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L_p-Theory of Boundary Integral Equations on a Contour with Inward Peak

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Dedicated to Prof. E. Meister on the occasion of his retirement

Abstract. Boundary integral equations of the second kind in the logarithmic potential theory are studied under the assumption that the contour has an inward peak. For each equation we find a pair of function spaces such that the corresponding operator bijectively maps one of them onto another.

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1. Introduction

In this paper we prove the unique solvability of boundary integral equations of the Dirichlet problem

$$\left.\begin{aligned} &\Delta u = 0 \ \text{ in } \ \Omega \\ &u\big|_{\Gamma} = \varphi \end{aligned}\right\} \tag{1.1}$$

and the Neumann problem

$$\left.\begin{aligned} &\Delta u = 0 \ \text{ in } \ \Omega \\ &\frac{\partial u}{\partial n}\Big|_{\Gamma} = \psi \end{aligned}\right\} \tag{1.2}$$

in a bounded plane simply connected domain Ω with inward peak $z = 0$ on the boundary Γ. Here and elsewhere we assume that the normal n is directed outwards.

We look for a solution of the problem (1.1) in the form

$$u(z) = (W\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k(z) \qquad (z = x + iy \in \Omega)$$

where $W\sigma$ is the double layer potential

$$(W\sigma)(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q\,,$$

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$t^{(k)}$ are real numbers and $\mathcal{I}_k(z) = \operatorname{Im} z^{k-\frac{1}{2}}$. The function σ and the vector $t = (t^{(1)}, \dots, t^{(m)})$ satisfy the equation

$$\pi\sigma - T\sigma + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k = -\varphi \qquad \text{on } \Gamma \setminus \{O\}, \tag{1.3}$$

where $T\sigma$ is the value of the potential $W\sigma$ at a boundary point.

A solution of problem (1.2) is sought in the form

$$u(z) = (V\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Omega),$$

where $V\sigma$ is defined by

$$(V\sigma)(z) = \int_\Gamma \sigma(q) \log \frac{|z|}{|z-q|} ds_q$$

and $\mathcal{R}_k(z) = \operatorname{Re} z^{k-\frac{1}{2}}$. Then the function σ and the vector $t = (t^{(1)}, \dots, t^{(m)})$ satisfy the equation

$$\pi\sigma + S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k = \varphi \qquad \text{on } \Gamma \setminus \{O\}, \tag{1.4}$$

where

$$(S\sigma)(z) = \int_\Gamma \sigma(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \qquad (z \in \Gamma \setminus \{O\}).$$

Let $\Gamma \setminus \{O\}$ belong to the class C^2. We say that O is an *outward* (*inward*) peak if Ω (the complementary domain Ω^c) is given near the peak by the inequalities $\kappa_-(x) < y < \kappa_+(x)$, $0 < x < \delta$, where $x^{-\mu-1}\kappa_\pm(x) \in C^2[0,\delta]$ and $\lim_{x\to+0} x^{-\mu-1}\kappa_\pm(x) = \alpha_\pm$ with $\mu > 0$ and $\alpha_+ > \alpha_-$. By $\Gamma_\pm$ we denote the arcs $\{(x, \kappa_\pm(x)) : x \in [0,\delta]\}$. Points on Γ_+ and Γ_- with equal abscissas will be denoted by q_+ and q_-.

In our previous articles [5 - 7], where we also studied boundary integral equations of logarithmic potential theory on contours with peaks, the solutions and boundary data were characterized by their asymptotic behaviour near the peaks. Here, for every integral operator under consideration we find a pair of function spaces such that the operator maps isomorphically one space onto another.

If $|q|^\beta \varphi \in L_p(\Gamma)$, then we say that φ belongs to $\mathcal{L}_{p,\beta}(\Gamma)$. We define the norm in this space by $\|\varphi\|_{\mathcal{L}_{p,\beta}(\Gamma)} = \| |q|^\beta \varphi \|_{L_p(\Gamma)}$. We shall make use of the same definition with Γ replaced by arcs of $\Gamma_\pm$ and intervals of $\mathbb{R}$.

Let $\mathcal{L}^1_{p,\beta}(\Gamma)$ be the space of absolutely continuous functions on $\Gamma \setminus \{O\}$ with finite norm $\|\varphi\|_{\mathcal{L}^1_{p,\beta}(\Gamma)} = \|\varphi'_s\|_{\mathcal{L}_{p,\beta}(\Gamma)} + \|\varphi\|_{\mathcal{L}_{p,\beta-1}(\Gamma)}$. It is an easy exercise to check the density in $\mathcal{L}^1_{p,\beta}(\Gamma)$ of the set of smooth functions on Γ vanishing near O.

We introduce the pair of spaces $\mathfrak{N}^{(\pm)}_{p,\beta}(\Gamma)$ of absolutely continuous functions φ on $\Gamma \setminus \{O\}$ with finite norms

$$\|\varphi\|_{\mathfrak{N}^{(\pm)}_{p,\beta}(\Gamma)} = \left(\int_{\Gamma_+ \cup \Gamma_-} |\varphi(q_+) \pm \varphi(q_-)|^p |q|^{p(\beta-\mu)} ds_q \right)^{\frac{1}{p}} + \|\varphi\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)}.$$

Let $\mathfrak{P}(\Gamma)$ denote the space of restrictions to $\Gamma \setminus \{O\}$ of real functions of the form $p(z) = \sum_{k=0}^{m} t^{(k)} \mathrm{Re}\, z^k$, where $m = [\mu - \beta - \frac{1}{p} + \frac{1}{2}]$. A norm of p is defined by $\|p\|_{\mathfrak{P}(\Gamma)} = \sum_{k=0}^{m} |t^{(k)}|$. The space $\mathfrak{M}_{p,\beta}(\Gamma)$ is defined as the direct sum of $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $\mathfrak{P}(\Gamma)$.

By $\mathfrak{Y}_{p,\beta}(\Gamma)$ we denote the space of functions on $\Gamma \setminus \{O\}$ represented in the form $\varphi = \frac{d}{ds}\psi$, where $\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$ and $\psi(z_0) = 0$ for a fixed point $z_0 \in \Gamma \setminus \{O\}$. We supply $\mathfrak{Y}_{p,\beta}(\Gamma)$ with the norm $\|\varphi\|_{\mathfrak{Y}_{p,\beta}(\Gamma)} = \|\psi\|_{\mathfrak{M}_{p,\beta}(\Gamma)}$.

In the following short description of our results we assume that $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$. In Theorem 1 we prove that the operator

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma - T\sigma + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k \in \mathfrak{M}_{p,\beta}(\Gamma) \tag{1.5}$$

is continuous. As is shown in Theorem 2, the range of operator (1.5) coincides with the space $\mathfrak{M}_{p,\beta}(\Gamma)$ if $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. For the exceptional case $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ we find that operator (1.5) is not Fredholm (see Proposition 4). In Theorem 4 we show that the operator

$$\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma) \tag{1.6}$$

is onto if $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Under the assumption $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ we prove in Proposition 5 that operator (1.6) is not Fredholm. In Theorems 3 and 5 we show that operators (1.5) and (1.6) are injective. The boundary integral equations of the exterior Dirichlet and Neumann problems for a domain Ω with outward peak are discussed in Theorems 6 - 9 and Proposition 6.

2. Continuity of the operator $\pi I - T$

Let the operator $\mathcal{K}$ be defined by

$$\mathcal{K}f(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy,$$

where

$$|K(x,y)| \le c \frac{1}{|x-y|} \frac{1}{(1 + |x-y|^J)} \qquad (J \ge 0).$$

Here and elsewhere by c we denote different positive constants.

We introduce the space $L_{p,\alpha}(\mathbb{R})$ of functions on $\mathbb{R}$ with the norm

$$\|\varphi\|_{L_{p,\alpha}(\mathbb{R})} = \|\,(1+x^2)^{\frac{\alpha}{2}} \varphi\,\|_{L_p(\mathbb{R})}.$$

The following lemma was formulated in [4].

Lemma 1. *If $\mathcal{K} : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ $(1 < p < \infty)$ is bounded and $-J < \alpha + \frac{1}{p} < J+1$, then $\mathcal{K}$ is continuous in $L_{p,\alpha}(\mathbb{R})$.*

We shall also use the following technical lemma.

Lemma 2. *Let $\rho(u) = \kappa_+(u) - \kappa_-(u)$ and let the function h be specified by*

$$\int_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\nu)} = \tau \qquad (\tau > 0). \tag{2.1}$$

If $\left|1 - \frac{\xi}{\tau}\right| < \varepsilon_0$, where ε_0 is a sufficiently small number, then

$$\left|\frac{\rho(h(\xi))\rho(h(\tau))}{(h(\xi) - h(\tau))^2 + (\rho(h(\xi)))^2} - \frac{1}{(\xi - \tau)^2 + 1}\right| \le \frac{c}{\tau}\left(\frac{1}{(\xi - \tau)^2 + 1} + \frac{1}{\xi}\right).$$

Proof. By the asymptotic formula

$$h(\tau) \sim (\mu(\alpha_+ - \alpha_-)\tau)^{-\frac{1}{\mu}} \qquad \text{as } \tau \to \infty,$$

which can be differentiated three times, we obtain the estimates

$$\left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2 - 1 = O\left(\frac{\xi - \tau}{\tau}\right) \tag{2.2}$$

and

$$\frac{\rho(h(\xi))\rho(h(\tau))}{(\alpha(\xi,\tau))^2} - 1 = O\left(\frac{(\xi - \tau)^2}{\xi\tau}\right), \tag{2.3}$$

where $\alpha(\xi,\tau) = (h(\xi) - h(\tau))(\xi - \tau)^{-1}$. We represent

$$\frac{\rho(h(\xi))\rho(h(\tau))}{(h(\xi) - h(\tau))^2 + (\rho(h(\xi)))^2} - \frac{1}{(\xi - \tau)^2 + 1}$$

in the form

$$\frac{\rho(h(\xi))\rho(h(\tau))}{(\alpha(\xi,\tau))^2} \frac{1 - \left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2}{\left((\xi - \tau)^2 + 1\right)\left((\xi - \tau)^2 + \frac{(\rho(h(\xi)))}{(\alpha(\xi,\tau))^2}\right)} - \frac{1 - \rho(h(\xi))\frac{\rho(h(\tau))}{(\alpha(\xi,\tau))^2}}{(\xi - \tau)^2 + 1} \tag{2.4}$$

From (2.2) and (2.3) it follows that (2.4) does not exceed $\frac{c}{\tau}\left(\frac{1}{(\xi-\tau)^2+1} + \frac{1}{\xi}\right)$ ∎

Theorem 1. *Let Ω have an inward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$. Then the operator*

$$(\pi I - T) : \mathcal{L}^1_{p,\beta+1}(\Gamma) \ni \sigma \longmapsto (\pi I - T)\sigma \in \mathfrak{M}_{p,\beta}(\Gamma)$$

is continuous.

Proof. Let ε be so small that $|\kappa_\pm(x) - \kappa_\mp(u)| \ge c\, u^{\mu+1}$ for all u satisfying $|x - u| < \varepsilon x$. The arcs of $\Gamma_\pm$ projected onto the segments $[0,(1-\varepsilon)x]$, $[(1-\varepsilon)x,(1+\varepsilon)x]$ and

$[(1+\varepsilon)x, \delta]$ will be denoted by $\Gamma^{\ell}_{\pm}(x)$, $\Gamma^{c}_{\pm}(x)$ and $\Gamma^{r}_{\pm}(x)$. Set $\sigma(u+i\kappa_{\pm}(u)) = \sigma_{\pm}(u)$, $u \in [0,\delta]$.

