# L<sub>p</sub>-Theory of Boundary Integral Equations **on a Contour with Inward Peak**

V. **Maz'ya and** A. **Soloviev** 

*Dedicated to Prof. E. Meister on the occasion of his retirement* 

Abstract. Boundary integral equations of the second kind in the logarithmic potential theory are studied under the assumption that the contour has an inward peak. For each equation we find a pair of function spaces such that the corresponding operator bijectively maps one of them onto another.

Keywords: *Boundary integral equations, logarithmic potential, asymptotics of solutions* AMS subject **classification:** Primary 31 A 10, secondary 45A05

## I. Introduction

In this paper we prove the unique solvability of boundary integral equations of the Dirichlet problem

at the contour has an inward peak. For each equation  
that the corresponding operator bijectively maps one of  
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$$
x = 31
$$
 (1.1)  

$$
2u = 0
$$
 in  $\Omega$   

$$
u|_{\Gamma} = \varphi
$$

and the Neumann problem

*Au* =0 in ci *au (1.2) ônr*  J *a(q* 

in a bounded plane simply connected domain  $\Omega$  with inward peak  $z = 0$  on the boundary  $\Gamma$ . Here and elsewhere we assume that the normal *n* is directed outwards.

We look for a solution of the problem (1.1) in the form

in a bounded plane simply connected domain 
$$
\Omega
$$
 with inward peak  $z = 0$  on the boundary  $\Gamma$ . Here and elsewhere we assume that the normal n is directed outwards. We look for a solution of the problem (1.1) in the form\n
$$
u(z) = (W\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k(z) \qquad (z = x + iy \in \Omega)
$$
\nwhere  $W\sigma$  is the double layer potential\n
$$
(W\sigma)(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q,
$$
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where  $W\sigma$  is the double layer potential

$$
(W\sigma)(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q,
$$

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 $t^{(k)}$  are real numbers and  $\mathcal{I}_k(z) = \text{Im } z^{k - \frac{1}{2}}$ . The function  $\sigma$  and the vector  $t =$  $(t^{(1)}, \ldots, t^{(m)})$  satisfy the equation

1 A. Soloviev

\nAs and 
$$
\mathcal{I}_k(z) = \text{Im } z^{k - \frac{1}{2}}
$$
. The function  $\sigma$  and the vector  $t = r$  the equation

\n
$$
\pi \sigma - T \sigma + \sum_{k=1}^m t^{(k)} \mathcal{I}_k = -\varphi \quad \text{on } \Gamma \setminus \{O\},
$$
 (1.3)

\nFor  $n = 0$  and  $n = 1$  and  $n = 0$  and  $n = 0$  and  $n = 1$ .

where  $T\sigma$  is the value of the potential  $W\sigma$  at a boundary point.

A solution of problem (1.2) is sought in the form

$$
\overline{k=1}
$$
  
\nWe get 
$$
\overline{k=1}
$$
  
\n
$$
\text{where } (1.2) \text{ is sought in the form}
$$
  
\n
$$
u(z) = (V\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Omega),
$$
  
\n
$$
\text{by}
$$
  
\n
$$
(V\sigma)(z) = \int_{\Gamma} \sigma(q) \log \frac{|z|}{|z-q|} ds_q
$$

where  $V\sigma$  is defined by

$$
(V\sigma)(z) = \int_{\Gamma} \sigma(q) \log \frac{|z|}{|z-q|} ds_q
$$

and  $\mathcal{R}_k(z) = \text{Re } z^{k - \frac{1}{2}}$ . Then the function  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$  satisfy<br>
the equation<br>  $\pi \sigma + S \sigma + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k = \varphi$  on  $\Gamma \setminus \{O\}$ , (1.4)<br>
where<br>  $(S\sigma)(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n_z} \$ the equation

e of the potential 
$$
W\sigma
$$
 at a boundary point.  
\nblem (1.2) is sought in the form  
\n
$$
u(z) = (V\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Omega),
$$
\nby  
\n
$$
(V\sigma)(z) = \int_{\Gamma} \sigma(q) \log \frac{|z|}{|z-q|} ds_q
$$
\n
$$
\frac{1}{2}
$$
. Then the function  $\sigma$  and the vector  $t = (t^{(1)}, \dots, t^{(m)})$  satisfy  
\n
$$
\pi\sigma + S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k = \varphi \qquad \text{on } \Gamma \setminus \{O\}, \qquad (1.4)
$$
\n
$$
u(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n} \log \frac{|z|}{|z-q|} ds, \qquad (z \in \Gamma \setminus \{O\}).
$$

where

$$
(S\sigma)(z) = \int_{\Gamma} \sigma(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \qquad (z \in \Gamma \setminus \{O\}).
$$

Let  $\Gamma \setminus \{O\}$  belong to the class  $C^2$ . We say that O is an *outward (inward)* peak if  $\Omega$  (the complementary domain  $\Omega^c$ ) is given near the peak by the inequalities  $\kappa$ <sub>-</sub>(x) <  $y < \kappa_+(x)$ ,  $0 < x < \delta$ , where  $x^{-\mu-1}\kappa_+(x) \in C^2[0,\delta]$  and  $\lim_{x\to+0} x^{-\mu-1}\kappa_+(x) = \alpha_+$ with  $\mu > 0$  and  $\alpha_+ > \alpha_-$ . By  $\Gamma_{\pm}$  we denote the arcs  $\{(x, \kappa_{\pm}(x)) : x \in [0, \delta]\}.$  Points on  $\Gamma_+$  and  $\Gamma_-$  with equal abscissas will be denoted by  $q_+$  and  $q_-$ .

In our previous articles [5 - 7], where we also studied boundary integral equations of logarithmic potential theory on contours with peaks, the solutions and boundary data were characterized by their asymptotic behaviour near the peaks. Here, for every integral operator under consideration we find a pair of function spaces such that the operator maps isomorphically one space onto another.

If  $|q|^{\beta} \varphi \in L_p(\Gamma)$ , then we say that  $\varphi$  belongs to  $\mathcal{L}_{p,\beta}(\Gamma)$ . We define the norm in this space by  $\|\varphi\|_{C_{p,\beta}(\Gamma)} = \|\,|q|^\beta \varphi\,\|_{L_p(\Gamma)}$ . We shall make use of the same definition with  $\Gamma$ replaced by arcs of  $\Gamma_{\pm}$  and intervals of R.

Let  $\mathcal{L}_{p,\beta}^1(\Gamma)$  be the space of absolutely continuous functions on  $\Gamma \setminus \{O\}$  with finite norm  $\|\varphi\|_{\mathcal{L}^1_{p,\theta}(\Gamma)} = \|\varphi'_s\|_{\mathcal{L}_{p,\theta}(\Gamma)} + \|\varphi\|_{\mathcal{L}_{p,\theta-1}(\Gamma)}$ . It is an easy exercise to check the density in  $\mathcal{L}_{p,\beta}^{1}( \Gamma)$  of the set of smooth functions on  $\Gamma$  vanishing near O. nder conside<br>
orphically o<br>
, then we sa<br>  $= ||q||^{\beta} \varphi ||$ <br>  $\Gamma_{\pm}$  and inte<br>
the space of<br>  $= ||\varphi'_{s}||_{\mathcal{L}_{p,\beta}}$ <br>
of the set of<br>
the pair of s<br>
norms<br>  $\left( \int_{\Gamma_{\pm} \cup \Gamma_{-}}$ 

We introduce the pair of spaces  $\mathfrak{N}_{p,\beta}^{(\pm)}(\Gamma)$  of absolutely continuous functions  $\varphi$  on  $\Gamma \setminus \{O\}$  with finite norms

$$
\|\varphi\|_{\mathfrak{N}_{p,\beta}^{(\pm)}(\Gamma)} = \left( \int_{\Gamma_+\cup \Gamma_-} |\varphi(q_+) \pm \varphi(q_-)|^p |q|^{p(\beta-\mu)} ds_q \right)^{\frac{1}{p}} + \|\varphi\|_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}.
$$

Let  $\mathfrak{P}(\Gamma)$  denote the space of restrictions to  $\Gamma \setminus \{O\}$  of real functions of the form Let  $\mathfrak{P}(\Gamma)$  denote the space of restrict<br>  $p(z) = \sum_{k=0}^{m} t^{(k)} \text{Re } z^k$ , where  $m = [\mu$ <br>  $||p||_{\mathfrak{P}(\Gamma)} = \sum_{k=0}^{m} |t^{(k)}|$ . The space  $\mathfrak{M}_{p,\beta}(\Gamma)$ ions to  $\Gamma \setminus \{O\}$  c<br>  $-\beta - \frac{1}{p} + \frac{1}{2}$ .  $\sum_{k=0}^{\infty} t^{(k)} \text{Re } z^k$ , where  $m = \left[ \mu - \beta - \frac{1}{p} + \frac{1}{2} \right]$ . A norm of p is defined by  $= \sum_{k=0}^{m} |t^{(k)}|$ . The space  $\mathfrak{M}_{p,\beta}(\Gamma)$  is defined as the direct sum of  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and  $\mathfrak{B}(\Gamma)$ . Let  $\mathfrak{P}(\Gamma)$  denote the space  $p(z) = \sum_{k=0}^{m} t^{(k)} \text{Re } z^k$ , where<br>  $||p||_{\mathfrak{P}(\Gamma)} = \sum_{k=0}^{m} |t^{(k)}|$ . The space<br>  $\mathfrak{P}(\Gamma)$ .<br>  $\text{By } \mathfrak{D}_{p,\beta}(\Gamma)$  we denote the s<br>  $\rho = \frac{d}{ds} \psi$ , where  $\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$  an<br> ightharpoonup is the form  $x = \left[\mu - \beta - \frac{1}{p} + \frac{1}{2}\right]$ . A norm of p is defined by<br>
The space  $\mathfrak{M}_{p,\beta}(\Gamma)$  is defined as the direct sum of  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and<br>
ote the space of functions on  $\Gamma \setminus \{O\}$  represente

By  $\mathfrak{D}_{p,\beta}(\Gamma)$  we denote the space of functions on  $\Gamma \setminus \{O\}$  represented in the form<br>=  $\frac{d}{ds}\psi$ , where  $\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$  and  $\psi(z_0) = 0$  for a fixed point  $z_0 \in \Gamma \setminus \{O\}$ . We supply  $\varphi = \frac{d}{ds} \psi$ , where  $\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$  and  $\psi(z_0) = 0$  for a fixed point  $z_0 \in \Gamma \setminus \{O\}$ . We supply  $\mathfrak{D}_{p,\beta}(\Gamma)$  with the norm  $\|\varphi\|_{\mathfrak{D}_{p,\beta}(\Gamma)} = \|\psi\|_{\mathfrak{D}_{p,\beta}(\Gamma)}$ .

In the following short description of our results we assume that  $0 < \beta + \frac{1}{n}$  $min\{\mu, 1\}$ . In Theorem 1 we prove that the operator

$$
\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m \ni (\sigma,t) \longmapsto \pi\sigma - T\sigma + \sum_{k=1}^m t^{(k)} \mathcal{I}_k \in \mathfrak{M}_{p,\beta}(\Gamma) \tag{1.5}
$$

is continuous. As is shown in Theorem 2, the range of operator (1.5) coincides with the space  $\mathfrak{M}_{p,\beta}(\Gamma)$  if  $\mu-\beta-\frac{1}{p}+\frac{1}{2}\notin\mathbb{N}$ . For the exceptional case  $\mu-\beta-\frac{1}{p}+\frac{1}{2}\in\mathbb{N}$  we find the operator Nown in Theorem 2, the range of operator (1.5) coincides with the<br>  $\theta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . For the exceptional case  $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$  we find<br>
not Fredholm (see Proposition 4). In Theorem 4 we show tha

that operator (1.5) is not Fredholm (see Proposition 4). In Theorem 4 we show that  
the operator\n
$$
\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma,t) \longmapsto \pi\sigma + S\sigma + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma) \tag{1.6}
$$

is onto if  $\mu-\beta-\frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Under the assumption  $\mu-\beta-\frac{1}{p} + \frac{1}{2} \in \mathbb{N}$  we prove in Proposition 5 that operator (1.6) is not Fredhoim. In Theorems 3 and 5 we show that operators (1.5) and (1.6) are injective. The boundary integral equations of the exterior Dirichlet and Neumann problems for a domain  $\Omega$  with outward peak are discussed in Theorems 6 - 9 and Proposition 6.

#### 2. Continuity of the operator  $\pi I - T$

Let the operator  $K$  be defined by

$$
\mathcal{K}f(x)=\int_{\mathbb{R}}K(x,y)f(y)\,dy,
$$

where

$$
\mathcal{K}f(x) = \int_{\mathbb{R}} K(x, y) f(y) dy,
$$
  
where  

$$
|K(x, y)| \le c \frac{1}{|x - y|} \frac{1}{(1 + |x - y|^J)}
$$
 ( $J \ge 0$ ).  
Here and elsewhere by c we denote different positive constants.

We introduce the space  $L_{p,\alpha}(\mathbb{R})$  of functions on  $\mathbb{R}$  with the norm

$$
\|\varphi\|_{L_{p,\alpha}(\mathbb{R})} = \|\left(1+x^2\right)^{\frac{\alpha}{2}}\varphi\|_{L_p(\mathbb{R})}.
$$

The following lemma was formulated in [4).

**Lemma 1.** *If*  $K: L_p(\mathbb{R}) \to L_p(\mathbb{R})$   $(1 < p < \infty)$  is bounded and  $-J < \alpha + \frac{1}{p} < J+1$ , *then* K is continuous in  $L_{p,\alpha}(\mathbb{R})$ .

We shall also use the following technical lemma.

**Lemma 2.** Let  $\rho(u) = \kappa_+(u) - \kappa_-(u)$  and let the function h be specified by

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\nLemma 1. If 
$$
K: L_p(\mathbb{R}) \to L_p(\mathbb{R})
$$
  $(1 < p < \infty)$  is bounded and  $-J < \alpha + \frac{1}{p} < J+1$ ,  
\nthen K is continuous in  $L_{p,\alpha}(\mathbb{R})$ .  
\nWe shall also use the following technical lemma.  
\nLemma 2. Let  $\rho(u) = \kappa_+(u) - \kappa_-(u)$  and let the function h be specified by  
\n
$$
\int_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\nu)} = \tau \qquad (\tau > 0).
$$
\n(2.1)  
\nIf  $|1 - \frac{\xi}{\tau}| < \varepsilon_0$ , where  $\varepsilon_0$  is a sufficiently small number, then

is continuous in 
$$
L_{p,\alpha}(\mathbb{R})
$$
.  
\nshall also use the following technical lemma.  
\nn
$$
\text{numa 2.} \quad Let \rho(u) = \kappa_{+}(u) - \kappa_{-}(u) \quad \text{and let the function } h \text{ be specified by}
$$
\n
$$
\int_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\nu)} = \tau \qquad (\tau > 0).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\nu)} = \tau \qquad (\tau > 0).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\nu)} = \tau \qquad (\tau > 0).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\kappa)} = \tau \qquad (\tau > 0).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\kappa)} = \frac{1}{(\xi - \tau)^2 + 1} \left| \leq \frac{c}{\tau} \left( \frac{1}{(\xi - \tau)^2 + 1} + \frac{1}{\xi} \right).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\kappa)} = \frac{1}{(\xi - \tau)^2 + 1} \left| \leq \frac{c}{\tau} \left( \frac{1}{(\xi - \tau)^2 + 1} + \frac{1}{\xi} \right).
$$
\n
$$
\sum_{h(\tau)}^{\delta} \frac{d\nu}{\rho(\kappa)} = \frac{1}{(\xi - \tau)^2 + 1} \left| \leq \frac{c}{\tau} \left( \frac{1}{(\xi - \tau)^2 + 1} + \frac{1}{\xi} \right).
$$

**Proof.** By the asymptotic formula

$$
h(\tau) \sim (\mu(\alpha_+ - \alpha_-)\tau)^{-\frac{1}{\mu}} \quad \text{as} \quad \tau \to \infty,
$$

which can be differentiated three times, we obtain the estimates

$$
\left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2 - 1 = O\left(\frac{\xi-\tau}{\tau}\right)
$$
\n(2.2)

and

$$
\left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2 - 1 = O\left(\frac{\xi-\tau}{\tau}\right)
$$
\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(\alpha(\xi,\tau))^2} - 1 = O\left(\frac{(\xi-\tau)^2}{\xi\tau}\right),
$$
\n
$$
h(\tau)(\xi-\tau)^{-1}.
$$
 We represent\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(\xi)-h(\tau))^2 + (\rho(h(\xi)))^2} - \frac{1}{(\xi-\tau)^2 + 1}
$$
\n(2.3)

where 
$$
\alpha(\xi, \tau) = (h(\xi) - h(\tau))(\xi - \tau)^{-1}
$$
. We represent  
\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(h(\xi) - h(\tau))^2 + (\rho(h(\xi)))^2} - \frac{1}{(\xi - \tau)^2 + 1}
$$

in the form

$$
\left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2 - 1 = O\left(\frac{\xi-\tau}{\tau}\right) \tag{2.2}
$$
\nand

\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(\alpha(\xi,\tau))^2} - 1 = O\left(\frac{(\xi-\tau)^2}{\xi\tau}\right), \tag{2.3}
$$
\nwhere

\n
$$
\alpha(\xi,\tau) = (h(\xi) - h(\tau))(\xi-\tau)^{-1}.
$$
\nWe represent

\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(h(\xi) - h(\tau))^2 + (\rho(h(\xi)))^2} - \frac{1}{(\xi-\tau)^2 + 1}
$$
\nin the form

\n
$$
\frac{\rho(h(\xi))\rho(h(\tau))}{(\alpha(\xi,\tau))^2} \frac{1 - \left(\frac{\rho(h(\xi))}{\alpha(\xi,\tau)}\right)^2}{((\xi-\tau)^2 + 1)((\xi-\tau)^2 + \frac{(\rho(h(\xi))}{(\alpha(\xi,\tau))^2})} - \frac{1 - \rho(h(\xi))\frac{\rho(h(\tau))}{(\alpha(\xi,\tau))^2}}{(\xi-\tau)^2 + 1} \tag{2.4}
$$
\nFrom (2.2) and (2.3) it follows that (2.4) does not exceed

\n
$$
\frac{\varepsilon}{\tau} \left(\frac{1}{(\xi-\tau)^2 + 1} + \frac{1}{\xi}\right) \blacksquare
$$
\nTheorem 1. Let  $\Omega$  have an inward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ . Then the operator

\n
$$
\left(\frac{\tau}{\tau} - \frac{\tau}{\tau}\right) \circ \mathcal{L} \left(\frac{\tau}{\tau}\right) \circ \mathcal{L} \left(\frac{\tau}{\tau}\right)
$$

From (2.2) and (2.3) it follows that (2.4) does not exceed  $\frac{c}{r}(\frac{1}{(\xi-r)^2+1}+\frac{1}{\xi})$ 

**Theorem 1.** Let  $\Omega$  have an inward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ . Then the *operator*

$$
(\pi I - T) : \mathcal{L}_{p,\beta+1}^1(\Gamma) \ni \sigma \longmapsto (\pi I - T)\sigma \in \mathfrak{M}_{p,\beta}(\Gamma)
$$

*is continuous.* 

**Proof.** Let  $\varepsilon$  be so small that  $|\kappa_{\pm}(x) - \kappa_{\mp}(u)| \geq c u^{\mu+1}$  for all *u* satisfying  $|x - u| <$  $\varepsilon x$ . The arcs of  $\Gamma_{\pm}$  projected onto the segments  $[0, (1 - \varepsilon)x]$ ,  $[(1 - \varepsilon)x, (1 + \varepsilon)x]$  and

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be denoted by  $\Gamma_{\pm}^{\ell}(x)$ ,  $\Gamma_{\pm}^{c}(x)$  and  $\Gamma_{\pm}^{r}(x)$ . Set  $\sigma(u + i\kappa_{\pm}(u)) = \sigma_{\pm}(u)$  $[(1+\varepsilon)x,\delta]$  will be denoted by  $\Gamma_{\pm}^{\ell}(x)$ ,  $\Gamma_{\pm}^{\epsilon}(x)$  and  $\Gamma_{\pm}^{\epsilon}(x)$ . Set  $\sigma(u+i\kappa_{\pm}(u))=\sigma_{\pm}(u)$ ,  $u \in [0,\delta].$ 

(i) We prove the continuity of the operator  $\frac{\partial}{\partial s}(\pi I - T) : L^1_{p,\beta+1}(\Gamma) \to L_{p,\beta+1}(\Gamma)$ . Since

$$
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$$
\nbe denoted by  $\Gamma_{\pm}^{\ell}(x)$ ,  $\Gamma_{\pm}^c(x)$  and  $\Gamma_{\pm}^r(x)$ . Set  $\sigma(u + i\kappa)$   
\ne the continuity of the operator  $\frac{\partial}{\partial s}(\pi I - T) : L_{p,\beta+1}^1(\Gamma \frac{\partial}{\partial s}(\pi \sigma - T\sigma)(z) = \pi \sigma'(z) + \int_{\Gamma} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q$ ,  
\neestimate the norm in  $\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})$  of the func  
\n
$$
T_{\tau}\sigma(z) = \int_{\Gamma_{\pm}\cup\Gamma_{-}} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q
$$

it is sufficient to estimate the norm in  $\mathcal{L}_{p,\theta+1}(\Gamma\cap\{|q|<\frac{\delta}{2}\})$  of the function

$$
T_*\sigma(z)=\int_{\Gamma_+\cup\Gamma_-}\sigma'(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q.
$$

We represent  $T_*\sigma(z)$  in the form

$$
T_{\ast}\sigma(z) = \int_{\Gamma_{+}\cup\Gamma_{-}} \sigma'(q) \frac{\partial}{\partial n_{\ast}} \log \frac{|z|}{|z-q|} ds_{q}.
$$
  
\nz) in the form  
\n
$$
\left(\int_{\Gamma_{-}^{\epsilon}(z)} + \int_{\Gamma_{+}^{\epsilon}(z)} \sigma'(q) \frac{\partial}{\partial n_{\ast}} \log \frac{|z|}{|z-q|} ds_{q} + I(z),
$$
  
\nn on  $\Gamma_{+} \cup \Gamma_{-}$  admits the estimate

where the last term on  $\Gamma_+ \cup \Gamma_-$  admits the estimate

$$
T_{\ast}\sigma(z) = \int_{\Gamma_{+}\cup\Gamma_{-}} \sigma'(q) \frac{\partial}{\partial n_{z}} \log \frac{|z|}{|z-q|} ds_{q}.
$$
  
We represent  $T_{\ast}\sigma(z)$  in the form  

$$
\left(\int_{\Gamma_{-}^{c}(z)} + \int_{\Gamma_{+}^{c}(z)}\right) \sigma'(q) \frac{\partial}{\partial n_{z}} \log \frac{|z|}{|z-q|} ds_{q} + I(z),
$$
  
where the last term on  $\Gamma_{+} \cup \Gamma_{-}$  admits the estimate  

$$
|I(z)| \leq \frac{c}{x^{2}} \int_{0}^{x} (|\sigma'_{+}(u)| + |\sigma'_{-}(u)|)u \, du + \frac{c}{x} \int_{z}^{\delta} (|\sigma'_{+}(u)| + |\sigma'_{-}(u)|) \, du.
$$
  
It follows by Hardy's inequality [2: Section 9.9] that  

$$
||I||_{\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$
  
It is sufficient to assume that  $z \in \Gamma_{+}$ . Since  $\left|\frac{\partial}{\partial n_{z}} \log \frac{|z|}{|z-q|}\right| \leq \frac{c}{z}$  for  $q \in \Gamma_{+}^{c}(x)$ ,

It follows by Hardy's inequality [2: Section 9.9] that

$$
||I||_{\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$

t term on 
$$
\Gamma_+ \cup \Gamma_-
$$
 admits the estimate  
\n
$$
|\xi| \leq \frac{c}{x^2} \int_0^x (|\sigma'_+(u)| + |\sigma'_-(u)|) u du + \frac{c}{x} \int_x^{\delta} (|\sigma'_+(u)| + |\sigma'_-(u)|) du.
$$
\nHardy's inequality [2: Section 9.9] that  
\n
$$
||I||_{\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$
\nto assume that  $z \in \Gamma_+$ . Since  $\left|\frac{\partial}{\partial n_z}\right| \log \frac{|z|}{|z-q|}\right| \leq \frac{c}{x}$  for  $q \in \Gamma_+^c(x)$ , we obtain  
\n
$$
\left| \int_{\Gamma_+^c(x)} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \right| \leq \frac{c}{x} \int_0^{\frac{1}{z} + c} |\sigma'_+(u)| du.
$$
\nestimate for the left-hand side follows from Hardy's inequality.  
\n $\int_{\Gamma_-^c(x)} \infty$  have  
\n $\frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} (1 + (\kappa'_-(x))^2)^{\frac{1}{2}} = -\frac{\kappa_+(u) - \kappa_-(u)}{|z-q|^2} + O\left(\frac{1}{x}\right).$   
\n $\geq c((x-u)^2 + (\rho(u))^2)$ , we obtain

The required estimate for the left-hand side follows from Hardy's inequality.

