On the Stabilization of the Inversion of Some Kontorovich-Lebedev Like Integral Transforms

H.-J. Glaeske and S. **B. Yakubovich**

Abstract. In this paper we construct a special type of regularization operators for the Kontorovich-Lebedev type integral transforms to stabilize their inversion in weighted $L_{\nu,p}$ -spaces. Some estimates of norms in these spaces are obtained.

Keywords: *Kontorovich-Lebedev transform, Lebedev-Skalskaya transforms, index transforms, Macdonald function*

AMS subject **classification:** 44 A 15

1. Introduction

Following ideas of regularization for ill-posed problems, especially in the theory of integral equations, we will show that the inversion formulas of certain integral transforms can be used to stabilize the original in the sense that small changes of the transforms lead to small changes of the originals. More precisely: If the transform g with *g*(*x*) = $\pi f(f(x)) = \int_0^{\infty} f(x) \, dx$ *g*(*x*) = $\pi f(f(x)) = \int_0^{\infty} f(x, y) f(y) \, dy$
 g(*x*) = $\pi f(f)(x) = \int_0^{\infty} f(x, y) f(y) \, dy$
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 bed function or ill-posed problems, especially in the theory of inte-
the inversion formulas of certain integral transforms
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als. More precisely: If the transform g with
 $\Gamma[f](x) = \int_{0}^{\infty} H(x$

$$
g(x) = T[f](x) = \int_{0}^{\infty} H(x, y) f(y) dy
$$
 (1)

is substituted by a perturbed function *h,* not necessarily in the range of *T,* and if one defines a regularization operator *I* by means of a slight modification of the inversion operator T^{-1} with

$$
T^{-1}[g](x) = f(x) \tag{2}
$$

one can find stability estimates with respect to $f = T^{-1}[g]$ and I[h]. In this sense the notion "stabilization" is used.

Our main goal is to consider a variety of the Kontorovich-Lebedev type integral transformations being connected by composition structure and mapping properties with

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the Kontorovich-Lebedev transform $[5, 13, 15 \cdot 16]$

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\nch-Lebedev transform [5, 13, 15 - 16]

\n
$$
K_{iz}[f](x) = \cosh(\pi x) \int_{0}^{\infty} K_{iz}(y)f(y) \, dy = g(x) \qquad (x \ge 0)
$$
\n(3)

\nthe Macdonald function (see [1: Vol.2]) and the Mehler-Fock transform [2,

\n
$$
P_{iz}[f](x) = \int_{0}^{\infty} P_{-\frac{1}{2}+iz}(\cosh y)f(y) \, dy = g(x) \qquad (4)
$$
\nical Legendre function of the first kind $P_{-\frac{1}{2}+iz}(\cosh \cdot)$ (see [1: Vol. 2]).

where $K_{i\tau}$ is the Macdonald function (see [1: Vol.2]) and the Mehler-Fock transform [2, 4, 10, 16]

$$
\mathbf{P}_{iz}[f](x) = \int_{0}^{\infty} P_{-\frac{1}{2}+iz}(\cosh y) f(y) \, dy = g(x) \tag{4}
$$

with the spherical Legendre function of the first kind $P_{-\frac{1}{2}+i\mathbf{r}}(\cosh \cdot)$ (see [1: Vol. 2]).

In Section 2 we deal with the Kontorovich-Let
 L_p -space $L_{\nu,p}(\mathbb{R}_+)$ $(1 \le p \le \infty, \nu \in \mathbb{R})$ normed by

$$
K_{iz}[f](x) = \cosh(\pi x) \int_{0}^{1} K_{iz}(y)f(y) dy = g(x) \qquad (x \ge 0)
$$
\n
$$
F_{iz}(x) = \int_{0}^{\infty} F_{iz}[f](x) dy = g(x) \qquad (4)
$$
\nthe spherical Legendre function of the first kind $F_{-\frac{1}{2}+iz}(\cosh \cdot)$ (see [1: Vol. 2]).

