Omega^0$ and use the following abbreviations for $k, m \in \mathbb{N}$ with $m > k$:

$$\begin{aligned} G_k &\equiv \{x \in \Omega^1 : k \le x^1_3 < k+1\} \\ G_{-k} &\equiv \{x \in \Omega^2 : k \le x^2_3 < k+1\} \\ G^*_k &\equiv G_{k-1} \cup G_k \cup G_{k+1} \\ \Omega_k &\equiv \{x \in \Omega : |x^j_3| < k \ \ (j = 1,2)\} \cup \Omega^0 \\ \omega_{k,m} &\equiv \cup_{i=k}^{m-1} G_i \\ \omega_{-k,-m} &\equiv \cup_{i=-k}^{-m+1} G_i. \end{aligned} \tag{3}$$



Moreover, we introduce the cut-off function $0 \leq \eta_k = \eta_k(x_3^j) \leq 1, \eta_k \in C_0^\infty(\Omega)$,

$$\eta_k = \begin{cases} 1 & \text{if } x \in \Omega_k \\ 0 & \text{if } x \in \Omega \backslash \Omega_{k+1}. \end{cases} \tag{4}$$

Since $\eta_k \mathbf{v}$ is not solenoidal, as test function in (WF) we take

$$\mathbf{v}_k \equiv \eta_k \mathbf{v} + \mathbf{h}_k \tag{5}$$

where $\mathbf{h}_k \in \mathring{W}^{1,2}(\Omega)$ is the solution to the problem

$$\operatorname{div} \mathbf{h}_k = -\operatorname{div}(\eta_k \mathbf{v}) \tag{6}$$

in $G_k \cup G_{-k}$. This problem is indeed solvable (see [13, 14] and [15: Lemma 1.1]) since because of the zero flux condition and $\operatorname{div} \mathbf{v} = 0$ we know $\int_{G_k \cup G_{-k}} \operatorname{div}(\eta_k \mathbf{v})\, dx = 0$. There holds the estimate

$$\|\nabla \mathbf{h}_k; L^2(G_k \cup G_{-k})\| \leq M \,\|\nabla \mathbf{v}; L^2(G_k \cup G_{-k})\| \tag{7}$$

(because $\partial\Omega$ is uniformly Lipschitz M can be chosen independently of k, see [6: p. 736]). A basic tool to prove Lemma 1 below is the following Proposition 1 which is proved in [18: Lemma 3.2]:

Proposition 1. *Let μ_1, μ_2, $\nu > 0$, $b_{k+1} \geq b_k$, $b_{k+l} \leq \nu^l b_k$ for $k, l \in \mathbb{N}_0$ and*

$$z_k \leq \mu_1(z_{k+1} - z_k) + \mu_2 b_k \qquad \textit{together with} \quad \nu < \frac{1+\mu_1}{\mu_1}.$$

(i) *If $z_k(\frac{\mu_1}{1+\mu_1})^k \to 0$ for $k \to \infty$, then for all $k \in \mathbb{N}_0$ it holds $z_k \leq c\,\mu_2 b_k$.*

(ii) *If $z_K \leq c_K b_K$ for some $K \in \mathbb{N}$, then for all $k < K$ it holds $z_k \leq c\,\mu_2 b_k + c_K$.*

The constants c depend on μ_1 and ν.

Lemma 1. *Consider problem $(\mathrm{S})_0$ in Ω. Let $\partial\Omega$ be uniformly Lipschitz,* $\operatorname{supp} \mathbf{f} \subset G_k$ *for some $k \in \mathbb{Z}$ and moreover $\mathbf{f} \in W^{-1,2}(\Omega)$. Then there exist some $\varepsilon_0 > 0$ and $\beta^* > 0$ such that for $|\beta| < \beta^*$ the weak solution $\mathbf{v}$ of problem $(\mathrm{S})_0$ fulfils*

$$\|\nabla \mathbf{v}; L^2_\beta(G_m)\| \leq c\, e^{-\varepsilon_0 ||m|-|k||} \|\mathbf{f}; W_\beta^{-1,2}(G_k)\|. \tag{8}$$

Proof. We proceed in three steps. First we will prove (9), an estimate for the norm of $\nabla \mathbf{v}$ in Ω_k. After that we consider G_k. In these investigations $\mathbf{f}$ need not have bounded support. Finally, in Step 3 we prove (8). We will mainly use the solvability theory for $\mathbf{f} \in W^{-1,2}(\Omega)$ and Proposition 1 provided $|\beta|$ is sufficiently small.

Step I (see also [15: Theorem 2.3]): Let $k \in \mathbb{N}$, $m \in \mathbb{Z}$, $\partial\Omega$ be uniformly Lipschitz and $\mathbf{f} \in W_\beta^{-1,2}(\Omega) \cap W^{-1,2}(\Omega)$. Then

$$\|\nabla \mathbf{v}; L^2(\Omega_k)\| \le \begin{cases} c\,\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\| & \text{for } \beta \ge 0 \\ c\,e^{-\beta(k+1)}\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\| & \text{for } \beta < 0. \end{cases} \tag{9}$$

The case $\beta \ge 0$ is trivial. Let $\beta < 0$ and $z_k \equiv \|\nabla \mathbf{v}; L^2(\Omega_k)\|^2$. In (WF) we set $\Psi = \mathbf{v}_k$ ($\mathbf{v}_k$ defined in (5)) and collect all terms different from z_k on the right-hand side. For the force term we obtain applying (2), Hölder's, Young's and the Poincaré inequality

$$\begin{aligned} \left|\int_{\Omega_{k+1}} \mathbf{f}\,\mathbf{v}_k\,dx\right| &\equiv \left|\int_{\Omega_{k+1}} \mathbf{f}_0 \mathbf{v}_k\,dx\right| + \sum_{i=1}^{3}\left|\int_{\Omega_{k+1}} \mathbf{f}_i \nabla \mathbf{v}_k\,dx\right| \\ &\le c(\tilde\varepsilon)\sum_{i=0}^{3}\|\mathbf{f}_i; L^2(\Omega_{k+1})\|^2 + \tilde\varepsilon(c_p+3)\|\nabla \mathbf{v}_k; L^2(\Omega_{k+1})\|^2 \end{aligned}$$

for some (small) $\tilde\varepsilon > 0$. Since $1 < x_3^j < k+1$ we see

$$\begin{aligned} \sum_{i=0}^{3}\|\mathbf{f}_i; L^2(\Omega_{k+1})\|^2 &\le \|\mathbf{f}; W^{-1,2}(\Omega_{k+1})\|^2 \\ &= \|\mathbf{f}\,\rho_\beta(\rho_\beta)^{-1}; W^{-1,2}(\Omega_{k+1})\|^2 \\ &\le e^{-2\beta(k+1)}\|\mathbf{f}; W_\beta^{-1,2}(\Omega_{k+1})\|^2 \\ &\le e^{-2\beta(k+1)}\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2. \end{aligned}$$

Together with the estimates $\eta_k \le 1$, $|\nabla\eta_k| \le c$ as well as (7) for $\mathbf{h}_k$ we deduce (the details are similar to the estimates (2.31) - (2.37) in [15])

$$z_k \le \tfrac{1+\varepsilon}{1-\varepsilon}\,\hat c\,(z_{k+1} - z_k) + c\,\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2\,e^{-2\beta(k+1)}.$$

Here $\hat c \sim 1 + c_p + M$. Proposition 1/(i) for $\mu_1 = \frac{1+\varepsilon}{1-\varepsilon}\hat c$, $\mu_2 = c\,\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2$, $\nu = e^{-2\beta}$, $b_k = e^{-2\beta(k+1)}$ then yields (9) if the condition $\nu < \frac{1+\mu_1}{\mu_1}$ is fulfilled. That $z_k(\frac{\mu_1}{1+\mu_1})^k \to 0$ for $k \to \infty$ is easy to see and the condition for ν is true if $\beta < \beta^* < \frac{1}{2}\ln(1+\frac{1}{\hat c})$.

Step II (see also [15: Theorem 2.4]): Let now $k \in \mathbb{Z}$. Then

$$\|\nabla \mathbf{v}; L^2(G_k)\|^2 \le c\,e^{-2\beta|k|}\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2. \tag{10}$$

In this step $\beta \ge 0$ is the interesting case. Without loss of generality let $k \ge 2$, $l \in \mathbb{N}_0$, $l < k$. We use (WF) with $\Psi = \mathbf{v}_{k,l}$ which has its support in $\omega_{k-l-1,k+l+2}$ and is equal to $\mathbf{v}$ in $\omega_{k-l,k+l+1}$. Precisely, we define

$$\mathbf{v}_{k,l} \equiv \begin{cases} (1-\eta_{k-l-1})\mathbf{v} - \mathbf{h}_{k-l-1} & \text{if } x \in G_{k-l-1} \\ \mathbf{v} & \text{if } x \in \omega_{k-l,k+l+1} \\ \eta_{k+l+1}\mathbf{v} + \mathbf{h}_{k+l+1} & \text{if } x \in G_{k+l+1} \\ 0 & \text{elsewhere.} \end{cases}$$

Here η_k is defined in (4) and $\mathbf{h}_k$ is the solution to (6) in G_k. Analogously to z_k in Step I we define $z_{k,l} \equiv \|\nabla \mathbf{v}; L^2(\omega_{k-l,k+l+1})\|^2$ and deduce for the force term

$$\sum_{j=0}^{3} \|\mathbf{f}_j; L^2(\omega_{k-l-1,k+l+2})\|^2 \leq \sum_{i=-l-1}^{l+1} e^{-2\beta(k+i)} \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2.$$