For  $q \in \Gamma^c_{-}(x)$  we have

$$
\left| \int_{\Gamma_+^c(x)} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \right| \leq \frac{c}{x} \int_{(1-\epsilon)z}^{(1+\epsilon)z} |\sigma'_+(u)| du.
$$
  
leestimate for the left-hand side follows from Hardy's inequality:  

$$
\int_{-\infty}^{\infty} (x) \text{ we have}
$$

$$
\frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} \left(1 + (\kappa'_-(x))^2\right)^{\frac{1}{2}} = -\frac{\kappa_+(u) - \kappa_-(u)}{|z-q|^2} + O\left(\frac{1}{x}\right).
$$

Since  $|z - q|^2 \ge c((x - u)^2 + (\rho(u))^2)$ , we obtain

we have  
\n
$$
g \frac{|z|}{|z-q|} (1 + (\kappa'_{-}(x))^2)^{\frac{1}{2}} = -\frac{\kappa_{+}(u) - \kappa_{-}(u)}{|z-q|^2} + C
$$
\n
$$
(x-u)^2 + (\rho(u))^2
$$
, we obtain\n
$$
\left| \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} \right| \leq c \left( \frac{1}{x} + \frac{\rho(u)}{(x-u)^2 + (\rho(u))^2} \right).
$$

This implies

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\nThis implies  
\n
$$
\left| \int_{\Gamma_-^c(x)} \sigma'(q) \frac{\partial}{\partial n_z} \log \frac{|z|}{|z-q|} ds_q \right|
$$
\n
$$
\leq \frac{c}{x} \int_{(1-\epsilon)x}^{(1+\epsilon)x} |\sigma_-'(u)| du + \int_{(1-\epsilon)x}^{(1+\epsilon)x} \frac{\rho(u) |\sigma_-'(u)|}{(x-u)^2 + (\rho(u))^2} du
$$
\n
$$
= J_1 + J_2.
$$
\nThe integral  $J_1$  can be estimated in  $\mathcal{L}_{p,\beta+1}(0, \frac{\delta}{2})$  by Hardy's inequality.  
\nchange of variables  $\tau = u^{-\mu}$ ,  $\xi = x^{-\mu}$  in  $J_2$  we obtain  
\n
$$
\int_0^{\delta/2} x^{p(\beta+1)} |J_2(x)|^p dx \leq c \left( \int_{(\delta/2)^{-\mu}}^{\infty} \xi^{p(1-\alpha)} \left( \int_{(\delta/2)^{-\mu}}^{\infty} \frac{|\sigma_-'(\tau^{-\frac{1}{\mu}})|\tau^{-1-\frac{1}{\mu}}}{(\xi-\tau)^2+1} \right) du
$$

The integral  $J_1$  can be estimated in  $\mathcal{L}_{p,\beta+1}(0,\frac{\delta}{2})$  by Hardy's inequality. By making the

$$
\leq \frac{c}{x} \int_{(1+\epsilon)x}^{(1+\epsilon)x} |\sigma_{-}(u)| du + \int_{(1-\epsilon)x}^{(1+\epsilon)x} \frac{\rho(u) |\sigma_{-}'(u)|}{(x-u)^2 + (\rho(u))^2} du
$$
\n
$$
= J_1 + J_2.
$$
\nThe integral  $J_1$  can be estimated in  $\mathcal{L}_{p,\beta+1}(0, \frac{\delta}{2})$  by Hardy's inequality. By making the change of variables  $\tau = u^{-\mu}, \xi = x^{-\mu}$  in  $J_2$  we obtain\n
$$
\int_{0}^{\delta/2} x^{p(\beta+1)} |J_2(x)|^p dx \leq c \left( \int_{(\delta/2)^{-\mu}}^{\infty} \frac{\rho(1-\alpha)}{\left(\int_{(\delta/2)^{-\mu}}^{\infty} \frac{|\sigma_{-}'(\tau^{-\frac{1}{\mu}})|\tau^{-1-\frac{1}{\mu}}}{(\xi-\tau)^2+1} d\tau \right)^p d\xi \right)^{\frac{1}{p}},
$$
\nwhere  $\beta + \frac{1}{p} = \mu(\alpha - \frac{1}{p})$ . From Lemma 1 it follows that the integral on the right is majorized by\n
$$
c \int_{(\delta/2)^{-\mu}}^{\infty} |\sigma_{-}'(\tau^{-\frac{1}{\mu}})|^p \tau^{-\frac{p(\beta+1)}{\mu}} \tau^{-1-\frac{1}{\mu}} d\tau = c \int_{0}^{\delta/2} |\sigma_{-}'(u)|^p u^{p(\beta+1)} du.
$$
\nThus,\n
$$
||T \cdot \sigma||_{\mathcal{L}_{p,\beta+1}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma||_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}.
$$
\n(i) Now we estimate the norm in  $\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})$  of  $(\pi I - T)\sigma(z_+) + (\pi I - T)\sigma(z_+)$ . We represent  $(\pi I - T)\sigma(z)$  for  $z = x + i\kappa_+(x) \in \Gamma_{\pm}$  in the form

$$
c\int_{(\delta/2)^{-\mu}}^{\infty}|\sigma'_{-}(\tau^{-\frac{1}{\mu}})|^{p}\tau^{\frac{-p(\beta+1)}{\mu}}\tau^{-1-\frac{1}{\mu}}d\tau=c\int_{0}^{\delta/2}|\sigma'_{-}(u)|^{p}u^{p(\beta+1)}du.
$$

Thus,

$$
\|T_*\sigma\|_{\mathcal{L}_{p,\beta+1}(\Gamma\cap\{|q|<\frac{\delta}{2}\})}\leq c\|\sigma\|_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}.\tag{2.5}
$$

(ii) Now we estimate the norm in  $\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})$  of  $(\pi I - T)\sigma(z_+) + (\pi I)$  $(-T)\sigma(z)$ . We represent  $(\pi I - T)\sigma(z)$  for  $z = x + i\kappa_{\pm}(x) \in \Gamma_{\pm}$  in the form

$$
||T_*\sigma||_{\mathcal{L}_{p,\beta+1}(\Gamma\cap\{|q|<\frac{\delta}{2}\})} \leq c||\sigma||_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}
$$
  
\n
$$
\text{estimate the norm in } \mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q|<\frac{\delta}{2}\}) \text{ of } (\pi
$$
  
\n
$$
\text{present } (\pi I - T)\sigma(z) \text{ for } z = x + i\kappa_{\pm}(x) \in \Gamma_{\pm} \text{ in } t
$$
  
\n
$$
\pm \pi[\sigma(z_+) - \sigma(z_-)]
$$
  
\n
$$
-\int_{\Gamma_{\pm}^{\ell}(z)\cup\Gamma_{\pm}^{\ell}(z)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q
$$
  
\n
$$
+\left[\pi\sigma(z_{\mp}) - \int_{\Gamma_{\pm}^{\epsilon}(z)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q\right]
$$
  
\n
$$
-\int_{\Gamma_{\pm}^{\epsilon}(z)\cup\Gamma_{\pm}^{\epsilon}(z)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q
$$
  
\n
$$
-\int_{\Gamma_{\pm}^{\epsilon}(z)\cup\Gamma_{\pm}^{\epsilon}(z)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = \sum_{k=1}^{6} I_k(z).
$$

For any  $q \in \Gamma_+^c(x) \cup \Gamma_+^{\ell}(x) \cup \Gamma_-^{\ell}(x)$  the inequality

$$
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$$
\n
$$
\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \right| \leq c \, x^{\mu-1} \tag{2.6}
$$

is valid. In fact,

$$
L_p\text{-Theory of Boundary Integral Equations}
$$
\n
$$
q \in \Gamma_+^c(x) \cup \Gamma_+^{\ell}(x) \cup \Gamma_-^{\ell}(x) \text{ the inequality}
$$
\n
$$
\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \right| \leq c x^{\mu-1}
$$
\n
$$
\left| \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \right| \leq c x^{\mu-1}
$$
\n
$$
\frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \left( 1 + (\kappa_+^{\prime}(u)^2)^{\frac{1}{2}} \right| \leq \frac{1}{|z-q|^2} \left| \int_u^r \kappa_+^{\prime\prime}(\tau)(x-u) d\tau \right| \leq c x^{\mu-1},
$$
\n
$$
\frac{c}{4}(x), \text{ and}
$$
\n
$$
g \frac{1}{|z-q|} \left( 1 + (\kappa_+^{\prime}(u)^2)^{\frac{1}{2}} \right| \leq \frac{1}{|z-q|} \left( |\kappa_+^{\prime}(u)| + \frac{|\kappa_+^{\prime}(u) - \kappa_+^{\prime}(x)|}{|q-z|} \right) \leq c \frac{u^{\mu}}{4}(x).
$$
\n
$$
\frac{\ell_+^{\prime}(x)}{2}.
$$
\n
$$
\left| \frac{\ell_+^{\prime}(x)}{2} \right| \leq c \frac{u^{\mu}}{2}
$$
\n
$$
\frac{u^{\mu}}{2}.
$$

if  $q \in \Gamma_+^c(x)$ , and

$$
\left|\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|}\right| \le c x^{\mu-1} \tag{2.6}
$$
\nis valid. In fact,  
\n
$$
\left|\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|}\left(1+(\kappa'_+(u)^2)^{\frac{1}{2}}\right) \le \frac{1}{|z-q|^2} \left|\int_u^z \kappa''_+(r)(x-u)dr\right| \le c x^{\mu-1},
$$
\n
$$
\text{if } q \in \Gamma_+^c(x), \text{ and}
$$
\n
$$
\left|\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|}\left(1+(\kappa'_+(u)^2)^{\frac{1}{2}}\right) \le \frac{1}{|z-q|}\left(|\kappa'_\pm(u)|+\frac{|\kappa_\pm(u)-\kappa_+(x)|}{|q-z|}\right) \le c \frac{u^\mu+x^\mu}{x},
$$
\n
$$
\text{if } q \in \Gamma_\pm^{\ell}(x). \text{ Inequality (2.6) follows. Therefore}
$$
\n(2.6)

if  $q \in \Gamma_{\pm}^{\ell}(x)$ . Inequality (2.6) follows. Therefore

$$
\frac{1}{|z-q|} \left(1 + (\kappa'_+(u)^2)^{\frac{1}{2}}\right) \le \frac{1}{|z-q|^2} \left| \int_u \kappa''_+(r)(x-u) d\tau \right|
$$
\n
$$
\left(1 + (\kappa'_+(u)^2)^{\frac{1}{2}}\right) \le \frac{1}{|z-q|} \left(|\kappa'_\pm(u)| + \frac{|\kappa_\pm(u) - \kappa_\pm(x)|}{|q-z|}\right)
$$
\n
$$
\text{quality (2.6) follows. Therefore}
$$
\n
$$
|I_2(z)| + |I_3(z)| \le c \, x^{\mu-1} \int_0^{(1+\epsilon)x} (|\sigma_+(u)| + |\sigma_-(u)|) du.
$$
\n
$$
\text{at } \epsilon
$$

Hence, the estimate

$$
||I_2||_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} + ||I_3||_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}
$$

results from Hardy's inequality.

Now we estimate  $I_4$ . We can assume that  $z \in \Gamma_+$ . In the sequel we shall use the estimate equa<br> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 

$$
|I_2(z)| + |I_3(z)| \leq c x^{\mu-1} \int_{0}^{(1+\epsilon)x} (|\sigma_+(u)| + |\sigma_-(u)|) du.
$$
  
\nHence, the estimate  
\n
$$
||I_2||_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\epsilon}{2}\})} + ||I_3||_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\epsilon}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}
$$
  
\nresults from Hardy's inequality.  
\nNow we estimate  $I_4$ . We can assume that  $z \in \Gamma_+$ . In the sequel we shall use the estimate  
\n
$$
\left|\frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} + (1 + (\kappa'_+(u))^2)^{-\frac{1}{2}} \frac{\rho(x)}{(x-u)^2 + (\rho(x))^2}\right|
$$
\n
$$
\leq c \left(x^{\mu-1} + \frac{x^{2\mu+1}}{(x-u)^2 + x^{2\mu+1}}\right) \quad (q = u + iv \in \Gamma^c_-(x))
$$
\nwhere  $\rho(x) = \kappa_+(x) - \kappa_-(x)$ . In order to obtain (2.7) we notice that  
\n
$$
|\kappa_-(x) - \kappa_-(u) - \kappa'_-(u)(x-u)| \leq c x^{\mu-1}(x-u)^2.
$$
  
\nTherefore  
\n
$$
-\frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} \left(1 + (\kappa'_+(u)^2)\right)^{\frac{1}{2}}
$$
\n
$$
= \frac{(u-x)\kappa'_-(u) - (\kappa_-(u) - \kappa_+(x))}{(\kappa_-(u) - \kappa_-(u))^2}
$$
\n(2.8)

where  $\rho(x) = \kappa_+(x) - \kappa_-(x)$ . In order to obtain (2.7) we notice that

$$
|\kappa_-(x) - \kappa_-(u) - \kappa_-'(u)(x-u)| \leq c x^{\mu-1}(x-u)^2.
$$

$$
\log \frac{1}{|z-q|} + (1 + (\kappa'_{+}(u))^{2})^{-\frac{1}{2}} \frac{\rho(x)}{(x-u)^{2} + (\rho(x))^{2}}|
$$
\n
$$
\leq c \left( x^{\mu-1} + \frac{x^{2\mu+1}}{(x-u)^{2} + x^{2\mu+1}} \right) \qquad (q = u + iv \in \Gamma_{-}^{c}(x))
$$
\n
$$
+(x) - \kappa_{-}(x). \text{ In order to obtain (2.7) we notice that}
$$
\n
$$
|\kappa_{-}(x) - \kappa_{-}(u) - \kappa'_{-}(u)(x-u)| \leq c x^{\mu-1}(x-u)^{2}.
$$
\n
$$
-\frac{\partial}{\partial n_{q}} \log \frac{1}{|z-q|} \left( 1 + (\kappa'_{+}(u)^{2})^{\frac{1}{2}} \right)
$$
\n
$$
= \frac{(u-x)\kappa'_{-}(u) - (\kappa_{-}(u) - \kappa_{+}(x))}{(x-u)^{2} + (\kappa_{-}(u) - \kappa_{+}(x))^{2}} \qquad (2.8)
$$
\n
$$
= \frac{\kappa_{+}(x) - \kappa_{-}(x)}{(x-u)^{2} + (\kappa_{-}(u) - \kappa_{+}(x))^{2}} + O(x^{\mu-1}).
$$

Taking into account  $|\kappa_-(x) - \kappa_-(u)| \leq c x^\mu |x-u|$  we obtain

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\nount 
$$
|\kappa_-(x) - \kappa_-(u)| \le c x^{\mu} |x - u|
$$
 we obtain  
\n
$$
\frac{\kappa_+(x) - \kappa_-(x)}{(x - u)^2 + (\kappa_-(u) - \kappa_+(x))^2}
$$
\n
$$
= \frac{\rho(x)}{(x - u)^2 + (\rho(x))^2} + O\left(\frac{x^{2\mu+1}}{(x - u)^2 + x^{2\mu+2}}\right).
$$
\n(2.9)  
\nws from (2.8) and (2.9). By making the change of variables  $\tau = u^{-\mu}$ ,  
\nive at  
\n
$$
\int_0^{\delta/2} x^{p(\beta-\mu)} \left( \int_0^{(1+\epsilon)x} \frac{x^{2\mu+1}\sigma_-(u)}{(x - u)^2 + x^{2\mu+2}} du \right)^p dx
$$

Now (2.7) follows from (2.8) and (2.9). By making the change of variables 
$$
\tau = \xi = x^{-\mu}
$$
 we arrive at  
\n
$$
\int_{0}^{\delta/2} x^{p(\beta-\mu)} \left( \int_{(1-\epsilon)z}^{(1+\epsilon)z} \frac{x^{2\mu+1}\sigma_{-}(u)}{(x-u)^2 + x^{2\mu+2}} du \right)^p dx
$$
\n
$$
\leq c \int_{(\delta/2)^{-\mu}}^{+\infty} \xi^{p(1-\alpha)} \left( \int_{\mathbb{R}} \frac{\mu |\sigma_{-}(\tau^{-\frac{1}{\mu}})|}{(\xi-\tau)^2 + \mu^2} \frac{d\tau}{\tau} \right)^p d\xi,
$$
\nwhere  $\beta + \frac{1}{p} = \mu(\alpha - \frac{1}{p})$ . From Lemma 1 it follows that the right-hand side is estir  
\nfrom above by  
\n
$$
\int_{(\delta/2)^{-\mu}}^{+\infty} \tau^{-p\alpha} |\sigma_{-}(\tau^{-\frac{1}{\mu}})|^p d\tau \leq c \int_{0}^{\delta} |\sigma'_{-}(u)|^p u^{p(\beta+1)} du.
$$
\n(6/2) $\int_{0}^{+\infty} |\sigma'_{-}(u)|^p u^{p(\beta+1)} du$ 

where  $\beta + \frac{1}{p} = \mu(\alpha - \frac{1}{p})$ . From Lemma 1 it follows that the right-hand side is estimated from above by

$$
c\int\limits_{(\delta/2)^{-\mu}}^{+\infty} \tau^{-p\alpha}|\sigma_{-}(\tau^{-\frac{1}{\mu}})|^p d\tau \leq c\int\limits_{0}^{\delta} |\sigma'_{-}(u)|^p u^{p(\beta+1)} du.
$$

Thus, it is sufficient to estimate

$$
|\theta - (1 - \epsilon)| u| \leq C \int_{0}^{1} |\theta - (u)|^{2} u^{2}
$$
  
stimate
$$
\pi \sigma_{-}(x) - \int_{(1-\epsilon)x}^{(1+\epsilon)x} \frac{\rho(x) \sigma_{-}(u)}{(x-u)^{2} + (\rho(x))^{2}} du.
$$

We make the change of variables  $u = h(\tau)$ ,  $x = h(\xi)$ , where *h* is specified by (2.1). By Lemma 2 we have

$$
\leq c \int_{(\delta/2)^{-\mu}} \xi^{p(1-\alpha)} \Big( \int_{\mathbb{R}} \overline{(\xi-\tau)^2 + \mu^2} \overline{\tau} \Big) d\xi,
$$
\n
$$
\int_{(\delta/2)^{-\mu}} \mu(\alpha - \frac{1}{p}). \text{ From Lemma 1 it follows that the right-hand side is estimated}
$$
\n
$$
c \int_{(\delta/2)^{-\mu}}^{\infty} \tau^{-p\alpha} |\sigma_{-}(\tau^{-\frac{1}{\mu}})|^p d\tau \leq c \int_{0}^{\delta} |\sigma'_{-}(u)|^p u^{p(\beta+1)} du.
$$
\n
$$
\text{Hicient to estimate}
$$
\n
$$
\pi \sigma_{-}(x) - \int_{(1-\epsilon)x}^{\infty} \frac{\rho(x) \sigma_{-}(u)}{(x-u)^2 + (\rho(x))^2} du.
$$
\n
$$
\text{change of variables } u = h(\tau), x = h(\xi), \text{ where } h \text{ is specified by (2.1). By}
$$
\n
$$
\int_{0}^{\delta/2} x^{p(\beta-\mu)} \left| \pi \sigma_{-}(x) - \int_{(1-\epsilon)x}^{\infty} \frac{\rho(x) \sigma_{-}(u)}{(x-u)^2 + (\rho(x))^2} du \right|^p dx
$$
\n
$$
\leq c \int_{h^{-1}(\delta/2)}^{\infty} \xi^{p(1-\alpha)} \left| \pi \sigma_{-}(h(\xi)) - \int_{\mathbb{R}} \frac{\sigma_{-}(h(\tau))}{(\xi-\tau)^2 + 1} d\tau \right|^p d\xi \qquad (2.10)
$$
\n
$$
+ \int_{h^{-1}(\delta/2)}^{\infty} \xi^{p(1-\alpha)} |I(\xi)|^p d\xi,
$$

where  $I(\xi)$  admits the estimate

$$
L_p\text{-Theory of Boundary Integral Equations}
$$
\n
$$
\text{tr } I(\xi) \text{ admits the estimate}
$$
\n
$$
|I(\xi)| \leq \frac{c}{\xi} \int_{h^{-1}(\delta/2)}^{\xi} |\sigma_{-}(h(\tau))| \frac{d\tau}{\tau} + c \int_{\xi}^{\delta} |\sigma_{-}(h(\tau))| \frac{d\tau}{\tau^2} + c \int_{h^{-1}(\delta/2)}^{\delta} \frac{|\sigma_{-}(h(\tau))|}{(\xi - \tau)^2 + 1} \frac{d\tau}{\tau}
$$
\nm Lemma 1 and Hardy's inequality it follows that the last integral in (2.10) exceed\n
$$
c \int_{h^{-1}(\delta)}^{\infty} \left| \frac{d}{d\tau} \sigma_{-}(h(\tau)) \right|^p \tau^{p(1-\alpha)} d\tau = c \int_{0}^{\delta} |\sigma_{-}'(u)|^p u^{p(\beta+1)} du.
$$
\n
$$
\text{Fourier transform of}
$$

not exceed

From Lemma 1 and Hardy's inequality it follows that the last integral in (2.10) does  
not exceed  

$$
c \int_{h^{-1}(\delta)}^{\infty} \left| \frac{d}{d\tau} \sigma_{-}(h(\tau)) \right|^{p} \tau^{p(1-\alpha)} d\tau = c \int_{0}^{\delta} |\sigma'_{-}(u)|^{p} u^{p(\beta+1)} du.
$$

The Fourier transform of

$$
\pi\sigma_{-}(h(\xi))-\int_{\mathbb{R}}\frac{\sigma_{-}(h(\tau))}{(\xi-\tau)^2+1}d\tau
$$

equals

$$
-\pi i \operatorname{sign}(\nu) \left(\widehat{\sigma \circ h} \right)'(\nu) \frac{1-\exp(-|\nu|)}{|\nu|}.
$$

Since the function  $(1 - \exp(-|\nu|))|\nu|^{-1}$  is the Fourier transform of  $\log(1 + \xi^{-2})$  up to a real factor, it follows from the boundedness of the Hilbert transform in  $\mathcal{L}_{p,1-\alpha}(\mathbb{R})$  (see [3]) and from Lemma 1 that the first integral on the right in (2.10) does not exceed  $c \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}^p$ .  $\pi\sigma_{-}(h(\xi)) - \int_{\mathbb{R}} \frac{\sigma_{-}(h(\tau))}{(\xi - \tau)^2 + 1} d\tau$ <br>  $-\pi i \operatorname{sign}(\nu) (\widehat{\sigma \circ h})'(\nu) \frac{1 - \exp(-|\nu|)}{|\nu|}$ .<br>
ttion  $(1 - \exp(-|\nu|)) |\nu|^{-1}$  is the Fourier transform of  $\log(1 + \xi^{-2})$  up to a<br>
follows from the boundedness of the Hilbert tran

We turn to the integral  $I_5$ . We have

$$
\frac{\partial}{\partial n_q} \log \frac{1}{|q-z|} = -\text{Re} \frac{1}{q-z} \cos(n_q, x) + \text{Im} \frac{1}{q-z} \cos(n_q, y). \tag{2.11}
$$

For  $q \in \Gamma_+^r(x) \cup \Gamma_-^r(x)$  and  $z \in \Gamma \cap \{|q| < \frac{\delta}{2}\}$  we represent the first term in (2.11) in the form form

$$
\frac{d}{d\eta_q} \log \frac{1}{|q-z|} = -\text{Re}\frac{1}{q-z}\cos(n_q, x) + \text{Im}\frac{1}{q-z}\cos(n_q, y)
$$
\n
$$
\text{J}\Gamma_-(x) \text{ and } z \in \Gamma \cap \{|q| < \frac{\delta}{2}\} \text{ we represent the first term}
$$
\n
$$
-\text{Re}\sum_{k=0}^m \frac{z^k}{q^{k+1}}\cos(n_q, x) - \text{Re}\left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right)\cos(n_q, x).
$$
\n
$$
\left|\text{Re}\left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right)\cos(n_q, x)\right| \leq c\, x^{m+1}u^{\mu-m-2}.
$$

It is clear that

$$
\left| \operatorname{Re} \left( \frac{z^{m+1}}{q^{m+1}(q-z)} \right) \cos(n_q, x) \right| \leq c \, z^{m+1} u^{\mu-m-2}.
$$
\nRe second term in (2.11). To this end we make us

\n
$$
\operatorname{Im} \sum_{k=0}^{m} \frac{z^k}{q^{k+1}} + \operatorname{Im} \frac{1}{q^{m+1}} \operatorname{Re} \frac{z^{m+1}}{q-z}
$$

Now we consider the second term in (2.11). To this end we make use of the equality

$$
q \in \Gamma_{+}^{r}(x) \cup \Gamma_{-}^{r}(x) \text{ and } z \in \Gamma \cap \{|q| < \frac{\delta}{2}\} \text{ we represent the first term in (2.11) in the}
$$
\n
$$
-\text{Re} \sum_{k=0}^{m} \frac{z^{k}}{q^{k+1}} \cos(n_{q}, x) - \text{Re} \left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right) \cos(n_{q}, x).
$$
\ns clear that

\n
$$
\left| \text{Re} \left(\frac{z^{m+1}}{q^{m+1}(q-z)}\right) \cos(n_{q}, x) \right| \leq c \, x^{m+1} u^{\mu-m-2}.
$$
\nwe consider the second term in (2.11). To this end we make use of the equality