\nIn Section 2 we deal with the Kontorovich-Lebedev transform (3) in the weighted space $L_{\nu,p}(\mathbb{R}_+)$ (1 $\leq p \leq \infty, \nu \in \mathbb{R}$) normed by

\n
$$
||f||_{\nu,p} = \left(\int_{0}^{\infty} x^{\nu p-1} |f(x)|^{p} dx\right)^{1/p} \qquad (5)
$$

Related problems were considered in $[3, 8 \cdot 12, 14]$ and in two monographs $[15 \cdot 16]$. Generalizing our results from the mentioned items we give a special construction of the inversion of the Kontorovich-Lebedev transform (3) that gives a solution of equation (3) and allows to stabilize it in the space $L_{\nu,p}(\mathbb{R}_+)$, when $\nu = 1 + \frac{1}{p}, 1 < p < 2$.

Finally, in Section 3 we will give such a construction of the inversion formula concerning the Lebedev-Skalskaya transformation [6] being related to the Kontorovich-Lebedev transform (see $[15 - 16]$).

2. The Kontorovich-Lebedev transform

Let us consider the inversion of the Kontorovich-Lebedev transform (3) as the equation where $f \in L_{1+\frac{1}{2},p}(\mathbb{R}_+)$ and suppose that instead of the exact function *g* on the righthand side we have a perturbed one $h \in L_{1+\frac{1}{n},p}(\mathbb{R}_+),$ not necessarily in the range of the operator K*¹* . Let Figure 1.100 (6) being related to the Kontorovich-Lebedev
 Iev transform
 I Kontorovich-Lebedev transform (3) as the equation

² that instead of the exact function *g* on the right-
 Ih $\in L_{1+\frac{1}{p},p}(\mathbb{R}_{+})$, **Example 15.4**
 Compare 17.4 Compare 17.4 Compare 17.4 Compare 17.4 Compare 17.4 Compare 17.4 (R₊), not necessarily in the range of the $\mathbf{r}_0(\mathbb{R}_+)$ **, not necessarily in the range of the** $\mathbf{r}_0(\mathbb{R}_+)$ *K*_{iz}(y) = K_{i} *K*_{iz}(y) = $o(\log y)$ (y) \rightarrow 0+) (8) \rightarrow *K*_{iz}(y) = $o(\log y)$ (8) \rightarrow *K*_{iz}(y) = K_{i} (y) = $o(\log y)$ (y) + 0+) (8) (8) (8) (8) (7) (7) (7) (7) (7) (7) (7) (7) (8) (8) (8) (8) (9) (7) (7) (7) (7) (

$$
||h - g||_{1 + \frac{1}{2}, p} < \varepsilon \tag{6}
$$

for some $\varepsilon > 0$. From the uniform estimate (see [16: Formula (1.100) with $\delta = 0$])

$$
|K_{ix}(y)| \leq K_0(y),\tag{7}
$$

where K_0 is the Macdonald function of zero index, and the asymptotic formulas (see [1: Vol. 2/ Subsection 7.2.2, Formula (12), resp. Subsection 7.13.1, Formula (7)])

$$
K_{ix}(y) = o(\log y) \qquad (y \to 0+)
$$
 (8)

and

$$
K_{ix}(y) = O\left(e^{-y}\sqrt{\pi/(2y)}\right) \qquad (y \to \infty)
$$
 (9)

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 $K_{ix}(y) = O(e^{-y}\sqrt{\pi/(2y)})$ $(y \to \infty)$ (9)

tegral (3) by the Hölder inequality converges at infinity for all

1. To guarantee the convergence of the integral in (3) at zero for we observe that the integral (3) by the Holder inequality converges at infinity for all values of $x \ge 0$ and $p \ge 1$. To guarantee the convergence of the integral in (3) at zero for all values of $p \ge 1$ according to (8) we restrict the class of solutions *f* by the additional condition $\tilde{f} \in L_p([0,1])$, where \tilde{f} denotes the restriction of f to the interval [0, 1]. In the sequel we simply write $f \in L_p([0, 1])$ for this. Some integral transforms θ *K*
 $(y \rightarrow \infty)$ (9)

ality converges at infinity for all

ce of the integral in (3) at zero for

s of solutions *f* by the additional

ion of *f* to the interval [0, 1]. In
 *K*_{ir}(*x*)*h*(*r* On Some Integral Transfor

and
 $K_{ix}(y) = O(e^{-y}\sqrt{\pi/(2y)})$ $(y \to \infty)$

we observe that the integral (3) by the Hölder inequality converges at inf

values of $x \ge 0$ and $p \ge 1$. To guarante the convergence of the integral in (3