(i) We prove the continuity of the operator $\frac{\partial}{\partial s}(\pi I - T) : \mathcal{L}^1_{p,\beta+1}(\Gamma) \to \mathcal{L}_{p,\beta+1}(\Gamma)$. Since

$$\frac{\partial}{\partial s}(\pi\sigma - T\sigma)(z) = \pi\sigma'(z) + \int_\Gamma \sigma'(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q,$$

it is sufficient to estimate the norm in $\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})$ of the function

$$T_*\sigma(z) = \int_{\Gamma_+\cup\Gamma_-}\sigma'(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q .$$

We represent $T_*\sigma(z)$ in the form

$$\left(\int_{\Gamma^c_-(x)} + \int_{\Gamma^c_+(x)}\right)\sigma'(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q + I(z),$$

where the last term on $\Gamma_+ \cup \Gamma_-$ admits the estimate

$$|I(z)| \le \frac{c}{x^2}\int_0^x \left(|\sigma'_+(u)| + |\sigma'_-(u)|\right)u\,du + \frac{c}{x}\int_x^\delta \left(|\sigma'_+(u)| + |\sigma'_-(u)|\right)du .$$

It follows by Hardy's inequality [2: Section 9.9] that

$$\|I\|_{\mathcal{L}_{p,\beta+1}(\Gamma\cap\{|q|<\frac{\delta}{2}\})} \le c\,\|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}.$$

It is sufficient to assume that $z \in \Gamma_+$. Since $\left|\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}\right| \le \frac{c}{x}$ for $q \in \Gamma^c_+(x)$, we obtain

$$\left|\int_{\Gamma^c_+(x)}\sigma'(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q\right| \le \frac{c}{x}\int_{(1-\varepsilon)x}^{(1+\varepsilon)x}|\sigma'_+(u)|du .$$

The required estimate for the left-hand side follows from Hardy's inequality.

For $q \in \Gamma^c_-(x)$ we have

$$\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}\left(1+(\kappa'_-(x))^2\right)^{\frac{1}{2}} = -\frac{\kappa_+(u)-\kappa_-(u)}{|z-q|^2} + O\left(\frac{1}{x}\right).$$

Since $|z-q|^2 \ge c\left((x-u)^2 + (\rho(u))^2\right)$, we obtain

$$\left|\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}\right| \le c\left(\frac{1}{x} + \frac{\rho(u)}{(x-u)^2+(\rho(u))^2}\right).$$

This implies

$$\begin{aligned}
&\left| \int_{\Gamma^c_-(x)} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \right| \\
&\quad \le \frac{c}{x} \int\limits_{(1-\varepsilon)x}^{(1+\varepsilon)x} |\sigma'_-(u)| du + \int\limits_{(1-\varepsilon)x}^{(1+\varepsilon)x} \frac{\rho(u)\, |\sigma'_-(u)|}{(x-u)^2 + (\rho(u))^2} du \\
&\quad = J_1 + J_2.
\end{aligned}$$

The integral J_1 can be estimated in $\mathcal{L}_{p,\beta+1}(0, \frac{\delta}{2})$ by Hardy's inequality. By making the change of variables $\tau = u^{-\mu}$, $\xi = x^{-\mu}$ in J_2 we obtain

$$\int\limits_0^{\delta/2} x^{p(\beta+1)} |J_2(x)|^p dx \le c \left(\int\limits_{(\delta/2)^{-\mu}}^{\infty} \xi^{p(1-\alpha)} \left(\int\limits_{(\delta/2)^{-\mu}}^{\infty} \frac{|\sigma'_-(\tau^{-\frac{1}{\mu}})| \tau^{-1-\frac{1}{\mu}}}{(\xi - \tau)^2 + 1} d\tau \right)^p d\xi \right)^{\frac{1}{p}},$$

where $\beta + \frac{1}{p} = \mu(\alpha - \frac{1}{p})$. From Lemma 1 it follows that the integral on the right is majorized by

$$c \int\limits_{(\delta/2)^{-\mu}}^{\infty} |\sigma'_-(\tau^{-\frac{1}{\mu}})|^p \tau^{\frac{-p(\beta+1)}{\mu}} \tau^{-1-\frac{1}{\mu}} d\tau = c \int\limits_0^{\delta/2} |\sigma'_-(u)|^p u^{p(\beta+1)} du\,.$$

Thus,

$$\|T_* \sigma\|_{\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \le c \|\sigma\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)}. \tag{2.5}$$

(ii) Now we estimate the norm in $\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})$ of $(\pi I - T)\sigma(z_+) + (\pi I - T)\sigma(z_-)$. We represent $(\pi I - T)\sigma(z)$ for $z = x + i\kappa_\pm(x) \in \Gamma_\pm$ in the form

$$\begin{aligned}
&\pm \pi[\sigma(z_+) - \sigma(z_-)] \\
&- \int_{\Gamma^\ell_+(x) \cup \Gamma^\ell_-(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q \\
&- \int_{\Gamma^c_\pm(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q \\
&+ \left[\pi \sigma(z_\mp) - \int_{\Gamma^c_\mp(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q \right] \\
&- \int_{\Gamma^r_+(x) \cup \Gamma^r_-(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q \\
&- \int_{\Gamma \setminus (\Gamma_+ \cup \Gamma_-)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = \sum_{k=1}^{6} I_k(z).
\end{aligned}$$

For any $q \in \Gamma_+^c(x) \cup \Gamma_+^\ell(x) \cup \Gamma_-^\ell(x)$ the inequality

$$\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \right| \le c\, x^{\mu-1} \tag{2.6}$$

is valid. In fact,

$$\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \left(1 + (\kappa_+'(u)^2\right)^{\frac{1}{2}} \right| \le \frac{1}{|z-q|^2} \left| \int_u^x \kappa_+''(\tau)(x-u)d\tau \right| \le c\, x^{\mu-1},$$

if $q \in \Gamma_+^c(x)$, and

$$\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \left(1 + (\kappa_+'(u)^2\right)^{\frac{1}{2}} \right| \le \frac{1}{|z-q|} \left(|\kappa_\pm'(u)| + \frac{|\kappa_\pm(u) - \kappa_+(x)|}{|q-z|} \right) \le c\, \frac{u^\mu + x^\mu}{x},$$

if $q \in \Gamma_\pm^\ell(x)$. Inequality (2.6) follows. Therefore

$$|I_2(z)| + |I_3(z)| \le c\, x^{\mu-1} \int_0^{(1+\varepsilon)x} (|\sigma_+(u)| + |\sigma_-(u)|)du\,.$$

Hence, the estimate

$$\|I_2\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q|<\frac{\delta}{2}\})} + \|I_3\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q|<\frac{\delta}{2}\})} \le c\, \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}$$

results from Hardy's inequality.

Now we estimate I_4. We can assume that $z \in \Gamma_+$. In the sequel we shall use the estimate

$$\begin{aligned} &\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} + \left(1 + (\kappa_+'(u))^2\right)^{-\frac{1}{2}} \frac{\rho(x)}{(x-u)^2 + (\rho(x))^2} \right| \\ &\quad \le c \left(x^{\mu-1} + \frac{x^{2\mu+1}}{(x-u)^2 + x^{2\mu+1}} \right) \qquad (q = u + iv \in \Gamma_-^c(x)) \end{aligned} \tag{2.7}$$

where $\rho(x) = \kappa_+(x) - \kappa_-(x)$. In order to obtain (2.7) we notice that

$$\left| \kappa_-(x) - \kappa_-(u) - \kappa_-'(u)(x-u) \right| \le c\, x^{\mu-1}(x-u)^2.$$

Therefore

$$\begin{aligned} -\frac{\partial}{\partial n_q} &\log \frac{1}{|z-q|} \left(1 + (\kappa_+'(u)^2\right)^{\frac{1}{2}} \\ &= \frac{(u-x)\kappa_-'(u) - \left(\kappa_-(u) - \kappa_+(x)\right)}{(x-u)^2 + \left(\kappa_-(u) - \kappa_+(x)\right)^2} \\ &= \frac{\kappa_+(x) - \kappa_-(x)}{(x-u)^2 + \left(\kappa_-(u) - \kappa_+(x)\right)^2} + O\left(x^{\mu-1}\right). \end{aligned} \tag{2.8}$$

Taking into account $|\kappa_-(x) - \kappa_-(u)| \le c\, x^\mu |x-u|$ we obtain

$$\begin{aligned} &\frac{\kappa_+(x) - \kappa_-(x)}{(x-u)^2 + \big(\kappa_-(u) - \kappa_+(x)\big)^2} \\ &\qquad = \frac{\rho(x)}{(x-u)^2 + \big(\rho(x)\big)^2} + O\Big(\frac{x^{2\mu+1}}{(x-u)^2 + x^{2\mu+2}}\Big). \end{aligned} \tag{2.9}$$

Now (2.7) follows from (2.8) and (2.9). By making the change of variables $\tau = u^{-\mu}$, $\xi = x^{-\mu}$ we arrive at

$$\begin{aligned} &\int_0^{\delta/2} x^{p(\beta-\mu)} \left(\int_{(1-\varepsilon)x}^{(1+\varepsilon)x} \frac{x^{2\mu+1}\sigma_-(u)}{(x-u)^2 + x^{2\mu+2}} du \right)^p dx \\ &\qquad \le c \int_{(\delta/2)^{-\mu}}^{+\infty} \xi^{p(1-\alpha)} \Big(\int_{\mathbb{R}} \frac{\mu |\sigma_-(\tau^{-\frac{1}{\mu}})|}{(\xi-\tau)^2 + \mu^2} \frac{d\tau}{\tau} \Big)^p d\xi \,, \end{aligned}$$

where $\beta + \frac{1}{p} = \mu(\alpha - \frac{1}{p})$. From Lemma 1 it follows that the right-hand side is estimated from above by

$$c \int_{(\delta/2)^{-\mu}}^{+\infty} \tau^{-p\alpha} |\sigma_-(\tau^{-\frac{1}{\mu}})|^p d\tau \le c \int_0^{\delta} |\sigma_-'(u)|^p u^{p(\beta+1)} du \,.$$

Thus, it is sufficient to estimate

$$\pi\sigma_-(x) - \int_{(1-\varepsilon)x}^{(1+\varepsilon)x} \frac{\rho(x)\,\sigma_-(u)}{(x-u)^2 + (\rho(x))^2} du \,.$$

We make the change of variables $u = h(\tau)$, $x = h(\xi)$, where h is specified by (2.1). By Lemma 2 we have

$$\begin{aligned} &\int_0^{\delta/2} x^{p(\beta-\mu)} \left| \pi\sigma_-(x) - \int_{(1-\varepsilon)x}^{(1+\varepsilon)x} \frac{\rho(x)\,\sigma_-(u)}{(x-u)^2 + (\rho(x))^2} du \right|^p dx \\ &\qquad \le c \int_{h^{-1}(\delta/2)}^{\infty} \xi^{p(1-\alpha)} \left| \pi\sigma_-(h(\xi)) - \int_{\mathbb{R}} \frac{\sigma_-(h(\tau))}{(\xi-\tau)^2 + 1} d\tau \right|^p d\xi \\ &\qquad\quad + \int_{h^{-1}(\delta/2)}^{\infty} \xi^{p(1-\alpha)} |I(\xi)|^p d\xi \,, \end{aligned} \tag{2.10}$$

where $I(\xi)$ admits the estimate

$$|I(\xi)| \le \frac{c}{\xi} \int\limits_{h^{-1}(\delta/2)}^{\xi} |\sigma_-(h(\tau))| \frac{d\tau}{\tau} + c \int\limits_{\xi}^{\delta} |\sigma_-(h(\tau))| \frac{d\tau}{\tau^2} + c \int\limits_{h^{-1}(\delta/2)}^{\delta} \frac{|\sigma_-(h(\tau))|}{(\xi-\tau)^2+1} \frac{d\tau}{\tau}.$$

From Lemma 1 and Hardy's inequality it follows that the last integral in (2.10) does not exceed

$$c \int\limits_{h^{-1}(\delta)}^{\infty} \left| \frac{d}{d\tau} \sigma_-(h(\tau)) \right|^p \tau^{p(1-\alpha)} d\tau = c \int\limits_0^{\delta} |\sigma'_-(u)|^p u^{p(\beta+1)} du.$$

The Fourier transform of

$$\pi\sigma_-(h(\xi)) - \int_{\mathbb{R}} \frac{\sigma_-(h(\tau))}{(\xi-\tau)^2+1} d\tau$$

equals

$$-\pi i \operatorname{sign}(\nu) \widehat{(\sigma \circ h)'}(\nu) \frac{1-\exp(-|\nu|)}{|\nu|}.$$

Since the function $(1-\exp(-|\nu|))|\nu|^{-1}$ is the Fourier transform of $\log(1+\xi^{-2})$ up to a real factor, it follows from the boundedness of the Hilbert transform in $\mathcal{L}_{p,1-\alpha}(\mathbb{R})$ (see [3]) and from Lemma 1 that the first integral on the right in (2.10) does not exceed $c\,\|\sigma'\|^p_{\mathcal{L}_{p,\beta+1}(\Gamma)}$.