\n
$$
\text{Im} (q-z)^{-1} = \text{Im} \sum_{k=0}^{m} \frac{z^{k}}{q^{k+1}} + \text{Im} \frac{1}{q^{m+1}} \text{Re} \frac{z^{m+1}}{q-z} + \text{Re} \frac{1}{q^{m+1}} \text{Im} \ z^{m+1} \text{Re} \frac{1}{q-z}.
$$
\n(2.12)

Since  $\text{Im } z^k \text{Re } q^{-k-1} = O(z^{\mu}u^{-1})$  and  $\text{Re } z^k \text{Im } q^{-k-1} = x^k \text{Im } q^{-k-1} + O(x^{\mu}u^{-1}),$  we obtain **M**  650 V. Maz'ya and A. Soloviev<br>
Since Im z<sup>k</sup>Req<sup>-k-1</sup> =  $O(x^{\mu}u^{-1})$  and Rez<sup>k</sup>Imq<sup>-k-1</sup> =  $x^{k}$ Imq<sup>-k-1</sup> +  $O(x^{\mu}u^{-1})$ , we<br>
obtain<br>  $\lim_{k\to 0} \frac{m}{q^{k+1}} = \sum_{k=0}^{m} x^{k} \text{Im } q^{-k-1} + O(x^{\mu}u^{-1})$ .<br>
The second term in (2.12

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\n
$$
= O(x^{\mu}u^{-1}) \text{ and Re } z^{k}\text{Im } q^{-k-1} = x^{k}\text{Im } q^{-k}
$$
\n
$$
\text{Im } \sum_{k=0}^{m} \frac{z^{k}}{q^{k+1}} = \sum_{k=0}^{m} x^{k}\text{Im } q^{-k-1} + O(x^{\mu}u^{-1}).
$$

 $x^{\mu}u^{-1}$ , we obtain for the third term in (2.12) Im  $\sum_{k=0} \frac{1}{q^{k+1}} = \sum_{k=0} x^k \text{Im } q^{-1}$ <br>
Perm in (2.12) does not exceed  $cx^mu$ <br>
btain for the third term in (2.12)<br>  $\left[ \text{Re } q^{-m-1} \text{Re } z^{m+1} \text{Im } (q-z)^{-1} \right] \leq 1$ <br>
p. in (2.12) satisfies  $\sum_{k=0}^{n} q^{n+1}$ <br>
.12) does n<br>
the third<br>  $\sum_{k=0}^{n} (q^{n+1})^2$ <br>
2) satisfies<br>  $\left[ \text{Re } q^{-m-1} \right]$ <br>  $\Gamma_+^r(x) \cup \Gamma$ 

$$
|\text{Re } q^{-m-1} \text{Re } z^{m+1} \text{Im } (q-z)^{-1}| \leq c (x^{m+1} u^{\mu-m-2} + x^{\mu} u^{-1}).
$$

The last term in (2.12) satisfies

$$
|\text{Re } q^{-m-1} \text{Im } z^{m+1} \text{Re } (q-z)^{-1}| \leq c z^{\mu} u^{-1}.
$$

Thus, we have for  $q \in \Gamma^r_+(x) \cup \Gamma^r_-(x)$ 

The last term in (2.12) satisfies  
\n
$$
|\text{Re } q^{-m-1} \text{Re } z^{m+1} \text{Im } (q-z)^{-1}| \leq c (x^{m+1}u^{\mu-m-2} + x^{\mu}u^{-1}).
$$
\nThe last term in (2.12) satisfies  
\n
$$
|\text{Re } q^{-m-1} \text{Im } z^{m+1} \text{Re } (q-z)^{-1}| \leq c x^{\mu}u^{-1}.
$$
\nThus, we have for  $q \in \Gamma_+^r(x) \cup \Gamma_-^r(x)$   
\n
$$
\frac{\partial}{\partial n_q} \log \frac{1}{|q-z|} = \sum_{k=0}^m x^k \left( -\text{Re } \frac{1}{q^{k+1}} \cos(n_q, x) + \text{Im } \frac{1}{q^{k+1}} \cos(n_q, y) \right) + I(q, z),
$$
\nwhere  
\n
$$
|I(q, z)| \leq c (x^{m+1}u^{\mu-m-2} + x^{\mu}u^{-1}).
$$
\nTherefore  
\n
$$
\int_{\Gamma_+^r(x) \cup \Gamma_-^r(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = \sum_{k=0}^m c^{(k)}(\sigma) x^k + (R_1 \sigma)(z),
$$
\nwhere  
\n
$$
c^{(k)}(\sigma) = \int_{\Gamma_+ \cup \Gamma_-} \sigma(q) \left( -\text{Re } \frac{1}{q^{k+1}} \cos(n_q, x) + \text{Im } \frac{1}{q^{k+1}} \cos(n_q, y) \right) ds_q
$$
\nand  $R_1 \sigma$  admits the estimate  
\n
$$
||R_1 \sigma||_{\mathcal{L}_{p,\sigma-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\sigma+1}(\Gamma)}.
$$
\nIt follows from Hardy's inequality that  $c^{(k)}(\sigma)$   $(k = 1, ..., m)$  are linear contin functions  
\nfunctions in  $\mathcal{L}_{p,\beta-\mu}(\Gamma)$ .

 $(x^{m+1}u^{\mu-m-2} + x^{\mu}u^{-1})$ . Therefore

$$
g\frac{1}{|q-z|} = \sum_{k=0} x^k \left( -\operatorname{Re} \frac{1}{q^{k+1}} \cos(n_q, x) + \operatorname{Im} \frac{1}{q^{k+1}} \cos(n_q, y) \right) + .
$$
  
\n
$$
|z| \le c \left( x^{m+1} u^{\mu-m-2} + x^{\mu} u^{-1} \right). \text{ Therefore}
$$
  
\n
$$
\int_{\Gamma_+^*(x) \cup \Gamma_-^*(x)} \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = \sum_{k=0}^m c^{(k)}(\sigma) x^k + (R_1 \sigma)(z),
$$

where

$$
J\Gamma_{+}^{r}(x)\cup\Gamma_{-}^{r}(x) \qquad \text{or} \qquad |z - q| \qquad \frac{1}{k=0}
$$
\n
$$
c^{(k)}(\sigma) = \int_{\Gamma_{+}\cup\Gamma_{-}} \sigma(q) \Big( -\text{Re}\frac{1}{q^{k+1}}\cos(n_{q}, x) + \text{Im}\frac{1}{q^{k+1}}\cos(n_{q}, y) \Big) ds_{q}
$$
\nadmits the estimate

and  $R_1\sigma$  admits the estimate

$$
||R_1\sigma||_{\mathcal{L}_{p,\beta-\mu}(\Gamma\cap\{|q|<\frac{\delta}{2}\})}\leq c||\sigma'||_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$

It follows from Hardy's inequality that  $c^{(k)}(\sigma)$   $(k = 1, ..., m)$  are linear continuous functionals in  $\mathcal{L}_{p,\beta-\mu}(\Gamma)$ .

It is clear that *16* is represented in the form

$$
\sum_{k=0}^m c^{(k)}x^k + (R_2\sigma)(z),
$$

where

and 
$$
c^{(k)}(\sigma)
$$
  $(k = 1, \ldots, m)$  are  $\beta - \mu(\Gamma)$ .

\n $I_6$  is represented in the form

\n
$$
\sum_{k=0}^{m} c^{(k)} x^k + (R_2 \sigma)(z),
$$
\n
$$
\sum_{k=0}^{m} |c^{(k)}| + \|R_2 \sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$

Finally,

$$
\lim_{n \to \infty} \lim_{L_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} - \lim_{n \to \infty} \lim_{L_{p,\beta+1}(\Gamma)} \lim_{L_{p,\beta+\mu}(\Gamma)}
$$
\nwe find  $L_{p,\beta-\mu}(\Gamma)$ .

\ns clear that  $I_6$  is represented in the form

\n
$$
\sum_{k=0}^{m} c^{(k)} x^k + (R_2 \sigma)(z),
$$
\n
$$
\sum_{k=0}^{m} |c^{(k)}| + \|R_2 \sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)}.
$$
\nwhere  $(\pi\sigma - T\sigma)(z) = \pm \pi (\sigma(z_+) - \sigma(z_-)) + \sum_{k=0}^{m} c^{(k)} x^k + (R\sigma)(z)$  (where  $z \in \Gamma_+),$ 

\n
$$
\sum_{k=0}^{m} |c^{(k)}| + \|R\sigma\|_{\mathcal{L}_{p,\beta-\mu}(\Gamma \cap \{|q| < \frac{\delta}{2}\})} \leq c \|\sigma'\|_{\mathcal{L}_{p,\beta+1}(\Gamma)},
$$

where

$$
\sum_{k=0}^{\infty} |c^{(k)}| + ||R\sigma||_{\mathcal{L}_{p,\theta-\mu}(\Gamma \cap \{|q| < \frac{\theta}{2}\})} \leq c ||\sigma'||_{\mathcal{L}_{p,\theta+1}(\Gamma)},
$$

with

$$
(R\sigma)(z)=\sum_{k=2}^4 I_k(z)+(R_1\sigma)(z)+(R_2\sigma)(z).
$$

Hence and by (2.5) we obtain the boundedness of  $(\pi I - T) : \mathcal{L}_{p,\beta+1}^1(\Gamma) \longrightarrow \mathfrak{M}_{p,\beta}(\Gamma)$ 

By changing the direction of the normal *n* we obtain the following assertion from Theorem 1.

**Corollary.** Let  $\Omega$  have an outward peak, and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ . Then the *operator*

$$
(\pi I + T) : \mathcal{L}_{p,\beta+1}^1(\Gamma) \ni \sigma \longmapsto (\pi I + T)\sigma \in \mathfrak{M}_{p,\beta}(\Gamma)
$$

i*s continuous.* 

In passing, we have proved the following statements.

**Proposition 1.** Let  $\Omega$  have either inward or outward peak and let  $0 < \beta + \frac{1}{p}$  $\min\{\mu,1\}$ . Then the operators

$$
T: \mathcal{L}_{p,\beta+1}^1(\Gamma) \ni \sigma \longmapsto T\sigma \in \mathcal{L}_{p,\beta+1}^1(\Gamma)
$$
  

$$
S: \mathcal{L}_{p,\beta+1}(\Gamma) \ni \sigma \longmapsto S\sigma \in \mathcal{L}_{p,\beta+1}(\Gamma)
$$

*are continuous.* 

#### **3. Asymptotic representation of a conformal mapping**

We shall make use of the following lemma.

**Lemma 3.** *A conformal mapping*  $\theta$  *of*  $\mathbb{R}^2_+ = {\mathcal{C} = \xi + i\eta : \eta > 0}$  *onto*  $\Omega$ ,  $\theta(0) = 0$ , *has the representation* 

continuous.

\nasymptotic representation of a conformal mapping

\ncall make use of the following lemma.

\nerman 3. A conformal mapping θ of ℝ<sup>2</sup><sub>+</sub> = {ζ = ξ + iη : η > 0} onto Ω, θ(0) = 0,

\nsee representation

\n
$$
\theta(\xi) = \begin{cases}\n\sum_{k=2}^{[2\mu]+1} B^{(k)}\xi^{k} + B^{([2\mu]+2)}\xi^{[2\mu]+2} \log |\xi| + B^{(\pm)}|\xi|^{2\mu+2} \\
\sum_{k=2}^{[2\mu]+2} B^{(k)}\xi^{k} + B^{(\pm)}|\xi|^{2\mu+2} + B^{([2\mu]+3)}\xi^{[2\mu]+3}\n\end{cases}
$$
\n
$$
\theta(\xi) = \begin{cases}\n\sum_{k=2}^{[2\mu]+2} B^{(k)}\xi^{k} + B^{(\pm)}|\xi|^{2\mu+2} + B^{([2\mu]+3)}\xi^{[2\mu]+3} \\
\sum_{k=2}^{[2\mu]+2} B^{(k)}\xi^{k} + B^{(\pm)}|\xi|^{2\mu+2} + B^{([2\mu]+3)}\xi^{[2\mu]+3}\n\end{cases}
$$
\n
$$
\Rightarrow \pm 0, \text{ where } B^{(k)} \text{ (k = 2, ... , } [2\mu] + 2) \text{ are real coefficients. Decomposition (3.1)}
$$
\nto find that each once.

\nProof. By *D* we denote the image of Ω under the mapping  $u + iv = (x + iy)^{\frac{1}{2}}$ . The  
\narray *∂D* near the origin is the graph of the function  $\kappa(u)$  such that

\n
$$
\kappa(u) = \pm \frac{1}{2} \alpha_{\pm} |u|^{2\mu+1} (1 + O(u^{2\min\{2\mu, 1\}})) \qquad (u \to \pm 0).
$$
\nasymptotic decomposition can be differentiated twice.

\net  $\vec{\theta}$  denote a conformal mapping of ℝ<sup>2</sup> onto *D*, normalized by  $\vec{\theta}(0) = 0$ .

 According

as  $\xi \to \pm 0$ , where  $B^{(k)}$   $(k = 2, ..., [2\mu] + 2)$  are real coefficients. Decomposition (3.1) *can be differentiated at least once.* 

**Proof.** By *D* we denote the image of  $\Omega$  under the mapping  $u + iv = (x + iy)^{\frac{1}{2}}$ . The boundary  $\partial D$  near the origin is the graph of the function  $\kappa(u)$  such that Fre  $B^{(1)}$   $(\kappa = 2,...,|Z\mu| + Z)$  are real coefficies<br>iated at least once.<br>D we denote the image of  $\Omega$  under the mappin<br>lear the origin is the graph of the function  $\kappa(u) = \pm \frac{1}{2}\alpha_{\pm}|u|^{2\mu+1}(1 + O(u^{2\min\{2\mu,1\}}))$ 

$$
\kappa(u) = \pm \frac{1}{2} \alpha_{\pm} |u|^{2\mu+1} \left( 1 + O\big( u^{2\min\{2\mu,1\}} \big) \right) \qquad (u \to \pm 0). \tag{3.2}
$$

This asymptotic decomposition can be differentiated twice.

Let  $\widetilde{\theta}$  denote a conformal mapping of  $\mathbb{R}_+^2$  onto  $D,$  normalized by  $\widetilde{\theta}(0) = 0.$  According to Kellogg's conformal mapping theorem [4] we have Figure 1 in the differentiate<br>
Re opting theorem [4] we have the proof  $\theta$ <br>
Re  $\widetilde{\theta}(\xi) = \xi + \psi(\xi)$ *(a)*  $\forall i \in \mathbb{N}$ ,<br> *(a)* coefficients. Decomposition (3.1)<br>
he mapping  $u + iv = (x + iy)^{\frac{1}{2}}$ . The<br>
metion  $\kappa(u)$  such that<br>  $\kappa(u)$  such that<br>  $\kappa(u)$  and that<br>  $\kappa(u)$ ,  $(u \to \pm 0)$ . (3.2)<br>
d twice.<br>
, normalized by  $\widetilde$ 

$$
\operatorname{Re}\tilde{\theta}(\xi)=\xi+\psi(\xi)\qquad(\xi\in\mathbb{R}),\tag{3.3}
$$

where  $\psi(0) = 0$ , and  $\psi'$  satisfies the Hölder condition

\n
$$
\forall y \text{a}
$$
 and A. Soloviev\n

\n\n $\text{and } \psi'$  satisfies the Hölder condition\n

\n\n $|\psi'(\xi_1) - \psi'(\xi_2)| \leq c |\xi_1 - \xi_2|^{\gamma}$ \n $(0 < \gamma < \min\{2\mu, 1\})$ \n

\n\n $\text{(3.3) to (3.2) we obtain}$ \n

By substituting (3.3) to (3.2) we obtain

$$
|\psi'(\xi_2)| \le c |\xi_1 - \xi_2|^\gamma \qquad (0 < \gamma < \min\{2\})
$$
  
to (3.2) we obtain  

$$
\frac{d}{d\xi} \operatorname{Im} \tilde{\theta}(\xi) = |\xi|^{2\mu} (\beta_{\pm} + \lambda(\xi)) \qquad (\xi \to \pm 0),
$$

where  $\lambda$  satisfies the Hölder condition with a small exponent  $\gamma$  and  $\lambda(0) = 0$ .

The derivative of  $\tilde{\theta}(\xi)$  belongs to the Hardy space  $H^1$  in the upper half-plane  $\mathbb{R}^2_+$ . Therefore

$$
\frac{d}{d\xi}\operatorname{Re}\widetilde{\theta}(\xi)=\mathfrak{H}\Big(\frac{d}{d\xi}\operatorname{Im}\widetilde{\theta}\Big)(\xi)+c,
$$

form

where 
$$
\lambda
$$
 satisfies the Hölder condition with a small exponent  $\gamma$  and  $\lambda(0) = 0$ .  
\nThe derivative of  $\tilde{\theta}(\xi)$  belongs to the Hardy space  $H^1$  in the upper half-plane  $\mathbb{R}^2_+$ .  
\nTherefore  
\n
$$
\frac{d}{d\xi} \text{Re } \tilde{\theta}(\xi) = 5 \left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right) (\xi) + c,
$$
\nwhere  $5$  denotes the Hilbert transform. We represent  $5 \left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right) (\xi)$  for  $\xi > 0$  in the form  
\n
$$
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d\xi} \text{Im } \tilde{\theta}(t) \frac{dt}{\xi - t}
$$
\n
$$
= \frac{1}{\pi} \int_{0}^{\xi} \left( \frac{d}{d\xi} \text{Im } \tilde{\theta}(\xi - t) - \frac{d}{d\xi} \text{Im } \tilde{\theta}(\xi + t) \right) \frac{dt}{t} + \frac{1}{\pi} \int_{0}^{2\xi} \frac{d}{d\xi} \text{Im } \tilde{\theta}(-t) \frac{dt}{\xi + t}
$$
\n
$$
+ \frac{2}{\pi} \int_{2\xi}^{\infty} \left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right)^{(-)}(t) \frac{tdt}{\xi^2 - t^2} + \frac{2}{\pi} \xi \int_{2\xi}^{\infty} \left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right)^{(+)}(t) \frac{dt}{\xi^2 - t^2}
$$
\nwhere  
\n
$$
\left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right)^{(-)}(\xi) = \frac{1}{2} \left( \frac{d}{d\xi} \text{Im } \tilde{\theta}(\xi) - \frac{d}{d\xi} \text{Im } \tilde{\theta}(-\xi) \right),
$$
\n
$$
\left( \frac{d}{d\xi} \text{Im } \tilde{\theta} \right)^{(+)}(\xi) = \frac{1}{2} \left( \frac{d}{d\xi} \text{Im } \tilde{\theta}(\xi) + \frac{d}{d
$$

where

$$
\left(\frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}\right)^{(-)}(\xi) = \frac{1}{2}\left(\frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}(\xi) - \frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}(-\xi)\right),
$$

$$
\left(\frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}\right)^{(+)}(\xi) = \frac{1}{2}\left(\frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}(\xi) + \frac{d}{d\xi}\mathrm{Im}\,\tilde{\theta}(-\xi)\right).
$$

By this representation we obtain

$$
\frac{d}{d\xi} \text{Im}(\theta) \qquad (1) \frac{d}{\xi^2 - t^2} + \frac{1}{\pi} \sum_{\xi} \int \frac{d}{d\xi} \text{Im}(\theta) \qquad (1) \frac{d}{\xi^2 - t^2}
$$
\nwhere\n
$$
\left(\frac{d}{d\xi} \text{Im}(\tilde{\theta})\right)^{(-)}(\xi) = \frac{1}{2} \left(\frac{d}{d\xi} \text{Im}(\tilde{\theta}(\xi)) - \frac{d}{d\xi} \text{Im}(\tilde{\theta}(-\xi))\right),
$$
\n
$$
\left(\frac{d}{d\xi} \text{Im}(\tilde{\theta})\right)^{(+)}(\xi) = \frac{1}{2} \left(\frac{d}{d\xi} \text{Im}(\tilde{\theta}(\xi)) + \frac{d}{d\xi} \text{Im}(\tilde{\theta}(-\xi)\right).
$$
\nBy this representation we obtain\n
$$
\frac{d}{d\xi} \text{Re}(\tilde{\theta}(\xi)) = \begin{cases}\n\sum_{k=0}^{\lfloor 2\mu \rfloor - 1} a^{(k)} \xi^k + a^{(\lfloor 2\mu \rfloor)} \xi^{\lfloor 2\mu \rfloor} \log |\xi| + a^{(\pm)} |\xi|^{2\mu} + O(\xi^{2\mu + \gamma}) & \text{if } 2\mu \in \mathbb{N} \\
\sum_{k=0}^{\lfloor 2\mu \rfloor} a^{(k)} \xi^k + a^{(\pm)} |\xi|^{2\mu} + a^{(\lfloor 2\mu \rfloor + 1)} \xi^{\lfloor 2\mu \rfloor + 1} + O(\xi^{2\mu + \gamma}) & \text{if } 2\mu \notin \mathbb{N}\n\end{cases}
$$
\n
$$
\tilde{\theta}(\xi) = \begin{cases}\n\sum_{k=1}^{\lfloor 2\mu \rfloor} b^{(k)} \xi^k + b^{(\lfloor 2\mu \rfloor + 1)} \xi^{\lfloor 2\mu \rfloor + 1} \log |\xi| + b^{(\pm)} |\xi|^{2\mu + 1} + O(\xi^{2\mu + 1 + \gamma}) & \text{if } 2\mu \in \mathbb{N} \\
\sum_{k=1}^{\lfloor 2\mu \rfloor + 1} b^{(k)} \xi^k + b^{(\pm)} |\xi|^{2\mu + 1} + b^{(\lfloor 2\mu \rf
$$

as  $\xi \rightarrow \pm 0$ . Hence,

as 
$$
\xi \to \pm 0
$$
. Hence,  
\n
$$
\tilde{\theta}(\xi) = \begin{cases}\n\sum_{k=1}^{[2\mu]} b^{(k)} \xi^k + b^{([2\mu]+1)} \xi^{[2\mu]+1} \log |\xi| + b^{(\pm)} |\xi|^{2\mu+1} + O(\xi^{2\mu+1+\gamma}) & \text{if } 2\mu \in \mathbb{N} \\
\sum_{k=1}^{[2\mu]+1} b^{(k)} \xi^k + b^{(\pm)} |\xi|^{2\mu+1} + b^{([2\mu]+2)} \xi^{[2\mu]+2} + O(\xi^{2\mu+1+\gamma}) & \text{if } 2\mu \notin \mathbb{N}.\n\end{cases}
$$

Squaring the preceding representation we arrive at  $(3.1)$ 

The inverse mapping  $\theta^{-1}(z)$  on  $\Gamma_{\pm}$  has the form

$$
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$$
\n
$$
\text{erse mapping }\theta^{-1}(z) \text{ on } \Gamma_{\pm} \text{ has the form}
$$
\n
$$
\xi = \sum_{k=1}^{\lfloor 2\mu \rfloor} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} + \left(\pm 1\right)^{\lfloor 2\mu \rfloor + 1} \beta^{(\lfloor 2\mu \rfloor + 1)} x^{\frac{(\lfloor 2\mu \rfloor + 1)}{2}} \log \frac{1}{x} + \beta^{(\pm)} x^{\mu + \frac{1}{2}} + o(x^{\mu + \frac{1}{2}})
$$
\nand

\n
$$
\xi = \sum_{k=1}^{\lfloor 2\mu \rfloor + 1} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} + \beta^{(\pm)} x^{\mu + \frac{1}{2}} + o(x^{\mu + \frac{1}{2}})
$$
\n(3.5)

\nHere  $\beta^{(k)}$   $(k = 1, \ldots, m + 1)$  are real coefficients.

if  $2\mu \in \mathbb{N}$ , and

$$
\xi = \sum_{k=1}^{\left[2\mu\right]+1} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} + \beta^{(\pm)} x^{\mu+\frac{1}{2}} + o(x^{\mu+\frac{1}{2}})
$$
(3.5)  

$$
\mu \notin \mathbb{N}.
$$
 Here  $\beta^{(k)}$   $(k = 1, ..., m + 1)$  are real coefficients.  
We notice that there exists a function of the form

if  $2\mu \notin \mathbb{N}$ . Here  $\beta^{(k)}$   $(k = 1, ..., m + 1)$  are real coefficients.

$$
k=1
$$
\n
$$
+ (\pm 1)^{[2\mu]+1} \beta^{([2\mu]+1)} x^{\frac{([2\mu]+1)}{2}} \log \frac{1}{x} + \beta^{(\pm)} x^{\mu+\frac{1}{2}} + o(x^{\mu+\frac{1}{2}})
$$
\n
$$
\zeta = \sum_{k=1}^{[2\mu]+1} (\pm 1)^k \beta^{(k)} x^{\frac{k}{2}} + \beta^{(\pm)} x^{\mu+\frac{1}{2}} + o(x^{\mu+\frac{1}{2}})
$$
\n
$$
= e \beta^{(k)} (k = 1, ..., m + 1) \text{ are real coefficients.}
$$
\n
$$
k=1
$$
\n
$$
k=1
$$
\n
$$
d(\zeta) = \begin{cases} \zeta + \sum_{k=1}^{[2\mu]} a^{(k)} \zeta^k + a^{([2\mu]+1)} \zeta^{[2\mu]+1} \log \zeta & \text{if } 2\mu \in \mathbb{N} \\ \zeta + \sum_{k=1}^{[2\mu]+1} a^{(k)} \zeta^k & \text{if } 2\mu \notin \mathbb{N} \end{cases}
$$
\n
$$
k^2 + \zeta^2 + \zeta^2 + \zeta^2 + o(\zeta^2 + o(\zeta^2
$$

defined on  $\mathbb{R}^2_+$  and satisfying

$$
(\widetilde{\theta}\circ d)(\zeta)=\zeta+O(\zeta^{2\mu+1}).
$$

It is clear that  $\theta_0 = (\widetilde{\theta} \circ d)^2$  is the conformal mapping of a neighbourhood of  $\zeta = 0$  in  $\mathbb{R}^2_+$  onto a neighbourhood of peak in  $\Omega$  and has the representation  $(\widetilde{\theta} \circ d)(\zeta) = \zeta + O(\zeta^{2\mu+1})$ <br>  $(\widetilde{\theta} \circ d)^2$  is the conformal mapping<br>
rhood of peak in  $\Omega$  and has the rep<br>  $x = \text{Re } \theta_0(\xi) = \xi^2 + O(\xi^{2\mu+2})$ 

$$
x = \text{Re}\,\theta_0(\xi) = \xi^2 + O(\xi^{2\mu+2})
$$
 as  $\xi \to \pm 0$ . (3.6)

The inverse mapping  $\theta_0^{-1}$  has the form

$$
\xi = \text{Re}\,\theta_0^{-1}(z) = \pm x^{\frac{1}{2}} + O(x^{\mu + \frac{1}{2}})
$$
 on  $\Gamma_{\pm}$ .