In a similar manner as in Step I we arrive at

$$z_{k,l} \leq \mu_1(z_{k,l+1} - z_{k,l}) + c\, e^{-\epsilon\beta k} \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2 e^{2\beta l}.$$

We apply Proposition 1/(ii) with respect to l choosing $K = k-2$, μ_1 as in Step I, $\mu_2 = ce^{-2\beta k}\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2$, $\nu = e^{2\beta}$, $b_l = e^{2\beta l}$. Estimate (9) yields

$$z_{k,k-2} \leq z_{2k-1} \leq c\, \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2 = c\, e^{4\beta} e^{-2\beta k} \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2 e^{2\beta(k-2)}.$$

Therefore, especially for $l = 0$, we observe

$$\|\nabla \mathbf{v}; L^2(G_k)\|^2 = z_{k,0} \leq c_1 e^{-2\beta k} \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2 e^0 + c_2 e^{-2\beta k} \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2$$

and this yields (10). Repeating word by word the same arguments for the other outlet we prove (10) for $k \leq -2$ and the cases $k \in \{-1, 0, 1\}$ are trivial. An easy consequence of (10) is

$$\|\nabla \mathbf{v}; L^2_\beta(G_k)\|^2 \leq c\, \|\mathbf{f}; W_\beta^{-1,2}(\Omega)\|^2 \tag{11}$$

where c does not depend on k.

Step III (see also [15: Theorem 2.5]): Take β and γ with $|\beta| < \beta^*$ and $|\gamma| < \beta^*$. We multiply $\|\nabla \mathbf{v}; L^2_\beta(G_m)\|$ by $\rho_\gamma \cdot (\rho_\gamma)^{-1}$ which means $e^{2\gamma x_3^j} \cdot e^{-2\gamma x_3^j}$ if G_m lies in the j^{th} outlet. After that we use that in G_m it holds $e^{x_3^j} \sim e^{|m|}$ since $|m| \leq x_3^j < |m+1|$ and e^x is a monotone function. Then we apply (11) keeping in mind that now $\|\mathbf{f}; W_\beta^{-1,2}(\Omega)\| = \|\mathbf{f}; W_\beta^{-1,2}(G_k)\|$ since $\operatorname{supp} \mathbf{f} \subset G_k$ and repeat this procedure for $\|\mathbf{f}; W_\gamma^{-1,2}(G_k)\|$ once more making use of $1 = \rho_\beta \cdot (\rho_\beta)^{-1}$. Precisely,

$$\begin{aligned} \|\nabla \mathbf{v}; L^2_\beta(G_m)\|^2 &= \int_{G_m} |\nabla \mathbf{v}|^2 e^{2\beta x_3^j} e^{2\gamma x_3^j} e^{-2\gamma x_3^j}\, dx \\ &\leq c\, e^{2\beta|m| - 2\gamma|m|} \int_{G_m} |\nabla \mathbf{v}|^2 e^{2\gamma x_3^j}\, dx \\ &\leq c\, e^{2|m|(\beta-\gamma)} \|\mathbf{f}; W_\gamma^{-1,2}(G_k)\|^2 \\ &\leq c\, e^{2|m|(\beta-\gamma)} e^{2|k|(\gamma-\beta)} \|\mathbf{f}; W_\beta^{-1,2}(G_k)\|^2 \\ &\leq c\, e^{2(|m|-|k|)(\beta-\gamma)} \|\mathbf{f}; W_\beta^{-1,2}(G_k)\|^2. \end{aligned}$$

We choose

$$\gamma = \begin{cases} \beta - \varepsilon_0 & \text{if } |k| \geq |m| \\ \beta + \varepsilon_0 & \text{if } |k| < |m|. \end{cases}$$

Here ε_0 is a certain possibly small number such that $|\gamma| < \beta^*$ is fulfilled. This yields (8) and completes the proof of Lemma 1 ∎

Up to now we considered weighted spaces generated by L^2. For the rest of this section we continue in treating bounded subdomains of Ω but we allow $1 < q < \infty$. Then the weak formulation of problem $(\mathrm{S})_0$ becomes

$$\textbf{(WF)} \quad \int_\Omega \nabla \mathbf{v} : \nabla \Psi \, dx = (\mathbf{f}, \Psi)_\Omega \qquad \text{for all } \Psi \in \mathring{W}^{1,q'}(\Omega) \text{ with } \operatorname{div}\Psi = 0.$$

We call $\mathbf{v} \in \mathring{W}^{1,q}(\Omega)$ *generalized* weak solution of problem $(\mathrm{S})_0$. Here and in what follows $\frac{1}{q} + \frac{1}{q'} = 1$. As long as we consider the problem in G_m^* the pressure $p \in L^q(G_m^*)$ exists due to [15: Lemma 1.2] (see also [13, 14]):

Proposition 2. *Let G be a bounded domain with Lipschitz boundary. Then any bounded linear functional $f(\mathbf{u})$ defined on vectors in $\overline{C_0^\infty(G)}^{\|\nabla\cdot;L^q(G)\|}$ and vanishing on all divergence free functions can be represented in the form $f(\mathbf{u}) = \int_G s \operatorname{div}\mathbf{u}\, dx$ where $s \in L^{q'}(G)$ and $\int_G s\, dx = 0$.*

We need estimates of Cattabriga type which are proved in [15: Theorems 3.1 and 3.2]:

Lemma 2. *Let Ω have uniform C^{l+1}-regularity, $l \in \mathbb{N}_0$, $m \in \mathbb{Z}$.*

(i) *If $l = 0$, $\mathbf{f} \in W_\beta^{-1,q}(\Omega) \cap W^{-1,2}(\Omega)$, $q \geq 2$, then the weak solution of problem* $(\mathrm{S})_0$ *satisfies*

$$\|\nabla \mathbf{v}; L_\beta^q(G_m)\| \leq c \left(\|\mathbf{f}; W_\beta^{-1,q}(G_m^*)\| + \|\nabla \mathbf{v}; L_\beta^2(G_m^*)\| \right). \tag{12}$$

(ii) *If $\mathbf{f} \in W_{loc}^{l-1,q}(\Omega)$, $1 < q < \infty$ and $l \in \mathbb{N}$, then each solution to problem* (S) *satisfies*

$$\begin{aligned} &\|\mathbf{v}; W_\beta^{l+1,q}(G_m)\| + \|\nabla p; W_\beta^{l-1,q}(G_m)\| \\ &\qquad \leq c \left(\|\mathbf{f}; W_\beta^{l-1,q}(G_m^*)\| + \|\nabla \mathbf{v}; L_\beta^q(G_m^*)\| \right). \end{aligned} \tag{13}$$

3. The Stokes problem with zero flux

In this section we prove that the Stokes system with zero flux through all outlets has a unique solution which is as regular as the data and the smoothness of the boundary allow. The proof of Theorem 1 is carried out in two main steps.

Step I considers $q \geq 2$. In this step first it is shown that there exists a generalized weak solution $\mathbf{v} \in \mathring{W}^{1,q}_\beta(\Omega)$ to problem $(S)_0$ if $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$ (Substep I.I). This is technical and lengthy. We need two partial steps: Initially we assume that $\mathbf{f}$ has *compact support* and prove the existence of a generalized weak solution in this case. The two inequalities (12) and (8) (shown in Lemmas 1 and 2) then help to go over from $q = 2$ to $2 \leq q < \infty$. With this result for $\mathbf{f}$ with a compact support we may treat *arbitrary* $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$ in a second partial step using a partition of unity. After existence the *uniqueness* of this solution is shown in Substep I.II. We may show more regularity for $\mathbf{v}$ if $\mathbf{f} \in W^{l,q}_\beta$, $l \in \mathbb{N}_0$ (see Substep I.III). Finally, Step II considering $1 < q < 2$ is carried out by duality and density arguments.

Remark. The Poincaré inequality applied to $\rho_\beta \mathbf{v}$ yields

$$\|\rho_\beta \mathbf{v}; L^q(\Omega)\| \leq c_p \|\nabla(\rho_\beta \mathbf{v}); L^q(\Omega)\| \leq c_p(\|(\nabla \rho_\beta)\mathbf{v}; L^q(\Omega)\| + \|\rho_\beta(\nabla \mathbf{v}); L^q(\Omega)\|).$$

The special form of ρ_β in each outlet induces $\nabla \rho_\beta = (0, 0, \beta\rho_\beta)$ and we see

$$(1 - q^{1/q} c_p \beta)\|\mathbf{v}; L^q_\beta(\Omega)\|^q \leq c_p^q \|\nabla \mathbf{v}; L^q_\beta(\Omega)\|^q.$$

Since $|\beta| < \beta^* < \frac{1}{2c_p+2+2M} < \frac{1}{q^{1/q}c_p}$ we are sure that $1 > q^{1/q}c_p\beta$ and from the boundedness of $\|\nabla \mathbf{v}; L^q_\beta(\Omega)\|$ we may infer the boundedness of $\|\mathbf{v}; L^q_\beta(\Omega)\|$ and $\|\mathbf{v}; W^{1,q}_\beta(\Omega)\|$.