Let us introduce a regularization operator
\n
$$
(I_{\delta}h)(x) = \frac{2}{\pi^2 x} \int_{0}^{\infty} \frac{\tau \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} K_{i\tau}(x)h(\tau) d\tau
$$
\n(10)

where $x > 0$ and $\delta \in (0, \frac{\pi}{2})$. Assuming that $h \in L_{1+\frac{1}{\pi},p}(\mathbb{R}_+)$ let us estimate the norm of the operator I_{δ} in this space. By means of the generalized Minkowski inequality, namely $(I_{\delta}h)(x) = \frac{2}{\pi^2 x} \int_{0}^{\infty} \frac{\tau \sinh((\pi \cosh(\tau)))}{\cosh(\tau)}$
 $\vec{b} \in (0, \frac{\pi}{2})$. Assuming that *h*

in this space. By means c
 $\int_{0}^{b} \left| \int_{0}^{d} f(x, y) dy \right|^p dx \right)^{1/p} \leq 1$ $\frac{(-\delta)\tau}{\pi\tau}$ *K*_{ir}(*x*)
 $\in L_{1+\frac{1}{p},p}(\mathbb{R}_{+})$
 f the general
 d $\int_{0}^{d} \left(\int_{0}^{b} |f(x,y)| \right)$

$$
\int_{0}^{h} \int_{0}^{x} \cosh(u \cdot f)
$$

and $\delta \in (0, \frac{\pi}{2})$. Assuming that $h \in L_{1+\frac{1}{p}, p}(\mathbb{R}_{+})$ let us est

$$
\int_{\delta}^{h} \left| \int_{c}^{d} f(x, y) dy \right|^{p} dx \bigg|^{1/p} \le \int_{c}^{d} \left(\int_{a}^{b} |f(x, y)|^{p} dx \right)^{1/p} dy
$$

and (1.10)]), we obtain

$$
\|_{1+\frac{1}{p}, p} \le \frac{2}{\pi^{2}} \int_{0}^{\infty} \frac{\pi \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} |h(\tau)| \left(\int_{0}^{\infty} |K_{ir}(x)|^{p} dx \right)
$$

lölder inequality and estimate (7) we continue $(q = \frac{p}{p-1})$

(see $[16: Formula (1.10)]$), we obtain

$$
\left(\int_{a}^{b} \left| \int_{c}^{d} f(x, y) dy \right|^{p} dx\right)^{1/p} \leq \int_{c}^{d} \left(\int_{a}^{b} |f(x, y)|^{p} dx\right)^{1/p} dy
$$

Formula (1.10)]), we obtain

$$
||(I_{\delta}h)||_{1+\frac{1}{p}, p} \leq \frac{2}{\pi^{2}} \int_{0}^{\infty} \frac{\tau \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} |h(\tau)| \left(\int_{0}^{\infty} |K_{i\tau}(x)|^{p} dx\right)^{1/p} d\tau. \tag{11}
$$

Hence by the Hölder inequality and estimate (7) we continue ($q =$

$$
\| (I_{\delta}h) \|_{1+\frac{1}{p},p} \leq \frac{2}{\pi^2} \int_0^{\infty} \frac{\tau \sinh((\pi-\delta)\tau)}{\cosh(\pi\tau)} |h(\tau)| \left(\int_0^{\infty} |K_{i\tau}(x)|^p dx \right)^{1/p} d\tau. \tag{11}
$$

ce by the Hölder inequality and estimate (7) we continue $(q = \frac{p}{p-1})$

$$
\| (I_{\delta}h) \|_{1+\frac{1}{p},p} \leq \frac{2}{\pi^2} \left(\int_0^{\infty} |K_0(x)|^p dx \right)^{1/p} \|h\|_{1+\frac{1}{p},p} \left(\int_0^{\infty} \frac{\sinh^q((\pi-\delta)\tau)}{\cosh^q(\pi\tau)} d\tau \right)^{1/q}. \tag{12}
$$

mating the integrals in (12) as

$$
\left(\int_0^{\infty} |K_0(x)|^p dx \right)^{1/p} = \left(\int_0^{\infty} \left(\int_0^{\infty} e^{-x \cosh y} dy \right)^p dx \right)^{1/p}
$$