By diminishing  $\delta$  in the definition of  $\Gamma_{\pm}$  we can assume that  $\theta_0$  is defined on  $\Gamma_{+} \cup \Gamma_{-}$ .

#### 4. Auxiliary boundary value problems for a domain with outward peak

**4. Auxiliary boundary value problems for a domain w outward peak**<br>Let  $n_0$  be the integer subject to the inequalities  $n_0 - 1 \leq 2(\mu - \beta - \frac{1}{p})$ <br> $m = [\mu - \beta - \frac{1}{p} + \frac{1}{2}]$  is the largest integer satisfying  $2m \leq n_0$ . Let  $n_0$  be the integer subject to the inequalities  $n_0 - 1 \leq 2(\mu - \beta - \frac{1}{n}) < n_0$ . Then

We shall make use of the following proposition proved in [4].

**Proposition 2.** Let  $\Omega$  have an outward peak and let  $\varphi$  belong to  $\mathfrak{N}_{\mathbf{p},\beta}^{(-)}(\Gamma)$ , where  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ . Then there exists a harmonic extension *h* onto  $\Omega$  of  $\varphi$  with *normal derivative in*  $\mathcal{L}_{p,\beta+1}(\Gamma)$  *satisfying* 

$$
\left\|\frac{\partial}{\partial n}h\right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \left\|\varphi\right\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}.
$$

Now we prove the following existence result.

**Proposition 3.** Let  $\Omega$  have an inward peak and let  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ , where  $0 < \beta + \frac{1}{p}$ **654** V. Maz'ya and A. Soloviev<br> **Proposition 3.** Let  $\Omega$  have an inward peak and let  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ , where  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$  and  $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Then there exists a harmonic extension  $\Omega^c$  with normal derivative in the space  $\mathcal{L}_{p,\beta+1}(\Gamma)$  such that the conjugate function g,  $g(z_0) = 0$  with a fixed point  $z_0 \in \Gamma \setminus \{O\}$ , has the representation **654** V. Maz'ya and A. Soloviev<br> **Proposition 3.** Let  $\Omega$  have an inward peak and let  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ , where <br>  $\min\{\mu,1\}$  and  $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Then there exists a harmonic extens<br>  $\Omega^c$  with no

$$
\sum_{k=1}^{m} c_k(\varphi) \text{Re} z^{k-\frac{1}{2}} + g^{\#}(z),
$$

$$
\|g^\#\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c \, \|\varphi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}
$$

with c independent of  $\varphi$ .

**Proof.** (i) We start with the case when  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  vanishes outside  $\Gamma_+ \cup \Gamma_-$ . We extend the function  $\Phi(\tau) = (\varphi \circ \theta_0)(\tau)$  by zero outside a small neighbourhood of *O*.

We first prove the estimate

where 
$$
c_k(\varphi)
$$
 are linear continuous functionals in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and  $g^{\#}$  satisfies  
\n
$$
||g^{\#}||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}
$$
\nwith *c* independent of  $\varphi$ .  
\nProof. (i) We start with the case when  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  vanishes outside  $\Gamma_{+} \cup \Gamma_{-}$ . We  
\nextend the function  $\Phi(\tau) = (\varphi \circ \theta_{0})(\tau)$  by zero outside a small neighbourhood of *O*.  
\nWe first prove the estimate  
\n
$$
c_1 ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)} \leq ||\Phi^{(+)}||_{L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{'}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{'}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}
$$
\nwhere  $\Phi^{(+)}(\xi) = \frac{\Phi(\xi) + \Phi(-\xi)}{2}$ . Let  $\tau$  be a measurable function on  $(0, \infty)$  subject to  
\n $|r(\xi)| \leq \xi^{2\mu+1}$ . We choose  $\ell \in [0, 1]$  such that  $\frac{\ell}{2} < \beta + \frac{1}{p} < \frac{\ell+1}{2}$ . Then, from the  
\nboundedness of the Hardy-Littlewood maximal operator in  $\Gamma$ .

boundedness of the Hardy-Littlewood maximal operator in  $\mathcal{L}_{p,2\beta-\ell+\frac{1}{p}}(\mathbb{R})$  (see [9]), we obtain

(i) We start with the case when 
$$
\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)
$$
 vanishes outside  $\Gamma_{+} \cup \Gamma_{-}$ . We  
\nunction  $\Phi(\tau) = (\varphi \circ \theta_{0})(\tau)$  by zero outside a small neighborhood of *O*.  
\nprove the estimate  
\n
$$
\mathfrak{N}_{p,\beta}^{(+)}(\Gamma) \leq ||\Phi^{(+)}||_{L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{'}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{'}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}
$$
\n
$$
\leq c_{2} ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)},
$$
\n
$$
\xi) = \frac{\Phi(\xi) + \Phi(-\xi)}{2}.
$$
 Let  $\tau$  be a measurable function on  $(0, \infty)$  subject to  
\n<sup>+</sup>1. We choose  $\ell \in [0,1]$  such that  $\frac{\ell}{2} < \beta + \frac{1}{p} < \frac{\ell+1}{2}$ . Then, from the  
\ns of the Hardy-Littlewood maximal operator in  $\mathcal{L}_{p,2\beta-\ell+\frac{1}{p}}(\mathbb{R})$  (see [9]), we  
\n
$$
\int_{\mathbb{R}} |\Phi(\xi) - \Phi(\xi + r(\xi))|^p |\xi|^{2\beta p - 2\mu p + 1} d\xi
$$
\n
$$
\leq c \int_{\mathbb{R}} \left( \frac{1}{|\xi|^{2\mu+1}} \int_{\xi - c\xi^{2\mu+1}}^{\xi + c\xi^{2\mu+1}} |r^{1+\ell} \frac{d}{d\tau} \Phi(\tau) |d\tau \right)^p |\xi|^{2\beta p - \ell p + 1} d\xi \qquad (4.2)
$$
\n
$$
\leq c \int_{\mathbb{R}} |\frac{d}{d\xi} \Phi(\xi)|^p |\xi|^{2\beta p + p + 1} d\xi.
$$

For  $z \in \Gamma_+$  we have  $|\theta_0^{-1}(z) + \theta_0^{-1}(z_-)| \le c \xi^{2\mu+1}$ . Hence and by (4.2) the left inequality in (4.1) follows.<br>Let *h* be a measurable function on  $[0, \delta]$  such that  $|h(x)| \le x^{\mu+1}$ . As in (4.2) we have in (4.1) follows.

Let h be a measurable function on  $[0, \delta]$  such that  $|h(x)| \leq x^{\mu+1}$ . As in (4.2) we have

$$
c \int_{\mathbb{R}} \left| \frac{d}{d\xi} \Phi(\xi) \right|^p |\xi|^{2\beta p + p + 1} d\xi.
$$
  
\n
$$
|\theta_0^{-1}(z) + \theta_0^{-1}(z_-)| \le c \xi^{2\mu + 1}.
$$
 Hence and by (4.2) the left inequality  
\nasurable function on  $[0, \delta]$  such that  $|h(x)| \le x^{\mu + 1}$ . As in (4.2) we  
\n
$$
\int_0^{\delta} |\varphi(x) - \varphi(x + h(x))|^p x^{(\beta - \mu)p} dx
$$
  
\n
$$
\le c \int_0^{\delta} \left( \frac{1}{x^{\mu + 1}} \int_{x - c x^{\mu + 1}}^{x + c x^{\mu + 1}} t \left| \frac{d}{dt} \varphi(t) \right| dt \right)^p x^{\beta} dx \qquad (4.3)
$$
  
\n
$$
\le c \int_0^{\delta} \left| \frac{d}{dt} \varphi(t) \right|^p x^{(\beta + 1)p} dx.
$$

By using (3.6) we obtain that for  $\xi$  in a small neighbourhood of the origin the distance By using (3.6) we obtain that for  $\xi$  in a small neighbourhood of the origin the distance<br>between  $\theta_0(\xi)$  and  $\theta_0(-\xi)$  does not exceed  $c x^{\mu+1}$ . Hence and by (4.3) the right-hand<br>inequality in (4.1) follows.<br>We int inequality in (4.1) follows.

We introduce a function  $H$  by

$$
\mathcal{H}(\zeta) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\Phi}{d\tau}(\tau) \operatorname{Re} \log \frac{\zeta - \tau}{\zeta} d\tau \qquad (\zeta = \xi + i\eta \in \mathbb{R}^2_+).
$$

From the norm inequality for the Hilbert transform of even functions in  $\mathcal{L}_{p,2\beta-1+\frac{1}{2}}(\mathbb{R})$ (see  $[1]$ ) it follows that the function

$$
\frac{\partial}{\partial \xi} \mathcal{H}^{(+)}(\xi) = \frac{1}{\pi \xi^2} \int_{\mathbb{R}} \frac{d}{d\tau} \Phi^{(-)}(\tau) \frac{\tau^2 d\tau}{\xi - \tau}
$$

satisfies

$$
\text{Hence, } \mathcal{H}(\zeta) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\Phi}{d\tau}(\tau) \text{Re} \log \frac{\zeta - \tau}{\zeta} d\tau \quad (\zeta = \xi + i\eta \in \mathbb{R}^2_+).
$$
\nFrom the norm inequality for the Hilbert transform of even functions in  $\mathcal{L}_{p,2\beta-1+\frac{1}{p}}(\mathbb{R})$  (see [1]) it follows that the function

\n
$$
\frac{\partial}{\partial \xi} \mathcal{H}^{(+)}(\xi) = \frac{1}{\pi \xi^2} \int_{\mathbb{R}} \frac{d}{d\tau} \Phi^{(-)}(\tau) \frac{\tau^2 d\tau}{\xi - \tau}
$$
\nsatisfies

\n
$$
\left\| \frac{\partial}{\partial \xi} \mathcal{H}^{(+)} \right\|_{\mathcal{L}_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{\mathcal{L}_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})}, \qquad (4.4)
$$
\nwhere  $\Phi^{(-)}(\xi) = \frac{\Phi(\xi) - \Phi(-\xi)}{\xi \xi}$ .

\nWe represent  $\frac{\partial}{\partial \xi} \mathcal{H}^{(-)}$  in the form

\n
$$
-\frac{1}{\pi \xi} \int_{\mathbb{R}} \Phi^{(-)}(\tau) \frac{d\tau}{\xi - \tau} + \frac{1}{\pi \xi} \int_{\mathbb{R}} \frac{d}{d\tau} (\tau \Phi^{(-)}(\tau)) \frac{d\tau}{\xi - \tau} = (T_1 \Phi^{(-)})(\xi) + (T_2 \Phi^{(-)})(\xi).
$$
\nFrom the norm inequality for the Hilbert transform of odd functions in  $\mathcal{L}_{p,2\beta+\frac{1}{p}}(\mathbb{R})$  (see [1]) it follows that

\n
$$
\|T_1 \Phi^{(-)}\|_{\mathcal{L}_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \|\Phi^{(-)}\|_{\mathcal{L}_{p,2\beta+\frac{1}{p}}(\mathbb{R})}.
$$
\nLet  $\chi$  be a  $C^{\infty}$ -function vanishing outside a neighbourhood of  $\xi = 0$ 

$$
\frac{\partial}{\partial \xi} \mathcal{H}^{(+)}(\xi) = \frac{1}{\pi \xi^2} \int_{\mathbb{R}} \frac{d}{d\tau} \Phi^{(-)}(\tau) \frac{\tau^2 d\tau}{\xi - \tau}
$$
\nsfies

\n
$$
\left\| \frac{\partial}{\partial \xi} \mathcal{H}^{(+)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})}, \qquad (4)
$$
\nHere

\n
$$
\Phi^{(-)}(\xi) = \frac{\Phi(\xi) - \Phi(-\xi)}{2}.
$$
\nWe represent

\n
$$
\frac{\partial}{\partial \xi} \mathcal{H}^{(-)} \text{ in the form}
$$
\n
$$
-\frac{1}{\pi \xi} \int_{\mathbb{R}} \Phi^{(-)}(\tau) \frac{d\tau}{\xi - \tau} + \frac{1}{\pi \xi} \int_{\mathbb{R}} \frac{d}{d\tau} (\tau \Phi^{(-)}(\tau)) \frac{d\tau}{\xi - \tau} = (T_1 \Phi^{(-)})(\xi) + (T_2 \Phi^{(-)})(\xi).
$$
\nm the norm inequality for the Hilbert transform of odd functions in

\n
$$
\mathcal{L}_{p,2\beta+\frac{1}{p}}(\mathbb{R})
$$
\nit follows that

From the norm inequality for the Hilbert transform of odd functions in  $\mathcal{L}_{p,2\beta+\frac{1}{2}}(\mathbb{R})$  (see [1]) it follows that

$$
||T_1\Phi^{(-)}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}.
$$
\n(4.5)

Let  $\chi$  be a  $C^{\infty}$ -function vanishing outside a neighbourhood of  $\xi = 0$  and subject to  $f_{\mathbb{R}} \chi(u)du = 1$ . The Fourier transform of  $\xi(T_2 \Phi^{(-)}) (\xi)$  with respect to  $\xi$  is given by

$$
\pi i \left[ | \tau | \widehat{\chi}(\tau) \widehat{\Psi}(\tau) + \text{sign}(\tau) (1 - \widehat{\chi}(\tau) (\tau \widehat{\Psi}(\tau)) \right] = \widehat{S_1 \Psi}(\tau) + \widehat{S_2 \Psi}(\tau),
$$

where  $\widehat{\Psi}(\tau)$  is the Fourier transform of  $\xi(\Phi^{(-)})(\xi)$ . Since  $|\tau|\widehat{\chi}(\tau)$  is the Fourier transform of a smooth function admitting the estimate  $O(\xi^{-2})$  as  $\xi \to \pm \infty$ , it follows by Lemma 1 that  $\begin{aligned} \mathcal{F}(r) + \text{sign}(\tau)(1 - \hat{\chi}(\tau)\big(\tau\hat{\Psi}(\tau))\big) &= \hat{\mathcal{F}}(r) \text{transform of } \mathcal{F}\big(\Phi^{(-)}\big)(\xi). \text{ Since } |\tau| \text{ 0} \text{ in } \mathcal{F}\big) \text{ in } \mathcal{F}\big(\mathcal{F}\big) \text{ in } \mathcal{F}\big(\mathcal{F}\big) \text{ as } \mathcal{F}\big(\mathcal{F}\big) \leq c \|\Phi^{(-)}\|_{L_{p,2p+\frac{1}{p}}} \text{ as } \hat{\mathcal{F}}\big(\mathcal{F}\$ orm of  $\xi(\Phi^{(-)})(\xi)$ <br>the estimate  $O(\xi)$ <br> $L_{p,2\theta+\frac{1}{p}}(\mathbb{R}) \leq c||\Phi^{(0)}|$ <br> $h(\tau)(1-\hat{\chi}(\tau))$  is<br> $\int_{\mathbb{R}} \frac{\chi(u)}{\xi-u} du = -\int_{\mathbb{R}}$ ansform of  $\xi(T_2\Phi^{(-)})(\xi)$  with<br>  $g(n(\tau)(1-\hat{\chi}(\tau)(\tau\hat{\Psi}(\tau))) = \xi$ <br>
form of  $\xi(\Phi^{(-)})(\xi)$ . Since  $|\tau|$ <br>  $g$  the estimate  $O(\xi^{-2})$  as  $\xi$ <br>  $|L_{p,2\theta+\frac{1}{p}}(\mathbb{R}) \leq c||\Phi^{(-)}||_{L_{p,2\theta+\frac{1}{p}}}$ <br>  $\lim_{\tau \to 0}(\tau)(1-\hat{\chi}(\tau))$  is the Four

$$
||S_1 \Psi||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \leq c||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}.
$$

Taking into account that  $\pi i$  sign( $\tau$ )(1 –  $\hat{\chi}(\tau)$ ) is the Fourier transform of the function

$$
S_1 \Psi \parallel_{L_{p,2\theta+\frac{1}{p}}(\mathbb{R})} \le c \|\Phi^{(-)}\|_{L_{p,2\theta+\frac{1}{p}}(\mathbb{R})}
$$
  

$$
\pi i \operatorname{sign}(\tau)(1-\widehat{\chi}(\tau)) \text{ is the Fourier t}
$$
  

$$
\frac{1}{x} - \int_{\mathbb{R}} \frac{\chi(u)}{\xi-u} du = - \int_{\mathbb{R}} \frac{u \chi(u)}{\xi(\xi-u)} du
$$

which admits the estimate  $O(\xi^{-2})$  as  $\xi \to \pm \infty$ , we obtain by Lemma 1

h function admitting the estimate 
$$
O(\xi^{-2})
$$
 as  $\xi \to \pm \infty$ , it follows  
\n
$$
||S_1 \Psi||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \le c||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}.
$$
\no account that  $\pi i \operatorname{sign}(\tau)(1-\hat{\chi}(\tau))$  is the Fourier transform of the  
\n
$$
\frac{1}{x} - \int_{\mathbb{R}} \frac{\chi(u)}{\xi - u} du = - \int_{\mathbb{R}} \frac{u \chi(u)}{\xi(\xi - u)} du
$$
\nuits the estimate  $O(\xi^{-2})$  as  $\xi \to \pm \infty$ , we obtain by Lemma 1  
\n
$$
||S_2 \Psi||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \le c \left( \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right).
$$

Thus,  $T_2\Phi^{(-)}$  satisfies

V. Maz'ya and A. Soloviev  
\n
$$
_{2}\Phi^{(-)}
$$
 satisfies  
\n
$$
||T_{2}\Phi^{(-)}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \bigg(||\frac{d}{d\xi}\Phi^{(-)}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})}\bigg).
$$
\nng (4.5) imply

This along (4.5) imply

V. Maz'ya and A. Soloviev  
\n
$$
T_{2}\Phi^{(-)} \text{ satisfies}
$$
\n
$$
||T_{2}\Phi^{(-)}||_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \left( \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right).
$$
\nalong (4.5) imply\n
$$
\left\| \frac{\partial}{\partial\xi} \mathcal{H}^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \left( \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi^{(-)}||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right).
$$
\n(4.6) represent the odd function  $\mathcal{H}^{(-)}$  on **R** in the form

We represent the odd function  $\mathcal{H}^{(-)}$  on R in the form

$$
\|\mathbf{y}\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} \leq c \left( \left\| \frac{d}{d\xi} \Phi^{(-)} \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + \|\Phi^{(-)}\|_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right)
$$
\nthe odd function

\n
$$
\mathcal{H}^{(-)}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d}{d\tau} \Phi^{(+)}(\tau) \log \left| \frac{\xi-\tau}{\xi} \right| d\tau
$$
\n
$$
= \frac{\xi^{n_0}}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi-\tau)} d\tau - \sum_{k=0}^{n_0-1} \frac{\xi^k}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{k+1}} d\tau.
$$
\nL

\nL

\n
$$
\mathcal{H}^{(-)}(\mathbb{R}) \text{ and since } 0 < 2\beta - 2\mu + n_0 + \frac{2}{\pi} < 1 \text{ for } n_0 + 1 \text{ for } n_0
$$

Since  $\Phi^{(+)} \in L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})$  and since  $0 < 2\beta - 2\mu + n_0 + \frac{2}{p} < 1$  for even  $n_0$  and  $0 < 2\beta - 2\mu + n_0 + \frac{2}{p} < 2$  for odd  $n_0$ , it follows from the boundedness of the Hilbert transform in weighted  $L_p$ -spaces (see [1]) that the norm in  $L_{p,2\beta-2\mu+\frac{1}{n}}(\mathbb{R})$  of since  $0 < 2\beta - 2\mu + n_0 + \frac{2}{p}$ <br> *d*  $n_0$ , it follows from the bounde<br>
see [1]) that the norm in  $L_{p,2\beta-2}$ <br>  $\frac{n_0}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi - \tau)} d\tau$ <br>
((R)<br>
in that the function  $h(z) = \mathcal{H} \circ \theta$ <br>  $a_k(\varphi) \text{Re} z^{k-\$ 

$$
\frac{\xi^{n_0}}{\pi}\int_{\mathbb{R}}\frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi-\tau)}d\tau
$$

does not exceed  $c \|\Phi^{(+)}\|_{L_{p,2\beta-2\mu+\frac{1}{p}}(\mathbb{R})}$ .

Hence by (4.4), (4.6) we obtain that the function  $h(z) = \mathcal{H} \circ \theta_0^{-1}(z)$  is represented in the form

$$
\sum_{k=1}^{m} a_k(\varphi) \text{Re} z^{k-\frac{1}{2}} + h^{\#}(z) \tag{4.7}
$$

for  $z \in \Omega$  situated in a small neighbourhood of the peak. Here

$$
\sum_{k=1}^{m} a_k(\varphi) \text{Re} z^{k - \frac{1}{2}} + h^{\#}(z)
$$
\na small neighbourhood of the peak. Here

\n
$$
a_k(\varphi) = \int_{\mathbb{R}} \Phi^{(+)}(\tau) \tau^{-2k} d\tau \qquad (1 \leq k \leq m)
$$

are linear continuous functionals in  $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma),$  and  $h^{\#}$  belongs to  $\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$  and satisfies

$$
||h^{\#}||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma_{+}\cup\Gamma_{-})}\leq c||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}.
$$

 $a_k(\varphi) = \int_{\mathbb{R}} \Phi^{(+)}(\tau) \tau^{-2k} d\tau$  ( $1 \le k \le m$ )<br>
are linear continuous functionals in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ , and  $h^{\#}$  belongs to  $\mathfrak{N}_{p,\beta}^{(-)}$ <br>  $\|h^{\#}\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma_+\cup\Gamma_-)} \le c \|\varphi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}$ .<br>
It is cle

Now let  $\kappa \in C^{\infty}(\mathbb{R}^2)$  be equal to 1 for  $|z| < \delta$  and vanish for  $|z| > \delta$ . We extend  $\kappa h$  is the conduction of  $\theta$  and set<br>  $\psi_1(z) = -\Delta(\kappa h)(z)$   $(z \in \Omega^c)$ <br>  $\varphi_1(z) = \frac{\partial}{\partial s} \varphi(z) - \frac{\partial}{\partial n}(\kappa h)(z)$   $(z \in \Gamma)$ . by zero outside a small neighbourhood of *0* and set

$$
a_k(\varphi) = \int_{\mathbb{R}} \Phi^{(+)}(\tau) \tau^{-2k} d\tau \qquad (1 \le k \le m)
$$
  
functionals in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ , and  $h^{\#}$  belongs to  $!\|h^{\#}\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma_+\cup\Gamma_-)} \le c \|\varphi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}$ .  
 $\frac{\partial}{\partial \tau} \Phi \in L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})$ . Therefore  $\frac{\partial}{\partial n} h$  belong  
(2) be equal to 1 for  $|z| < \delta$  and vanish for  $|\Gamma|$  neighborhood of  $O$  and set  
 $\psi_1(z) = -\Delta(\kappa h)(z) \qquad (z \in \Omega^c)$   
 $\varphi_1(z) = \frac{\partial}{\partial s} \varphi(z) - \frac{\partial}{\partial n}(\kappa h)(z) \qquad (z \in \Gamma)$ .