Theorem 1. *Consider problem* $(S)_0$ *in* Ω *with uniform* C^1*-regularity. If* $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$ *where* $|\beta| < \beta^*$ *(this constant is fixed in Lemma 1), then there is a unique solution and*

$$\|\mathbf{v}; W^{1,q}_\beta(\Omega)\| \leq c\, \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\|. \tag{14}$$

If moreover $\mathbf{f} \in W^{l-1,q}_\beta(\Omega)$ $(l \in \mathbb{N})$ *and* Ω *has the uniform* C^{l+1}*-property, then*

$$\|\mathbf{v}; W^{l+1,q}_\beta(\Omega)\| + \|\nabla p; W^{l-1,q}_\beta(\Omega)\| \leq c\, \|\mathbf{f}; W^{l-1,q}_\beta(\Omega)\|. \tag{15}$$

Proof. Step I: The case $q \geq 2$. **Substep I.I**: Existence of solutions if $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$. **Part I.I.I**: The case $\operatorname{supp} \mathbf{f} \subset G_k$ (compare also [15: Theorem 4.1]). In this part we assume $\mathbf{f}$ has compact support, i.e. $\mathbf{f} \in W^{-1,q}_\beta(G_k)$. Therefore $\mathbf{f} \in W^{-1,2}(\Omega)$ and there is a unique weak solution $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$ with $\operatorname{div}\mathbf{v} = 0$ (in the weak sense). We want to deduce from estimate (12) in Lemma 2 that for this solution in addition

$$\|\nabla \mathbf{v}; L^q_\beta(\Omega)\| \leq c\, \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\| \qquad (q \geq 2).$$

holds. In (12) the term $\|\nabla \mathbf{v}; L^2_\beta(G^*_m)\|$ is disturbing. With the help of inequality (8) in Lemma 1 we may conclude that

$$\|\nabla \mathbf{v}; L^2_\beta(G^*_m)\| \leq c\, e^{-\varepsilon_0 ||m|-|k||} \|\mathbf{f}; W^{-1,2}_\beta(G^*_k)\| = c\, e^{-\varepsilon_0 ||m|-|k||} \|\mathbf{f}; W^{-1,2}_\beta(G_k)\|.$$

Since G_k is bounded and $q \geq 2$, moreover $\|\mathbf{f}; W_\beta^{-1,2}(G_k)\| \leq c\,\|\mathbf{f}; W_\beta^{-1,q}(G_k)\|$ and

$$\|\nabla\mathbf{v}; L^2_\beta(G^*_m)\| \leq c\,e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}; W_\beta^{-1,q}(G_k)\|.$$

Together with this estimate (12) becomes

$$\|\nabla\mathbf{v}; L^q_\beta(G_m)\| \leq c\left(e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}; W_\beta^{-1,q}(G_k)\| + \overbrace{\|\mathbf{f}; W_\beta^{-1,q}(G^*_m)\|}^{=0 \text{ if } m\notin\{k-1,k,k+1\}}\right).$$

Summing over m (k is fixed) we obtain since $\varepsilon_0 > 0$

$$\|\nabla\mathbf{v}; L^q_\beta(\Omega)\| \leq c\,\|\mathbf{f}; W_\beta^{-1,q}(G_k)\| = c\,\|\mathbf{f}; W_\beta^{-1,q}(\Omega)\|. \tag{16}$$

Moreover, using the same arguments as in the proof of Lemma 1 but starting with (16) we may deduce that (8) holds also for $q \geq 2$ and with the Remark before Theorem 1 we conclude

$$\|\mathbf{v}; W^{1,q}_\beta(G_m)\| \leq c\,e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}; W_\beta^{-1,q}(G_k)\|. \tag{17}$$

Part I.I.II: The case supp $\mathbf{f}$ arbitrary (compare also [15: Theorem 4.3]). We choose a partition of unity $\{\chi_k\}_{k=-\infty}^{\infty}$ such that supp $\chi_k = G^*_k$ and define $\mathbf{f}_k \equiv \chi_k\mathbf{f}$. Let $\mathbf{v}_k$ be the solution to problem $(S)_0$ with right-hand side $\mathbf{f}_k$. From Part I.I.I it is clear that $\mathbf{v}_k \in \mathring{W}^{1,q}_\beta(\Omega)$ and $\operatorname{div}\mathbf{v}_k = 0$ in the weak sense. We define

$$\mathbf{v}^N \equiv \sum_{k=-N}^{N}\mathbf{v}_k.$$

Obviously, this is the solution of problem $(S)_0$ to the right-hand side $\mathbf{f}^N \equiv \sum_{k=-N}^{N}\mathbf{f}_k$. By definition $\mathbf{f}^N \to \mathbf{f}$ as $N \to \infty$. To prove that $\mathbf{v}^N \to \mathbf{v}$ we must show that $\|\mathbf{v}^N; W^{1,q}_\beta(\Omega)\|$ is bounded independently of N. For $\mathbf{v}^N = \sum_m \chi_m\mathbf{v}^N$ we calculate

$$\begin{aligned}\|\mathbf{v}^N; W^{1,q}_\beta(\Omega)\|^q &\leq c\sum_{m=-\infty}^{\infty}\|\mathbf{v}^N; W^{1,q}_\beta(G^*_m)\|^q\\ &\leq c\sum_{m=-\infty}^{\infty}\left(\sum_{k=-N}^{N}\|\mathbf{v}_k; W^{1,q}_\beta(G^*_m)\|\right)^q.\end{aligned}$$

To the right-hand side of this inequality we will apply inequality (17). This is possible since $\mathbf{f}_k$ has compact support. We obtain writing $\|\mathbf{f}_k\|$ instead of $\|\mathbf{f}_k; W_\beta^{-1,q}(G^*_k)\|$

$$\sum_{k=-N}^{N}\|\mathbf{v}_k; W^{1,q}_\beta(G^*_m)\| \leq c\sum_{k=-N}^{N}e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}_k\| \leq c\sum_{k=-\infty}^{\infty}e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}_k\|.$$

After that we use the following arguments for $a_k = e^{-\varepsilon_0||m|-|k||}$ and $b_k = \|\mathbf{f}_k\|$:

$$\sum_k a_kb_k = \sum_k(a_k^{1/q}b_k)\cdot a_k^{1/q'} \leq \left(\sum_k a_k^{q/q}b_k^q\right)^{1/q}\cdot\left(\sum_k a_k^{q'/q'}\right)^{1/q'}$$

hence

$$\left(\sum_k a_k b_k\right)^q \le \sum_k a_k b_k^q \cdot \left(\sum_k a_k\right)^{q/q'} = \sum_k a_k b_k^q \cdot \left(\sum_k a_k\right)^{q-1}.$$

As usual $q' \equiv \frac{q}{q-1}$. We obtain taking up the above series of estimates

$$\begin{aligned}\|\mathbf{v}^N; W_\beta^{1,q}(\Omega)\|^q &\le c \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-\varepsilon_0||m|-|k||}\|\mathbf{f}_k\|^q \cdot \overbrace{\left(\sum_{k=-\infty}^{\infty} e^{-\varepsilon_0||m|-|k||}\right)^{q-1}}^{=c \textit{ for any fixed } m} \\ &\le c \sum_{k=-\infty}^{\infty} \|\mathbf{f}_k\|^q \sum_{m=-\infty}^{\infty} e^{-\varepsilon_0||m|-|k||} \\ &\le c\,\|\mathbf{f}; W_\beta^{-1,q}(\Omega)\|^q.\end{aligned}$$

Therefore we see $\mathbf{v}^N \to \mathbf{v}$ in $W_\beta^{1,q}(\Omega)$ and it holds

$$\|\mathbf{v}; W_\beta^{1,q}(\Omega)\| \le c\,\|\mathbf{f}; W_\beta^{-1,q}(\Omega)\|. \tag{18}$$

Substep I.II: Uniqueness for $\mathbf{f} \in W_\beta^{-1,q}(\Omega)$. We will prove that $\mathbf{v} \equiv 0$ is the only solution to the homogeneous problem, i.e. for $\mathbf{f} = 0$. Then from Substep I.I it is clear that $\mathbf{v} \in \mathring{W}_\beta^{1,q}(\Omega)$. We take $\mathbf{v}_k$ from (5) and estimate (7) leads to

$$\|\nabla \mathbf{h}_k; L_\beta^q(G_k \cup G_{-k})\| \le c\,\|\nabla \mathbf{v}; L_\beta^q(G_k \cup G_{-k})\|. \tag{19}$$

We calculate

$$-\Delta(\eta_k \mathbf{v} + \mathbf{h}_k) + \nabla(\eta_k p) = -(\Delta\eta_k)\mathbf{v} - 2\nabla\eta_k \cdot \nabla\mathbf{v} - \Delta\mathbf{h}_k + p\nabla\eta_k.$$

So we may interprete the situation such that $\mathbf{v}_k$ solves problem $(\mathrm{S})_0$ in the weak sense for the right-hand side $\mathbf{F}_k$ defined for $\Psi \in C_0^\infty(\Omega)$ as

$$\langle \mathbf{F}_k, \Psi\rangle_\Omega \equiv \int_\Omega \nabla\eta_k(\nabla\Psi \cdot \mathbf{v} - \nabla\mathbf{v} \cdot \Psi)\,dx + \int_\Omega (p - \bar{p})\nabla\eta_k \cdot \Psi\,dx - \int_\Omega \nabla\mathbf{h}_k \cdot \nabla\Psi\,dx.$$