We turn to the integral I_5. We have

$$\frac{\partial}{\partial n_q} \log \frac{1}{|q-z|} = -\mathrm{Re} \frac{1}{q-z} \cos(n_q, x) + \mathrm{Im} \frac{1}{q-z} \cos(n_q, y). \tag{2.11}$$

For $q \in \Gamma^r_+(x) \cup \Gamma^r_-(x)$ and $z \in \Gamma \cap \{|q| < \frac{\delta}{2}\}$ we represent the first term in (2.11) in the form

$$-\mathrm{Re} \sum_{k=0}^{m} \frac{z^k}{q^{k+1}} \cos(n_q, x) - \mathrm{Re}\left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right) \cos(n_q, x).$$

It is clear that

$$\left| \mathrm{Re}\left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right) \cos(n_q, x) \right| \le c\, x^{m+1} u^{\mu-m-2}.$$

Now we consider the second term in (2.11). To this end we make use of the equality

$$\begin{aligned} \mathrm{Im}\,(q-z)^{-1} &= \mathrm{Im} \sum_{k=0}^{m} \frac{z^k}{q^{k+1}} + \mathrm{Im} \frac{1}{q^{m+1}} \mathrm{Re} \frac{z^{m+1}}{q-z} \\ &+ \mathrm{Re} \frac{1}{q^{m+1}} \mathrm{Re}\, z^{m+1} \mathrm{Im} \frac{1}{q-z} + \mathrm{Re} \frac{1}{q^{m+1}} \mathrm{Im}\, z^{m+1} \mathrm{Re} \frac{1}{q-z}. \end{aligned} \tag{2.12}$$

Since $\operatorname{Im} z^k \operatorname{Re} q^{-k-1} = O(x^\mu u^{-1})$ and $\operatorname{Re} z^k \operatorname{Im} q^{-k-1} = x^k \operatorname{Im} q^{-k-1} + O(x^\mu u^{-1})$, we obtain

$$\operatorname{Im} \sum_{k=0}^{m} \frac{z^k}{q^{k+1}} = \sum_{k=0}^{m} x^k \operatorname{Im} q^{-k-1} + O(x^\mu u^{-1}).$$

The second term in (2.12) does not exceed $cx^m u^{\mu-m-2}$. Since $|\operatorname{Im}(q-z)^{-1}| \le c(u^{\mu-1} + x^\mu u^{-1})$, we obtain for the third term in (2.12)

$$\left|\operatorname{Re} q^{-m-1} \operatorname{Re} z^{m+1} \operatorname{Im}(q-z)^{-1}\right| \le c\left(x^{m+1} u^{\mu-m-2} + x^\mu u^{-1}\right).$$

The last term in (2.12) satisfies

$$\left|\operatorname{Re} q^{-m-1} \operatorname{Im} z^{m+1} \operatorname{Re}(q-z)^{-1}\right| \le c\, x^\mu u^{-1}.$$

Thus, we have for $q \in \Gamma_+^r(x) \cup \Gamma_-^r(x)$

$$\frac{\partial}{\partial n_q} \log \frac{1}{|q-z|} = \sum_{k=0}^{m} x^k \left(-\operatorname{Re} \frac{1}{q^{k+1}} \cos(n_q, x) + \operatorname{Im} \frac{1}{q^{k+1}} \cos(n_q, y)\right) + I(q,z),$$

where $|I(q,z)| \le c\,(x^{m+1} u^{\mu-m-2} + x^\mu u^{-1})$. Therefore

$$\int_{\Gamma_+^r(x) \cup \Gamma_-^r(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = \sum_{k=0}^{m} c^{(k)}(\sigma) x^k + (R_1\sigma)(z),$$

where

$$c^{(k)}(\sigma) = \int_{\Gamma_+ \cup \Gamma_-} \sigma(q) \Big(-\operatorname{Re} \frac{1}{q^{k+1}} \cos(n_q, x) + \operatorname{Im} \frac{1}{q^{k+1}} \cos(n_q, y)\Big) ds_q$$

and $R_1\sigma$ admits the estimate

$$\|R_1\sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \le c\, \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}.$$

It follows from Hardy's inequality that $c^{(k)}(\sigma)$ $(k = 1, \ldots, m)$ are linear continuous functionals in $\mathcal{L}_{p,\beta-\mu}(\Gamma)$.

It is clear that I_6 is represented in the form

$$\sum_{k=0}^{m} c^{(k)} x^k + (R_2\sigma)(z),$$

where

$$\sum_{k=0}^{m} |c^{(k)}| + \|R_2\sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \le c\, \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}.$$

Finally,

$$(\pi\sigma - T\sigma)(z) = \pm\pi\big(\sigma(z_+) - \sigma(z_-)\big) + \sum_{k=0}^{m} c^{(k)} x^k + (R\sigma)(z) \qquad (z \in \Gamma_+),$$

where

$$\sum_{k=0}^{m} |c^{(k)}| + \|R\sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \le c\, \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)},$$

with

$$(R\sigma)(z) = \sum_{k=2}^{4} I_k(z) + (R_1\sigma)(z) + (R_2\sigma)(z).$$

Hence and by (2.5) we obtain the boundedness of $(\pi I - T) : \mathcal{L}^1_{p,\beta+1}(\Gamma) \longmapsto \mathfrak{M}_{p,\beta}(\Gamma)$ ∎

By changing the direction of the normal n we obtain the following assertion from Theorem 1.

Corollary. *Let Ω have an outward peak, and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$. Then the operator*

$$(\pi I + T) : \mathcal{L}^1_{p,\beta+1}(\Gamma) \ni \sigma \longmapsto (\pi I + T)\sigma \in \mathfrak{M}_{p,\beta}(\Gamma)$$

is continuous.

In passing, we have proved the following statements.

Proposition 1. *Let Ω have either inward or outward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$. Then the operators*

$$T : \mathcal{L}^1_{p,\beta+1}(\Gamma) \ni \sigma \longmapsto T\sigma \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$$

$$S : \mathcal{L}_{p,\beta+1}(\Gamma) \ni \sigma \longmapsto S\sigma \in \mathcal{L}_{p,\beta+1}(\Gamma)$$

are continuous.

3. Asymptotic representation of a conformal mapping

We shall make use of the following lemma.

Lemma 3. *A conformal mapping θ of $\mathbb{R}^2_+ = \{\zeta = \xi + i\eta : \eta > 0\}$ onto Ω, $\theta(0) = 0$, has the representation*

$$\theta(\xi) = \begin{cases} \sum_{k=2}^{[2\mu]+1} B^{(k)}\xi^k + B^{([2\mu]+2)}\xi^{[2\mu]+2}\log|\xi| + B^{(\pm)}|\xi|^{2\mu+2} \\ \quad +O(\xi^{2\mu+2+\gamma}) & \text{if } 2\mu \in \mathbb{N}, \\ \sum_{k=2}^{[2\mu]+2} B^{(k)}\xi^k + B^{(\pm)}|\xi|^{2\mu+2} + B^{([2\mu]+3)}\xi^{[2\mu]+3} \\ \quad +O(\xi^{2\mu+2+\gamma}) & \text{if } 2\mu \notin \mathbb{N}, \end{cases} \tag{3.1}$$

as $\xi \to \pm 0$, where $B^{(k)}$ $(k = 2, \dots, [2\mu] + 2)$ are real coefficients. Decomposition (3.1) *can be differentiated at least once.*

Proof. By D we denote the image of Ω under the mapping $u + iv = (x + iy)^{\frac{1}{2}}$. The boundary ∂D near the origin is the graph of the function $\kappa(u)$ such that

$$\kappa(u) = \pm\frac{1}{2}\alpha_\pm|u|^{2\mu+1}\big(1 + O\big(u^{2\min\{2\mu,1\}}\big)\big) \qquad (u \to \pm 0). \tag{3.2}$$

This asymptotic decomposition can be differentiated twice.

Let $\widetilde{\theta}$ denote a conformal mapping of $\mathbb{R}^2_+$ onto D, normalized by $\widetilde{\theta}(0) = 0$. According to Kellogg's conformal mapping theorem [4] we have

$$\operatorname{Re}\widetilde{\theta}(\xi) = \xi + \psi(\xi) \qquad (\xi \in \mathbb{R}), \tag{3.3}$$

where $\psi(0) = 0$, and ψ' satisfies the Hölder condition

$$|\psi'(\xi_1) - \psi'(\xi_2)| \le c\,|\xi_1 - \xi_2|^\gamma \qquad (0 < \gamma < \min\{2\mu, 1\}).$$

By substituting (3.3) to (3.2) we obtain

$$\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(\xi) = |\xi|^{2\mu}(\beta_\pm + \lambda(\xi)) \qquad (\xi \to \pm 0),$$

where λ satisfies the Hölder condition with a small exponent γ and $\lambda(0) = 0$.

The derivative of $\widetilde{\theta}(\xi)$ belongs to the Hardy space H^1 in the upper half-plane $\mathbb{R}^2_+$. Therefore

$$\frac{d}{d\xi}\operatorname{Re}\widetilde{\theta}(\xi) = \mathfrak{H}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)(\xi) + c,$$

where $\mathfrak{H}$ denotes the Hilbert transform. We represent $\mathfrak{H}(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta})(\xi)$ for $\xi > 0$ in the form

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(t)\frac{dt}{\xi - t}\\
&\quad = \frac{1}{\pi}\int_0^{\xi}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(\xi - t) - \frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(\xi + t)\Big)\frac{dt}{t} + \frac{1}{\pi}\int_0^{2\xi}\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(-t)\frac{dt}{\xi + t}\\
&\qquad + \frac{2}{\pi}\int_{2\xi}^{\infty}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)^{(-)}(t)\frac{t\,dt}{\xi^2 - t^2} + \frac{2}{\pi}\xi\int_{2\xi}^{\infty}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)^{(+)}(t)\frac{dt}{\xi^2 - t^2}
\end{aligned}$$

where

$$\begin{aligned}
\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)^{(-)}(\xi) &= \frac{1}{2}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(\xi) - \frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(-\xi)\Big),\\
\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)^{(+)}(\xi) &= \frac{1}{2}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(\xi) + \frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}(-\xi)\Big).
\end{aligned}$$

By this representation we obtain

$$\frac{d}{d\xi}\operatorname{Re}\widetilde{\theta}(\xi) = \begin{cases} \sum_{k=0}^{[2\mu]-1} a^{(k)}\xi^k + a^{([2\mu])}\xi^{[2\mu]}\log|\xi| + a^{(\pm)}|\xi|^{2\mu} + O(\xi^{2\mu+\gamma}) & \text{if } 2\mu \in \mathbb{N}\\ \sum_{k=0}^{[2\mu]} a^{(k)}\xi^k + a^{(\pm)}|\xi|^{2\mu} + a^{([2\mu]+1)}\xi^{[2\mu]+1} + O(\xi^{2\mu+\gamma}) & \text{if } 2\mu \notin \mathbb{N}\end{cases}$$

as $\xi \to \pm 0$. Hence,

$$\widetilde{\theta}(\xi) = \begin{cases} \sum_{k=1}^{[2\mu]} b^{(k)}\xi^k + b^{([2\mu]+1)}\xi^{[2\mu]+1}\log|\xi| + b^{(\pm)}|\xi|^{2\mu+1} + O(\xi^{2\mu+1+\gamma}) & \text{if } 2\mu \in \mathbb{N}\\ \sum_{k=1}^{[2\mu]+1} b^{(k)}\xi^k + b^{(\pm)}|\xi|^{2\mu+1} + b^{([2\mu]+2)}\xi^{[2\mu]+2} + O(\xi^{2\mu+1+\gamma}) & \text{if } 2\mu \notin \mathbb{N}.\end{cases}$$

Squaring the preceding representation we arrive at (3.1) ∎

The inverse mapping $\theta^{-1}(z)$ on $\Gamma_\pm$ has the form

$$\xi = \sum_{k=1}^{[2\mu]} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} \\ + (\pm 1)^{[2\mu]+1} \beta^{([2\mu]+1)} x^{\frac{([2\mu]+1)}{2}} \log\frac{1}{x} + \beta^{(\pm)} x^{\mu+\frac{1}{2}} + o(x^{\mu+\frac{1}{2}}) \tag{3.4}$$

if $2\mu \in \mathbb{N}$, and

$$\xi = \sum_{k=1}^{[2\mu]+1} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} + \beta^{(\pm)} x^{\mu+\frac{1}{2}} + o(x^{\mu+\frac{1}{2}}) \tag{3.5}$$

if $2\mu \notin \mathbb{N}$. Here $\beta^{(k)}$ $(k = 1, \dots, m+1)$ are real coefficients.