We consider the boundary value problem

$$
L_p\text{-Theory of Boundary Integral Equations} \qquad 657
$$
\nby value problem

\n
$$
\Delta \mathcal{F}(\zeta) = \mathcal{Q}(\zeta) \qquad (\zeta \in \mathbb{R}_+^2) \Bigg\} \qquad (4.8)
$$
\n
$$
\frac{\partial}{\partial n} \mathcal{F}(\xi + i0) = \mathcal{T}(\xi) \qquad (\xi \in \mathbb{R}) \Bigg\} \qquad (4.8)
$$
\n
$$
|\theta'(\zeta)|^2 \text{ and } \mathcal{T}(\xi) = (\varphi_1 \circ \theta)(\xi + i0) |\theta'(\xi + i0)|. \text{ By using the}
$$
\n
$$
\int_{\mathbb{R}} \frac{\tau^2 (d\Phi/d\tau)(\tau)}{\zeta - \tau} d\tau \Bigg| \qquad \text{and} \qquad |\mathcal{H}(\zeta)| \le \frac{c}{|\zeta|} \Bigg| \int_{\mathbb{R}} \frac{\tau \Phi(\tau)}{\zeta - \tau} d\tau \Bigg|
$$

where  $Q(\zeta) = (\psi_1 \circ \theta)(\zeta) |\theta'(\zeta)|^2$  and  $\mathcal{T}(\xi) = (\varphi_1 \circ \theta)(\xi + i0) |\theta'(\xi + i0)|$ . By using the estimates

$$
\varphi(\zeta) = (\psi_1 \circ \theta)(\zeta) |\theta'(\zeta)|^2 \text{ and } \mathcal{T}(\xi) = (\varphi_1 \circ \theta)(\xi + i0) |\theta'(\xi + i0)|. \text{ By using}
$$
  
ates  

$$
|\text{grad } \mathcal{H}(\zeta)| \le \frac{c}{|\zeta|^2} \left| \int_{\mathbb{R}} \frac{\tau^2 (d\Phi/d\tau)(\tau)}{\zeta - \tau} d\tau \right| \quad \text{and} \quad |\mathcal{H}(\zeta)| \le \frac{c}{|\zeta|} \left| \int_{\mathbb{R}} \frac{\tau \Phi(\tau)}{\zeta - \tau} d\tau \right|
$$

and theorems on the boundedness of the Hardy- Littlewood maximal operator and

Hilbert transform in weighted 
$$
L_p
$$
-spaces [3, 9] we obtain  
\n
$$
||T||_{L_p(\mathbb{R})} + ||Q||_{L_p(\mathbb{R}_+^2)} \leq c \left( \left\| \frac{d}{d\tau} \Phi \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right).
$$

A solution of problem (4.8) *is* given by

$$
||T||_{L_p(\mathbb{R})} + ||Q||_{L_p(\mathbb{R}_+^2)} \le c \left( \left\| \frac{1}{d\tau} \Phi \right\|_{L_{p,2\beta+1+\frac{1}{p}}(\mathbb{R})} + ||\Phi||_{L_{p,2\beta+\frac{1}{p}}(\mathbb{R})} \right).
$$
  
to from (4.8) is given by  

$$
\mathcal{F}(\zeta) = \int_{\mathbb{R}} T(u)\Phi(u,\zeta)du - \int_{\mathbb{R}_+^2} Q(w)\Phi(w,\zeta)du dv \qquad (w = u + iv)
$$

with Green's function.

$$
\mathfrak{G}(w,\zeta) = \frac{1}{2\pi} \log \left| \left( 1 - \frac{w}{\zeta} \right) \left( 1 - \frac{w}{\overline{\zeta}} \right) \right|.
$$

We rewrite F on *R* in the form

$$
\mathfrak{G}(w,\zeta) = \frac{1}{2\pi} \log \left| \left( 1 - \frac{w}{\zeta} \right) \left( 1 - \frac{w}{\overline{\zeta}} \right) \right|.
$$
  

$$
\mathcal{F} \text{ on } \mathbb{R} \text{ in the form}
$$

$$
\mathcal{F}(\xi) = t_{-1}(\varphi) \log |\xi| + t_0(\varphi)
$$

$$
+ \frac{1}{\pi} \int_{\mathbb{R}} \mathcal{T}(u) \log \left| 1 - \frac{\xi}{u} \right| du - \frac{1}{\pi} \int_{\mathbb{R}^2_+} \mathcal{Q}(w) \log \left| 1 - \frac{\xi}{w} \right| du dv \qquad (4.9)
$$

where

$$
\frac{1}{\pi} \int_{\mathbb{R}} T(u) \log |1 - \frac{1}{u}| du - \frac{1}{\pi} \int_{\mathbb{R}^2_+} Q(w) \log |1 - \frac{1}{\pi} \int_{\mathbb{R}^2_+} T(u) du + \frac{1}{\pi} \int_{\mathbb{R}^2_+} Q(w) du dv
$$
\n
$$
= \frac{1}{\pi} \int_{\mathbb{R}} T(u) \log |u| du - \frac{1}{\pi} \int_{\mathbb{R}^2_+} Q(w) \log |w| dv
$$

and

$$
t_0(\varphi)=\frac{1}{\pi}\int_{\mathbb{R}}T(u)\log|u|du-\frac{1}{\pi}\int_{\mathbf{R}_+^2}\mathcal{Q}(w)\log|w|dudv.
$$

Hence we obtain

$$
t_0(\varphi) = \frac{1}{\pi} \int_{\mathbb{R}} I(u) \log |u| du - \frac{1}{\pi} \int_{\mathbb{R}^2_+} \mathcal{Q}(w) \log |w| du dv.
$$
  
tain  

$$
\frac{\partial \mathcal{F}}{\partial \xi}(\xi) - \frac{t_{-1}(\varphi)}{\xi} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{T(u)}{\xi - u} du - \frac{1}{\pi} \int_{\mathbb{R}^2_+} \frac{(\xi - u) \mathcal{Q}(w)}{|\xi - w|^2} du dv.
$$

By the boundedness of the Hilbert transform in  $\mathcal{L}_p(\mathbb{R})$  and the Minkovski inequality we prove that

let transform in 
$$
\mathcal{L}_p(\mathbb{R})
$$
 and the Minkowski inequality we

\n
$$
\frac{\partial}{\partial \xi} \mathcal{F}(\xi) - t_{-1}(\varphi) \frac{1}{\xi} \in L_p(\mathbb{R}) \tag{4.10}
$$
\nfunction does not exceed

\n
$$
\|T\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}.
$$
\nhood of infinity

\n
$$
\frac{\partial}{\partial \xi} \mathcal{F}(\xi) = R_{\infty}(\xi) \frac{1}{\xi^2} \tag{4.11}
$$
\n
$$
+ \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)} \text{ for large } |\xi|. \text{ Set } f = \mathcal{F} \circ \theta. \text{ From (4.10)}
$$

and that the  $L_p$ -norm of this function does not exceed

$$
\|T\|_{L_p(\mathbb{R})} + \|Q\|_{L_p(\mathbb{R}^2_+)}.
$$

It is clear that in a neighbourhood of infinity

$$
\frac{\partial}{\partial \xi} \mathcal{F}(\xi) = R_{\infty}(\xi) \frac{1}{\xi^2}
$$
\n(4.11)

where  $|R_{\infty}(\xi)| \le c \left( ||\mathcal{T}||_{L_p(\mathbb{R})} + ||\mathcal{Q}||_{L_p(\mathbb{R}_+^2)} \right)$  for large  $|\xi|$ . Set  $f = \mathcal{F} \circ \theta$ . From (4.10)<br>and (4.11) it follows that  $\frac{\partial}{\partial s} f$  belongs to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies<br> $\left\| \frac{\partial}{\partial s} f \right\|_{\mathcal$ and (4.11) it follows that  $\frac{\partial}{\partial s} f$  belongs to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies (4.11)<br>  $|\xi|$ . Set  $f = \mathcal{F} \circ \theta$ . From (4.10)<br>
nd satisfies<br>  $\theta_{+1}(\Gamma)$ . (4.12)

$$
\left\|\frac{\partial}{\partial s}f\right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \left\|\varphi\right\|_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}.\tag{4.12}
$$

By Taylor's decomposition of the integral terms in (4.9) we obtain

$$
|\leq c \left( \|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}_+^2)} \right) \text{ for large } |\xi|. \text{ Set } f = \mathcal{F} \circ \theta. \text{ From (4.10)}
$$
  
follows that  $\frac{\partial}{\partial s} f$  belongs to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies  

$$
\left\| \frac{\partial}{\partial s} f \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \|\varphi\|_{\mathcal{L}_{p,\beta+1}^1(\Gamma)}.
$$
  
(4.12)  
decomposition of the integral terms in (4.9) we obtain  

$$
\mathcal{F}(\xi) = t_{-1}(\varphi) \log |\xi| + t_0(\varphi) + \sum_{k=1}^{n_0 - 1} t_k(\varphi) \xi^k + |\xi|^{n_0} R_{n_0}(\xi),
$$
  

$$
\leq c \left( \|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2)} \right) \text{ for } k = -1, \quad n_0 = 1, \text{ and } |R_{-1}(\xi)| \leq c \left( \|\mathcal{T}\|_{L_p(\mathbb{R}^2)} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2)} \right) \text{ for } k = -1, \quad n_0 = 1, \text{ and } |R_{-1}(\xi)| \leq c \left( \|\mathcal{T}\|_{L_p(\mathbb{R}^2)} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2)} \right) \text{ for } k = -1, \quad n_0 = 1, \text{ and } |R_{-1}(\xi)| \leq c \left( \|\mathcal{T}\|_{L_p(\mathbb{R}^2)} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2)} \right) \text{ for } k = -1, \quad n_0 = 1, \
$$

where  $|t_k(\varphi)| \le c \left(\|T\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right)$  for  $k = -1, ..., n_0 - 1$ , and  $|R_{n_0}(\xi)| \le$  $c(\|T\|_{L_p(\mathbb{R})} + \|Q\|_{L_p(\mathbb{R}^2_+)})$  for small  $|\xi|$ . Taking into account the asymptotic representations (3.4), (3.5) of  $\theta^{-1}$  and the inequality  $2(\mu - \beta - \frac{1}{n}) < n_0$ , it follows from (4.12) and (4.13) that *f* is represented in the form  $\frac{1}{2}$  *(* $\mathbb{R}^{2}$ <sub>1</sub>) for small  $|\xi|$ . Takin<br> *f*  $\theta^{-1}$  and the inequality 2<br> *f* (*z*) =  $\sum_{k=1}^{m} b_k(\varphi) \text{Re } z^{k-\frac{1}{2}}$  $\sum_{k=1}^{n_0-1} t_k(\varphi) \xi^k + |\xi|^{n_0} R_{n_0}(\xi),$  (4.13)<br>
or  $k = -1, ..., n_0 - 1$ , and  $|R_{n_0}(\xi)| \le$ <br> *ng* into account the asymptotic represen-<br>  $2(\mu - \beta - \frac{1}{p}) < n_0$ , it follows from (4.12)<br>  $+ f^{\#}(z)$  ( $z \in \Omega$ ) (4.14)

$$
f(z) = \sum_{k=1}^{m} b_k(\varphi) \operatorname{Re} z^{k - \frac{1}{2}} + f^{\#}(z) \qquad (z \in \Omega)
$$
 (4.14)

where  $f^{\#} \in \mathfrak{N}^{(-)}_{p,\beta}(\Gamma)$ , and  $b_k(\varphi)$   $\ (k = 1, \ldots, m)$  are linear combinations of the coefficients  $t_{\ell}(\varphi)$   $(\ell = 1, ..., n_0 - 1)$  in (4.13).

According to (4.7) and (4.14) the function  $g = \kappa h + f$  is harmonic in  $\Omega$  and can be written as

), and 
$$
b_k(\varphi)
$$
  $(k = 1, ..., m)$  are linear combin  
\n...,  $n_0 - 1$  in (4.13).  
\n7) and (4.14) the function  $g = \kappa h + f$  is harm  
\n
$$
g(z) = \sum_{k=1}^{m} c_k(\varphi) \operatorname{Re} z^{k - \frac{1}{2}} + g^{\#}(z) \qquad (z \in \Omega^c)
$$
\n+  $b_k(\varphi)$ . Moreover,  
\n
$$
\sum_{k=1}^{m} |c^{(k)}| + ||g^{\#}||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}
$$

with  $c_k(\varphi) = a_k(\varphi) + b_k(\varphi)$ . Moreover,

$$
\sum_{k=1}^m |c^{(k)}| + ||g^{\#}||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}
$$

and by the definition of g it follows that  $(g \circ \theta)(\infty) = 0$ . Because  $\frac{\partial}{\partial n} g = \frac{\partial}{\partial n} \varphi$  on  $\Gamma \setminus \{O\}$ ,  $L_p$ -Theory of Boundary Integral Equa<br>and by the definition of g it follows that  $(g \circ \theta)(\infty) = 0$ . Because  $\frac{\partial}{\partial n}g = \frac{\partial}{\partial n}$ <br>it is clear that one of the functions conjugate to g is the harmonic extens it is clear that one of the functions conjugate to g is the harmonic extension of  $\varphi$  onto  $\Omega$  with normal derivative in  $\mathcal{L}_{p,\theta+1}(\Gamma)$ . L<sub>p</sub>-Theory of Bound<br>
1 of g it follows that  $(g \circ \theta)(\infty) = 0$ . Between the functions conjugate to g is the<br>
ative in  $\mathcal{L}_{p,\beta+1}(\Gamma)$ .<br>
elong to  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and let  $\varphi$  vanish on<br>  $(\varphi \circ \theta)(\frac{1}{\xi})$   $(\xi \in \mathbb{R})$ dary Integral Equations 659<br>
ecause  $\frac{\partial}{\partial n}g = \frac{\partial}{\partial n}\varphi$  on  $\Gamma \setminus \{O\}$ ,<br>
harmonic extension of  $\varphi$  onto<br>  $\Gamma \cap \{ |q| < \frac{\delta}{2} \}$ . We introduce<br>
outside a certain interval. Set<br>  $(\zeta \in \mathbb{R}^2_+)$ . (4.15)<br>
ic extensio and by the definition of g it follows that  $(g \circ \theta)(\infty) = 0$ . Because<br>it is clear that one of the functions conjugate to g is the harms<br> $\Omega$  with normal derivative in  $\mathcal{L}_{p,\beta+1}(\Gamma)$ .<br>(ii) Now let  $\varphi$  belong to  $\mathfrak{N}_{$ 

(ii) Now let  $\varphi$  belong to  $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$  and let  $\varphi$  vanish on  $\Gamma\cap\{|q|<\frac{\delta}{2}\}.$  We introduce the function  $\Phi(\xi) = (\varphi \circ \theta)(\frac{1}{\xi})$   $(\xi \in \mathbb{R})$  which equals zero outside a certain interval. Set

$$
\mathcal{G}(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi(\tau) \operatorname{Re} \frac{\zeta}{\tau(\zeta - \tau)} d\tau \qquad (\zeta \in \mathbb{R}^2_+). \tag{4.15}
$$

It is clear that one of conjugate functions  $\tilde{G}$  is a harmonic extension of  $\Phi$  onto  $\mathbb{R}^2_+$ . It It is clear that one of conjugate functions  $G$  is a harmonic extension of  $\Phi$  onto  $\mathbb{R}^2_+$ . It<br>follows from the boundedness of Hilbert transform in  $L_p$ -spaces that  $g = G \circ \theta^{-1}$  belongs<br>to  $\mathcal{L}^1_{p,\beta+1}(\Gamma)$  and  $\Phi(\tau) \text{Re} \frac{1}{\tau(\zeta)}$ <br>
unctions  $\widetilde{\mathcal{G}}$  is<br>
ilbert transi<br>  $\sum_{\mathbf{p},\mathbf{p+1}}(\Gamma) \leq c$ <br>
ne form

$$
\| g \|_{\mathcal{L}_{p,\beta+1}^1(\Gamma)} \leq c \| \varphi \|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)} \ . \tag{4.16}
$$

Further, we represent  $\mathcal G$  on  $\mathbb R$  in the form

$$
\mathcal{G}(\zeta) = \frac{1}{\pi} \int \Phi(\tau) \text{Re} \frac{1}{\tau(\zeta - \tau)} d\tau \qquad (\zeta \in \mathbb{R}^2_+).
$$
  
at one of conjugate functions  $\tilde{\mathcal{G}}$  is a harmonic extension of  $\Phi$  on  
the boundedness of Hilbert transform in  $L_p$ -spaces that  $g = \mathcal{G} \circ \theta$ –  
and satisfies  

$$
\|g\|_{\mathcal{L}_{p,\rho+1}^1(\Gamma)} \le c \|\varphi\|_{\mathfrak{N}_{p,\rho}^{(+)}}(\Gamma) .
$$
  
represent  $\mathcal{G}$  on  $\mathbb{R}$  in the form  

$$
\mathcal{G}(\xi) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \Phi(\tau) \tau^{-1} d\tau
$$

$$
+ \sum_{k=1}^{n_0-1} \frac{1}{\pi \xi^k} \int_{\mathbb{R}} \Phi(\tau) \tau^{k-1} d\tau + \frac{1}{\pi \xi^{n_0-1}} \int_{\mathbb{R}} \frac{\Phi(\tau) \tau^{n_0-1}}{\xi - \tau} d\tau
$$

$$
= \sum_{k=0}^{n_0-1} \frac{t_k(\varphi)}{\xi^k} + \mathcal{G}^{\#}(\xi)
$$

$$
t_k(\varphi) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi(\tau) \tau^{k-1} d\tau
$$

$$
\mathcal{G}^{\#}(\xi) = \frac{1}{\pi \xi^{n_0-1}} \int_{\mathbb{R}} \frac{\Phi(\tau) \tau^{n_0-1}}{\xi - \tau} d\tau \quad (\xi \in \mathbb{R}).
$$
  
 $2\mu - 2\beta - \frac{3}{p} - n_0 + 1 < 1 - \frac{1}{p}$ , we have

where

$$
k=0
$$
  
\n
$$
t_{k}(\varphi) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi(\tau) \tau^{k-1} d\tau
$$
  
\n
$$
\mathcal{G}^{#}(\xi) = \frac{1}{\pi \xi^{n_{0}-1}} \int_{\mathbb{R}} \frac{\Phi(\tau) \tau^{n_{0}-1}}{\xi - \tau} d\tau \quad (\xi \in \mathbb{R}).
$$
  
\n
$$
-\frac{3}{p} - n_{0} + 1 < 1 - \frac{1}{p}, \text{ we have}
$$
  
\n
$$
\sum_{k=0}^{n_{0}-1} |t_{k}(\varphi)| + ||\mathcal{G}^{#}||_{L_{p,2\mu - 2\beta - \frac{3}{p}}(\mathbb{R})} \leq c ||\Phi||_{L_{p}(\mathbb{R})}.
$$
  
\n6) it follows that *g* is represented in the form

Since  $-\frac{1}{p} < 2\mu - 2\beta - \frac{3}{p} - n_0 + 1 < 1 - \frac{1}{p}$ , we have  $\frac{n_0-1}{n_0}$ 

$$
\pi \xi^{n_0 - 1} \int_{\mathbb{R}} \xi - \tau
$$
  
\n
$$
-\frac{3}{p} - n_0 + 1 < 1 - \frac{1}{p}, \text{ we have}
$$
  
\n
$$
\sum_{k=0}^{p_0 - 1} |t_k(\varphi)| + ||\mathcal{G}^{\#}||_{L_{p, 2p - 2\beta - \frac{3}{p}}(\mathbb{R})} \leq c ||\Phi||_{L_p(\mathbb{R})}
$$

Hence and from (4.16) it follows that  $g$  is represented in the form

$$
\sum_{k=0}^{\infty} |t_k(\varphi)| + ||\mathcal{G}^{\#}||_{L_{p,2\mu-2\beta-\frac{3}{p}}(\mathbb{R})} \leq c ||\Phi||_{L_p(\mathbb{R})}
$$
  
3) it follows that  $g$  is represented in the form  

$$
g(z) = \sum_{k=1}^m c_k(\varphi) \operatorname{Re} z^{k-\frac{1}{2}} + g^{\#}(z) \qquad (z \in \Omega)
$$

where  $g^{\#} \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ ,  $c_k(\varphi)$   $(k = 1,...,m)$  are linear combinations of coefficients  $t_{\ell}(\varphi)$  ( $\ell = 1, \ldots, n_0 - 1$ ) in (4.17). These coefficients and the function  $q^{\#}$  satisfy  $g(z) = \sum_{k=1}^{m} c_k(\varphi) \text{Re } z^{k-\frac{1}{2}} + g^{\#}(z)$   $(z \in \Omega)$ <br>
where  $g^{\#} \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ ,  $c_k(\varphi)$   $(k = 1, ..., m)$  are linear combinations of coefficient<br>  $t_{\ell}(\varphi)$   $(\ell = 1, ..., n_0 - 1)$  in (4.17). These coefficients and the func

$$
\sum_{k=1}^m |c_k(\varphi)| + \|g^{\#}\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c \, \|\varphi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}.
$$

According to (4.15) we have  $(g \circ \theta)(\infty) = 0$ , and the conjugate function  $\tilde{g} = \tilde{\mathcal{G}}\left(\frac{1}{\theta-1}\right)$  is the harmonic extension of  $\varphi$  onto  $\Omega$  with normal derivative in  $\mathcal{L}_{p,\theta+1}(\Gamma)$ 

### **5. Boundary integral equation of the Dirichiet problem**

Here we prove the unique solvability of equation (1.3) on the contour  $\Gamma$  with inward peak.

**Theorem 2.** Let  $\Omega$  have an inward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}, \ \mu - \beta - \frac{1}{p}$  $\frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Then the operator

<sup>R</sup>*m* (, *i) - To, + E t(k)Ik E 1(r), (5.1)* 

*where*  $\mathcal{I}_k(z) = \text{Im} z^{k-\frac{1}{2}}$ , *is surjective.* 

**Proof.** (i) Let  $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$  and  $\varphi = 0$  in a neighbourhood of the peak. We consider the harmonic extension  $h^i$  of  $\varphi$  onto  $\Omega$  and its conjugate function  $g^i$ , normalized by the where  $\mathcal{I}_k(z) = \text{Im} z^{k-\frac{1}{2}}$ , is surjective.<br> **Proof.** (i) Let  $\varphi \in \mathfrak{N}_{r,\beta}^{(+)}(\Gamma)$  and  $\varphi = 0$  in a neighbourhood of the peak. We consider<br>
the harmonic extension  $h^i$  of  $\varphi$  onto  $\Omega$  and its conjugate func Proposition 3, *as*  Let  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and  $\varphi = 0$  in a neighbourhood of the peak. We consider extension  $h^i$  of  $\varphi$  onto  $\Omega$  and its conjugate function  $g^i$ , normalized by the  $\varphi$   $)= 0$   $(z_0 \in \Gamma \setminus \{O\})$  which were introdu such that the function *g*<sup>1</sup>, normalized by the<br> *n* Proposition 3. By<br> *k*)( $\varphi$ ) ( $k = 1, ..., m$ )<br> *k*)( $\varphi$ ) ( $k = 1, ..., m$ )<br>
(5.2)  $(z_0 \in \Gamma \setminus \{O\})$  whi<br>  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and there<br>  $g_0^i(z) = g^i(z) - \sum_k$ <br>  $-\frac{1}{2}$ , belongs to  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ <br>
satisfy<br>  $\sum_{k=1}^m |c^{(k)}(\varphi)| + ||g_0^i||_{\mathfrak{N}_{p,\beta}^0}$ 

$$
g_0^i(z)=g^i(z)-\sum_{k=1}^m c^{(k)}(\varphi)\mathcal{R}_k(z),
$$

where  $\mathcal{R}_k(z) = \text{Re } z^{k-\frac{1}{2}}$ , belongs to  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . The coefficients  $c^{(k)}(\varphi)$   $(k = 1, \ldots, m)$ and the function  $g_0^i(z)$  satisfy

$$
\sum_{k=1}^{m} |c^{(k)}(\varphi)| + ||g_0^i||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \le c ||\varphi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}.
$$
\n(5.2)\n
$$
h_0^i(z) = h^i(z) + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_k(z) \qquad (z \in \Omega)
$$
\nision of

\n
$$
\varphi + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_k \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma).
$$
\nis antisymmetric of  $k$  is set of  $\mathbb{R}$  which is  $\text{res}^{\pm 1} (z) = O(|z|^{-\frac{1}{2}}).$  Hence

The function

$$
h_0^i(z) = h^i(z) + \sum_{k=1}^m c^{(k)}(\varphi) \mathcal{I}_k(z) \qquad (z \in \Omega)
$$

is the harmonic extension of

$$
\varphi + \sum_{k=1}^m c^{(k)}(\varphi) \mathcal{I}_k \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma).
$$

Let  $h_0^{\epsilon}$  be the harmonic extension of  $h_0^i$  onto  $\Omega^{\epsilon}$  subject to grad  $h_0^{\epsilon}(z)=O(|z|^{-\frac{1}{2}}).$  Hence and by the estimate  $h_0^i(z) = O(|z|^{-\frac{1}{2}})$   $(z \in \Omega)$  we obtain

$$
\varphi + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_{k} \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma).
$$
  
is be the harmonic extension of  $h_0^i$  onto  $\Omega^c$  subject to grad  $h_0^*(z) = O(|z|^{-\frac{1}{2}})$ . Hence  
by the estimate  $h_0^i(z) = O(|z|^{-\frac{1}{2}})$  ( $z \in \Omega$ ) we obtain  

$$
h_0^i(z) = \frac{1}{2\pi} \int_{\Gamma} \left( \frac{\partial h_0^i}{\partial n} - \frac{\partial h_0^{\varepsilon}}{\partial n} \right) \log \frac{|z|}{|z - q|} ds_q + h_0^{\varepsilon}(\infty) \qquad (z \in \Gamma \setminus \{0\}).
$$
 (5.3)