Estimating the integrals in (12) as

$$
\rho \leq \frac{2}{\pi^2} \left(\int_0^\infty |K_0(x)|^p dx \right)^{1/p} ||h||_{1+\frac{1}{p},p} \left(\int_0^\infty \frac{\sinh^q((\pi-\delta)\tau)}{\cosh^q(\pi\tau)} d\tau \right)^{1/q}.
$$
 (12)
integrals in (12) as

$$
\left(\int_0^\infty |K_0(x)|^p dx \right)^{1/p} = \left(\int_0^\infty \left(\int_0^\infty e^{-x \cosh y} dy \right)^p dx \right)^{1/p}
$$

$$
\leq \frac{1}{p^{1/p}} \int_0^\infty \frac{dy}{\cosh^{1/p} y}
$$
 (13)
$$
= \frac{2^{1/p-2}}{p^{1/p}} \frac{\left[\Gamma(\frac{1}{2p}) \right]^2}{\Gamma(\frac{1}{p})}
$$

and

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\n
$$
\left(\int_{0}^{\infty} \frac{\sinh^{q}((\pi - \delta)\tau)}{\cosh^{q}(\pi \tau)} d\tau\right)^{1/q} \le \left(\int_{0}^{\infty} e^{-q\delta\tau} d\tau\right)^{1/q} = \frac{1}{(q\delta)^{1/q}}
$$
\n
$$
\|[(I_{\delta}h)\|_{1+\frac{1}{p},p} \le \frac{2^{1/p}}{2\pi^{2}p^{1/p}(q\delta)^{1/q}} \frac{[\Gamma(1/2p)]^{2}}{\Gamma(1/p)} \|h\|_{1+\frac{1}{p},p}.
$$
\n(15)
\n
$$
\text{mposition } (I_{\delta}K_{iz}[f]) \text{ the following assertion is valid.}
$$

we have finally

$$
\| (I_{\delta}h) \|_{1+\frac{1}{p},p} \leq \frac{2^{1/p}}{2\pi^2 p^{1/p} (q\delta)^{1/q}} \frac{[\Gamma(1/2p)]^2}{\Gamma(1/p)} \|h\|_{1+\frac{1}{p},p}.
$$
 (15)

For the composition $(I_{\delta} \mathbf{K}_{i\mathbf{z}}[f])$ the following assertion is valid.

Lemma 1. Let $f \in L_p([0,1]) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+)$ $(1 \leq p \leq \infty)$. Then

\n We finally\n
$$
\| (I_{\delta}h) \|_{1 + \frac{1}{p}, p} \leq \frac{2^{1/p}}{2\pi^2 p^{1/p} (q\delta)^{1/q}} \frac{[\Gamma(1/2p)]^2}{\Gamma(1/p)} \, \|h\|_{1 + \frac{1}{p}, p}.\n
$$
\n

\n\n For the composition\n $(I_{\delta} \mathbf{K}_{i\mathbf{z}}[f])$ the following assertion is valid.\n

\n\n lemma 1. Let $f \in L_p([0, 1]) \cap L_{1 + \frac{1}{p}, p}(\mathbb{R}_+) \quad (1 \leq p \leq \infty). \quad \text{Then}$ \n

\n\n $(I_{\delta} \mathbf{K}_{i\mathbf{r}}[f]) = \frac{\sin \delta}{\pi} \int_{0}^{\infty} \frac{K_1 \left((x^2 + y^2 - 2xy \cos \delta)^{1/2} \right)}{(x^2 + y^2 - 2xy \cos \delta)^{1/2}} \, yf(y) \, dy \quad (x > 0).$ \n

\n\n For the standard data, the integral $f(x)$ is the formula of the string, the integral $f(x)$ is the formula of the string.\n

where K_1 is the Macdonald function of order one.

Proof. Substituting the value of $K_{ir}[f]$ by formula (3) into (10), interchanging the order of integration, and then calculating the integral with respect to τ invoking formula (2.16.51.8) in [7: Vol. 2], we arrive at the representation (16). The absolute convergence of the iterated integral for arbitrary $\delta \in (0, \frac{\pi}{2})$ can easily be shown using the Holder inequality in the inner integral and the uniform estimate (see [16: Formula $(1.100)]$ on, and then calculating the integra.