Here again $\bar{p} = \frac{1}{|G_k \cup G_{-k}|}\int_{G_k \cup G_{-k}} p\,dx$. Obviously, $\operatorname{supp}\mathbf{F}_k \subseteq G_k \cup G_{-k}$. Moreover, from the equations above we may conclude with the help of Hölder's inequality that

$$\|\mathbf{F}_k; W_\beta^{-1,q}(G_k \cup G_{-k})\| \le c\,\|\mathbf{v}; W_\beta^{1,q}(G_k \cup G_{-k})\| \tag{20}$$

since for $\mathbf{h}_k$ we apply (19), Ψ and $\nabla\eta_k$ are bounded and we may estimate the pressure term as in Lemma 2. From (18) and (20) we may infer

$$\begin{aligned}\|\mathbf{v}_k; W_\beta^{1,q}(\Omega)\| &\le c\,\|\mathbf{F}_k; W_\beta^{-1,q}(\Omega)\| \\ &= c\,\|\mathbf{F}_k; W_\beta^{-1,q}(G_k \cup G_{-k})\| \\ &\le c\,\|\mathbf{v}; W_\beta^{1,q}(G_k \cup G_{-k})\|.\end{aligned}$$

Since $\mathbf{v} \in W^{1,q}_\beta(\Omega)$ the right-hand side of this inequality tends to zero for $k \to \infty$. Moreover, because of (19) we know that $\lim_{k\to\infty} \|\nabla \mathbf{h}_k; L^q_\beta(G_k \cup G_{-k})\| = 0$ and we conclude $0 = \lim_{k\to\infty} \|\eta_k \mathbf{v}; W^{1,q}_\beta(\Omega)\| = \|\mathbf{v}; W^{1,q}_\beta(\Omega)\|$ and thus $\mathbf{v} \equiv 0$.

Substep I.III: Regularity for $\mathbf{f} \in W^{l,q}_\beta(\Omega)$, $l \in \mathbb{N}_0$ (see also [15: Theorem 4.4]). Until now we considered weak solutions of problem $(\mathrm{S})_0$ assuming $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$. Now we take a more regular right-hand side. The local estimates of Lemma 2 are independent of the location due to the uniform C^{l+2}-property.

Indeed, if $\mathbf{f} \in W^{l,q}_\beta(\Omega)$ $(l \in \mathbb{N}_0)$, then in particular $\mathbf{f} \in L^q_\beta(\Omega)$ and $L^q_\beta(\Omega) \subset W^{-1,q}_\beta(\Omega)$. Hence, there exists a unique solution $\mathbf{v} \in W^{1,q}_\beta(\Omega)$ and it holds because of (18) and the embedding $L^q_\beta(\Omega) \subset W^{-1,q}_\beta(\Omega)$

$$\|\mathbf{v}; W^{1,q}_\beta(\Omega)\| \le c\,\|\mathbf{f}; W^{-1,q}_\beta(\Omega)\| \le c\,\|\mathbf{f}; L^q_\beta(\Omega)\| \le c\,\|\mathbf{f}; W^{l,q}_\beta(\Omega)\|. \tag{21}$$

Summing the local estimates (13) over m we obtain

$$\begin{aligned}\|\mathbf{v}; W^{l+2,q}_\beta(\Omega)\| + \|\nabla p; W^{l,q}_\beta(\Omega)\| &\le c\,\Big(\|\mathbf{f}; W^{l,q}_\beta(\Omega)\| + \|\mathbf{v}; L^q_\beta(\Omega)\|\Big)\\ &\le c\,\Big(\|\mathbf{f}; W^{l,q}_\beta(\Omega)\| + \|\mathbf{v}; W^{1,q}_\beta(\Omega)\|\Big)\\ &\le c\,\Big(\|\mathbf{f}; W^{l,q}_\beta(\Omega)\| + \|\mathbf{f}; W^{l,q}_\beta(\Omega)\|\Big)\end{aligned}$$

and the last inequality follows from (21).

Step II: The case $q < 2$ (see also [15: Theorem 4.5]). We will show that also for $1 < q < 2$ it holds $\|\mathbf{v}; W^{1,q}_\beta(\Omega)\| \le c\,\|\mathbf{f}; W^{-1,q}_\beta(\Omega)\|$, and if $l \in \mathbb{N}_0$, moreover,

$$\|\mathbf{v}; W^{l+2,q}_\beta(\Omega)\| + \|\nabla p; W^{l,q}_\beta(\Omega)\| \le c\,\|\mathbf{f}; W^{l,q}_\beta(\Omega)\|.$$

For this, in a certain sense we proceed similar to Step I: Initially we derive the estimate for weak solutions and prove then more regularity for more regular right-hand sides.

We exploit the following two main ideas:

Approximation of $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$ by functions in $W^{-1,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$ for $\varepsilon > 0$ and duality between $\mathring{W}^{1,q}_\beta(\Omega)$ and $W^{-1,q'}_{-\beta}(\Omega)$. Precisely, we consider first $\mathbf{f} \in W^{-1,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$. In this case on one hand we may use the L^2-result and on the other hand derive results for a certain $\mathbf{g} \in W^{-1,q'}_{-\beta}(\Omega)$ because $q' > 2$ and Step I can be applied. By duality arguments these results may be transferred to $\mathbf{f}$. Then we show that each $\mathbf{f} \in W^{-1,q}_\beta(\Omega)$ can be approximated by functions in $W^{-1,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$. The procedure is similar to Part I.I.II. Regularity then follows with the same arguments as in Substep I.III. At the end we show uniqueness also for the so constructed solutions.

Substep II.I: The case $\mathbf{f} \in W^{-1,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$. Because of Step I for such $\mathbf{f}$ there exists a unique solution $\mathbf{v} \in \mathring{W}^{1,2}_{\beta+\varepsilon}(\Omega)$ to problem $(\mathrm{S})_0$. This $\mathbf{v}$ fulfils (WF) for $\Psi \in \mathring{W}^{1,2}_{-\beta-\varepsilon}(\Omega)$ with $\operatorname{div}\Psi = 0$ in the weak sense:

$$\int_\Omega \nabla \mathbf{v} \cdot \nabla \Psi \, dx = \langle \mathbf{f}, \Psi \rangle_\Omega. \tag{22}$$

Let $\mathbf{g} \in W^{-1,q'}_{-\beta}(\Omega)$. Because $q' > 2$ Step I applies and there is a unique solution $\mathbf{w} \in \mathring{W}^{1,q'}_{-\beta}$ to problem $(\mathrm{S})_0$ with right-hand side $\mathbf{g}$, satisfying for $\Phi \in \mathring{W}^{1,q}_{\beta}(\Omega)$ with $\operatorname{div}\Phi = 0$ in the weak sense

$$\int_\Omega \nabla \mathbf{w} \cdot \nabla \Phi \, dx = \langle \mathbf{g}, \Phi \rangle_\Omega, \qquad \|\mathbf{w}; W^{1,q'}_{-\beta}(\Omega)\| \le c\, \|\mathbf{g}; W^{-1,q'}_{-\beta}(\Omega)\|. \tag{23}$$

With the generalized Hölder inequality one can easily prove that for $q < 2$, $\varepsilon > 0$

$$W^{1,2}_{\beta+\varepsilon}(\Omega) \subset W^{1,q}_{\beta}(\Omega) \qquad \text{and} \qquad W^{1,q'}_{-\beta}(\Omega) \subset W^{1,2}_{-\beta-\varepsilon}(\Omega).$$

Therefore we may take $\Psi = \mathbf{w}$ in (22) and $\Phi = \mathbf{v}$ in (23). This yields

$$\int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, dx = \langle \mathbf{f}, \mathbf{w} \rangle_\Omega \qquad \text{and} \qquad \int_\Omega \nabla \mathbf{w} \cdot \nabla \mathbf{v} \, dx = \langle \mathbf{g}, \mathbf{v} \rangle_\Omega.$$

We deduce that $\langle \mathbf{f}, \mathbf{w} \rangle_\Omega = \langle \mathbf{g}, \mathbf{v} \rangle_\Omega$. Now we take $\mathbf{g} \equiv \operatorname{div} \mathcal{G}$ where $\mathcal{G} \equiv \rho_\beta |\nabla \mathbf{v}|^{q-2} \nabla \mathbf{v}$. Here ρ_β is the weight function from (1). Relation (23) and direct calculation yield

$$\|\mathbf{w}; W^{1,q'}_{-\beta}(\Omega)\| \le c\, \|\mathbf{g}; W^{-1,q'}_{-\beta}(\Omega)\| = c\, \|\nabla \mathbf{v}; L^q_\beta(\Omega)\|^{q-1}. \tag{24}$$

Moreover, for this $\mathbf{g}$ we observe

$$\begin{aligned} \|\nabla \mathbf{v}; L^q_\beta(\Omega)\|^q = \langle \mathbf{g}, \mathbf{v} \rangle_\Omega &= \langle \mathbf{f}, \mathbf{w} \rangle_\Omega \\ &\le \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\| \, \|\mathbf{w}; W^{1,q'}_{-\beta}(\Omega)\| \\ &\le c \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\| \, \|\nabla \mathbf{v}; L^q_\beta(\Omega)\|^{q-1}. \end{aligned}$$

This means

$$\|\nabla \mathbf{v}; L^q_\beta(\Omega)\| \le c\, \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\| \qquad \text{and} \qquad \|\mathbf{v}; W^{1,q}_\beta(\Omega)\| \le c\, \|\mathbf{f}; W^{-1,q}_\beta(\Omega)\|.$$

Using word by word the arguments of Substep I.III for $\mathbf{f} \in W^{l,q}_\beta(\Omega)$, $l \in \mathbb{N}_0$, we conclude

$$\|\mathbf{v}; W^{l+2,q}_\beta(\Omega)\| + \|\nabla p; W^{l,q}_\beta(\Omega)\| \le c\, \|\mathbf{f}; W^{l,q}_\beta(\Omega)\|.$$

Substep II.II: The case $\mathbf{f} \in W^{l,q}_\beta(\Omega)$ arbitrary. We will show that it is possible to approximate each $\mathbf{f} \in W^{l,q}_\beta(\Omega)$ in the norm of this space by functions $\mathbf{f} \in W^{l,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$.