We notice that there exists a function of the form

$$d(\zeta) = \begin{cases} \zeta + \sum_{k=1}^{[2\mu]} a^{(k)}\zeta^k + a^{([2\mu]+1)}\zeta^{[2\mu]+1}\log\zeta & \text{if } 2\mu \in \mathbb{N} \\ \zeta + \sum_{k=1}^{[2\mu]+1} a^{(k)}\zeta^k & \text{if } 2\mu \notin \mathbb{N} \end{cases}$$

defined on $\mathbb{R}^2_+$ and satisfying

$$(\widetilde{\theta} \circ d)(\zeta) = \zeta + O(\zeta^{2\mu+1}).$$

It is clear that $\theta_0 = (\widetilde{\theta} \circ d)^2$ is the conformal mapping of a neighbourhood of $\zeta = 0$ in $\mathbb{R}^2_+$ onto a neighbourhood of peak in Ω and has the representation

$$x = \operatorname{Re}\theta_0(\xi) = \xi^2 + O(\xi^{2\mu+2}) \qquad \text{as } \xi \to \pm 0. \tag{3.6}$$

The inverse mapping θ_0^{-1} has the form

$$\xi = \operatorname{Re}\theta_0^{-1}(z) = \pm x^{\frac{1}{2}} + O(x^{\mu+\frac{1}{2}}) \qquad \text{on } \Gamma_\pm.$$

By diminishing δ in the definition of $\Gamma_\pm$ we can assume that θ_0 is defined on $\Gamma_+ \cup \Gamma_-$.

4. Auxiliary boundary value problems for a domain with outward peak

Let n_0 be the integer subject to the inequalities $n_0 - 1 \le 2(\mu - \beta - \frac{1}{p}) < n_0$. Then $m = [\mu - \beta - \frac{1}{p} + \frac{1}{2}]$ is the largest integer satisfying $2m \le n_0$.

We shall make use of the following proposition proved in [4].

Proposition 2. *Let Ω have an outward peak and let φ belong to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$, where $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$. Then there exists a harmonic extension h onto Ω of φ with normal derivative in $\mathcal{L}_{p,\beta+1}(\Gamma)$ satisfying*

$$\left\| \frac{\partial}{\partial n} h \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \le c\, \|\varphi\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)}.$$

Now we prove the following existence result.

Proposition 3. *Let Ω have an inward peak and let $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$, where $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ and $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Then there exists a harmonic extension of φ onto Ω^c with normal derivative in the space $\mathcal{L}_{p,\beta+1}(\Gamma)$ such that the conjugate function g, $g(z_0) = 0$ with a fixed point $z_0 \in \Gamma \setminus \{O\}$, has the representation*

$$\sum_{k=1}^{m} c_k(\varphi) \mathrm{Re}\, z^{k-\frac{1}{2}} + g^{\#}(z),$$

where $c_k(\varphi)$ are linear continuous functionals in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $g^{\#}$ satisfies

$$\|g^{\#}\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} \le c \, \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)},$$

with c independent of φ.

Proof. (i) We start with the case when $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ vanishes outside $\Gamma_+ \cup \Gamma_-$. We extend the function $\Phi(\tau) = (\varphi \circ \theta_0)(\tau)$ by zero outside a small neighbourhood of O.

We first prove the estimate

$$\begin{aligned} c_1 \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} &\le \|\Phi^{(+)}\|_{L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})} + \|\Phi'\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \\ &\le c_2 \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}, \end{aligned} \tag{4.1}$$

where $\Phi^{(+)}(\xi) = \frac{\Phi(\xi)+\Phi(-\xi)}{2}$. Let r be a measurable function on $(0,\infty)$ subject to $|r(\xi)| \le \xi^{2\mu+1}$. We choose $\ell \in [0,1]$ such that $\frac{\ell}{2} < \beta + \frac{1}{p} < \frac{\ell+1}{2}$. Then, from the boundedness of the Hardy-Littlewood maximal operator in $\mathcal{L}_{p,2\beta-\ell+\frac{1}{p}}(\mathbb{R})$ (see [9]), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} |\Phi(\xi) - \Phi(\xi + r(\xi))|^p |\xi|^{2\beta p - 2\mu p + 1} d\xi \\ &\le c \int_{\mathbb{R}} \left(\frac{1}{|\xi|^{2\mu+1}} \int_{\xi - c\xi^{2\mu+1}}^{\xi + c\xi^{2\mu+1}} \left| \tau^{1+\ell} \frac{d}{d\tau} \Phi(\tau) \right| d\tau \right)^p |\xi|^{2\beta p - \ell p + 1} d\xi \\ &\le c \int_{\mathbb{R}} \left| \frac{d}{d\xi} \Phi(\xi) \right|^p |\xi|^{2\beta p + p + 1} d\xi \,. \end{aligned} \tag{4.2}$$

For $z \in \Gamma_+$ we have $|\theta_0^{-1}(z) + \theta_0^{-1}(z_-)| \le c\,\xi^{2\mu+1}$. Hence and by (4.2) the left inequality in (4.1) follows.

Let h be a measurable function on $[0,\delta]$ such that $|h(x)| \le x^{\mu+1}$. As in (4.2) we have

$$\begin{aligned} &\int_0^\delta |\varphi(x) - \varphi(x + h(x))|^p x^{(\beta-\mu)p} dx \\ &\le c \int_0^\delta \left(\frac{1}{x^{\mu+1}} \int_{x - cx^{\mu+1}}^{x + cx^{\mu+1}} t \left| \frac{d}{dt} \varphi(t) \right| dt \right)^p x^\beta dx \\ &\le c \int_0^\delta \left| \frac{d}{dt} \varphi(t) \right|^p x^{(\beta+1)p} dx \,. \end{aligned} \tag{4.3}$$

By using (3.6) we obtain that for ξ in a small neighbourhood of the origin the distance between $\theta_0(\xi)$ and $\theta_0(-\xi)$ does not exceed $c\,x^{\mu+1}$. Hence and by (4.3) the right-hand inequality in (4.1) follows.

We introduce a function $\mathcal{H}$ by

$$\mathcal{H}(\zeta) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\Phi}{d\tau}(\tau)\,\mathrm{Re}\,\log\frac{\zeta-\tau}{\zeta}\,d\tau \qquad (\zeta = \xi + i\eta \in \mathbb{R}^2_+).$$

From the norm inequality for the Hilbert transform of even functions in $\mathcal{L}_{p,2\beta-1+\frac{1}{p}}(\mathbb{R})$ (see [1]) it follows that the function

$$\frac{\partial}{\partial\xi}\mathcal{H}^{(+)}(\xi) = \frac{1}{\pi\xi^2}\int_{\mathbb{R}} \frac{d}{d\tau}\Phi^{(-)}(\tau)\frac{\tau^2\,d\tau}{\xi-\tau}$$

satisfies

$$\left\|\frac{\partial}{\partial\xi}\mathcal{H}^{(+)}\right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \le c\left\|\frac{d}{d\xi}\Phi^{(-)}\right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})}, \tag{4.4}$$

where $\Phi^{(-)}(\xi) = \frac{\Phi(\xi)-\Phi(-\xi)}{2}$.

We represent $\frac{\partial}{\partial\xi}\mathcal{H}^{(-)}$ in the form

$$-\frac{1}{\pi\xi}\int_{\mathbb{R}} \Phi^{(-)}(\tau)\frac{d\tau}{\xi-\tau} + \frac{1}{\pi\xi}\int_{\mathbb{R}} \frac{d}{d\tau}\big(\tau\Phi^{(-)}(\tau)\big)\frac{d\tau}{\xi-\tau} = (T_1\Phi^{(-)})(\xi) + (T_2\Phi^{(-)})(\xi).$$

From the norm inequality for the Hilbert transform of odd functions in $\mathcal{L}_{p,2\beta+\frac{1}{p}}(\mathbb{R})$ (see [1]) it follows that

$$\|T_1\Phi^{(-)}\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \le c\|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}. \tag{4.5}$$

Let χ be a C^∞-function vanishing outside a neighbourhood of $\xi = 0$ and subject to $\int_{\mathbb{R}}\chi(u)du = 1$. The Fourier transform of $\xi(T_2\Phi^{(-)})(\xi)$ with respect to ξ is given by

$$\pi i\left[|\tau|\widehat{\chi}(\tau)\widehat{\Psi}(\tau) + \mathrm{sign}(\tau)(1-\widehat{\chi}(\tau)\big(\tau\widehat{\Psi}(\tau)\big)\right] = \widehat{S_1\Psi}(\tau) + \widehat{S_2\Psi}(\tau),$$

where $\widehat{\Psi}(\tau)$ is the Fourier transform of $\xi\big(\Phi^{(-)}\big)(\xi)$. Since $|\tau|\widehat{\chi}(\tau)$ is the Fourier transform of a smooth function admitting the estimate $O(\xi^{-2})$ as $\xi \to \pm\infty$, it follows by Lemma 1 that

$$\|S_1\Psi\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \le c\|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}.$$

Taking into account that $\pi i\,\mathrm{sign}(\tau)(1-\widehat{\chi}(\tau))$ is the Fourier transform of the function

$$\frac{1}{x} - \int_{\mathbb{R}} \frac{\chi(u)}{\xi-u}du = -\int_{\mathbb{R}} \frac{u\,\chi(u)}{\xi\,(\xi-u)}du$$

which admits the estimate $O(\xi^{-2})$ as $\xi \to \pm\infty$, we obtain by Lemma 1

$$\|S_2\Psi\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \le c\left(\left\|\frac{d}{d\xi}\Phi^{(-)}\right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}\right).$$

Thus, $T_2\Phi^{(-)}$ satisfies

$$\|T_2\Phi^{(-)}\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \le c\Big(\Big\|\frac{d}{d\xi}\Phi^{(-)}\Big\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}\Big).$$

This along (4.5) imply

$$\Big\|\frac{\partial}{\partial\xi}\mathcal{H}^{(-)}\Big\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \le c\Big(\Big\|\frac{d}{d\xi}\Phi^{(-)}\Big\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}\Big). \tag{4.6}$$

We represent the odd function $\mathcal{H}^{(-)}$ on $\mathbb{R}$ in the form

$$\begin{aligned}\mathcal{H}^{(-)}(\xi) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{d}{d\tau}\Phi^{(+)}(\tau)\,\log\Big|\frac{\xi-\tau}{\xi}\Big|d\tau \\ &= \frac{\xi^{n_0}}{\pi}\int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi-\tau)}d\tau - \sum_{k=0}^{n_0-1}\frac{\xi^k}{\pi}\int_{\mathbb{R}}\frac{\Phi^{(+)}(\tau)}{\tau^{k+1}}d\tau\,.\end{aligned}$$

Since $\Phi^{(+)} \in L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})$ and since $0 < 2\beta - 2\mu + n_0 + \frac{2}{p} < 1$ for even n_0 and $0 < 2\beta - 2\mu + n_0 + \frac{2}{p} < 2$ for odd n_0, it follows from the boundedness of the Hilbert transform in weighted L_p-spaces (see [1]) that the norm in $L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})$ of

$$\frac{\xi^{n_0}}{\pi}\int_{\mathbb{R}}\frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi-\tau)}d\tau$$

does not exceed $c\|\Phi^{(+)}\|_{L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})}$.

Hence by (4.4), (4.6) we obtain that the function $h(z) = \mathcal{H}\circ\theta_0^{-1}(z)$ is represented in the form

$$\sum_{k=1}^{m} a_k(\varphi)\mathrm{Re} z^{k-\frac{1}{2}} + h^{\#}(z) \tag{4.7}$$

for $z \in \Omega$ situated in a small neighbourhood of the peak. Here

$$a_k(\varphi) = \int_{\mathbb{R}} \Phi^{(+)}(\tau)\tau^{-2k}d\tau \qquad (1 \le k \le m)$$

are linear continuous functionals in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$, and $h^{\#}$ belongs to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$ and satisfies

$$\|h^{\#}\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma_+\cup\Gamma_-)} \le c\,\|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}.$$

It is clear that $\frac{\partial}{\partial\eta}\mathcal{H} = \frac{\partial}{\partial\tau}\Phi \in L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})$. Therefore $\frac{\partial}{\partial n}h$ belongs to $L_{p,\beta+1}(\Gamma_+\cup\Gamma_-)$.