According to Proposition 2 the Dirichlet problem in  $\Omega^c$  with the boundary data  $g_0^i$  has a solution  $f^e$  such that  $\frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies

$$
\varphi + \sum_{k=1} c^{(k)}(\varphi) \mathcal{I}_k \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma).
$$
  
\nension of  $h_0^i$  onto  $\Omega^c$  subject to grad  $h_0^{\epsilon}(z) = O(|z|^{-\frac{1}{2}})$ . Hence  
\n
$$
= O(|z|^{-\frac{1}{2}}) \quad (z \in \Omega) \text{ we obtain}
$$
\n
$$
- \frac{\partial h_0^{\epsilon}}{\partial n} \log \frac{|z|}{|z - q|} ds_q + h_0^{\epsilon}(\infty) \qquad (z \in \Gamma \setminus \{0\}). \qquad (5.3)
$$
\n
$$
\therefore \text{ the Dirichlet problem in } \Omega^c \text{ with the boundary data } g_0^i \text{ has}
$$
\n
$$
f^{\epsilon} \in \mathcal{L}_{p,\beta+1}(\Gamma) \text{ and satisfies}
$$
\n
$$
\left\| \frac{\partial}{\partial n} f^{\epsilon} \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \|g_0^i\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}.
$$
\n(5.4)

Let  $g^e$  denote the harmonic function conjugate to  $f^e$  and vanishing at infinity. We have Let  $g^e$  denote the harmonic function  $\frac{\partial}{\partial s}g^e = \frac{\partial}{\partial n}f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Since

$$
L_p\text{-Theory of Boundary Integral E}
$$
  
harmonic function conjugate to  $f^e$  and vanishing at  $p, \beta+1(\Gamma)$ . Since  

$$
\frac{\partial}{\partial n}g^e = -\frac{\partial}{\partial s}f^e = -\frac{\partial}{\partial s}g^i_0 = -\frac{\partial}{\partial n}h^i_0 \quad \text{on } \Gamma \setminus \{O\},
$$

Let  $g^{\epsilon}$  denote the harmonic function conjugate to  $f^{\epsilon}$  and vanishing at infinity. We have  $\frac{\partial}{\partial s}g^{\epsilon} = \frac{\partial}{\partial n}f^{\epsilon} \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Since<br>  $\frac{\partial}{\partial n}g^{\epsilon} = -\frac{\partial}{\partial s}f^{\epsilon} = -\frac{\partial}{\partial s}g^{\epsilon}_{0} = -\frac{\partial}{\partial n}h^{\epsilon}_{0}$ it follows that  $g^c$  belongs to  $\mathcal{L}^1_{p,\beta+1}(\Gamma)$  and satisfies the Neumann problem in  $\Omega^c$  with<br>boundary data  $-\frac{\partial}{\partial n}h^i_0$ . By the integral representation of a harmonic function in  $\Omega^c$  and<br>by (5.4) we obtain by (5.4) we obtain *I*<sup>*s*</sup> and vanishing at infinity. We have<br> *I*<sup>*n*</sup><sub>*n*</sub><sup>*h*<sub>0</sub></sub><sup>*i*</sup> on  $\Gamma \setminus \{0\}$ ,<br>
fies the Neumann problem in  $\Omega^c$  with<br>
thion of a harmonic function in  $\Omega^c$  and<br>  $\|\mathfrak{m}_{p,\theta}^{(-)}(\Gamma)$ . (5.5)<br>
inction  $w = -g^e - h$ in conjugate<br>  $-\frac{\partial}{\partial s}g_0^i = -1$ <br>  $\frac{1}{2}(\Gamma)$  and sat<br>  $\frac{1}{2}$  represent  $\frac{1}{2}$ <br>  $\frac{1}{2}$ ,  $\frac{1}{2}$ , then the Let  $g^{\epsilon}$  denote the harmonic function conjugate to  $f^{\epsilon}$  and vanishing<br>  $\frac{\partial}{\partial s}g^{\epsilon} = \frac{\partial}{\partial n}f^{\epsilon} \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Since<br>  $\frac{\partial}{\partial n}g^{\epsilon} = -\frac{\partial}{\partial s}f^{\epsilon} = -\frac{\partial}{\partial s}g^{\epsilon}_{0} = -\frac{\partial}{\partial n}h^{\epsilon}_{0}$  on  $\Gamma \setminus \{\infty\}$ <br> ws that  $g^e$  belongs to  $L^1_{p,\beta+1}(\Gamma)$  and satisfies the  $\Gamma$ <br>
ary data  $-\frac{\partial}{\partial n} h_0^i$ . By the integral representation of a<br>
1) we obtain<br>  $||g^e||_{L^1_{p,\beta+1}(\Gamma)} \le c||g_0^i||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}$ <br>
grad  $g^e = O(|z|^{-\mu-\frac{1}{2}})$   $rac{\partial}{\partial n}g^e = -\frac{\partial}{\partial s}f$ <br>  $g^e$  belongs to  $\mathcal{L}_p^1$ <br>  $1 - \frac{\partial}{\partial n}h_0^1$ . By the i<br>
tain<br>  $||g^i$ <br>  $= O(|z|^{-\mu - \frac{1}{2}})$  (z<br>
tion<br>  $\frac{1}{2\pi} \int_{\Gamma} \left( w(q) \frac{\partial}{\partial n_q} \right)$ <br>  $\therefore$  relation for the

$$
||g^e||_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} \le c||g_0^i||_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)}.
$$
\n(5.5)

Since grad  $g^e = O(|z|^{-\mu - \frac{1}{2}})$   $(z \in \Omega^c)$ , then the function  $w = -g^e - h_0^e + h_0^e(\infty)$  admits<br>
the representation<br>  $w(z) = \frac{1}{2\pi} \int_{\Gamma} \left( w(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} - \frac{\partial w}{\partial n_q}(q) \log \frac{|z|}{|z-q|} \right) ds_q$   $(z \in \Omega^c)$ .

$$
w(z) = \frac{1}{2\pi} \int_{\Gamma} \left( w(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} - \frac{\partial w}{\partial n_q}(q) \log \frac{|z|}{|z-q|} \right) ds_q \qquad (z \in \Omega^c).
$$
  
the limit relation for the double layer potential and from (5.3) it follows the  

$$
w - \pi^{-1}Tw = -2(h_0^i - h_0^c(\infty)) \qquad \text{in } \Gamma \setminus \{O\}.
$$

From the limit relation for the double layer potential and from (5.3) it follows that

$$
w-\pi^{-1}Tw=-2(h_0^i-h_0^{\epsilon}(\infty)) \quad \text{in } \Gamma \setminus \{O\}.
$$

Since  $T1 = -\pi$ , we obtain that the function

$$
\int_{\Gamma} \sqrt{\frac{m_q}{\pi}} \left| \frac{z - q}{z - q} \right| \, d\mathbf{r}
$$
\nrelation for the double layer potential and from (5.3)

\n
$$
w - \pi^{-1} \mathbf{T} w = -2 \big( h_0^i - h_0^e(\infty) \big) \qquad \text{in } \Gamma \setminus \{O\}.
$$
\nwe obtain that the function

\n
$$
\sigma = -(2\pi)^{-1} \left( g^e + \varphi + \sum_{k=1}^m c^{(k)}(\varphi) \mathbf{T}_k \right) \in \mathcal{L}_{p,\beta+1}^1(\Gamma)
$$

satisfies

$$
\sigma = -(2\pi)^{-1} \left( g^{\epsilon} + \varphi + \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_{k} \right) \in \mathcal{L}_{p,\beta+1}^{1}(\Gamma)
$$
  

$$
\pi \sigma(z) - T\sigma(z) = -\varphi(z) - \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_{k}(z) \qquad (z \in \Gamma \setminus \{O\}).
$$

We set  $t^{(k)} = c^{(k)}(\varphi)$   $(k = 1, \ldots, m)$ . From (5.2) and (5.5) it follows that

sfies  
\n
$$
\pi\sigma(z) - T\sigma(z) = -\varphi(z) - \sum_{k=1}^{m} c^{(k)}(\varphi) \mathcal{I}_k(z) \qquad (z \in \Gamma \setminus \{0\}).
$$
\nset  $t^{(k)} = c^{(k)}(\varphi)$   $(k = 1, ..., m)$ . From (5.2) and (5.5) it follows that  
\n
$$
\sum_{k=1}^{m} |t^{(k)}| + ||\sigma||_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} \le c||\varphi||_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}.
$$
\n(i) Now let  $\varphi$  be an arbitrary function in  $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ . There exists a sequence  $\{\varphi_r\}_{r\geq 1}$ 

of smooth functions on  $\Gamma \setminus \{O\}$  vanishing near the peak, which tends to  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ . Let  $(\sigma_r, t_r) \in \mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$  be the solution of equation (1.3) with the right-hand side  $-\varphi_r$  which is constructed as in (i). <sup>2</sup>/+  $\|\sigma\|_{\mathcal{L}^1_{p,\beta+1}}(r) \leq c \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(r)}$  (5.6)<br> *ary* function in  $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ . There exists a sequence  $\{\varphi_r\}_{r\geq 1}$ <br> *a* vanishing near the peak, which tends to  $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ .

According to (5.6) the sequence  $\{(\sigma_r,t_r)\}_{r\geq 1}$  converges in  $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  to a limit  $(\sigma, t)$ . Since the operator  $T: \mathcal{L}_{p,\beta+1}^1(\Gamma) \mapsto \mathcal{L}_{p,\beta+1}^1(\Gamma)$  is continuous (see Proposition 1), it follows, by taking the limit, that

$$
\pi\sigma - T\sigma + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_k = -\varphi. \tag{5.7}
$$

Consequently, equation (1.3) is solvable in  $\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$  for every  $\varphi \in$ <br>(iii) We turn to the case  $\varphi(z) = \text{Re } z^k$  ( $z \in \Gamma \setminus \{O\}$ ). As the harmed  $\varphi$  onto  $\Omega$  we take the function  $h^i(z) = \text{Re } z^k$ (iii) We turn to the case  $\varphi(z) = \text{Re } z^k$  ( $z \in \Gamma \setminus \{O\}$ ). As the harmonic extension of  $\varphi$  onto  $\Omega$  we take the function  $h^{i}(z) = \text{Re } z^{k}$ . It is clear that the conjugate harmonic function  $g^{i}(z) = \text{Im } z^{k}$  belongs to  $\mathfrak{N}_{p,q}^{(-)}(\Gamma)$ . According to Proposition 2,  $g^{i}$  admits the harmonic extension  $f^e$  onto  $\Omega^c$  such that  $\frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Let  $g^e$  be the harmonic function conjugate to  $f^e$  and vanishing at infinity. We have ake the function  $h^i(z) = \text{Re } z^k$ . It is clear that  $\text{Im } z^k$  belongs to  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . According to Propo<br>
ion  $f^e$  onto  $\Omega^c$  such that  $\frac{\partial}{\partial n} f^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Let to  $f^e$  and vanishing at infinity. W

$$
\frac{\partial g^{\epsilon}}{\partial n} = -\frac{\partial h^i}{\partial n} \text{ on } \Gamma \setminus \{O\} \quad \text{and} \quad g^{\epsilon} \in \mathcal{L}_{p,\beta+1}^1(\Gamma)
$$

Arguing as in (i) we prove that the pair  $(\sigma, 0)$ , where  $\sigma(z) = -(2\pi)^{-1} (g^{\epsilon}(z) + \text{Re } z^{\mathbf{k}})$ , belongs to  $\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  and satisfies (1.3). This and (5.7) imply

$$
\mathfrak{M}_{p,\beta}(\Gamma) \subset \left(\pi I - T + \sum_{k=1}^m t^{(k)} \mathcal{I}_k\right) \left(\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m\right).
$$

The converse inclusion was proved in Theorem **ii** 

**Theorem 3.** Let  $\Omega$  have an inward peak. Then operator (5.1) is injective for  $0 <$  $\beta + \frac{1}{p} < \min\{\mu, 1\}.$ 

converse inclusion was proved in Theorem 1  $\blacksquare$ <br> **Theorem 3.** Let  $\Omega$  have an inward peak. Then operator (5.1) is injective for  $0 < \frac{1}{p} < \min\{\mu, 1\}$ .<br> **Proof.** Let  $(\sigma, t) \in \mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$  be an element o Then the harmonic function

$$
\sum_{p,\beta+1}^{n}(\Gamma) \times \mathbb{R}^{m}
$$
 be an element of Ker  
ion  

$$
(W\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{I}_{k}(z) \qquad (z \in \Omega)
$$

vanishes on the contour  $\Gamma \setminus \{O\}$ . By  $(W\sigma)$  we denote an arbitrary function conjugate to  $W\sigma$  in  $\Omega$ . We introduce the holomorphic function

$$
(W\sigma)(z) + \sum_{k=1}^{m} t^{(k)} T_k(z) \qquad (z \in \Omega)
$$
  
ee contour  $\Gamma \setminus \{O\}$ . By  $(\widetilde{W\sigma})$  we denote an arbitrary func  
We introduce the holomorphic function  

$$
W(z) = -(\widetilde{W\sigma})(z) + i(W\sigma)(z) + \sum_{k=1}^{m} t^{(k)} z^{k-\frac{1}{2}} \qquad (z \in \Omega).
$$

Let  $\zeta = \gamma(z)$  be a conformal mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0) = 0$ . The function  $F(z) = (W \circ \gamma^{-1})(\frac{1}{\zeta})$  is holomorphic in the lower half-plane  $\mathbb{R}^2 = {\zeta = \xi + i\eta : \eta <}$ , continuous up to the boundary and Im  $F = 0$  on the real axis. We notice that the function  $W\sigma$  admits the estimate<br>  $|(W\sigma)(z)| \le c |z|^{-N}$ function  $W\sigma$  admits the estimate

$$
|(W\sigma)(z)|\leq c |z|^{-N}
$$

for an integer N. Therefore the holomorphic extension  $F<sup>ext</sup>$  of F onto C is an entire function with real part satisfying

$$
|\text{Re}F^{ext}(\zeta)| \leq c |\zeta|^{2N}.
$$

From the Schwarz integral formula it follows that  $F^{ext}$  has the same order of growth at infinity as  $\mathbb{R}e^{i\pi t}$ . In particular, there exists a polynomial P with real coefficients such that *L<sub>p</sub>*-Theory of Boundary Integral Econdary Integral Equivalent Equivalent Equivalent Equivalent Critical II is the same or  $-(\widetilde{W\sigma})(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) = \text{Re } P\left(\frac{1}{\gamma(z)}\right) \qquad (z \in \Omega).$ <br>  $\overline{V\sigma}(z) = \int \frac{d\sigma}{\sigma} ($ warz integral formula it follows that  $F^{ext}$  has the same order  $F^{ext}$ . In particular, there exists a polynomial P with real coef<br>  $-(\widetilde{W\sigma})(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) = \text{Re } P\left(\frac{1}{\gamma(z)}\right) \qquad (z \in \Omega).$ <br>  $(\widetilde{W\sigma})(z) = \int_{\Gamma$ *Jesumina it f*<br> *J*<sub> $k=1$ </sub>  $\sum_{k=1}^{m} t^{(k)} \mathcal{R}$ <br> *J*<sub> $\Gamma$ </sub>  $\frac{d\sigma}{ds}(q) \log \frac{1}{2}$ bilows that  $F$ <br> *i* exists a poly<br>  $\frac{|z|}{|z-q|}$  ds<sub>q</sub> = *g (* $\overline{W}$ *er'*. In particular, there exists a polynomial *P* with real coefficients such<br>  $-(\widetilde{W}\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) = \text{Re } P\left(\frac{1}{\gamma(z)}\right) \qquad (z \in \Omega).$ <br>  $(\widetilde{W}\sigma)(z) = \int_{\Gamma} \frac{d\sigma}{ds}(q) \log \frac{|z|}{|z-q|} ds_q = \left(V\frac{d\sigma$ 

$$
-(\widetilde{W\sigma})(z)+\sum_{k=1}^m t^{(k)}\mathcal{R}_k(z)=\text{Re }P\Big(\frac{1}{\gamma(z)}\Big) \qquad (z\in\Omega).
$$

Since

$$
(\widetilde{W\sigma})(z) = \int_{\Gamma} \frac{d\sigma}{ds}(q) \log \frac{|z|}{|z-q|} ds_q = \left(V\frac{d\sigma}{ds}\right)(z) \qquad (z \in \Omega)
$$

the equality

$$
(\widetilde{W\sigma})(z) = \int_{\Gamma} \frac{d\sigma}{ds}(q) \log \frac{|z|}{|z-q|} ds_q = (V\frac{d\sigma}{ds})(z) \qquad (z \in \Omega)
$$

$$
(V\frac{d\sigma}{ds})(z) = \text{Re } P\left(\frac{1}{\gamma(z)}\right) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Gamma \setminus \{O\}) \tag{5.8}
$$

the equality<br>  $\left(V\frac{d\sigma}{ds}\right)(z) = \text{Re } P\left(\frac{1}{\gamma(z)}\right) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Gamma \setminus \{O\})$  (5.8)<br>
holds. Taking into account that  $V\left(\frac{d}{ds}\sigma\right) \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$  (see [4: Theorem 1]) we obtain that<br>
the right-han the right-hand side in (5.8) belongs to  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . However, since  $m - \frac{1}{2} + \beta - \mu < -\frac{1}{p}$ , the functions  $\mathcal{R}_k$   $(k = 1, ..., m)$  and positive integer powers of  $\frac{1}{\alpha}$  do not belong to holds. Taking into account that  $V(\frac{d}{ds}\sigma) \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$  (see [4: Theorem 1]) we obtain that<br>the right-hand side in (5.8) belongs to  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . However, since  $m - \frac{1}{2} + \beta - \mu < -\frac{1}{p}$ ,<br>the functions  $\math$  $\mathfrak{N}_{n}^{(-)}(\Gamma)$ . Hence, it follows that the coefficients  $t^{(k)}$   $(k = 1, \ldots, m)$  are equal to zero and the polynomial  $P$  is constant. Therefore  $\widetilde{W\sigma}$  and  $W\sigma$  are constant in  $\Omega.$ 

By  $W_{-\sigma}$  we denote the harmonic function conjugate to  $(W_{\sigma})(z)$  ( $z \in \Omega^c$ ) which equals  $\widetilde{W\sigma}$  on  $\Gamma \setminus \{O\}$ . Since  $\widetilde{W_{-\sigma}}$  admits the estimate Solution the coefficients  $\ell^{N-1}$  ( $k = 1$ )<br>
constant. Therefore  $\widetilde{W\sigma}$  and  $W\sigma$  are harmonic function conjugate to<br>
nce  $\widetilde{W-\sigma}$  admits the estimate<br>  $|(\widetilde{W-\sigma})(z)| \leq c |z|^{-N}$  as  $z \to 0$ 

$$
|(\widetilde{W_{-}\sigma})(z)| \leq c |z|^{-N} \quad \text{as } z \to 0
$$

for an integer N and is constant on  $\Gamma \backslash \{O\}$ , it follows that  $\widetilde{W_{-\sigma}} = \text{const in } \Omega^c$ . Therefore  $W\sigma$  is constant in  $\Omega^c$ .

From the jump formula for  $W\sigma$  we obtain that  $\sigma = \text{const}$  on  $\Gamma \setminus \{O\}$ . Since the non-zero constant does not satisfy the homogeneous equation (1.3), it follows that  $\sigma$ equals zero  $\blacksquare$ 

**Proposition 4.** Let  $\Omega$  have an inward peak, and let  $0 < \beta + \frac{1}{n} < \min\{\mu, 1\}$ ,  $\mu - \beta - \frac{1}{n} + \frac{1}{2} \in \mathbb{N}$ . Then operator (5.1) is not Fredholm.

**Proof.** Let

$$
\Phi(\xi) = |\xi|^{n_0 - 1} (-\log|\xi|)^{-\gamma}
$$

in a small neighbourhood of the origin and let supp  $\Phi$  be in the domain of the mapping  $\theta_0$  introduced in Section 3. Let  $\gamma$  be such that  $\frac{1}{p} < \gamma < 1$ . By  $\varphi$  we denote the function non-zero constant does not satisfy the homogeneous equation (1.5), it ionows that  $\theta$ <br>equals zero **E**<br>**Proposition 4.** Let  $\Omega$  have an inward peak, and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ ,<br> $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ . The Proposition 3. Let  $g^i$  be the harmonic function conjugate to  $h^i$  from Proposition 3. We have<br>  $g^i(z) = \sum_{k=1}^m c^{(k)} \mathcal{R}_k(z) + g^{\#}(z)$ .<br>
Here  $g^{\#}(z) = c \operatorname{Re}\left(z^{\frac{n_0}{2} - \frac{1}{2}} (\log z)^{-\gamma + 1}\right) + g^{\#}_{0}(z)$ , have tant does not satisfy the homogeneous equals<br>
ion 4. Let  $\Omega$  have an inward peak, and let<br>  $\Phi(\xi) = |\xi|^{n_0 - 1}(-\log|\xi|)^{-\gamma}$ <br>
ghbourhood of the origin and let supp  $\Phi$  be in<br>
in Section 3. Let  $\gamma$  be such that  $\frac{1}{p} < \gamma$ **Proposition 4.** Let  $\Omega$  have an inward peak, and let  $0 < \beta + \frac{1}{p} < \min\{\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ . Then operator (5.1) is not Fredholm.<br> **Proof.** Let<br>  $\Phi(\xi) = |\xi|^{n_0 - 1}(-\log|\xi|)^{-\gamma}$ <br>
in a small neighbourhood of the o

$$
g^{i}(z) = \sum_{k=1}^{m} c^{(k)} \mathcal{R}_{k}(z) + g^{\#}(z)
$$

$$
g^{\#}(z) = c \operatorname{Re} \left( z^{\frac{n_0}{2} - \frac{1}{2}} (\log z)^{-\gamma + 1} \right) + g_0^{\#}(z),
$$

where  $g_0^{\#} \in \mathfrak{N}_{p,\beta}(\Gamma)$ . The harmonic extension  $f^e$  of  $g^{\#}$  to  $\Omega^c$  described in Proposition 2 has the form<br>  $f^e(z) = c_1 \text{Im}\left[\left(\frac{z z_0}{z_0 - z}\right)^{-\mu + \frac{n_0}{2} - \frac{1}{2}} (\log z)^{-\gamma + 1}\right]$ 2 has the form

$$
f^{2}(T)
$$
. The harmonic extension  $f^{e}$  of  $g^{\#}$  to  $\Omega^{c}$  describe  

$$
f^{e}(z) = c_{1} \text{Im} \left[ \left( \frac{z z_{0}}{z_{0} - z} \right)^{-\mu + \frac{n_{0}}{2} - \frac{1}{2}} \left( \log z \right)^{-\gamma + 1} \right] + c_{2} \text{Re} \left[ \left( \frac{z z_{0}}{z_{0} - z} \right)^{\frac{n_{0}}{2} - \frac{1}{2}} \left( \log z \right)^{-\gamma + 1} \right] + f_{0}^{e}(z),
$$

where  $c_1, c_2 \in \mathbb{R}$ ,  $z_0$  is a fixed point of  $\Omega$ , and  $\frac{\partial}{\partial n} f_0^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ .

The function  $g^e$ ,  $g^e(\infty) = 0$ , conjugate to  $f^e$ , has the representation

$$
g^e(z) \sim c \, x^{-\beta - \frac{1}{p}} (\log x)^{-\gamma + 1}.
$$

It is clear that  $g^c \notin \mathcal{L}_{p,\beta+1}^1(\Gamma)$  and  $g^e \in \mathcal{L}_{p,\beta'+1}^1(\Gamma)$  for  $\beta' > \beta$ . By Theorem 2 the pair  $(\sigma, t)$ , where  $t = (c^{(1)}, \ldots, c^{(m)})$  and

$$
g^{\epsilon}(z) \sim c x^{-\beta - \frac{1}{p}} (\log x)^{-\gamma + 1}.
$$
  
\n
$$
\epsilon \notin \mathcal{L}_{p,\beta+1}^{1}(\Gamma) \text{ and } g^{\epsilon} \in \mathcal{L}_{p,\beta'+1}^{1}(\Gamma) \text{ for } \beta' > \beta. \text{ By Th}
$$
  
\n
$$
(c^{(1)}, \dots, c^{(m)}) \text{ and}
$$
  
\n
$$
\sigma = -(2\pi)^{-1} \left( g^{\epsilon} + \varphi + \sum_{k=1}^{m} c^{(k)} \mathcal{I}_{k} \right) \qquad \text{on } \Gamma \setminus \{O\},
$$

is the solution of equation (1.3) in  $\mathcal{L}^1_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$  for  $\beta' > \beta$ . From Theorem 3 it follows that equation (1.3) is not solvable in  $\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$ .

According to Theorem 2 equation (1.3) with a right-hand side in  $\mathfrak{N}_{p,\beta'}^{(+)}(\Gamma), \beta' < \beta$ , is solvable in  $\mathcal{L}_{p,\beta'+1}^1(\Gamma) \times \mathbb{R}^m$ . Since the set of smooth functions vanishing near the peak is dense in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and  $\mathcal{L}_{p,\beta'+1}^1(\Gamma) \times \mathbb{R}^m$  is embedded to  $\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$ , it follows that the range of operator (5.1) is not closed in  $\mathfrak{M}_{p,\beta}(\Gamma)$ 

# **6. Boundary integral equation of the Neumann problem**

In this section we prove the unique solvability of equation (1.4) on the contour  $\Gamma$  with inward peak.