By definition functions in $W^{l,q}_\beta(\Omega)$ are approximated by functions $\mathbf{f}_j \in C^\infty_0(\overline{\Omega})$, i.e. for $\mathbf{f} \in W^{l,q}_\beta(\Omega)$ it holds $\mathbf{f} = \lim_{j\to\infty} \mathbf{f}_j$ and $\lim_{j\to\infty} \|\mathbf{f}_j; W^{l,q}_\beta(\Omega)\| = \|\mathbf{f}; W^{l,q}_\beta(\Omega)\|$. Taking the partition of unity $\{\chi_k\}_{k=-\infty}^{\infty}$ of Part I.I.II we define, for $j \in \mathbb{N}_0$, $\mathbf{f}_j^N \equiv \sum_{-N}^{N} \chi_k \mathbf{f}_j$. These $\mathbf{f}_j^N$ have compact support and therefore it holds $\mathbf{f}_j^N \in W^{l,q}_\beta(\Omega) \cap W^{-1,2}_{\beta+\varepsilon}(\Omega)$. Hence, $\mathbf{f} \in W^{l,q}_\beta(\Omega)$ is approximated as $\mathbf{f} = \lim_{j\to\infty} \lim_{N\to\infty} \mathbf{f}_j^N$. Thus, by density the estimate proved in Substep II.I is valid for all $\mathbf{f} \in W^{l,q}_\beta(\Omega)$.

Let us finally show uniqueness. For this we consider the solution $\mathbf{u} \in \mathring{W}^{1,q}_\beta(\Omega)$ of the homogeneous problem. It holds for all $\Phi \in \mathring{W}^{1,q'}_{-\beta}(\Omega)$ with $\operatorname{div}\Phi = 0$, in the weak sense $\int_\Omega \nabla \mathbf{u} \cdot \nabla \Phi \, dx = 0$. Take $\mathbf{F} \in W^{-1,q'}_{-\beta}(\Omega)$ and let $\mathbf{v} \in \mathring{W}^{1,q'}_{-\beta}(\Omega)$ be the corresponding solution to problem $(\mathrm{S})_0$ with right-hand side $\mathbf{F}$. It holds $\int_\Omega \nabla \mathbf{v} \cdot \nabla \Psi \, dx = \langle \mathbf{F}, \Psi \rangle_\Omega$ for all $\Psi \in \mathring{W}^{1,q}_\beta(\Omega)$ with $\operatorname{div}\Psi = 0$, in the weak sense. Taking $\Phi = \mathbf{v}$ and $\Psi = \mathbf{u}$ yields $\langle \mathbf{u}, \mathbf{F} \rangle_\Omega = 0$ for all $\mathbf{F} \in W^{-1,q'}_{-\beta}(\Omega)$ and consequently $\mathbf{u} \equiv 0$ since $\mathbf{F}$ and $\mathbf{u}$ are from dual spaces. This finishes the proof of Theorem 1 ∎

4. The Stokes system with prescribed fluxes. Setting in weighted spaces

To investigate problem (S) with prescribed fluxes different from zero we look for solutions

$$\mathbf{v} = \mathbf{u} + \mathbf{a}.$$

Here $\mathbf{a}$ is constructed as a solenoidal vector field which carries the flux, i.e.

$$\left.\begin{aligned} \operatorname{div}\mathbf{a} = 0 \quad &\text{in } \Omega \\ \mathbf{a} = 0 \quad &\text{on } \partial\Omega \\ \textstyle\int_{\Sigma^j(x_3^j)} a_3\, dx_1^j dx_2^j = \mathcal{F}^j & \end{aligned}\right\}$$

and $\mathbf{u}$ is found as solution of problem $(\mathrm{S})_0$ with right-hand side $\mathbf{f} + \Delta\mathbf{a}$. In view of Theorem 1 the solvability of problem $(\mathrm{S})_0$ for this right-hand side is apparent if $\mathbf{f} + \Delta\mathbf{a} \in W_\beta^{-1,q}(\Omega)$ for some $|\beta| < \beta^*$. More regularity of the solution follows if $\mathbf{f} + \Delta\mathbf{a} \in W_\beta^{l,q}(\Omega)$ $(l \in \mathbb{N}_0)$ provided $\partial\Omega$ is regular enough. Thus, the main job to be done in this section is the choice of an adequate flux carrier and the discussion of its regularity and decay properties in the frame of $W_\beta^{l,q}(\Omega)$-spaces.

The crucial point in this investigation is that our flux carrier does not explicitly depend on x_3^j. We will see that $|\mathbf{a}|$ together with the moduli of all derivatives in $\Sigma^j(x_3^j)$ is bounded by a constant independent of x_3^j which leads to the knowledge that $\mathbf{a}$ belongs to all $W_\beta^{l,q}(\Omega)$-spaces for $\beta < 0$. This agrees with the results known for similar domains: Problems with non-zero flux cannot have solutions which belong to usual Sobolev spaces (i.e. $\beta = 0$), especially they do not have a finite Dirichlet integral (see [6, 15] and the literature quoted there).

We set $\mathbf{a} = \sum_{i=1}^{J-1} \alpha_i \mathbf{a}^{i,i+1}$, where $\mathbf{a}^{i,i+1}$ carries a unit flux from outlet number i to the next one. To fulfil the prescribed flux conditions in the outlets it is evident that $\alpha_i \equiv \sum_{j=1}^{i} \mathcal{F}^j$. To construct $\mathbf{a}^{i,i+1}$ consider a flow from outlet number i to outlet number j. We define (see also [18: (2.29)]

$$\mathbf{a}^{ij} \equiv \operatorname{curl}(\zeta^{ij}\mathbf{b}^{ij}) = \nabla\zeta^{ij} \times \mathbf{b}^{ij}, \tag{25}$$

where $\mathbf{b}^{ij}$ shall carry the flux, $\operatorname{curl}\mathbf{b}^{ij} = 0$ and ζ^{ij} is a monotonic smooth cut-off function which ensures that $\mathbf{a}^{ij} = 0$ on the boundary and in a neighbourhood of the singularity of the field $\mathbf{b}^{ij}$. By construction as curl we know that $\operatorname{div}\mathbf{a}^{ij} = 0$. We choose

$$\mathbf{b}^{ij}(x) \equiv -\frac{1}{4\pi}\int_{\gamma^{ij}} \frac{x-y}{|x-y|^3} \times dl$$

knowing that this describes a magnetic field which generates passing through the path γ^{ij} an electric flow of unit density. In our frame this is fixed such that the distance between $\partial\Omega$ and this line shall be greater than some number $d_0 < \frac{d}{2}$ (remember, d is the lower bound for the diameters of the outlets). From the physical interpretation of $\mathbf{b}^{ij}$ it is clear that $\operatorname{curl}\mathbf{b}^{ij} = 0$ in $\Omega\backslash\gamma^{ij}$ and

$$\oint_\Gamma \mathbf{b}^{ij}\, ds = \begin{cases} \pm 1 & \text{if } \Gamma \text{ encircles } \gamma^{ij} \\ 0 & \text{if } \Gamma \text{ does not encircle } \gamma^{ij}. \end{cases}$$

The sign corresponds to the right-hand rule. The cut-off function is fixed as

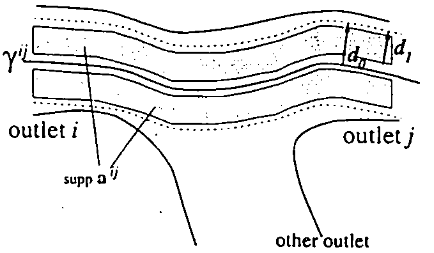
$$\zeta^{ij} \equiv \Phi\left(\ln \frac{d_1}{d_0 - k_{ij}(x)}\right),$$

where

$$\begin{cases} d_1 \in \mathbb{R} & \text{is such that } 0 < d_1 < d_0 \\ k_{ij}(x) & \text{is the regularized distance between } x \text{ and } \gamma^{ij} \\ \Phi & \text{is a smooth function such that } \Phi(t) = 0 \text{ if } t \le 0, \Phi(t) = 1 \text{ if } t \ge 1. \end{cases}$$

This definition is such that

$$\zeta^{ij} = \begin{cases} 1 & \text{if } k_{ij} > d_0 - \frac{d_1}{e} \\ 0 & \text{if } k_{ij} < d_0 - d_1 \end{cases} \quad \text{and} \quad \operatorname{supp}\nabla\zeta^{ij} \subset \left\{x : d_0 - d_1 < k_{ij} < d_0 - \frac{d_1}{e}\right\}.$$



With Stoke's law one checks that $\int_{\Sigma^j(x_3^j)} a_3^{ij}\, dx_1^j dx_2^j \mathbf{b}^{ij}\, ds = 1$ (see also [18: Formula (2.22)]. Besides that,

$$\int_{\Sigma^j(x_3^j)} a_3\, dx_1^j dx_2^j = \oint_{\partial\Sigma^j(x_3^j)} \sum_{i=1}^{J-1} \alpha_i \mathbf{b}^{i,i+1}\, ds = \alpha_j - \alpha_{j-1} = \mathcal{F}^j.$$