Now let $\kappa \in C^\infty(\mathbb{R}^2)$ be equal to 1 for $|z| < \delta$ and vanish for $|z| > \delta$. We extend κh by zero outside a small neighbourhood of O and set

$$\begin{aligned}\psi_1(z) &= -\Delta(\kappa h)(z) && (z \in \Omega^c)\\ \varphi_1(z) &= \frac{\partial}{\partial s}\varphi(z) - \frac{\partial}{\partial n}(\kappa h)(z) && (z \in \Gamma).\end{aligned}$$

We consider the boundary value problem

$$\left.\begin{aligned} &\Delta\mathcal{F}(\zeta) = \mathcal{Q}(\zeta) && (\zeta \in \mathbb{R}^2_+) \\ &\frac{\partial}{\partial n}\mathcal{F}(\xi + i0) = \mathcal{T}(\xi) && (\xi \in \mathbb{R}) \end{aligned}\right\} \tag{4.8}$$

where $\mathcal{Q}(\zeta) = (\psi_1 \circ \theta)(\zeta)\,|\theta'(\zeta)|^2$ and $\mathcal{T}(\xi) = (\varphi_1 \circ \theta)(\xi + i0)\,|\theta'(\xi + i0)|$. By using the estimates

$$|\mathrm{grad}\,\mathcal{H}(\zeta)| \le \frac{c}{|\zeta|^2}\left|\int_{\mathbb{R}} \frac{\tau^2 (d\Phi/d\tau)(\tau)}{\zeta - \tau}d\tau\right| \quad \text{and} \quad |\mathcal{H}(\zeta)| \le \frac{c}{|\zeta|}\left|\int_{\mathbb{R}} \frac{\tau\,\Phi(\tau)}{\zeta - \tau}d\tau\right|$$

and theorems on the boundedness of the Hardy-Littlewood maximal operator and Hilbert transform in weighted L_p-spaces [3, 9] we obtain

$$\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)} \le c\left(\left\|\frac{d}{d\tau}\Phi\right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}\right).$$

A solution of problem (4.8) is given by

$$\mathcal{F}(\zeta) = \int_{\mathbb{R}} \mathcal{T}(u)\mathfrak{G}(u,\zeta)du - \int_{\mathbb{R}^2_+} \mathcal{Q}(w)\mathfrak{G}(w,\zeta)dudv \qquad (w = u + iv)$$

with Green's function

$$\mathfrak{G}(w,\zeta) = \frac{1}{2\pi}\log\left|\left(1 - \frac{w}{\zeta}\right)\left(1 - \frac{w}{\overline{\zeta}}\right)\right|.$$

We rewrite $\mathcal{F}$ on $\mathbb{R}$ in the form

$$\begin{aligned} \mathcal{F}(\xi) &= t_{-1}(\varphi)\log|\xi| + t_0(\varphi) \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}} \mathcal{T}(u)\log\left|1 - \frac{\xi}{u}\right|du - \frac{1}{\pi}\int_{\mathbb{R}^2_+} \mathcal{Q}(w)\log\left|1 - \frac{\xi}{w}\right|dudv \end{aligned} \tag{4.9}$$

where

$$t_{-1}(\varphi) = -\frac{1}{\pi}\int_{\mathbb{R}} \mathcal{T}(u)\,du + \frac{1}{\pi}\int_{\mathbb{R}^2_+} \mathcal{Q}(w)\,dudv$$

and

$$t_0(\varphi) = \frac{1}{\pi}\int_{\mathbb{R}} \mathcal{T}(u)\log|u|du - \frac{1}{\pi}\int_{\mathbb{R}^2_+} \mathcal{Q}(w)\log|w|dudv.$$

Hence we obtain

$$\frac{\partial\mathcal{F}}{\partial\xi}(\xi) - \frac{t_{-1}(\varphi)}{\xi} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathcal{T}(u)}{\xi - u}du - \frac{1}{\pi}\int_{\mathbb{R}^2_+} \frac{(\xi - u)\mathcal{Q}(w)}{|\xi - w|^2}dudv.$$

By the boundedness of the Hilbert transform in $\mathcal{L}_p(\mathbb{R})$ and the Minkovski inequality we prove that

$$\frac{\partial}{\partial \xi}\mathcal{F}(\xi) - t_{-1}(\varphi)\,\frac{1}{\xi} \in L_p(\mathbb{R}) \tag{4.10}$$

and that the L_p-norm of this function does not exceed

$$\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}.$$

It is clear that in a neighbourhood of infinity

$$\frac{\partial}{\partial \xi}\mathcal{F}(\xi) = R_\infty(\xi)\frac{1}{\xi^2} \tag{4.11}$$

where $|R_\infty(\xi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right)$ for large $|\xi|$. Set $f = \mathcal{F}\circ\theta$. From (4.10) and (4.11) it follows that $\frac{\partial}{\partial s}f$ belongs to $\mathcal{L}_{p,\beta+1}(\Gamma)$ and satisfies

$$\left\|\frac{\partial}{\partial s}f\right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \le c\,\|\varphi\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)}. \tag{4.12}$$

By Taylor's decomposition of the integral terms in (4.9) we obtain

$$\mathcal{F}(\xi) = t_{-1}(\varphi)\log|\xi| + t_0(\varphi) + \sum_{k=1}^{n_0-1} t_k(\varphi)\xi^k + |\xi|^{n_0}R_{n_0}(\xi)\,, \tag{4.13}$$

where $|t_k(\varphi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right)$ for $k = -1,\dots,n_0-1$, and $|R_{n_0}(\xi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right)$ for small $|\xi|$. Taking into account the asymptotic representations (3.4), (3.5) of θ^{-1} and the inequality $2(\mu-\beta-\frac{1}{p}) < n_0$, it follows from (4.12) and (4.13) that f is represented in the form

$$f(z) = \sum_{k=1}^{m} b_k(\varphi)\operatorname{Re} z^{k-\frac{1}{2}} + f^{\#}(z) \qquad (z\in\Omega) \tag{4.14}$$

where $f^{\#} \in \mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$, and $b_k(\varphi)$ $(k=1,\dots,m)$ are linear combinations of the coefficients $t_\ell(\varphi)$ $(\ell = 1,\dots,n_0-1)$ in (4.13).

According to (4.7) and (4.14) the function $g = \kappa h + f$ is harmonic in Ω and can be written as

$$g(z) = \sum_{k=1}^{m} c_k(\varphi)\operatorname{Re} z^{k-\frac{1}{2}} + g^{\#}(z) \qquad (z\in\Omega^c)$$

with $c_k(\varphi) = a_k(\varphi) + b_k(\varphi)$. Moreover,

$$\sum_{k=1}^{m}|c^{(k)}| + \|g^{\#}\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} \le c\,\|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}$$

and by the definition of g it follows that $(g\circ\theta)(\infty)=0$. Because $\frac{\partial}{\partial n}g=\frac{\partial}{\partial n}\varphi$ on $\Gamma\setminus\{O\}$, it is clear that one of the functions conjugate to g is the harmonic extension of φ onto Ω with normal derivative in $\mathcal{L}_{p,\beta+1}(\Gamma)$.

(ii) Now let φ belong to $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and let φ vanish on $\Gamma\cap\{|q|<\frac{\delta}{2}\}$. We introduce the function $\Phi(\xi)=(\varphi\circ\theta)(\frac{1}{\xi})$ $(\xi\in\mathbb{R})$ which equals zero outside a certain interval. Set

$$\mathcal{G}(\zeta)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\Phi(\tau)\mathrm{Re}\,\frac{\zeta}{\tau(\zeta-\tau)}d\tau \qquad (\zeta\in\mathbb{R}^2_+). \tag{4.15}$$

It is clear that one of conjugate functions $\widetilde{\mathcal{G}}$ is a harmonic extension of Φ onto $\mathbb{R}^2_+$. It follows from the boundedness of Hilbert transform in L_p-spaces that $g=\mathcal{G}\circ\theta^{-1}$ belongs to $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ and satisfies

$$\| g \|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)}\le c\ \| \varphi \|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}\ . \tag{4.16}$$

Further, we represent $\mathcal{G}$ on $\mathbb{R}$ in the form

$$\begin{aligned}\mathcal{G}(\xi)&=\frac{1}{\pi}P.V.\int_{\mathbb{R}}\Phi(\tau)\tau^{-1}d\tau\\&+\sum_{k=1}^{n_0-1}\frac{1}{\pi\xi^k}\int_{\mathbb{R}}\Phi(\tau)\tau^{k-1}d\tau+\frac{1}{\pi\xi^{n_0-1}}\int_{\mathbb{R}}\frac{\Phi(\tau)\,\tau^{n_0-1}}{\xi-\tau}d\tau\\&=\sum_{k=0}^{n_0-1}\frac{t_k(\varphi)}{\xi^k}+\mathcal{G}^{\#}(\xi)\end{aligned}$$

where

$$\begin{aligned}t_k(\varphi)&=\frac{1}{\pi}\int_{\mathbb{R}}\Phi(\tau)\tau^{k-1}d\tau\\ \mathcal{G}^{\#}(\xi)&=\frac{1}{\pi\xi^{n_0-1}}\int_{\mathbb{R}}\frac{\Phi(\tau)\,\tau^{n_0-1}}{\xi-\tau}d\tau \quad (\xi\in\mathbb{R}).\end{aligned} \tag{4.17}$$

Since $-\frac{1}{p}<2\mu-2\beta-\frac{3}{p}-n_0+1<1-\frac{1}{p}$, we have

$$\sum_{k=0}^{n_0-1}|t_k(\varphi)|+\left\|\mathcal{G}^{\#}\right\|_{L_{p,2\mu-2\beta-\frac{3}{p}}(\mathbb{R})}\le c\left\|\Phi\right\|_{L_p(\mathbb{R})}.$$

Hence and from (4.16) it follows that g is represented in the form

$$g(z)=\sum_{k=1}^{m}c_k(\varphi)\mathrm{Re}\,z^{k-\frac{1}{2}}+g^{\#}(z) \qquad (z\in\Omega)$$

where $g^{\#}\in\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$, $c_k(\varphi)$ $(k=1,\dots,m)$ are linear combinations of coefficients $t_\ell(\varphi)$ $(\ell=1,\dots,n_0-1)$ in (4.17). These coefficients and the function $g^{\#}$ satisfy

$$\sum_{k=1}^{m}|c_k(\varphi)|+\|g^{\#}\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)}\le c\,\|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}.$$

According to (4.15) we have $(g\circ\theta)(\infty)=0$, and the conjugate function $\widetilde{g}=\widetilde{\mathcal{G}}(\frac{1}{\theta-1})$ is the harmonic extension of φ onto Ω with normal derivative in $\mathcal{L}_{p,\beta+1}(\Gamma)$ ∎

5. Boundary integral equation of the Dirichlet problem

Here we prove the unique solvability of equation (1.3) on the contour Γ with inward peak.

Theorem 2. *Let Ω have an inward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$, $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Then the operator*

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma - T\sigma + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k \in \mathfrak{M}_{p,\beta}(\Gamma), \tag{5.1}$$

where $\mathcal{I}_k(z) = \operatorname{Im} z^{k-\frac{1}{2}}$, is surjective.