**Theorem 4.** Let  $\Omega$  have an inward peak and let  $0 < \beta + \frac{1}{n} < \min\{\mu, 1\}$  and  $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Then the operator

that the range of operator (5.1) is not closed in 
$$
\mathfrak{M}_{p,\beta}(\Gamma) \blacksquare
$$
  
\n6. Boundary integral equation of the Neumann problem  
\nIn this section we prove the unique solvability of equation (1.4) on the contour  $\Gamma$  with  
\ninward peak.  
\nTheorem 4. Let  $\Omega$  have an inward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$  and  
\n $\mu - \beta - \frac{1}{p} + \frac{1}{2} \notin \mathbb{N}$ . Then the operator  
\n
$$
\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + S\sigma + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma) \qquad (6.1)
$$
\nwith  $\mathcal{R}_k(z) = \text{Re } z^{k-\frac{1}{2}}$  is surjective.

*is surjective.* 

**Proof.** (i) Let  $\psi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and  $\psi = 0$  in a neighbourhood of the peak. By  $h^i$  we denote the harmonic extension of  $\psi$  onto  $\Omega$  which is introduced in Proposition 3. Let  $g^i$  be the function conjugate to  $h^i$  and normalized by the condition  $g^i(z_0) = 0$  with  $z_0 \in \Gamma \setminus \{O\}$ . By Proposition 3,  $g^i$  belongs to  $\mathcal{L}^1_{p,\beta+1}(\Gamma)$  and there exist real numbers  $c^{(k)}(\psi)$   $(k = 1, \ldots, m)$  such that

*L<sub>p</sub>*-Theory of Boundary Integration 3, 
$$
g^i
$$
 belongs to  $\mathcal{L}_{p,\beta+1}^1(\Gamma)$  and there  
\n*m*) such that  
\n
$$
g^i(z) = -\sum_{k=1}^m c^{(k)}(\psi) \mathcal{R}_k(z) + g_0^i(z) \qquad (z \in \Omega).
$$
\n*k*)( $\psi$ ) and the function  $g_0^i$  satisfy

These coefficients  $c^{(k)}(\psi)$  and the function  $g_0^i$  satisfy

$$
L_p\text{-Theory of Boundary Integral Equations} \qquad 665
$$
\noposition 3,  $g^i$  belongs to  $\mathcal{L}_{p,\beta+1}^1(\Gamma)$  and there exist real numbers

\ni) such that

\n
$$
i(z) = -\sum_{k=1}^m c^{(k)}(\psi)\mathcal{R}_k(z) + g_0^i(z) \qquad (z \in \Omega).
$$
\n
$$
i(\psi) \text{ and the function } g_0^i \text{ satisfy}
$$
\n
$$
\sum_{k=1}^m |c^{(k)}(\psi)| + ||g_0^i||_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)} \leq c ||\psi||_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)} \qquad (6.2)
$$
\n
$$
= -\frac{d}{ds}\psi, \text{ the function } -g_0^i \text{ solves the Neumann problem in } \Omega \text{ with}
$$

Since  $g^{i}(z) = -\frac{a}{z}$ <br>
coefficients  $c^{(k)}(\psi)$  and<br>  $\sum_{k=1}^{m} |c^{(k)}(\psi)|$ <br>  $\frac{\partial}{\partial n} g^{i} = -\frac{\partial}{\partial s} h^{i} = -\frac{d}{ds} \psi$ ,<br>
ary data the function  $-g_0^\epsilon$  solves the Neumann problem in  $\Omega$  with boundary data

$$
\psi, \text{ the function } -g_0^i \text{ solves the N}
$$
\n
$$
\frac{d}{ds}\psi - \sum_{k=1}^m c^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma).
$$
\nchlet problem in  $\Omega^c$  with bound

\n
$$
(\Gamma) \text{ and satisfies}
$$
\n
$$
\frac{\partial}{\partial n} h^c \Big\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \parallel g_0^i \parallel_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}
$$

By Proposition 2, the Dirichlet problem in  $\Omega^c$  with boundary data  $-g_0^i$  has a solution  $h^c$  such that  $\frac{\partial}{\partial n} h^c \in \mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies  $\left\| \frac{\partial}{\partial n} h^c \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \left\| g_0^i \right\|_{\mathfrak{N}_{p,\beta}^{$  $\frac{d}{ds}\psi - \sum_{k=1}^n c^{(k)}\frac{\partial}{\partial n}$ <br>By Proposition 2, the Dirichlet problem in  $h^e$  such that  $\frac{\partial}{\partial n}h^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies ichle<br>+1( $\Gamma$ )<br> $\|\frac{\partial}{\partial n}$ 

$$
\left\|\frac{\partial}{\partial n}h^e\right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \left\|g_0^i\right\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}.
$$
\n(6.3)

From the equality

$$
\frac{d}{ds}\psi - \sum_{k=1}^{m} c^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \in \mathfrak{Y}_{p,\beta}(\Gamma).
$$
\nposition 2, the Dirichlet problem in  $\Omega^c$  with boundary data  $-g_0^i$  has a s  
\ni that  $\frac{\partial}{\partial n} h^c \in \mathcal{L}_{p,\beta+1}(\Gamma)$  and satisfies\n
$$
\left\| \frac{\partial}{\partial n} h^e \right\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \left\| g_0^i \right\|_{\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)}.
$$
\nthe equality\n
$$
h^c(\infty) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial h^e}{\partial n_q}(q) \log \frac{|z_0|}{|z_0 - q|} ds_q - \frac{1}{2\pi} \int_{\Gamma} h^c(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z_0 - q|} ds_q,
$$
\n
$$
z_0 \text{ is a fixed point in } \Omega, \text{ and from (6.3) we obtain that the linear function
$$
\n
$$
f(\infty) \text{ is continuous in } \mathfrak{N}^{(-)}(\Gamma).
$$
 Therefore, we can choose  $a^i$  so that  $b^c$ .

where  $z_0$  is a fixed point in  $\Omega$ , and from (6.3) we obtain that the linear functional  $g^i \to h^e(\infty)$  is continuous in  $\mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . Therefore, we can choose  $g^i$  so that  $h^e(\infty) = 0$ and inequality (6.3) remains valid. Since grad  $g_0^1 = O(|z|^{-\frac{1}{2}})$  and grad  $h^e = O(|z|^{-\mu-\frac{1}{2}})$ , if following that it follows that

$$
g_0^i(z) = \frac{1}{2\pi} \int_{\Gamma} \left( \frac{\partial g_0^i}{\partial n}(q) + \frac{\partial h^{\epsilon}}{\partial n}(q) \right) \log \frac{|z|}{|z - q|} ds_q.
$$

Set

it follows that  
\n
$$
g_0^i(z) = \frac{1}{2\pi} \int_{\Gamma} \left( \frac{\partial g_0^i}{\partial n} (q) + \frac{\partial h^e}{\partial n} (q) \right) \log \frac{|z|}{|z - q|} ds_q.
$$
\nSet  
\n
$$
\sigma(z) = \frac{1}{2\pi} \left( \frac{d}{ds} \psi(z) - \sum_{k=1}^m c^{(k)}(\psi) \frac{\partial}{\partial n} \mathcal{R}_k(z) - \frac{\partial}{\partial n} h^e(z) \right) \qquad (z \in \Gamma \setminus \{0\}). \qquad (6.4)
$$
\nTaking into account that  $V\sigma(z) = O(|z|^{-(\beta + \frac{1}{p})})$   $(z \neq 0)$  as well as the boundedness of the functions  $g_0^i(z)$   $(z \in \Omega)$  and  $h^e(z)$   $(z \in \Omega^c)$ , we have

Taking into account that  $V\sigma(z) = O(|z|^{-(\beta + \frac{1}{p})})$   $(z \neq 0)$  as well as the boundedness of<br>the functions  $a^i(z)$   $(z \in \Omega)$  and  $b^e(z)$   $(z \in \Omega^c)$  we have

ng into account that 
$$
V\sigma(z) = O(|z|^{-(\beta + \frac{1}{p})})
$$
  $(z \neq 0)$  as well as the bounded  
functions  $g_0^1(z)$   $(z \in \Omega)$  and  $h^e(z)$   $(z \in \Omega^c)$ , we have  
 $g_0^i(z) + V\sigma(z) = c \operatorname{Im} \frac{1}{\gamma(z)}$   $(z \in \Omega)$  and  $h^e(z) + V\sigma(z) = 0$   $(z \in \Omega^c)$ 

where  $\gamma(z)$  is a conformal mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0)=0$ . From the jump formula for  $\frac{\partial}{\partial x}V\sigma$  we obtain

allowiev

\nmapping of 
$$
\Omega
$$
 onto  $\mathbb{R}^2_+$ ,  $\gamma(0) = 0$ . I

\n
$$
c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)} = 0 \qquad (z \in \Gamma \setminus \{0\}).
$$

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where  $\gamma(z)$  is a conformal mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0) = 0$ . From the jump formula for<br>  $\frac{\partial}{\partial n}V\sigma$  we obtain<br>  $c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)} = 0$   $(z \in \Gamma \setminus \{0\})$ .<br>
Since  $\frac{\partial}{\$  $\frac{1}{2}x^{-\frac{3}{2}}$ relation for the normal derivative of the simple layer potential implies

$$
\sigma n \quad \gamma(z)
$$
\n
$$
m \frac{1}{\gamma(z)} \sim \frac{1}{2} x^{-\frac{3}{2}} \text{ as } x \to 0, \text{ we have } c = 0. \text{ Thus, } V\sigma = -g_0^i \text{ on } \Omega
$$
\nor the normal derivative of the simple layer potential implies

\n
$$
\pi \sigma(z) + (S\sigma)(z) = \frac{d}{ds} \psi(z) - \sum_{k=1}^m c^{(k)}(\psi) \frac{\partial}{\partial n} \mathcal{R}_k(z) \qquad (z \in \Gamma \setminus \{0\}).
$$

We set  $t^{(k)} = c^{(k)}(\psi)$   $(k = 1, ..., m)$ . Hence, it follows that the pair  $(\sigma, t)$ , where  $t = (t^{(1)}, \ldots, t^{(m)})$  and  $\sigma$  is defined by (6.4), belongs to  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  and satisfies *is* a conformal mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0) = 0$ . From the jump formula for<br> *c*  $\frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)} = 0$  ( $z \in \Gamma \setminus \{0\}$ ).<br> *i*<sub> $\frac{1}{\gamma(z)} \sim \frac{1}{2} x^{-\frac{3}{2}}$  as  $x \to 0$ , we have  $c = 0$ . Thus,  $V\sigma = -g_0$ equation  $(1.4)$ . From  $(6.2)$  and  $(6.3)$  if follows that al derivative of the simple layer potential implies<br>  $\sigma$ )(z) =  $\frac{d}{ds}\psi(z) - \sum_{k=1}^{m} c^{(k)}(\psi) \frac{\partial}{\partial n} \mathcal{R}_k(z)$  (z  $\in \Gamma \setminus \{0\}$ )<br>
(k = 1,..., m). Hence, it follows that the pair<br>
(d  $\sigma$  is defined by (6.4), belongs t

$$
\begin{aligned} \n\text{a (6.2) and (6.3) if follows that} \\
\|\sigma\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} + \sum_{k=1}^{m} |c^{(k)}(\psi)| &\leq c \|\psi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)} \,. \n\end{aligned} \tag{6.5}
$$

Now let  $\psi$  be an arbitrary function in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ . There exits a sequence  $\{\psi_r\}_{r\geq 1}$  of smooth functions on  $\Gamma \setminus \{0\}$ , which vanishes in a neighbourhood of the peak and tends smooth functions on  $\Gamma \setminus \{0\}$ , which vanishes in a neighbourhood of the peak and tends<br>to  $\psi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ . Let  $(\sigma_r,t_r) \in \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  be the solution of (1.4) with right-hand side  $\frac{d}{ds}\psi_r$ . According to (6.5) the sequence  $\{(\sigma_r,t_r)\}_{r\geq 1}$  converges in  $\mathcal{L}_{p,\beta+1}(\Gamma)$  x side  $\frac{d}{ds}\psi_r$ . According to (6.5) the sequence  $\{(\sigma_r, t_r)\}_{r\geq 1}$  converges in  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ <br>to a limit  $(\sigma, t)$ . Since the operator  $S : \mathcal{L}_{p,\beta+1}(\Gamma) \longrightarrow \mathcal{L}_{p,\beta+1}(\Gamma)$  (see Proposition 1) is<br>continuous, continuous, it follows by taking the limit that  $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ .<br>es in a nei<sub>l</sub><br>c R<sup>m</sup> be th<br>e {( $\sigma_r, t_r$ )<br>hat<br>hat<br> $t^{(k)}\frac{\partial}{\partial n}\mathcal{R}_k$  $\|\psi\|_{\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)}$  (6.5)<br>
There exits a sequence  $\{\psi_r\}_{r\geq 1}$  of<br>
ghbourhood of the peak and tends<br>
e solution of (1.4) with right-hand<br>  $\}_r\geq_1$  converges in  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ <br>  $\rightarrow \mathcal{L}_{p,\beta+1}(\Gamma)$  (

$$
\pi\sigma + S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k = \frac{d}{ds} \psi.
$$
 (6.6)

Consequently, equation (1.4) is solvable in  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  for every  $\psi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$ .<br>
(ii) We turn to the case  $\psi(z) = \text{Re } z^k$  ( $z \in \Gamma \setminus \{0\}$ ). As a harmonic extension  $\psi$  onto  $\Omega$  we take the fu (ii) We turn to the case  $\psi(z) = \text{Re } z^k$   $(z \in \Gamma \setminus \{0\})$ . As a harmonic extension of  $\psi$  onto  $\Omega$  we take the function  $h^{i}(z) = \text{Re } z^{k}$ . The conjugate function  $g^{i}(z) = -\text{Im } z^{k}$ belongs to  $\mathfrak{N}_{n,\beta}^{(-)}(\Gamma)$ . By Proposition 2, the function  $-g^i$  has the harmonic extension *h*<sup>1</sup> Consequently, equation (1.4) is solvable in  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  for every<br>
(ii) We turn to the case  $\psi(z) = \text{Re } z^k$  ( $z \in \Gamma \setminus \{0\}$ ). As a harm<br>  $\psi$  onto  $\Omega$  we take the function  $h^i(z) = \text{Re } z^k$ . The conjugate

$$
\sigma(z) = \frac{1}{2\pi} \Big( \frac{d}{ds} \psi(z) - \frac{\partial}{\partial n} h^{\epsilon}(z) \Big) \qquad (z \in \Gamma \setminus \{O\}).
$$

Then the pair  $(\sigma, 0) \in \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$  is a solution of equation (1.4) with right-hand side  $\frac{d}{d s} \psi$  (see (i)).<br>
This and (6.6) imply<br>  $\mathfrak{Y}_{p,\beta}(\Gamma) \subset \left(\pi I + S + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k\right) (\mathcal{L}_{p,\beta+1}(\$ on  $\Omega^c$  such that  $\frac{\partial}{\partial n}h^e \in \Omega$ <br>  $\sigma(z) =$ <br>
Then the pair  $(\sigma, 0) \in \mathcal{L}$ <br>
side  $\frac{d}{ds}\psi$  (see (i)).<br>
This and (6.6) imply

$$
\begin{aligned}\n\mathcal{C}(\Gamma). \text{ By Proposition 2, the function } -g^i \text{ has the harmonic} \\
\mathcal{C}(\Gamma) &= \frac{\partial}{\partial n} h^e \in \mathcal{L}_{p,\beta+1}(\Gamma). \text{ Set} \\
\sigma(z) &= \frac{1}{2\pi} \left( \frac{d}{ds} \psi(z) - \frac{\partial}{\partial n} h^e(z) \right) \quad (z \in \Gamma \setminus \{O\}). \\
(\sigma, 0) & \in \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \text{ is a solution of equation (1.4) with} \\
\mathcal{D}(\sigma, 0) &= \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \text{ is a solution of equation (1.4).} \\
\mathcal{D}_{p,\beta}(\Gamma) & \subset \left( \pi I + S + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \right) \left( \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \right).\n\end{aligned}
$$

(iii) It remains to prove the converse inclusion. Clearly,

 

$$
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$$
  
converse inclusion. Cle:  

$$
\sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \subset \mathfrak{Y}_{p,\beta}(\Gamma)
$$

*L<sub>p</sub>*-Theory of Boundary Integral Equations 667<br>
(iii) It remains to prove the converse inclusion. Clearly,<br>  $\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \subset \mathfrak{Y}_{p,\beta}(\Gamma)$ <br>
for any  $t \in \mathbb{R}^m$ . Now let  $\sigma$  belong to  $\mathcal{L}_{p,\beta+1}(\$ defined by  $\frac{d}{ds}\mathfrak{S} = \sigma$  on  $\Gamma \setminus \{O\}$ . By  $\psi$  we denote the function for any  $t \in \mathbb{R}^m$ . Now let  $\sigma$  belong to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and let a function  $\mathfrak{S} \in \mathcal{L}_{n,\beta+1}^1(\Gamma)$  be

$$
\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k \subset \mathfrak{Y}_{p,\beta}(\Gamma)
$$
\n  
\n*r* let  $\sigma$  belong to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  and let a function  
\n $\ln \Gamma \setminus \{O\}$ . By  $\psi$  we denote the function  
\n
$$
\psi(z) = \pi \mathfrak{S}(z) - \int_{\Gamma} \mathfrak{S}(q) \frac{\partial}{\partial n_q} \log \frac{|z|}{|z-q|} ds_q
$$
\n.

From Theorem 1 it follows that  $\psi \in \mathfrak{M}_{n, \beta}(\Gamma)$ . Since

$$
\frac{d}{ds}\psi(z)=\pi\sigma(z)+\int_{\Gamma}\sigma(q)\frac{\partial}{\partial n_z}\log\frac{|z|}{|z-q|}ds_q,
$$

we obtain that the image of  $\mathcal{L}_{p,\beta+1}(\Gamma)$  under the mapping (6.1) is the space  $\mathfrak{Y}_{p,\beta}(\Gamma)$ 

**Theorem 5.** Let  $\Omega$  have an *inward peak. Then operator* (6.1) is injective provided  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}.$ 

**Proof.** Let  $(\sigma, t) \in \mathcal{L}_{p, \beta+1}(\Gamma) \times \mathbb{R}^m$ , where  $t = (t^{(1)}, \dots, t^{(m)})$ , belong to Ker  $(\pi I + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k)$ . Then the harmonic function<br> $v(z) = V\sigma(z) + \sum_{k=1}^m t^{(k)} \mathcal{R}_k(z)$   $(z \in \Omega)$ we obtain that the image of  $\mathcal{L}_{p,\beta+1}(\Gamma)$  under the r<br>
Theorem 5. Let  $\Omega$  have an inward peak. The<br>  $0 < \beta + \frac{1}{p} < \min{\{\mu, 1\}}$ .<br> **Proof.** Let  $(\sigma, t) \in \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ , where  $t =$ <br>  $S + \sum_{k=1}^m t^{(k)} \frac{\partial}{\partial n} \$ 

$$
\begin{aligned}\n\mathbb{E} \mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m, \text{ where } t &= (t^{(1)}, \dots, t^{(m)}) \\
\text{Then the harmonic function} \\
v(z) &= V\sigma(z) + \sum_{k=1}^m t^{(k)} \mathcal{R}_k(z) \qquad (z \in \Omega)\n\end{aligned}
$$

has zero Neumann boundary data on  $\Gamma \setminus \{O\}$ . Since

$$
\mathbf{a} \text{ on } \Gamma \setminus \{O\}. \text{ Since}
$$
  

$$
|v(z)| \le c |z|^{-(\beta + \frac{1}{p})}, \tag{6.7}
$$

we obtain by the integral representation for the harmonic function  $v(z)$  and a limit relation for the double layer potential

$$
v(z) = V \sigma(z) + \sum_{k=1} t^{N} K_k(z) \qquad (z \in \Omega)
$$
  
ann boundary data on  $\Gamma \setminus \{O\}$ . Since  

$$
|v(z)| \le c |z|^{-(\beta + \frac{1}{p})},
$$
  
he integral representation for the harmonic function  $v(\$   
double layer potential  

$$
\pi v(z) + \int_{\Gamma} v(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} ds_q = 0 \qquad (z \in \Gamma \setminus \{O\}).
$$
  
lution of the homogeneous integral equation of the Dirichlet

Thus, *v* is a solution of the homogeneous integral equation of the Dirichlet problem in  $\Omega^c$ .

The double layer potential  $(Wv)(z)$   $(z \in \Omega^c)$  grows not faster than a power function as  $z \to 0$ . Since the limit values of *Wv* vanish on  $\Gamma \setminus \{O\}$ , it follows that  $(Wv)(z) =$ *0* ( $z \in \Omega^c$ ). Therefore an arbitrary conjugate function  $\widetilde{Wv}$  is constant in  $\Omega^c$ . We set  $\widetilde{Wv}=C.$ 

Let  $W_+v$  be defined by  $W_+v = Wv$  in  $\Omega$  and let  $\widetilde{W_+v}$  be a conjugate function such that  $\widetilde{W_+v}=C$  on  $\Gamma\setminus\{O\}$ . We introduce the holomorphic-function *We are an arontrary conjugate function*  $Wv$  *is consided by*  $W_+v = Wv$  *in*  $\Omega$  *and let*  $\widetilde{W_+v}$  *be a contrary*  $\setminus \{O\}$ *. We introduce the holomorphic function*  $W(z) = (W_+v)(z) + i(\widetilde{W_+v} - C)$  $(z \in \Omega)$ *.* 

$$
W(z) = (W_{+}v)(z) + i(\widetilde{W_{+}v} - C) \qquad (z \in \Omega).
$$

Let  $\zeta = \gamma(z)$  be a conformal mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0) = 0$ . The function  $F(\zeta) =$  $(W \circ \gamma^{-1})\left(\frac{1}{\zeta}\right)$  is holomorphic in the lower half-plane  $\mathbb{R}^2$ , continuous up to the boundary, and Im  $F = 0$  on  $\partial \mathbb{R}^2$ . The holomorphic extension  $F^{ext}$  of F to C is the entire function, which grows not faster than a power function as  $\zeta \to \infty$ . It follows that  $W(z) = P(\frac{1}{\gamma(z)})$ , where *P* is a polynomial with real coefficients. This implies that mapping of  $\Omega$  onto  $\mathbb{R}^2_+$ ,  $\gamma(0)$ <br>
the lower half-plane  $\mathbb{R}^2_-$ , co<br>
omorphic extension  $F^{ext}$  of<br>
ower function as  $\zeta \to \infty$ . It is<br>
real coefficients. This implie<br>  $=\sum_{k=0}^{\ell} c^{(k)} \text{Re} \left(\frac{1}{\gamma(z)}\right)^{-k}$ 

$$
(W_+)v(z)=\sum_{k=0}^{\ell}c^{(k)}\mathrm{Re}\left(\frac{1}{\gamma(z)}\right)^{-k}\qquad (z\in\Omega).
$$

From the jump formula for *Wv* we obtain

$$
(W_+)v(z) = \sum_{k=0}^{\ell} c^{(k)} \text{Re}\left(\frac{1}{\gamma(z)}\right)^{-k} \qquad (z \in \Omega).
$$
  
formula for  $Wv$  we obtain  

$$
v(z) = -(2\pi)^{-1} \sum_{k=0}^{\ell} c^{(k)} \text{Re}\left(\frac{1}{\gamma(z)}\right)^{-k} \qquad (z \in \Gamma \setminus \{O\}).
$$

By (6.7) we have

$$
(z) = -(2\pi)^{-1} \sum_{k=0} c^{k} Re\left(\frac{1}{\gamma(z)}\right) \qquad (z \in \Gamma \setminus \{0\})
$$
  

$$
v(z) = -(2\pi)^{-1} \left(c^{(0)} + c^{(1)} Re\frac{1}{\gamma(z)}\right) \qquad \text{on } \Gamma \setminus \{0\}.
$$

Therefore

$$
v(z) = -(2\pi)^{-1} \sum_{k=0}^{l} c^{(k)} \text{Re}\left(\frac{1}{\gamma(z)}\right)^{-k} \qquad (z \in \Gamma \setminus \{0\}).
$$
  
we have  

$$
v(z) = -(2\pi)^{-1} \left(c^{(0)} + c^{(1)} \text{Re}\frac{1}{\gamma(z)}\right) \qquad \text{on } \Gamma \setminus \{0\}.
$$
  
re  

$$
(V\sigma)(z) + \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k(z) = -(2\pi)^{-1} c^{(0)} - (2\pi)^{-1} c^{(1)} \text{Re}\frac{1}{\gamma(z)} + c \text{Im}\frac{1}{\gamma(z)}
$$
  

$$
\Omega. \text{ By } h_k^{\epsilon} \quad (k = 1, ..., m) \text{ and } h_0^{\epsilon} \text{ we denote harmonic extensions of}
$$

for  $z \in \Omega$ . By  $h_k^e$   $(k = 1, ..., m)$  and  $h_0^e$  we denote harmonic extensions of  $\mathcal{R}_k$  and  $\text{Re}(\frac{1}{\gamma})$  onto  $\Omega^c$  which grow not faster than a power function as  $z \to 0$ . Since

$$
k = 1, ..., m) \text{ and } h_0^{\epsilon} \text{ we denote harmonic } c;
$$
  
grow not faster than a power function as  $z$  -  

$$
V\sigma + \sum_{k=1}^{m} t^{(k)} h_k^{\epsilon} + (2\pi)^{-1} c^{(1)} h_0^{\epsilon} + (2\pi)^{-1} c^{(0)}
$$

vanishes on  $\Gamma \setminus \{O\}$ , we have

$$
V\sigma + \sum_{k=1} t^{(k)} h_k^{\epsilon} + (2\pi)^{-1} c^{(1)} h_0^{\epsilon} + (2\pi)^{-1} c^{(0)}
$$
  
on  $\Gamma \setminus \{O\}$ , we have  

$$
(V\sigma)(z) = -\sum_{k=1}^{m} t^{(k)} h_k^{\epsilon}(z) - (2\pi)^{-1} c^{(1)} h_0^{\epsilon}(z) - (2\pi)^{-1} c^{(0)} \qquad (z \in \Omega^{\epsilon}).
$$
  
e jump formula for the normal derivative of  $V\sigma$  it follows  

$$
2\pi\sigma(z) = -\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k(z) + c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)}
$$

From the jump formula for the normal derivative of  $V\sigma$  it follows

$$
k=1
$$
  
\non  $\Gamma \setminus \{O\}$ , we have  
\n
$$
\sqrt{\sigma}(z) = -\sum_{k=1}^{m} t^{(k)} h_k^{\epsilon}(z) - (2\pi)^{-1} c^{(1)} h_0^{\epsilon}(z) - (2\pi)^{-1} c^{(0)}
$$
 ( $z \in \Omega^{\epsilon}$ )  
\njump formula for the normal derivative of  $V\sigma$  it follows  
\n
$$
2\pi\sigma(z) = -\sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k(z) + c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)}
$$
\n
$$
+ \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} h_k^{\epsilon}(z) + (2\pi)^{-1} c^{(1)} \frac{\partial}{\partial n} h_0^{\epsilon}(z)
$$
\n
$$
(z \in \Gamma \setminus \{O\})
$$

where

$$
L_p.\text{Theory of Boundary Integral}
$$
\n
$$
\frac{\partial}{\partial n} \mathcal{R}_k(z) \sim a_k \alpha_{\pm} |z|^{k+\mu-\frac{3}{2}} \qquad (k = 1, \dots, m)
$$
\n
$$
\frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma(z)} \sim a_0 |z|^{-\frac{3}{2}}
$$
\n
$$
\frac{\partial}{\partial n} h_k^{\epsilon}(z) \sim \pm b_k |z|^{k-\mu-\frac{3}{2}} \qquad (k = 0, 1, \dots, m).
$$
\n
$$
k = 0, 1, \dots, m \text{ are real coefficients. Since } m \le \mu - k - \mu - \frac{1}{2} + \beta < \frac{1}{p} \qquad \text{for } k = 0, 1, \dots, m.
$$

Here  $a_k$  and  $b_k$   $(k = 0, 1, ..., m)$  are real coefficients. Since  $m \leq \mu - \beta - \frac{1}{p}$ , we have

$$
k - \mu - \frac{1}{2} + \beta < \frac{1}{p}
$$
 for  $k = 0, 1, ..., m$ .