Lemma 3. *Let $1 < q < \infty$ and $l \in \mathbb{N}_0$. Then $\mathbf{a}^{ij}$ $(1 \le i,j \le J)$ defined in (25) is in $W^{l,q}_\beta(\Omega)$ for all $\beta < 0$.*

Proof. From [18: Formula (2.32)] we conclude

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathbf{a}^{ij}\right| \le \frac{c}{(d_0 - k_{ij})^{|\alpha|+1} k_{ij}} + \cdots + \frac{c}{k_{ij}^{1+|\alpha|}(d_0 - k_{ij})}. \tag{26}$$

The support of $\mathbf{a}^{ij}$ is such that $d_0 - d_1 < k_{ij} < d_0 - \frac{d_1}{e}$. Therefore all summands in (26) can be bounded from above by

$$\frac{c}{\min\left\{\frac{d_1}{e}, d_0 - d_1\right\}^{|\alpha|+2}}$$

and thus,

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathbf{a}^{ij}\right| \le \tilde{c}(\alpha, d).$$

Moreover, we see that

$$\begin{aligned}\int_{\Omega^j}\left|e^{\beta x_3^j}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathbf{a}^{ij}\right|^q dx^j &= \int_1^\infty e^{q\beta x_3^j}\int_{\Sigma^j(x_3^j)}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathbf{a}^{ij}\right|^q dx^j \\ &\le \int_1^\infty e^{q\beta x_3^j}|\Sigma^j(x_3^j)|\tilde{c}^q dx_3^j \\ &\le \lim_{x_3^j\to\infty}\frac{c}{q\beta}e^{q\beta x_3^j} - \frac{c}{q\beta}e^{q\beta}.\end{aligned}$$

This limit exists for $\beta < 0$. Summation over j leads to the result of the lemma ∎

Remark. In the case $n = 2$ one takes (see [18: p. 267] $\mathbf{a}^{ij} \equiv \frac{1}{2}\left(-\frac{\partial}{\partial x_2}\zeta^{ij}, \frac{\partial}{\partial x_1}\zeta^{ij}\right)$. Obviously, $\operatorname{div}\mathbf{a}^{ij} = 0$, $\mathbf{a}^{ij} = 0$ on $\partial\Omega$ and

$$\int_{\Sigma^j(x_2^j)} a_3^{ij}\,dx_1^j = \frac{1}{2}\int_{\Sigma^j(x_2^j)}\frac{\partial}{\partial x_1}\zeta^{ij}dx_1^j = \frac{1}{2}\cdot 2\cdot\zeta^{ij}\Big|_{x_2^j=0}^{x_2^j\in\partial\Omega} = 1.$$

The estimates in Lemma 3 change to

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathbf{a}^{ij}\right| \le \frac{1}{(d_0 - k_{ij})^{1+|\alpha|}} \le \tilde{c}(\alpha, d) \qquad (|\alpha| \ge 0).$$

Theorem 2. *In Ω consider problem (S) with prescribed fluxes $\mathcal{F}^j \in \mathbb{R}$ $(j = 1, \cdots, J)$ under the condition $\sum_{j=1}^J \mathcal{F}^j = 0$. Assume Ω has the uniform C^1-property. If $\mathbf{f} \in W_\beta^{-1,q}(\Omega)$ where $-\beta^* < \beta < 0$ (β^* is fixed in Lemma 1), then there is a unique solution $\mathbf{v} \in \mathring{W}_\beta^{1,q}(\Omega)$ and*

$$\|\mathbf{v}; W_\beta^{1,q}(\Omega)\| \le c\left(\|\mathbf{f}; W_\beta^{-1,q}(\Omega)\| + \textstyle\sum_{j=1}^J|\mathcal{F}^j|\right). \tag{27}$$

If moreover $\mathbf{f} \in W_\beta^{l-1,q}(\Omega)$ $(l \in \mathbb{N})$ and Ω has the uniform C^{l+1}-property, then

$$\|\mathbf{v}; W_\beta^{l+1,q}(\Omega)\| + \|\nabla p; W_\beta^{l-1,q}(\Omega)\| \le c\left(\|\mathbf{f}; W_\beta^{l-1,q}(\Omega)\| + \textstyle\sum_{j=1}^J|\mathcal{F}^j|\right). \tag{28}$$

Proof. From Lemma 3 we infer

$$\|\Delta \mathbf{a}; W_\beta^{-1,q}(\Omega)\| = \|\nabla \mathbf{a}; L_\beta^q(\Omega)\| = \left\| \nabla \left(\sum_{i=1}^{J-1} \alpha_i \mathbf{a}^{i,i+1} \right); L_\beta^q(\Omega) \right\| \le c \sum_{j=1}^{J} |\mathcal{F}^j|.$$

This means that

$$\|\mathbf{f} + \Delta \mathbf{a}; W_\beta^{-1,q}(\Omega)\| \le c \left(\|\mathbf{f}; W_\beta^{-1,q}(\Omega)\| + \sum_{j=1}^{J} |\mathcal{F}^j| \right)$$

and analogously one obtains (28) ∎

Remark. It is worth to mention an important special case: For straight cylindrical outlets we may take the so-called *Poiseuille flow* as a flux carrier. This is a special solution for problem (S) with zero force and unit flux in the infinite cylinder $\Sigma \times \mathbb{R}$:

$$\mathbf{v}_p \equiv \left(0, \cdots, 0, \frac{2}{\kappa(\Sigma)} v_p(x_1, \cdots, x_{n-1})\right), \qquad p_p = \frac{-4}{\kappa(\Sigma)} x_n + c,$$

where v_p is the solution of the problem

$$\left.\begin{aligned} -\Delta v_p &= 2 \quad \text{in } \Sigma \\ v_p &= 0 \quad \text{on } \partial\Sigma \\ \kappa(\Sigma) &\equiv \textstyle\int_\Sigma |\nabla v_p|^2 dx_1 dx_2. \end{aligned}\right\}$$

For each outlet we can take the Poiseuille solution corresponding to the cylinder $\Sigma^j \times \mathbb{R}$, say $\mathbf{v}_p^j$, and cut them off near Ω^0 with the help of a smooth cut-off function χ_j which is 1 in $\Omega^j \setminus \Omega_2$ and 0 in Ω^0. To make the result solenoidal we must add some function $\mathbf{h}$ which is the solution to

$$\operatorname{div} \mathbf{h} = -\operatorname{div} \sum_{j=1}^{J} \mathcal{F}^j \chi_j \mathbf{v}_p^j \qquad \text{in } \Omega_2 \setminus \Omega^0$$

and with zero boundary value. Then $\mathbf{a} = \mathbf{h} + \sum_{j=1}^{J} \mathcal{F}^j \chi_j \mathbf{v}_p^j$.

For $\mathbf{u}$ we have to solve problem $(S)_0$ with the right-hand side $\Delta \mathbf{a}$. Since in the outlets $\mathbf{v}_p^j$ fulfils the homogeneous Stokes system it differs only on Ω_2 from zero. This fact leads to a different decay estimate for $\mathbf{u} = \mathbf{v} - \mathbf{a}$ than in the general case: Due to the compact support of $\Delta \mathbf{a}$, $\mathbf{u}$ decays exponentially. (But this is *not* valid for $\mathbf{v}$ since $\|\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{a}\|$ and the norm of $\mathbf{v}_p$ enters the estimate of $\|\mathbf{a}\|$. This leads to a solution in $W_\beta^{l,q}(\Omega)$ for $\beta < 0$ with the same arguments as in the general case, since $\mathbf{v}_p$ is constant with respect to the x_n^j-direction.)

5. The Stokes system with prescribed fluxes. A setting in local spaces

It is always useful to emphasize how much different settings are physically suitable depending on the geometry and forces.

As we noticed in the last remark, the famous problem of Leray concerning the spatial stability of the Poiseuille flow in a cylindrical outlet can be approached in a very natural way by means of weighted spaces. They are helpful tools because Ω^0 can be regarded as a region of torsion of a channel and this is equivalent to a bounded support force on the right-hand side of problem (S) (therefore belonging to any $W^{l,q}_\beta(\Omega)$). Then, by choosing $\beta > 0$ it is easy to prove (see, for example, [6]) that the perturbation $\mathbf{u}$ to the Poiseuille flow tends to zero in the direction of any exit.

If one considers a more general domain, with exits which are neither cylindrical nor periodic, the choice of weighted spaces is less natural. Any weight is specifically related, at least in the present formulation, to the choice of the origin of the x_3^j-axes. In contrast to Leray's problem where the origin is in Ω^0 since Ω^0 breaks the symmetry for any more general domain all locations are equivalent.

Since the Stokes problem is linear, we can always find the solution as follows: $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{a}$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ solve problem $(\mathrm{S})_0$ with right-hand sides $\mathbf{f}$ and $\Delta\mathbf{a}$, respectively, and $\mathbf{a}$ is again the flux carrier introduced in the preceding section. In the following we investigate $\mathbf{u}_2$ calling it $\mathbf{u}$, for simplicity.