Proof. (i) Let $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $\varphi = 0$ in a neighbourhood of the peak. We consider the harmonic extension h^i of φ onto Ω and its conjugate function g^i, normalized by the condition $g(z_0) = 0 \quad (z_0 \in \Gamma \setminus \{O\})$ which were introduced in Proposition 3. By Proposition 3, $\frac{\partial}{\partial s} g^i \in \mathcal{L}_{p,\beta+1}(\Gamma)$ and there exist real numbers $c^{(k)}(\varphi) \quad (k = 1, \ldots, m)$ such that the function

$$g^i_0(z) = g^i(z) - \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{R}_k(z),$$

where $\mathcal{R}_k(z) = \operatorname{Re} z^{k-\frac{1}{2}}$, belongs to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. The coefficients $c^{(k)}(\varphi) \quad (k = 1, \ldots, m)$ and the function $g^i_0(z)$ satisfy

$$\sum_{k=1}^{m} |c^{(k)}(\varphi)| + \| g^i_0 \|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} \le c \, \| \varphi \|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} \,. \tag{5.2}$$

The function

$$h^i_0(z) = h^i(z) + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_k(z) \qquad (z \in \Omega)$$

is the harmonic extension of

$$\varphi + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_k \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma) \,.$$

Let h^e_0 be the harmonic extension of h^i_0 onto Ω^c subject to $\operatorname{grad} h^e_0(z) = O(|z|^{-\frac{1}{2}})$. Hence and by the estimate $h^i_0(z) = O(|z|^{-\frac{1}{2}}) \quad (z \in \Omega)$ we obtain

$$h^i_0(z) = \frac{1}{2\pi} \int_\Gamma \Big(\frac{\partial h^i_0}{\partial n} - \frac{\partial h^e_0}{\partial n} \Big) \log \frac{|z|}{|z-q|} ds_q + h^e_0(\infty) \qquad (z \in \Gamma \setminus \{O\}). \tag{5.3}$$

According to Proposition 2 the Dirichlet problem in Ω^c with the boundary data g^i_0 has a solution f^e such that $\frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ and satisfies

$$\Big\| \frac{\partial}{\partial n} f^e \Big\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \le c \| g^i_0 \|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} . \tag{5.4}$$

Let g^e denote the harmonic function conjugate to f^e and vanishing at infinity. We have $\frac{\partial}{\partial s} g^e = \frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$. Since

$$\frac{\partial}{\partial n} g^e = -\frac{\partial}{\partial s} f^e = -\frac{\partial}{\partial s} g_0^i = -\frac{\partial}{\partial n} h_0^i \qquad \text{on } \Gamma \setminus \{O\},$$

it follows that g^e belongs to $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ and satisfies the Neumann problem in Ω^c with boundary data $-\frac{\partial}{\partial n} h_0^i$. By the integral representation of a harmonic function in Ω^c and by (5.4) we obtain

$$\|g^e\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} \le c\|g_0^i\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)}\,. \tag{5.5}$$

Since $\operatorname{grad} g^e = O(|z|^{-\mu-\frac{1}{2}})$ $(z \in \Omega^c)$, then the function $w = -g^e - h_0^e + h_0^e(\infty)$ admits the representation

$$w(z) = \frac{1}{2\pi}\int_\Gamma \left(w(q)\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|} - \frac{\partial w}{\partial n_q}(q)\log\frac{|z|}{|z-q|}\right) ds_q \qquad (z \in \Omega^c).$$

From the limit relation for the double layer potential and from (5.3) it follows that

$$w - \pi^{-1} T w = -2\big(h_0^i - h_0^e(\infty)\big) \qquad \text{in } \Gamma \setminus \{O\}.$$

Since $T1 = -\pi$, we obtain that the function

$$\sigma = -(2\pi)^{-1}\left(g^e + \varphi + \sum_{k=1}^m c^{(k)}(\varphi)\mathcal{I}_k \right) \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$$

satisfies

$$\pi\sigma(z) - T\sigma(z) = -\varphi(z) - \sum_{k=1}^m c^{(k)}(\varphi)\mathcal{I}_k(z) \qquad (z \in \Gamma \setminus \{O\}).$$

We set $t^{(k)} = c^{(k)}(\varphi)$ $(k = 1, \ldots, m)$. From (5.2) and (5.5) it follows that

$$\sum_{k=1}^m |t^{(k)}| + \|\sigma\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} \le c\|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}\,. \tag{5.6}$$

(ii) Now let φ be an arbitrary function in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. There exists a sequence $\{\varphi_r\}_{r\ge1}$ of smooth functions on $\Gamma \setminus \{O\}$ vanishing near the peak, which tends to $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. Let $(\sigma_r, t_r) \in \mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ be the solution of equation (1.3) with the right-hand side $-\varphi_r$ which is constructed as in (i).

According to (5.6) the sequence $\{(\sigma_r, t_r)\}_{r\ge1}$ converges in $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ to a limit (σ, t). Since the operator $T : \mathcal{L}^1_{p,\beta+1}(\Gamma) \mapsto \mathcal{L}^1_{p,\beta+1}(\Gamma)$ is continuous (see Proposition 1), it follows, by taking the limit, that

$$\pi\sigma - T\sigma + \sum_{k=1}^m t^{(k)}\mathcal{I}_k = -\varphi\,. \tag{5.7}$$

Consequently, equation (1.3) is solvable in $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ for every $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$.

(iii) We turn to the case $\varphi(z) = \operatorname{Re} z^k \quad (z \in \Gamma \setminus \{O\})$. As the harmonic extension of φ onto Ω we take the function $h^i(z) = \operatorname{Re} z^k$. It is clear that the conjugate harmonic function $g^i(z) = \operatorname{Im} z^k$ belongs to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. According to Proposition 2, g^i admits the harmonic extension f^e onto Ω^c such that $\frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$. Let g^e be the harmonic function conjugate to f^e and vanishing at infinity. We have

$$\frac{\partial g^e}{\partial n} = -\frac{\partial h^i}{\partial n} \quad \text{on } \Gamma \setminus \{O\} \qquad \text{and} \qquad g^e \in \mathcal{L}^1_{p,\beta+1}(\Gamma).$$

Arguing as in (i) we prove that the pair $(\sigma, 0)$, where $\sigma(z) = -(2\pi)^{-1}\big(g^e(z) + \operatorname{Re} z^k\big)$, belongs to $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ and satisfies (1.3). This and (5.7) imply

$$\mathfrak{M}_{p,\beta}(\Gamma) \subset \left(\pi I - T + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k\right) \left(\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m\right).$$

The converse inclusion was proved in Theorem 1 ∎

Theorem 3. *Let Ω have an inward peak. Then operator* (5.1) *is injective for* $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$.

Proof. Let $(\sigma, t) \in \mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ be an element of $\operatorname{Ker}\big(\pi I - T + \sum_{k=0}^{m} t^{(k)} \mathcal{I}_k\big)$. Then the harmonic function

$$(W\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k(z) \qquad (z \in \Omega)$$

vanishes on the contour $\Gamma \setminus \{O\}$. By $\widetilde{(W\sigma)}$ we denote an arbitrary function conjugate to $W\sigma$ in Ω. We introduce the holomorphic function

$$W(z) = -\widetilde{(W\sigma)}(z) + i(W\sigma)(z) + \sum_{k=1}^{m} t^{(k)} z^{k-\frac{1}{2}} \qquad (z \in \Omega).$$

Let $\zeta = \gamma(z)$ be a conformal mapping of Ω onto $\mathbb{R}^2_+$, $\gamma(0) = 0$. The function $F(z) = (W \circ \gamma^{-1})(\frac{1}{\zeta})$ is holomorphic in the lower half-plane $\mathbb{R}^2_- = \{\zeta = \xi + i\eta : \eta <\}$, continuous up to the boundary and $\operatorname{Im} F = 0$ on the real axis. We notice that the function $W\sigma$ admits the estimate

$$|(W\sigma)(z)| \le c\,|z|^{-N}$$

for an integer N. Therefore the holomorphic extension F^{ext} of F onto $\mathbb{C}$ is an entire function with real part satisfying

$$|\operatorname{Re} F^{ext}(\zeta)| \le c\,|\zeta|^{2N}.$$

From the Schwarz integral formula it follows that F^{ext} has the same order of growth at infinity as $\mathrm{Re}F^{ext}$. In particular, there exists a polynomial P with real coefficients such that

$$-(\widetilde{W\sigma})(z)+\sum_{k=1}^{m} t^{(k)}\mathcal{R}_k(z)=\mathrm{Re}\,P\Big(\frac{1}{\gamma(z)}\Big) \qquad (z\in\Omega).$$

Since

$$(\widetilde{W\sigma})(z)=\int_\Gamma \frac{d\sigma}{ds}(q)\log\frac{|z|}{|z-q|}ds_q=\Big(V\frac{d\sigma}{ds}\Big)(z) \qquad (z\in\Omega)$$

the equality

$$\Big(V\frac{d\sigma}{ds}\Big)(z)=\mathrm{Re}\,P\Big(\frac{1}{\gamma(z)}\Big)-\sum_{k=1}^{m} t^{(k)}\mathcal{R}_k(z) \qquad (z\in\Gamma\setminus\{O\}) \tag{5.8}$$

holds. Taking into account that $V(\frac{d}{ds}\sigma)\in\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$ (see [4: Theorem 1]) we obtain that the right-hand side in (5.8) belongs to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. However, since $m-\frac{1}{2}+\beta-\mu<-\frac{1}{p}$, the functions $\mathcal{R}_k$ $(k=1,\dots,m)$ and positive integer powers of $\frac{1}{\gamma}$ do not belong to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. Hence, it follows that the coefficients $t^{(k)}$ $(k=1,\dots,m)$ are equal to zero and the polynomial P is constant. Therefore $\widetilde{W\sigma}$ and $W\sigma$ are constant in Ω.

By $\widetilde{W_-\sigma}$ we denote the harmonic function conjugate to $(W\sigma)(z)$ $(z\in\Omega^c)$ which equals $\widetilde{W\sigma}$ on $\Gamma\setminus\{O\}$. Since $\widetilde{W_-\sigma}$ admits the estimate

$$|(\widetilde{W_-\sigma})(z)|\le c\,|z|^{-N} \qquad \text{as } z\to 0$$

for an integer N and is constant on $\Gamma\setminus\{O\}$, it follows that $\widetilde{W_-\sigma}=$ const in Ω^c. Therefore $W\sigma$ is constant in Ω^c.

From the jump formula for $W\sigma$ we obtain that $\sigma=$ const on $\Gamma\setminus\{O\}$. Since the non-zero constant does not satisfy the homogeneous equation (1.3), it follows that σ equals zero ∎

Proposition 4. *Let Ω have an inward peak, and let $0<\beta+\frac{1}{p}<\min\{\mu,1\}$, $\mu-\beta-\frac{1}{p}+\frac{1}{2}\in\mathbb{N}$. Then operator* (5.1) *is not Fredholm.*

Proof. Let

$$\Phi(\xi)=|\xi|^{n_0-1}(-\log|\xi|)^{-\gamma}$$

in a small neighbourhood of the origin and let $\mathrm{supp}\,\Phi$ be in the domain of the mapping θ_0 introduced in Section 3. Let γ be such that $\frac{1}{p}<\gamma<1$. By φ we denote the function $\Phi\circ\theta_0^{-1}\in\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and introduce the harmonic extension h^i of φ onto Ω constructed in Proposition 3. Let g^i be the harmonic function conjugate to h^i from Proposition 3. We have

$$g^i(z)=\sum_{k=1}^{m} c^{(k)}\mathcal{R}_k(z)+g^{\#}(z).$$

Here

$$g^{\#}(z)=c\,\mathrm{Re}\big(z^{\frac{n_0}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\big)+g_0^{\#}(z),$$

where $g_0^\# \in \mathfrak{N}_{p,\beta}(\Gamma)$. The harmonic extension f^e of $g^\#$ to Ω^c described in Proposition 2 has the form

$$f^e(z) = c_1 \operatorname{Im}\Big[\Big(\frac{zz_0}{z_0 - z}\Big)^{-\mu+\frac{n_\Omega}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\Big] + c_2 \operatorname{Re}\Big[\Big(\frac{zz_0}{z_0 - z}\Big)^{\frac{n_\Omega}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\Big] + f_0^e(z),$$

where $c_1, c_2 \in \mathbb{R}$, z_0 is a fixed point of Ω, and $\frac{\partial}{\partial n} f_0^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$.

The function g^e, $g^e(\infty) = 0$, conjugate to f^e, has the representation

$$g^e(z) \sim c\, x^{-\beta-\frac{1}{p}}(\log x)^{-\gamma+1}.$$

It is clear that $g^e \notin \mathcal{L}^1_{p,\beta+1}(\Gamma)$ and $g^e \in \mathcal{L}^1_{p,\beta'+1}(\Gamma)$ for $\beta' > \beta$. By Theorem 2 the pair (σ, t), where $t = (c^{(1)}, \dots, c^{(m)})$ and

$$\sigma = -(2\pi)^{-1}\Big(g^e + \varphi + \sum_{k=1}^{m} c^{(k)} \mathcal{I}_k\Big) \qquad \text{on } \Gamma \setminus \{O\},$$

is the solution of equation (1.3) in $\mathcal{L}^1_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$ for $\beta' > \beta$. From Theorem 3 it follows that equation (1.3) is not solvable in $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$.