Here  $a_k$  and  $b_k$   $(k = 0, 1, ..., m)$  are real coefficients. Since  $m \le \mu - \beta$ <br>  $k - \mu - \frac{1}{2} + \beta < \frac{1}{p}$  for  $k = 0, 1, ..., m$ .<br>
It means that the function  $\frac{\partial}{\partial n} h_k^c$  does not belong to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  for  $k$ <br>
Therefore the ion  $\frac{\partial}{\partial n} h_k^e$  does not belong to  $\mathcal{L}_{p,\beta+1}$ ,<br>  $i^{(1)}, \ldots, i^{(m)}$  and  $c^{(1)}$  are equal to zero  $\sigma = (2\pi)^{-1} c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma}$  on  $\Gamma \setminus \{O\}$ .

$$
\sigma = (2\pi)^{-1} c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma} \quad \text{on } \Gamma \setminus \{O\}
$$

By the integral representation for the harmonic function Im  $\frac{1}{\infty}$  on  $\Omega$  we have

$$
k - \mu - \frac{1}{2} + \beta < \frac{1}{p} \qquad \text{for} \quad k = 0, 1, \dots, m.
$$
\nmeans that the function  $\frac{\partial}{\partial n} h_k^c$  does not belong to  $\mathcal{L}_{p,\beta+1}(\Gamma)$  for  $k = 0, 1, \dots$  refer to the coefficients  $t^{(1)}, \dots, t^{(m)}$  and  $c^{(1)}$  are equal to zero. Thus,

\n
$$
\sigma = (2\pi)^{-1} c \frac{\partial}{\partial n} \text{Im} \frac{1}{\gamma} \qquad \text{on } \Gamma \setminus \{O\}.
$$
\nthe integral representation for the harmonic function  $\text{Im} \frac{1}{\gamma}$  on  $\Omega$  we have

\n
$$
\frac{1}{2\pi} \int_{\Gamma} \frac{\partial}{\partial n_q} \left( \text{Im} \frac{1}{\gamma(q)} \right) \log \frac{|z|}{|z-q|} ds_q = \frac{1}{2\pi} \int_{\Gamma} \text{Im} \frac{1}{\gamma(q)} \frac{\partial}{\partial n_q} \log \frac{|z|}{|z-q|} ds_q + \text{Im} \frac{1}{\gamma(z)}
$$
\nand

\n
$$
z \in \Omega.
$$
\nSince  $\text{Im} \frac{1}{\gamma(z)} = 0$  on  $\Gamma \setminus \{O\}$ , it follows from a limit relation for the sum of  $\Gamma$  to be  $\gamma$ .

\nFor  $\Omega$  and  $\Gamma$  is  $\gamma$ .

for  $z \in \Omega$ . Since Im  $\frac{1}{\gamma(z)} = 0$  on  $\Gamma \setminus \{O\}$ , it follows from a limit relation for the simple layer potential that  $\pi\sigma - S\sigma = 0$  on  $\Gamma \setminus \{O\}$ . However, we have  $\pi\sigma + S\sigma = 0$  on  $\Gamma \setminus \{O\}$ . Hence, we obtain  $\sigma = 0$ 

**Proposition 5.** Let  $\Omega$  have an inward peak, and let  $0 < \beta + \frac{1}{p} < \min\{\mu,1\}$ ,  $-\beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ . Then operator (6.1) is not Fredholm.<br> **Proof.** Let  $\Psi(\xi) = |\xi|^{n_0-1} (-\log |\xi|)^{-1}$ 

**Proof.** Let

$$
\Psi(\xi) = |\xi|^{n_0-1} \big(-\log|\xi|\big)^{-\gamma}
$$

in a small neighbourhood of the origin and let supp $\Psi$  be in the domain of mapping  $\theta$ **i**ntroduced in Section 3. We assume  $\oint_C \phi = \int_C |\phi|^{n_0 - 1} (1 - \log |\phi|)^{-\gamma}$ <br>  $\phi = \int_C |\phi|^{n_0 - 1} (\cos |\phi|)^{-\gamma}$ <br>  $\phi = \int_C |\phi|^{n_0 - 1} (\cos |\phi|)^{-\gamma}$ <br>  $\phi = \int_C |\phi|^{n_0} \sin \phi$  introduced in Section 3. We assume  $\frac{1}{p} < \gamma < 1$  and set  $\psi = \$ we denote the harmonic extension of  $\psi$  on  $\Omega$  constructed in Proposition 3. Let  $g^i$  be a conjugate function from Proposition 3. We have

$$
g^{i}(z) = \sum_{k=1}^{m} c^{(k)} \mathcal{R}_{k}(z) + g^{\#}(z).
$$
  

$$
g^{\#}(z) = c \operatorname{Re}(z^{\frac{n_0}{2} - \frac{1}{2}} (\log z)^{-\gamma + 1}) + g^{\#}_{0}(z),
$$

Here

$$
g^{\#}(z) = c \operatorname{Re} \left( z^{\frac{n_0}{2} - \frac{1}{2}} (\log z)^{-\gamma + 1} \right) + g_0^{\#}(z),
$$

where  $g_0^{\#} \in \mathfrak{N}_{n,d}^{(-)}(\Gamma)$ .

By  $h^{\epsilon}$  we denote the harmonic extension of  $-g^{\text{\#}}$  on  $\Omega^{\epsilon}$  from Proposition 2. We have of  $-\frac{1}{2}$ 

<br/>  $\zeta$ 

By 
$$
h^e
$$
 we denote the harmonic extension of  $-g^{\#}$  on  $\Omega^c$  from Pr  
\nhave  
\n
$$
h^e(z) = c_1 \operatorname{Im} \left[ \left( \frac{z z_0}{z_0 - z} \right)^{-\mu + \frac{n_0}{2} - \frac{1}{2}} \left( \log z \right)^{-\gamma + 1} \right]
$$
\n
$$
+ c_2 \operatorname{Re} \left[ \left( \frac{z z_0}{z_0 - z} \right)^{\frac{n_0}{2} - \frac{1}{2}} \left( \log z \right)^{-\gamma + 1} \right] + h_0^e(z)
$$
\nwhere  $z_0$  is a fixed point of  $\Omega$  and  $\frac{\partial}{\partial n} h^e \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Since  
\n
$$
\frac{\partial}{\partial n} h^e(z) \sim c x^{-\beta - \frac{1}{p} - 1} (\log x)^{-\gamma - 1},
$$
\nit follows that  $\frac{\partial}{\partial n} h^e \notin \mathcal{L}_{p,\beta+1}(\Gamma)$  and  $\frac{\partial}{\partial n} h^e \in \mathcal{L}_{p,\beta'+1}(\Gamma)$  for  $\beta' > \beta$ .  
\nBy Theorem 4 the pair  $(\sigma, t)$ , where  $t = (c^{(1)}, \ldots, c^{(m)})$  and  
\n
$$
\sigma = (2\pi)^{-1} \left( \frac{d}{dz} \right)_{\beta} = \sum_{i=1}^{m} c^{(k)} \frac{\partial}{\partial z} \mathcal{L}_{i} \left( \frac{\partial}{\partial z} \right)_{\beta} \mathcal{L}_{i} \left( \frac{\partial}{\partial z} \right)_{\beta} \mathcal{L}_{j} \left( \frac{\partial}{\partial z} \right)_{\beta} \mathcal{L}_{j}
$$

$$
c_2 \operatorname{Re}\left[\left(\frac{1}{z_0 - z}\right)^{2} \left(\log z\right)^{-1+1}\right]
$$
  
if  $\Omega$  and  $\frac{\partial}{\partial n} h_0^c \in \mathcal{L}_{p,\beta+1}(\Gamma)$ . Since  

$$
\frac{\partial}{\partial n} h^c(z) \sim c \, z^{-\beta - \frac{1}{p}-1} (\log x)^{-\gamma - 1},
$$

 $\beta_{\beta'+1}(\Gamma)$  for  $\beta' > \beta$ .

By Theorem 4 the pair  $(\sigma, t)$ , where  $t = (c^{(1)}, \ldots, c^{(m)})$  and

= (27r)-'(- - - *--h0* on \ {O}, an on *k=I* 

belongs to  $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$  for  $\beta' > \beta$  and satisfies (6.6). From Theorem 5 it follows that the same equation is not solvable in  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ .

Equation (6.6) is solvable in  $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$ ,  $\beta' < \beta$ . Since the set of smooth functions vanishing near the peak is dense in  $\mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  and since  $\mathcal{L}_{p,\beta'+1}(\Gamma) \times \mathbb{R}^m$  is embedded to  $\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ , we obtain that the range of operator (6.1) is not closed in  $\mathfrak{Y}_{p,\beta}(\Gamma)\blacksquare$ 

# 7. Integral equations of the exterior Dirichlet and Neumann problems for a domain with outward peak

Now we shortly discuss the integral equations mentioned in the title of the section. Their proofs are similar to those of the corresponding results relating to the interior problems for a domain with inward peak, which were proved earlier.

Let  $\Omega$  have an outward peak. The solution of the Dirichlet problem

$$
\left.\begin{array}{l} \Delta u=0\;\;{\rm in}\;\;\Omega^c\\ \left.u\right|_\Gamma=\varphi \end{array}\right\}
$$

is sought in the form

$$
\Delta u = 0 \text{ in } \Omega^c
$$
  
\n
$$
u\big|_{\Gamma} = \varphi
$$
  
\n
$$
u(z) = (W^{ext}\sigma)(z) + \sum_{k=1}^m t^{(k)} \mathcal{I}_k^{ext}(z) \qquad (z \in \Omega^c).
$$
  
\n
$$
(W^{ext}\sigma)(z) = \int_{\Gamma} \sigma(q) \left(\frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} + 1\right) ds_q
$$

Here

$$
(W^{ext}\sigma)(z) = \int_{\Gamma} \sigma(q) \Big( \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} + 1 \Big) ds_q
$$

and

$$
L_p.\text{Theory of Boundary Intes}
$$
\n
$$
\mathcal{I}_k^{ext}(z) = \text{Re}\left(\frac{z z_0}{z_0 - z}\right)^{k - \frac{1}{2}} \qquad (z \in \Omega^c),
$$
\nat in  $\Omega$ . The density  $\sigma$  and the vector  $t =$ 

where  $z_0$  is a fixed point in  $\Omega$ . The density  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$  satisfy the equation

$$
L_p.\text{Theory of Boundary Integral Equations} \qquad 671
$$
\n
$$
T_k^{ext}(z) = \text{Re}\left(\frac{z z_0}{z_0 - z}\right)^{k - \frac{1}{2}} \qquad (z \in \Omega^c),
$$
\n1 point in  $\Omega$ . The density  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$  satisfy

\n
$$
\pi\sigma + T^{ext}\sigma + \sum_{k=1}^m t^{(k)} T_k^{ext} = \varphi \qquad \text{on } z \in \Gamma \setminus \{O\}, \qquad (7.1)
$$
\ne value of the potential  $W^{ext}\sigma$  at a point of  $\Gamma \setminus \{O\}$ .

\nLet  $\Omega$  have an outward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}, \mu - \beta -$  the operator

\n
$$
\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + T^{ext}\sigma + \sum_{k=1}^m t^{(k)} T_k^{ext} \in \mathfrak{M}_{p,\beta}(\Gamma) \qquad (7.2)
$$

where  $T^{ext}\sigma$  is the value of the potential  $W^{ext}\sigma$  at a point of  $\Gamma \setminus \{O\}$ .

**Theorem 6.** Let  $\Omega$  have an outward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ ,  $\mu - \beta - \frac{1}{p}$  $\pi\sigma + T^{ext}\sigma$ <br>nere  $T^{ext}\sigma$  is the value of the **Theorem 6.** Let  $\Omega$  have  $+\frac{1}{2} \notin \mathbb{N}$ . Then the operator

$$
\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi \sigma + T^{\epsilon x t} \sigma + \sum_{k=1}^m t^{(k)} \mathcal{I}_k^{\epsilon x t} \in \mathfrak{M}_{p,\beta}(\Gamma) \tag{7.2}
$$

*is surjective.* 

 $\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \longmapsto \pi\sigma + T^{ext}\sigma + \sum_{k=1}^m t^{(k)} \mathcal{I}_k^{ext} \in \mathfrak{M}_{p,\beta}(\Gamma)$  (7.2)<br>
is surjective.<br> **Proof.** Let  $h^e$  be the harmonic extension of  $\varphi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  on  $\Omega^c$  constructed in<br>
Proposi Proposition 3, and let  $g^e$  be a conjugate function vanishing at a fixed point on  $\Gamma \setminus \{O\}$ .<br>By Proposition 3 there exist real numbers  $c^{(k)}$   $(k = 1, ..., m)$  such that<br> $g^e = \sum_{k=1}^m c^{(k)} \mathcal{R}_k^{ext} + g_0^e$ ,

imjugate function vanis

\nl numbers 
$$
c^{(k)}
$$
  $(k = 1, 0)$ 

\n
$$
g^{\epsilon} = \sum_{k=1}^{m} c^{(k)} \mathcal{R}_k^{\epsilon t} + g_0^{\epsilon},
$$

\n
$$
e^{c t}(z) = \text{Re}\left(\frac{z z_0}{z_0 - z}\right)^{k-1}
$$

where  $g_0^e \in \mathfrak{N}_{n,\theta}^{(-)}(\Gamma)$  and

$$
\mathcal{R}_k^{ext}(z) = \text{Re}\left(\frac{z z_0}{z_0 - z}\right)^{k - \frac{1}{2}}.
$$

We set  $h_0^e = h^e + \sum_{k=1}^m t^{(k)} \mathcal{I}_k^{ext}$ .

The only change to be made in the proof of Theorem 2 is that the solution *g' of* the where  $g_0^e \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$  and<br>  $\mathcal{R}_k^{ext}(z) = \text{Re}\left(\frac{zz_0}{z_0 - z}\right)^{k - \frac{1}{2}}$ .<br>
We set  $h_0^e = h^e + \sum_{k=1}^m t^{(k)} \mathcal{I}_k^{ext}$ .<br>
The only change to be made in the proof of Theorem 2 is that the solution<br>
Neumann prob

$$
\int_{\Gamma} g^{\mathfrak{r}} ds = \int_{\Gamma} h_0^{\mathfrak{e}} ds - 2h_0^{\mathfrak{e}}(\infty).
$$

Then the pair  $(\sigma, t)$ , where  $t = (c^{(1)}, \ldots, c^{(m)})$  and

$$
J_{\Gamma} \qquad J_{\Gamma}
$$
  
we  $t = (c^{(1)}, \dots, c^{(m)})$  and  

$$
\sigma = (2\pi)^{-1} \left( \varphi + \sum_{k=1}^{m} c^{(k)} \mathcal{I}_{k}^{ext} - g^{i} \right)
$$

is a solution in  $\mathcal{L}_{p,\beta+1}^1(\Gamma) \times \mathbb{R}^m$  of equation (7.1). The case  $\varphi \in \mathfrak{P}(\Gamma)$  is considered as in Theorem 2

We represent the solution of the Neumann problem

$$
\left.\begin{array}{l}\n\Delta u = 0 \text{ in } \Omega^c \\
\frac{\partial u}{\partial n}\bigg|_{\Gamma} = \varphi\n\end{array}\right\}
$$

in the form

$$
\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \varphi \qquad \qquad \}
$$

$$
u(z) = (V\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k^{ext}(z) \qquad (z \in \Omega^c),
$$

where  $V\sigma$  is the simple layer potential. The density  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$ satisfy

The solution of the Neumann problem  
\n
$$
\Delta u = 0 \text{ in } \Omega^c
$$
\n
$$
\frac{\partial u}{\partial n}\Big|_{\Gamma} = \varphi
$$
\n
$$
u(z) = (V\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_k^{ext}(z) \qquad (z \in \Omega^c),
$$
\nmple layer potential. The density  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$ 

\n
$$
\pi\sigma - S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k^{ext} = -\varphi \qquad \text{on } \Gamma \setminus \{O\}. \tag{7.3}
$$
\nLet  $\Omega$  have an outward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}, \mu - \beta - \frac{1}{p} < \frac{1$ 

**Theorem 7.** Let  $\Omega$  have an outward peak and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ ,  $\mu - \beta \pi\sigma-S\sigma+$ <br>Theorem 7. Let  $\Omega$  have<br> $+\frac{1}{2}\notin\mathbb{N}$ . Then the operator

$$
\partial n \mid_{\Gamma} - \mathfrak{p}
$$
\n
$$
u(z) = (V\sigma)(z) - \sum_{k=1}^{m} t^{(k)} \mathcal{R}_{k}^{ext}(z) \qquad (z \in \Omega^{c}),
$$
\nis the simple layer potential. The density  $\sigma$  and the vector  $t = (t^{(1)}, \ldots, t^{(m)})$ \n
$$
\pi\sigma - S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_{k}^{ext} = -\varphi \qquad \text{on } \Gamma \setminus \{O\}. \tag{7.3}
$$
\n
$$
\text{rem 7. } \text{Let } \Omega \text{ have an outward peak and let } 0 < \beta + \frac{1}{p} < \min\{\mu, 1\}, \ \mu - \beta - \Omega.
$$
\n
$$
\text{Then the operator}
$$
\n
$$
\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R}^{m} \ni (\sigma, t) \longmapsto \pi\sigma - S\sigma + \sum_{k=1}^{m} t^{(k)} \frac{\partial}{\partial n} \mathcal{R}_{k}^{ext} \in \mathfrak{Y}_{p,\beta}(\Gamma) \tag{7.4}
$$
\n
$$
\text{we.}
$$

*is surjective.* 

**Proof.** Let  $h^e$  be the harmonic extension of  $\psi \in \mathfrak{N}_{p,\beta}^{(+)}(\Gamma)$  on  $\Omega^c$  and  $g^e$  be a conjugate function constructed in Proposition 3. Then there exist real numbers  $c^{(k)}$ such that t)  $\longmapsto \pi\sigma$  -<br>
monic extens<br>
in Propositie<br>  $g^e = \sum_{k=1}^m c^{(k)}$ 

$$
g^{\epsilon} = \sum_{k=1}^{m} c^{(k)} \mathcal{R}_k^{\epsilon \tau t} + g_0^{\epsilon},
$$

where  $g_0^{\epsilon} \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . We choose  $g_0^{\epsilon}$  to satisfy  $g_0^{\epsilon}(\infty) = 0$ . Here the function  $g^{\epsilon}$  plays the same role as  $g^i$  in the proof of Theorem 4.

Now we use the same argument as in Theorem 4. By  $h^i$  we denote the harmonic where  $g_0^e \in \mathfrak{N}_{p,\beta}^{(-)}(\Gamma)$ . We choose  $g_0^e$  to satisfy  $g_0^e(\infty) = 0$ . Here the function  $g^e$  plays the same role as  $g^i$  in the proof of Theorem 4.<br>Now we use the same argument as in Theorem 4. By  $h^i$  we deno  $(\sigma, t)$ , where  $t = (c^{(1)}, \ldots, c^{(m)})$  and if in the proof of Theorem 4.<br>
se the same argument as in Theorem 4. By  $h^i$ <br>
of on  $\Omega$  such that  $\frac{\partial}{\partial n}h^i \in \mathcal{L}_{p,\beta+1}(\Gamma)$  (see Propo $\epsilon = (c^{(1)}, \ldots, c^{(m)})$  and<br>  $\sigma = (2\pi)^{-1} \left( \frac{\partial}{\partial n} h^i - \frac{d}{ds} \psi + \sum_{k=1}^m c^{(k)} \$  $g^e = \sum_{k=1}^m c^{(k)} \mathcal{R}_k^{ex}$ <br>
bose  $g_0^e$  to satisfy  $g_0^e$ <br>
(of Theorem 4.<br>
argument as in Theorem 1.<br>
that  $\frac{\partial}{\partial n} h^i \in \mathcal{L}_{p,\beta+1}$ <br>
(m) and<br>  $\frac{\partial}{\partial n} h^i - \frac{d}{ds} \psi + \sum_{k=1}^m c^{(k)}$ 

use the same argument as in Theorem 4. By 
$$
h^*
$$
 we denote  $g_0^{\epsilon}$  on  $\Omega$  such that  $\frac{\partial}{\partial n}h^i \in \mathcal{L}_{p,\beta+1}(\Gamma)$  (see Proposition 2).  
\n $t = (c^{(1)}, \ldots, c^{(m)})$  and  
\n
$$
\sigma = (2\pi)^{-1} \left( \frac{\partial}{\partial n} h^i - \frac{d}{ds} \psi + \sum_{k=1}^m c^{(k)} \frac{\partial}{\partial n} \mathcal{R}_k^{\epsilon x i} \right) \in \mathcal{L}_{p,\beta+1}(\Gamma),
$$

solves equation (7.3).

The case  $\varphi \in \mathfrak{P}(\Gamma)$  is considered in the same way as in Theorem 4, one should only replace  $g^i$ ,  $h^i$  and  $h^e$  by  $g^e$ ,  $h^e$  and  $h^i$ , respectively  $\blacksquare$ 

Two following theorems can be proved in the same way as Theorems 3 and 5.

**Theorem 8.** Let  $\Omega$  have an outward peak. Then operator (7.2) is injective for  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}.$ 

**Theorem 9.** Let  $\Omega$  have an outward peak. Then operator (7.4) is injective for  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}.$ 

The proof of the following proposition is essentially the same as those of Propositions 4 ( the case of operator  $(7.2)$ ) and 5 (the case of operator  $(7.4)$ ).

Theorem 8. Let  $\Omega$  hav<br>  $< \beta + \frac{1}{p} < \min\{\mu, 1\}$ .<br>
Theorem 9. Let  $\Omega$  hav.<br>  $< \beta + \frac{1}{p} < \min\{\mu, 1\}$ .<br>
The proof of the following<br>
( the case of operator (7.2))<br>
Proposition 6. Let  $\Omega$  h<br>  $- \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ . **Proposition 6.** Let  $\Omega$  have an outward peak, and let  $0 < \beta + \frac{1}{p} < \min\{\mu, 1\}$ ,  $\mu - \beta - \frac{1}{p} + \frac{1}{2} \in \mathbb{N}$ . Then operators (7.2) and (7.4) are not Fredholm.

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