Now, instead of minding that $\mathbf{a} \in W^{l,q}_\beta(\Omega)$ with $\beta < 0$, we notice that $\mathbf{a} \in C^\infty(\Omega)$ has no global summability property but belongs to any $W^{l,q}_{loc}(\Omega)$. This leads to the definition of Banach spaces which we introduced in [11]. For the moment let the index k in the definition of G_k and Ω_k be free to be a continuous variable $z \in \mathbb{R}$. For $l \in \mathbb{N}$ and $1 < q < \infty$ we set [2)]

$$\mathcal{W}^{l,q}(\Omega) \equiv \left\{\mathbf{v} \in W^{l,q}_{loc}(\Omega) : \|\mathbf{v}; W^{l,q}(G_z)\| \le c \quad (z \in \mathbb{R}) \text{ for some } c \ge 0\right\}.$$

We will call these spaces *local* spaces and endow them with the norm

$$\|\mathbf{v}; \mathcal{W}^{l,q}(\Omega)\| \equiv \inf\left\{c \in \mathbb{R}^+ \cup \{0\} : \|\mathbf{v}; W^{l,q}(G_z)\| \le c \text{ for all } z \in \mathbb{R}\right\}.$$

Locally applying Sobolev embedding theorems one deduces that a function like the flux carrier $\mathbf{a}$ is in $\mathcal{W}^{l,q}(\Omega)$ and bounded even without global summability properties:

$$\|\mathbf{v}; C^m(\Omega)\| = \sup_{k\in\mathbf{Z}} \|\mathbf{v}; C^m(G_k)\| \le \sup_{k\in\mathbf{Z}} (c(k)\|\mathbf{v}; W^{l,q}(G_k)\|) \le c.$$

This holds for all $m \in \left[0, l - \frac{n}{q}\right)$ (note that the sequence $c(k)$ is uniformly bounded because Ω enjoys the uniform C^m-regularity property). Thus, $C^m(\Omega) \supset \mathcal{W}^{l,q}(\Omega)$.

---

2) One could also consider other spaces with adequate properties, for example $\|\mathbf{v}; W^{l,q}(G_z)\| \le c_1 z + c_2$ for all $z \in \mathbb{R}$. But this different choice would imply that the space is not homogeneous with respect to z, which is not physically senseful in the present case. In this respect see also the local spaces defined by Solonnikov in [18].

Before justifying that local spaces fit in well with our problem, we want to show the relation between the two settings: weighted and local spaces. For $\beta \geq 0$ we see

$$\|\mathbf{v}; \mathcal{W}^{0,q}(\Omega)\|^q \leq e^{q\beta|z|} \int_{G_z} |\mathbf{v}|^q dx \leq \int_{G_z} |\mathbf{v}|^q e^{q\beta x_3^j} dx \leq \|\mathbf{v}; L^q_\beta(\Omega)\|^q$$

while

$$\|\mathbf{v}; L^q_\beta(\Omega)\|^q \leq \sum_{k=-\infty}^{\infty} e^{q\beta|k|} \int_{G_k} |\mathbf{v}|^q dx \leq \|\mathbf{v}; \mathcal{W}^{0,q}(\Omega)\|^q \sum_{k=-\infty}^{\infty} e^{q\beta|k|}$$

for $\beta < 0$. Therefore, we can conclude

$$\begin{aligned} W^{l,q}_\beta(\Omega) &\subset \mathcal{W}^{l,q}(\Omega) && \text{for } \beta \geq 0 \\ \mathcal{W}^{l,q}(\Omega) &\subset W^{l,q}_\beta(\Omega) && \text{for } \beta < 0. \end{aligned}$$

Hence, the setting in $\mathcal{W}^{l,q}(\Omega)$ is weaker when dealing with forces having some decay ($\beta \geq 0$). But it is stronger and therefore more suitable when considering the problem with non-zero flux ($\beta < 0$).

Let now $q = 2$. The spaces $\mathcal{W}^{l,2}(\Omega)$ can be endowed with a Hilbert space structure by defining the scalar product for $\mathbf{v}, \mathbf{w} \in \mathcal{W}^{l,2}(\Omega)$ as

$$(\mathbf{v}, \mathbf{w})_l^* \equiv \sup_{z \in \mathbb{R}} (\mathbf{v}, \mathbf{w})_{l,G_z}$$

where $(\cdot, \cdot)_{l,G_z}$ denotes the usual scalar product in $W^{l,2}(G_z)$. Unfortunately, this structure cannot be used to prove existence by means of Riesz' representation theorem, because of the difficulties related to the pressure term. However, we can prove the following

Theorem 3. *Consider problem* $(\mathrm{S})_0$ *for the right-hand side* $\Delta\mathbf{a}$ *(this is the flux carrier defined in* (25)*). Let* Ω *have the uniform* C^1*-property. Then there is a unique solution, say* $\mathbf{u}$*, which is in* $\mathring{\mathcal{W}}^{1,2}(\Omega)$ *and*

$$\|\mathbf{u}; \mathcal{W}^{1,2}(\Omega)\| \leq c\, \|\mathbf{a}; \mathcal{W}^{1,2}(\Omega)\|.$$

If Ω *has the uniform* C^{l+1}*-regularity property* ($l \in \mathbb{N}$) *in the sense of Adams* [1: IV.4.6], *then* $(\mathbf{u}, \nabla p) \in (\mathcal{W}^{l+1,2}(\Omega) \cap \mathring{\mathcal{W}}^{1,2}(\Omega)) \times \mathcal{W}^{l-1,2}(\Omega)$ *and*

$$\|\mathbf{u}; \mathcal{W}^{l+1,2}(\Omega)\| + \|\nabla p; \mathcal{W}^{l-1,2}(\Omega)\| \leq c\, \|\mathbf{a}; \mathcal{W}^{l+1,2}(\Omega)\|.$$

The constants depend on Ω *and* l.

Proof. Again, let $z \in \mathbb{R}^+$ be the continuous counterpart to $k \in \mathbb{N}$ and we consider the domains Ω_z. Then, it is well-known (see, for example, [6, 12]) that a weak solution $\mathbf{u} \in \mathring{W}^{1,2}_{loc}(\Omega)$ exists and it obeys the estimate

$$\|\mathbf{u}; W^{1,2}(\Omega_z)\|^2 \leq c\, \|\mathbf{a}; \mathcal{W}^{1,2}(\Omega)\|^2 (z+1) \tag{29}$$

for all $z \in \mathbb{R}^+$. The constant c in (29) can be bounded depending on the supremum of the constants in the estimate for the divergence equation in the domains G_k $(k \in \mathbb{Z})$. This supremum exists because of the assumptions on the domain Ω (for details see [11]). Along the same lines as in [11] we can prove that this solution actually belongs to $\mathcal{W}^{1,2}(\Omega)$:

For the moment let us consider one channel with two exits. As in Section 4 we use an orientated path γ^{12} in place of a common x_n-axis like [6]. Let z be the curvilinear coordinate along γ^{12} in our setting. This name stands also for a fixed point z on γ^{12} as no confusion may arise. Next, we notice that we could have constructed a different solution by choosing a different origin for z on γ^{12}. In particular, the new construction is characterized by a different (shifted) family of domains $\Omega_{z'}$ which invade the whole domain for growing z' and the new solution verifies (29) by replacing z with the new variable z'. Also the constant in (29) is the same by the hypothesis on the domain. Then we can prove the result by a uniqueness argument, as follows.

Let us assume that we have uniqueness in the class of solutions verifying (29) for some choice of the origin. Since $\Omega_1 = G_0$ one can change continuously the origin of the reference frame and each time put $z = 1$ in (29). The result can be written in a unique reference frame, which can be fixed once for all, so yielding

$$\|\mathbf{u}; W^{1,2}(G_z)\| \le (2c)^{1/2}\|\mathbf{a}; \mathcal{W}^{1,2}(\Omega)\| \qquad \text{for all } z \in \mathbb{R}.$$

Thus, by definition $\mathbf{u}$ belongs to $\mathcal{W}^{1,2}(\Omega)$.

Now, to prove uniqueness, we consider the difference $\mathbf{u}_0$ between two solutions and notice that $c > 0$ exists such that

$$\|\mathbf{u}_0; W^{1,2}(\Omega_z)\|^2 \le c(z+1) \qquad \text{for all } z \in \mathbb{R}^+.$$

This means that the possible growth cannot be more than linear. On the other hand, we can directly write the equation for the difference, multiply it by $\mathbf{u}_0$, integrate by parts in Ω_z and then integrate again on $z \in (s-1, s)$. Since $\operatorname{div}\mathbf{u}_0 = 0$ in Ω and $\mathbf{u}_0 = 0$ on $\partial\Omega$, the result is

$$\int_{s-1}^{s} \|\nabla \mathbf{u}_0; L^2(\Omega_z)\|^2 dz + \int_{G_{s-1}} \left(p_0 u_{03} - \frac{\partial \mathbf{u}_0^2}{\partial z}\right) dx - \int_{G_{1-s}} \left(p_0 u_{03} - \frac{\partial \mathbf{u}_0^2}{\partial z}\right) dx = 0.$$

By means of Schwarz' and Poincaré's inequalities one immediately finds that

$$\int_{G_{s-1}\cup G_{1-s}} \left|\frac{\partial \mathbf{u}_0^2}{\partial z}\right| dx \le c\|\nabla \mathbf{u}_0; L^2(G_{s-1}\cup G_{1-s})\|^2 = \frac{d}{ds}\int_{s-1}^{s} \|\nabla \mathbf{u}_0; L^2(\Omega_z)\|^2 dz.$$

At this point, we notice that the problem

$$\left.\begin{aligned} \operatorname{div}\Psi &= u_{03} \quad \text{in } G_{s-1} \\ \Psi &= 0 \quad\ \ \text{on } \partial G_{s-1} \end{aligned}\right\}$$

allows a solution which satisfies

$$\|\nabla\Psi; L^2(G_{s-1})\| \le c\,\|u_{03}; L^2(G_{s-1})\| \tag{30}$$

where the constant does not depend on s. In fact, since $\mathbf{u}_0$ has zero flux the compatibility condition $\int_{G_{s-1}} u_{03}\,dx = 0$ is fulfilled.