According to Theorem 2 equation (1.3) with a right-hand side in $\mathfrak{N}^{(+)}_{p,\beta'}(\Gamma)$, $\beta' < \beta$, is solvable in $\mathcal{L}^1_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$. Since the set of smooth functions vanishing near the peak is dense in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $\mathcal{L}^1_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$ is embedded to $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$, it follows that the range of operator (5.1) is not closed in $\mathfrak{M}_{p,\beta}(\Gamma)$ ∎

6. Boundary integral equation of the Neumann problem

In this section we prove the unique solvability of equation (1.4) on the contour Γ with inward peak.

Theorem 4. *Let Ω have an inward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ and $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Then the operator*

$$\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n}\mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma) \qquad (6.1)$$

with $\mathcal{R}_k(z) = \operatorname{Re} z^{k-\frac{1}{2}}$ is surjective.

Proof. (i) Let $\psi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $\psi = 0$ in a neighbourhood of the peak. By h^i we denote the harmonic extension of ψ onto Ω which is introduced in Proposition 3. Let g^i be the function conjugate to h^i and normalized by the condition $g^i(z_0) = 0$ with

$z_0 \in \Gamma \setminus \{O\}$. By Proposition 3, g^i belongs to $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ and there exist real numbers $c^{(k)}(\psi)$ $(k = 1, \dots, m)$ such that

$$g^i(z) = -\sum_{k=1}^{m} c^{(k)}(\psi)\mathcal{R}_k(z) + g_0^i(z) \qquad (z \in \Omega).$$

These coefficients $c^{(k)}(\psi)$ and the function g_0^i satisfy

$$\sum_{k=1}^{m} |c^{(k)}(\psi)| + \| g_0^i \|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} \leq c \| \psi \|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} . \tag{6.2}$$

Since $\frac{\partial}{\partial n} g^i = -\frac{\partial}{\partial s} h^i = -\frac{d}{ds}\psi$, the function $-g_0^i$ solves the Neumann problem in Ω with boundary data

$$\frac{d}{ds}\psi - \sum_{k=1}^{m} c^{(k)} \frac{\partial}{\partial n}\mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma).$$

By Proposition 2, the Dirichlet problem in Ω^c with boundary data $-g_0^i$ has a solution h^e such that $\frac{\partial}{\partial n} h^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ and satisfies

$$\left\| \frac{\partial}{\partial n} h^e \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \| g_0^i \|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} . \tag{6.3}$$

From the equality

$$h^e(\infty) = \frac{1}{2\pi}\int_\Gamma \frac{\partial h^e}{\partial n_q}(q) \log \frac{|z_0|}{|z_0 - q|} ds_q - \frac{1}{2\pi}\int_\Gamma h^e(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z_0 - q|} ds_q ,$$

where z_0 is a fixed point in Ω, and from (6.3) we obtain that the linear functional $g^i \to h^e(\infty)$ is continuous in $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. Therefore, we can choose g^i so that $h^e(\infty) = 0$ and inequality (6.3) remains valid. Since $\operatorname{grad} g_0^i = O(|z|^{-\frac{1}{2}})$ and $\operatorname{grad} h^e = O(|z|^{-\mu-\frac{1}{2}})$, it follows that

$$g_0^i(z) = \frac{1}{2\pi}\int_\Gamma \Big(\frac{\partial g_0^i}{\partial n}(q) + \frac{\partial h^e}{\partial n}(q) \Big) \log \frac{|z|}{|z - q|} ds_q .$$

Set

$$\sigma(z) = \frac{1}{2\pi}\Big(\frac{d}{ds}\psi(z) - \sum_{k=1}^{m} c^{(k)}(\psi) \frac{\partial}{\partial n}\mathcal{R}_k(z) - \frac{\partial}{\partial n} h^e(z) \Big) \qquad (z \in \Gamma \setminus \{0\}). \tag{6.4}$$

Taking into account that $V\sigma(z) = O\big(|z|^{-(\beta+\frac{1}{p})}\big)$ $(z \neq 0)$ as well as the boundedness of the functions $g_0^i(z)$ $(z \in \Omega)$ and $h^e(z)$ $(z \in \Omega^c)$, we have

$$g_0^i(z) + V\sigma(z) = c \operatorname{Im} \frac{1}{\gamma(z)} \quad (z \in \Omega) \qquad \text{and} \qquad h^e(z) + V\sigma(z) = 0 \quad (z \in \Omega^c)$$

where $\gamma(z)$ is a conformal mapping of Ω onto $\mathbb{R}^2_+$, $\gamma(0)=0$. From the jump formula for $\frac{\partial}{\partial n}V\sigma$ we obtain

$$c\frac{\partial}{\partial n}\mathrm{Im}\frac{1}{\gamma(z)}=0 \qquad (z\in\Gamma\setminus\{0\}).$$

Since $\frac{\partial}{\partial n}\mathrm{Im}\frac{1}{\gamma(z)}\sim\frac{1}{2}x^{-\frac{3}{2}}$ as $x\to 0$, we have $c=0$. Thus, $V\sigma=-g_0^i$ on Ω. A limit relation for the normal derivative of the simple layer potential implies

$$\pi\sigma(z)+(S\sigma)(z)=\frac{d}{ds}\psi(z)-\sum_{k=1}^{m}c^{(k)}(\psi)\frac{\partial}{\partial n}\mathcal{R}_k(z) \qquad (z\in\Gamma\setminus\{0\}).$$

We set $t^{(k)}=c^{(k)}(\psi)$ $(k=1,\dots,m)$. Hence, it follows that the pair (σ,t), where $t=(t^{(1)},\dots,t^{(m)})$ and σ is defined by (6.4), belongs to $\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m$ and satisfies equation (1.4). From (6.2) and (6.3) if follows that

$$\|\sigma\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}+\sum_{k=1}^{m}|c^{(k)}(\psi)|\le c\|\psi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}. \tag{6.5}$$

Now let ψ be an arbitrary function in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. There exits a sequence $\{\psi_r\}_{r\ge1}$ of smooth functions on $\Gamma\setminus\{0\}$, which vanishes in a neighbourhood of the peak and tends to $\psi\in\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. Let $(\sigma_r,t_r)\in\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m$ be the solution of (1.4) with right-hand side $\frac{d}{ds}\psi_r$. According to (6.5) the sequence $\{(\sigma_r,t_r)\}_{r\ge1}$ converges in $\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m$ to a limit (σ,t). Since the operator $S:\mathcal{L}_{p,\beta+1}(\Gamma)\longmapsto\mathcal{L}_{p,\beta+1}(\Gamma)$ (see Proposition 1) is continuous, it follows by taking the limit that

$$\pi\sigma+S\sigma+\sum_{k=1}^{m}t^{(k)}\frac{\partial}{\partial n}\mathcal{R}_k=\frac{d}{ds}\psi. \tag{6.6}$$

Consequently, equation (1.4) is solvable in $\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m$ for every $\psi\in\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$.

(ii) We turn to the case $\psi(z)=\mathrm{Re}\,z^k$ $(z\in\Gamma\setminus\{0\})$. As a harmonic extension of ψ onto Ω we take the function $h^i(z)=\mathrm{Re}\,z^k$. The conjugate function $g^i(z)=-\mathrm{Im}\,z^k$ belongs to $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$. By Proposition 2, the function $-g^i$ has the harmonic extension h^e on Ω^c such that $\frac{\partial}{\partial n}h^e\in\mathcal{L}_{p,\beta+1}(\Gamma)$. Set

$$\sigma(z)=\frac{1}{2\pi}\Big(\frac{d}{ds}\psi(z)-\frac{\partial}{\partial n}h^e(z)\Big) \qquad (z\in\Gamma\setminus\{O\}).$$

Then the pair $(\sigma,0)\in\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m$ is a solution of equation (1.4) with right-hand side $\frac{d}{ds}\psi$ (see (i)).

This and (6.6) imply

$$\mathfrak{Y}_{p,\beta}(\Gamma)\subset\left(\pi I+S+\sum_{k=1}^{m}t^{(k)}\frac{\partial}{\partial n}\mathcal{R}_k\right)(\mathcal{L}_{p,\beta+1}(\Gamma)\times\mathbb{R}^m).$$

(iii) It remains to prove the converse inclusion. Clearly,

$$\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \subset \mathfrak{Y}_{p,\beta}(\Gamma)$$

for any $t \in \mathbb{R}^m$. Now let σ belong to $\mathcal{L}_{p,\beta+1}(\Gamma)$ and let a function $\mathfrak{S} \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$ be defined by $\frac{d}{ds}\mathfrak{S} = \sigma$ on $\Gamma \setminus \{O\}$. By ψ we denote the function

$$\psi(z) = \pi \mathfrak{S}(z) - \int_\Gamma \mathfrak{S}(q) \frac{\partial}{\partial n_q} \log \frac{|z|}{|z-q|} ds_q .$$

From Theorem 1 it follows that $\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$. Since

$$\frac{d}{ds}\psi(z) = \pi\sigma(z) + \int_\Gamma \sigma(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q ,$$

we obtain that the image of $\mathcal{L}_{p,\beta+1}(\Gamma)$ under the mapping (6.1) is the space $\mathfrak{Y}_{p,\beta}(\Gamma)$ ∎

Theorem 5. *Let Ω have an inward peak. Then operator* (6.1) *is injective provided* $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$.

Proof. Let $(\sigma, t) \in \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$, where $t = (t^{(1)}, \dots, t^{(m)})$, belong to $\operatorname{Ker}\big(\pi I + S + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k\big)$. Then the harmonic function

$$v(z) = V\sigma(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Omega)$$

has zero Neumann boundary data on $\Gamma \setminus \{O\}$. Since

$$|v(z)| \le c\,|z|^{-(\beta+\frac{1}{p})}, \tag{6.7}$$

we obtain by the integral representation for the harmonic function $v(z)$ and a limit relation for the double layer potential

$$\pi v(z) + \int_\Gamma v(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = 0 \qquad (z \in \Gamma \setminus \{O\}).$$

Thus, v is a solution of the homogeneous integral equation of the Dirichlet problem in Ω^c.

The double layer potential $(Wv)(z)$ $(z \in \Omega^c)$ grows not faster than a power function as $z \to 0$. Since the limit values of Wv vanish on $\Gamma \setminus \{O\}$, it follows that $(Wv)(z) = 0$ $(z \in \Omega^c)$. Therefore an arbitrary conjugate function $\widetilde{Wv}$ is constant in Ω^c. We set $\widetilde{Wv} = C$.

Let W_+v be defined by $W_+v = Wv$ in Ω and let $\widetilde{W_+v}$ be a conjugate function such that $\widetilde{W_+v} = C$ on $\Gamma \setminus \{O\}$. We introduce the holomorphic function

$$W(z) = (W_+v)(z) + i(\widetilde{W_+v} - C) \qquad (z \in \Omega).$$

Let $\zeta = \gamma(z)$ be a conformal mapping of Ω onto $\mathbb{R}^2_+$, $\gamma(0) = 0$. The function $F(\zeta) = (W \circ \gamma^{-1})(\frac{1}{\zeta})$ is holomorphic in the lower half-plane $\mathbb{R}^2_-$, continuous up to the boundary, and $\operatorname{Im} F = 0$ on $\partial\mathbb{R}^2_-$. The holomorphic extension F^{ext} of F to $\mathbb{C}$ is the entire function, which grows not faster than a power function as $\zeta \to \infty$. It follows that $W(z) = P(\frac{1}{\gamma(z)})$, where P is a polynomial with real coefficients. This implies that

$$(W_+)v(z) = \sum_{k=0}^{\ell} c^{(k)} \operatorname{Re}\left(\frac{1}{\gamma(z)}\right)^{-k} \qquad (z \in \Omega).$$

From the jump formula for Wv we obtain

$$v(z) = -(2\pi)^{-1} \sum_{k=0}^{\ell} c^{(k)} \operatorname{Re}\Big(\frac{1}{\gamma(z)}\Big)^{-k} \qquad (z \in \Gamma \setminus \{O\}).$$

By (6.7) we have

$$v(z) = -(2\pi)^{-1}\Big(c^{(0)} + c^{(1)} \operatorname{Re}\frac{1}{\gamma(z)}\Big) \qquad \text{on } \Gamma \setminus \{O\}.$$