We follow the ideas of [6] in estimating the integrals containing the pressure and insert (30) after the last term of the following chain of equalities

$$\int_{G_{s-1}} p_0 u_{03}\,dx = \int_{G_{s-1}} p_0 \operatorname{div}\Psi\,dx = -\int_{G_{s-1}} \nabla p_0 \cdot \Psi\,dx = \int_{G_{s-1}} \nabla \mathbf{u}_0 \cdot \nabla\Psi\,dx,$$

where the equation of motion written in the weak form in G_{s-1} has been used. Hence,

$$\left|\int_{G_{s-1}} p_0 u_{03}\,dx\right| + \left|\int_{G_{1-s}} p_0 u_{03}\,dx\right| \le c\,\|\nabla\mathbf{u}_0; L^2(G_{s-1}\cup G_{1-s})\|^2.$$

Finally, we obtain the differential inequality

$$\int_{s-1}^{s} \|\nabla\mathbf{u}_0; L^2(\Omega_z)\|^2 dz \le c\frac{d}{ds}\int_{s-1}^{s} \|\nabla\mathbf{u}_0; L^2(\Omega_z)\|^2 dz.$$

Setting $\varphi(s) \equiv \int_{s-1}^{s} \|\nabla\mathbf{u}_0; L^2(\Omega_z)\|^2 dz$, that differential inequality implies

$$\varphi(s) \ge \varphi(s_0) e^{\frac{1}{c}(s-s_0)},$$

for $0 \le s_0 \le s$ where s_0 and s are arbitrary. Since

$$\varphi(s) \le \|\nabla\mathbf{u}_0; L^2(\Omega_s)\|^2 \qquad \text{and} \qquad \varphi(s_0) \ge \|\nabla\mathbf{u}_0; L^2(\Omega_{s_0-1})\|$$

the exponential growth is in contradiction with the linear one unless $\varphi(s_0) = 0$ for any $s_0 \ge 0$. This implies $\mathbf{u}_0 = 0$.

It remains to prove the regularity. We start by recalling (13) in the case $\beta = 0$, and z in place of m (where now it is essential that z may be arbitrary). We insert on the right-hand side the estimate

$$\|\nabla\mathbf{u}; L^2(G_z^*)\| \le 3\|\mathbf{u}; \mathcal{W}^{1,2}(\Omega)\| \le 3c\,\|\mathbf{a}; \mathcal{W}^{1,2}(\Omega)\|,$$

we replace $\mathbf{f}$ with $\Delta\mathbf{a}$, and the proof is completed since z is arbitrary ∎

Remark. We have already pointed out that due to its regularity $\mathbf{a}$ belongs to all local spaces $\mathcal{W}^{l,q}(\Omega)$.

Since the spaces here considered are local it is immediate how to extend the results of Theorem 3 to local spaces with $q > 2$. In fact, it is sufficient to insert the first estimate of Theorem 3 on the right-hand side of (12), evaluated for $\beta = 0$, and $\mathbf{f} = \Delta\mathbf{a}$. Then, we use the inequality

$$\|\mathbf{a}; \mathcal{W}^{1,2}(\Omega)\| \leq \sup_{z\in\mathbb{R}} |G_z|^{\frac{q-2}{2q}} \|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|$$

where $|G_z|$ is the Lebesgue measure of G_z. The so found estimate can be used to increase the right-hand side of (13) for $\beta = 0$ and $\mathbf{f} = \Delta\mathbf{a}$, as

$$\|\nabla\mathbf{u}; \mathcal{W}^{0,q}(\Omega)\| \leq c\,\|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|.$$

Notice that the constants depend on q, Ω and l.

From the result for $q' > 2$ we can now deduce the result for $q < 2$ by means of usual duality arguments. However, those arguments have to be applied to the solution $\mathbf{u}_k$ of the "approximating" problem

$$\left.\begin{aligned} -\Delta\mathbf{u}_k + \nabla p_k &= \Delta\mathbf{a} && \text{in } \Omega_k \\ \operatorname{div}\mathbf{u}_k &= 0 && \text{in } \Omega_k \\ \mathbf{u}_k &= 0 && \text{on } \partial\Omega_k. \end{aligned}\right\} \tag{31}$$

In fact, the solution $\mathbf{u} = \lim_{k\to\infty}\mathbf{u}_k$ (in the weak sense) cannot be used as test function since it has no bounded support while $\mathbf{u}_k$ is zero on the boundary. Since $\Delta\mathbf{a} \in W^{-1,2}(\Omega_k) \cap W^{-1,q}(\Omega_k)$ the analogous of (22) is

$$\int_{\Omega_k} \nabla\mathbf{u}_k : \nabla\Psi\, dx = \langle\Delta\mathbf{a}, \Psi\rangle_{\Omega_k} \tag{32}$$

for all $\Psi \in \mathring{W}^{1,2}(\Omega_k)$ such that $\operatorname{div}\Psi = 0$. On the other hand (32) certainly holds true for any $\Psi \in \mathring{W}^{1,q'}(\Omega_k) \subset \mathring{W}^{1,2}(\Omega_k)$ for $q' > 2$. Such Ψ could be the solution of the analogous of (23)

$$\int_{\Omega_k} \nabla\mathbf{w}_k : \nabla\Phi\, dx = \langle\mathbf{g}_k, \Phi\rangle_{\Omega_k} \tag{33}$$

where Φ is any function in $\mathring{W}^{1,q}(\Omega_k)$ such that $\operatorname{div}\Phi = 0$ and $\mathbf{g}_k \equiv \operatorname{div}\mathcal{G}_k$ is defined by

$$\mathcal{G}_k \equiv |\nabla\mathbf{u}_k|^{q-2}\nabla\mathbf{u}_k.$$

For instance, one can choose $\Phi = \mathbf{u}_k$ in (33), because $\mathbf{u}_k$ belongs necessarily to $W^{1,q}(\Omega_k)$. Going on in the same way as for weighted spaces one proves

$$\|\mathbf{u}_k; W^{1,q}(\Omega_{k'})\|^q \leq c(q,\Omega)\|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|^q(k'+1),$$

for any $k, k' \in \mathbb{N}$. By fixing k' and varying k, one starts constructing a uniformly bounded sequence converging in $\Omega_{k'}$. By varying k', one has the series of "invading" domains. Finally, the weak limit $\mathbf{u}'$ will verify the analogous of (29):

$$\|\mathbf{u}'; W^{1,q}(\Omega_z)\|^q \leq c(q,\Omega)\|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|^q(z+1) \tag{34}$$

for all $z \in \mathbb{R}^+$. Moreover, it is obvious that $\mathbf{u} = \mathbf{u}'$: By compactness we may conclude

$$\|\mathbf{u}_k - \mathbf{u}; W^{1,q}(\Omega_z)\| \leq \sup_{z\in\mathbb{R}^+} |\Omega_z|^{\frac{2-q}{2q}} \|\mathbf{u}_k - \mathbf{u}; W^{1,2}(\Omega_z)\| \longrightarrow 0 \quad \text{as } k \to \infty.$$

Since the strong limit coincides with the weak one, and since the limit is unique, then $\mathbf{u} = \mathbf{u}'$. Therefore from (34) it follows

$$\|\mathbf{u}; \mathcal{W}^{1,q}(\Omega)\| \leq (2c(q,\Omega))^{1/q}\|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|.$$

In order to prove more regularity, it is sufficient to insert this last result in (13), with $q < 2$, $\beta = 0$, $\mathbf{f} = \Delta\mathbf{a}$ and arbitrary $z \in \mathbb{R}^+$ in place of m. Thus, we proved the following

Theorem 4. *Consider problem* $(S)_0$ *for the right-hand side* $\Delta\mathbf{a}$ *(this is the flux carrier defined in* (25)*). Let* Ω *have the uniform* C^1*-property and* $1 < q < \infty$*. Then there is a unique solution, say* $\mathbf{u}$*, which is in* $\mathring{\mathcal{W}}^{1,q}(\Omega)$ *and*

$$\|\mathbf{u}; \mathcal{W}^{1,q}(\Omega)\| \leq c\,\|\mathbf{a}; \mathcal{W}^{1,q}(\Omega)\|.$$

If Ω *has the uniform* C^{l+1}*-regularity property* $(l \in \mathbb{N})$ *in the sense of Adams* [1: IV.4.6], *then* $(\mathbf{u}, \nabla p) \in (\mathcal{W}^{l+1,q}(\Omega) \cap \mathring{\mathcal{W}}^{1,q}(\Omega)) \times \mathcal{W}^{l-1,q}(\Omega)$ *and*

$$\|\mathbf{u}; \mathcal{W}^{l+1,q}(\Omega)\| + \|\nabla p; \mathcal{W}^{l-1,q}(\Omega)\| \leq c\,\|\mathbf{a}; \mathcal{W}^{l+1,q}(\Omega)\|.$$

The constants depend on Ω*,* l *and* q*.*

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