Therefore

$$(V\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) = -(2\pi)^{-1} c^{(0)} - (2\pi)^{-1} c^{(1)} \operatorname{Re}\frac{1}{\gamma(z)} + c \operatorname{Im}\frac{1}{\gamma(z)}$$

for $z \in \Omega$. By h^e_k $(k = 1, \dots, m)$ and h^e_0 we denote harmonic extensions of $\mathcal{R}_k$ and $\operatorname{Re}(\frac{1}{\gamma})$ onto Ω^c which grow not faster than a power function as $z \to 0$. Since

$$V\sigma + \sum_{k=1}^{m} t^{(k)} h^e_k + (2\pi)^{-1} c^{(1)} h^e_0 + (2\pi)^{-1} c^{(0)}$$

vanishes on $\Gamma \setminus \{O\}$, we have

$$(V\sigma)(z) = -\sum_{k=1}^{m} t^{(k)} h^e_k(z) - (2\pi)^{-1} c^{(1)} h^e_0(z) - (2\pi)^{-1} c^{(0)} \qquad (z \in \Omega^c).$$

From the jump formula for the normal derivative of $V\sigma$ it follows

$$\begin{aligned} 2\pi\sigma(z) = &-\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k(z) + c\,\frac{\partial}{\partial n} \operatorname{Im}\frac{1}{\gamma(z)} \\ &+ \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} h^e_k(z) + (2\pi)^{-1} c^{(1)} \frac{\partial}{\partial n} h^e_0(z) \end{aligned} \qquad (z \in \Gamma \setminus \{O\})$$

where

$$\frac{\partial}{\partial n}\mathcal{R}_k(z) \sim a_k \alpha_\pm |z|^{k+\mu-\frac{3}{2}} \qquad (k = 1, \dots, m)$$
$$\frac{\partial}{\partial n}\mathrm{Im}\frac{1}{\gamma(z)} \sim a_0 |z|^{-\frac{3}{2}}$$
$$\frac{\partial}{\partial n}h_k^e(z) \sim \pm b_k |z|^{k-\mu-\frac{3}{2}} \qquad (k = 0, 1, \dots, m).$$

Here a_k and b_k $(k = 0, 1, \dots, m)$ are real coefficients. Since $m \le \mu - \beta - \frac{1}{p}$, we have

$$k - \mu - \frac{1}{2} + \beta < \frac{1}{p} \qquad \text{for } k = 0, 1, \dots, m.$$

It means that the function $\frac{\partial}{\partial n}h_k^e$ does not belong to $\mathcal{L}_{p,\beta+1}(\Gamma)$ for $k = 0, 1, \dots, m$. Therefore the coefficients $t^{(1)}, \dots, t^{(m)}$ and $c^{(1)}$ are equal to zero. Thus,

$$\sigma = (2\pi)^{-1} c \frac{\partial}{\partial n}\mathrm{Im}\frac{1}{\gamma} \qquad \text{on } \Gamma \setminus \{O\}.$$

By the integral representation for the harmonic function $\mathrm{Im}\,\frac{1}{\gamma}$ on Ω we have

$$\frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_q}\Big(\mathrm{Im}\frac{1}{\gamma(q)}\Big)\log\frac{|z|}{|z-q|}ds_q = \frac{1}{2\pi}\int_\Gamma \mathrm{Im}\frac{1}{\gamma(q)}\frac{\partial}{\partial n_q}\log\frac{|z|}{|z-q|}ds_q + \mathrm{Im}\frac{1}{\gamma(z)}$$

for $z \in \Omega$. Since $\mathrm{Im}\,\frac{1}{\gamma(z)} = 0$ on $\Gamma \setminus \{O\}$, it follows from a limit relation for the simple layer potential that $\pi\sigma - S\sigma = 0$ on $\Gamma \setminus \{O\}$. However, we have $\pi\sigma + S\sigma = 0$ on $\Gamma \setminus \{O\}$. Hence, we obtain $\sigma = 0$ ∎

Proposition 5. *Let Ω have an inward peak, and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$, $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$. Then operator* (6.1) *is not Fredholm.*

Proof. Let

$$\Psi(\xi) = |\xi|^{n_0-1}(-\log|\xi|)^{-\gamma}$$

in a small neighbourhood of the origin and let suppΨ be in the domain of mapping θ_0 introduced in Section 3. We assume $\frac{1}{p} < \gamma < 1$ and set $\psi = \Psi \circ \theta_0^{-1} \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$. By h^i we denote the harmonic extension of ψ on Ω constructed in Proposition 3. Let g^i be a conjugate function from Proposition 3. We have

$$g^i(z) = \sum_{k=1}^{m} c^{(k)}\mathcal{R}_k(z) + g^\#(z).$$

Here

$$g^\#(z) = c\,\mathrm{Re}\big(z^{\frac{n_0}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\big) + g_0^\#(z),$$

where $g_0^\# \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$.

By h^e we denote the harmonic extension of $-g^{\#}$ on Ω^c from Proposition 2. We have

$$h^e(z) = c_1 \operatorname{Im}\Big[\Big(\frac{zz_0}{z_0 - z}\Big)^{-\mu+\frac{n_0}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\Big] + c_2 \operatorname{Re}\Big[\Big(\frac{zz_0}{z_0 - z}\Big)^{\frac{n_0}{2}-\frac{1}{2}}(\log z)^{-\gamma+1}\Big] + h_0^e(z)$$

where z_0 is a fixed point of Ω and $\frac{\partial}{\partial n} h_0^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$. Since

$$\frac{\partial}{\partial n} h^e(z) \sim c\, x^{-\beta-\frac{1}{p}-1}(\log x)^{-\gamma-1},$$

it follows that $\frac{\partial}{\partial n} h^e \notin \mathcal{L}_{p,\beta+1}(\Gamma)$ and $\frac{\partial}{\partial n} h^e \in \mathcal{L}_{p,\beta'+1}(\Gamma)$ for $\beta' > \beta$.

By Theorem 4 the pair (σ, t), where $t = (c^{(1)}, \dots, c^{(m)})$ and

$$\sigma = (2\pi)^{-1}\left(\frac{d}{ds}\psi - \sum_{k=1}^{m} c^{(k)} \frac{\partial}{\partial n}\mathcal{R}_k - \frac{\partial}{\partial n} h^e\right) \qquad \text{on } \Gamma \setminus \{O\},$$

belongs to $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$ for $\beta' > \beta$ and satisfies (6.6). From Theorem 5 it follows that the same equation is not solvable in $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$.

Equation (6.6) is solvable in $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$, $\beta' < \beta$. Since the set of smooth functions vanishing near the peak is dense in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and since $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$ is embedded to $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$, we obtain that the range of operator (6.1) is not closed in $\mathfrak{Y}_{p,\beta}(\Gamma)$ ∎

7. Integral equations of the exterior Dirichlet and Neumann problems for a domain with outward peak

Now we shortly discuss the integral equations mentioned in the title of the section. Their proofs are similar to those of the corresponding results relating to the interior problems for a domain with inward peak, which were proved earlier.

Let Ω have an outward peak. The solution of the Dirichlet problem

$$\left.\begin{aligned} \Delta u &= 0 \ \text{ in } \ \Omega^c \\ u\,|_{\Gamma} &= \varphi \end{aligned}\right\}$$

is sought in the form

$$u(z) = (W^{ext}\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k^{ext}(z) \qquad (z \in \Omega^c).$$

Here

$$(W^{ext}\sigma)(z) = \int_\Gamma \sigma(q)\Big(\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|} + 1\Big) ds_q$$

and

$$\mathcal{I}_k^{ext}(z) = \mathrm{Re}\Big(\frac{zz_0}{z_0 - z}\Big)^{k-\frac{1}{2}} \qquad (z \in \Omega^c),$$

where z_0 is a fixed point in Ω. The density σ and the vector $t = (t^{(1)}, \dots, t^{(m)})$ satisfy the equation

$$\pi\sigma + T^{ext}\sigma + \sum_{k=1}^{m} t^{(k)}\mathcal{I}_k^{ext} = \varphi \qquad \text{on } z \in \Gamma \setminus \{O\}, \tag{7.1}$$

where $T^{ext}\sigma$ is the value of the potential $W^{ext}\sigma$ at a point of $\Gamma \setminus \{O\}$.

Theorem 6. *Let Ω have an outward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$, $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Then the operator*

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + T^{ext}\sigma + \sum_{k=1}^{m} t^{(k)}\mathcal{I}_k^{ext} \in \mathfrak{M}_{p,\beta}(\Gamma) \tag{7.2}$$

is surjective.

Proof. Let h^e be the harmonic extension of $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ on Ω^c constructed in Proposition 3, and let g^e be a conjugate function vanishing at a fixed point on $\Gamma \setminus \{O\}$. By Proposition 3 there exist real numbers $c^{(k)}$ $(k = 1, \dots, m)$ such that

$$g^e = \sum_{k=1}^{m} c^{(k)}\mathcal{R}_k^{ext} + g_0^e,$$

where $g_0^e \in \mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$ and

$$\mathcal{R}_k^{ext}(z) = \mathrm{Re}\Big(\frac{zz_0}{z_0 - z}\Big)^{k-\frac{1}{2}}.$$

We set $h_0^e = h^e + \sum_{k=1}^{m} t^{(k)}\mathcal{I}_k^{ext}$.

The only change to be made in the proof of Theorem 2 is that the solution g^i of the Neumann problem on Ω with boundary data $\frac{\partial}{\partial n}h_0^e$ should be chosen so that

$$\int_\Gamma g^i ds = \int_\Gamma h_0^e ds - 2h_0^e(\infty).$$

Then the pair (σ, t), where $t = (c^{(1)}, \dots, c^{(m)})$ and

$$\sigma = (2\pi)^{-1}\left(\varphi + \sum_{k=1}^{m} c^{(k)}\mathcal{I}_k^{ext} - g^i\right)$$

is a solution in $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ of equation (7.1). The case $\varphi \in \mathfrak{P}(\Gamma)$ is considered as in Theorem 2 ∎

We represent the solution of the Neumann problem

$$\left.\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega^c \\ &\frac{\partial u}{\partial n}\Big|_\Gamma = \varphi \end{aligned}\right\}$$

in the form

$$u(z) = (V\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k^{ext}(z) \qquad (z \in \Omega^c),$$

where $V\sigma$ is the simple layer potential. The density σ and the vector $t = (t^{(1)}, \ldots, t^{(m)})$ satisfy

$$\pi\sigma - S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k^{ext} = -\varphi \qquad \text{on } \Gamma \setminus \{O\}. \tag{7.3}$$

Theorem 7. *Let Ω have an outward peak and let $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$, $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$. Then the operator*

$$\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma - S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k^{ext} \in \mathfrak{Y}_{p,\beta}(\Gamma) \tag{7.4}$$

is surjective.

Proof. Let h^e be the harmonic extension of $\psi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ on Ω^c and g^e be a conjugate function constructed in Proposition 3. Then there exist real numbers $c^{(k)}$ such that

$$g^e = \sum_{k=1}^{m} c^{(k)} \mathcal{R}_k^{ext} + g_0^e,$$

where $g_0^e \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$. We choose g_0^e to satisfy $g_0^e(\infty) = 0$. Here the function g^e plays the same role as g^i in the proof of Theorem 4.

Now we use the same argument as in Theorem 4. By h^i we denote the harmonic extension of g_0^e on Ω such that $\frac{\partial}{\partial n} h^i \in \mathcal{L}_{p,\beta+1}(\Gamma)$ (see Proposition 2). Then the pair (σ, t), where $t = (c^{(1)}, \ldots, c^{(m)})$ and

$$\sigma = (2\pi)^{-1} \left(\frac{\partial}{\partial n} h^i - \frac{d}{ds}\psi + \sum_{k=1}^{m} c^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k^{ext} \right) \in \mathcal{L}_{p,\beta+1}(\Gamma),$$

solves equation (7.3).

The case $\varphi \in \mathfrak{P}(\Gamma)$ is considered in the same way as in Theorem 4, one should only replace g^i, h^i and h^e by g^e, h^e and h^i, respectively ∎

Two following theorems can be proved in the same way as Theorems 3 and 5.

Theorem 8. *Let Ω have an outward peak. Then operator* (7.2) *is injective for* $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$.

Theorem 9. *Let Ω have an outward peak. Then operator* (7.4) *is injective for* $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$.

The proof of the following proposition is essentially the same as those of Propositions 4 (the case of operator (7.2)) and 5 (the case of operator (7.4)).

Proposition 6. *Let Ω have an outward peak, and let* $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$, $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$. *Then operators* (7.2) *and* (7.4) *are not Fredholm.*

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