[7: Vol. 2], we arrive at the repres-

rated integral for arbitrary $\delta \in (0, \frac{\pi}{2})$

in the inner integral and the uniform
 $|K_{ir}(y)| \leq e^{-\mu \tau} K_0(y \cos \mu) \qquad (\mu \in \mathbb{R})$

6) is proved (v) dy $(x > 0)$ (16)

(3) into (10), interchanging

(3) into (10), interchanging

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in estimate (see [16: Formula

(0, $\frac{\pi}{2}$)). (17)
 Le value of K_{ir}[f] by formula (3) into (10), interchanging
 f—*f*</sup> (*n*). 2], we arrive at the representation (16). The absolute

integral for arbitrary $\delta \in (0, \frac{\pi}{2})$ can easily be shown using
 f inner integral

$$
|K_{i\tau}(y)| \leq e^{-\mu\tau} K_0(y \cos \mu) \qquad (\mu \in [0, \frac{\pi}{2})). \tag{17}
$$

The representation (16) is proved \blacksquare

Denoting by f_{ϵ} the left-hand side of (10), where ϵ depends upon δ (we will give such a dependence below), we write

$$
f_{\epsilon}-f=(I_{\delta}(h-g))+(I_{\delta}g)-f
$$
\n(18)

where $g = \mathbf{K}_{ix}[f]$ is the exact right-hand side of equation (3). So, by (15) and (6)

notation (16) is proved

\n
$$
\mathbf{g} \text{ by } f_{\epsilon} \text{ the left-hand side of (10), where } \epsilon \text{ depends upon } \delta \text{ (we will give such}
$$
\n
$$
f_{\epsilon} - f = (I_{\delta}(h - g)) + (I_{\delta}g) - f \qquad (18)
$$
\n
$$
\mathbf{K}_{i\mathbf{z}}[f] \text{ is the exact right-hand side of equation (3). So, by (15) and (6),}
$$
\n
$$
\|f_{\epsilon} - f\|_{1 + \frac{1}{p}, p} \le \frac{2^{1/p} \varepsilon}{2\pi^2 p^{1/p} (q\delta)^{1/q}} \frac{[\Gamma(1/2p)]^2}{\Gamma(1/p)} + \| (I_{\delta}g) - f \|_{1 + \frac{1}{p}, p}. \qquad (19)
$$
\npose now is to estimate the norm
$$
\|(I_{\delta}g) - f\|_{1 + \frac{1}{p}, p}
$$
. Note that our approach

Our purpose now is to estimate the norm $\|(I_{\delta}g) - f\|_{1+\frac{1}{p},p}$. Note that our approach alid for all $1 < p < 2$. Substituting $y = x(\cos \delta + t \sin \delta)$ in (16) we obtain the dity
 $(I_{\delta}g)(x) = \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{R(x,t,\delta)}{t^2+1} f$ is valid for all $1 < p < 2$. Substituting $y = x(\cos \delta + t \sin \delta)$ in (16) we obtain the equality
 $(I_{\delta}g)(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{R(x, t, \delta)}{t^2 + 1} f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) dt$ (20) equality

$$
\mathbf{1}_{\{x\}}[\mathbf{1}_{\{x\}}] \text{ is the exact right-hand side of equation (3). So, by (15) and (0),}
$$
\n
$$
\|f_{\epsilon} - f\|_{1 + \frac{1}{p}, p} \le \frac{2^{1/p} \varepsilon}{2\pi^2 p^{1/p} (q\delta)^{1/q}} \frac{\left[\Gamma(1/2p)\right]^2}{\Gamma(1/p)} + \|(I_{\delta}g) - f\|_{1 + \frac{1}{p}, p}. \qquad (19)
$$
\n
$$
\text{trpose now is to estimate the norm } \| (I_{\delta}g) - f\|_{1 + \frac{1}{p}, p}. \text{ Note that our approach: all } 1 < p < 2. \text{ Substituting } y = x(\cos \delta + t \sin \delta) \text{ in (16) we obtain the}
$$
\n
$$
(I_{\delta}g)(x) = \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{R(x, t, \delta)}{t^2 + 1} f(x(\cos \delta + t \sin \delta)) (\cos \delta + t \sin \delta) dt \qquad (20)
$$
\n
$$
R(x, t, \delta) = x \sin \delta (t^2 + 1)^{1/2} K_1 (x \sin \delta (t^2 + 1)^{1/2}). \qquad (21)
$$

where

$$
R(x,t,\delta) = x \sin \delta (t^2+1)^{1/2} K_1(x \sin \delta (t^2+1)^{1/2}).
$$
 (21)

Hence owing to the generalized Minkowski inequality and accounting the identity

On S
Minkowski inequality an

$$
\frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{dt}{t^2 + 1} = 1 - \frac{\delta}{\pi}
$$

we have the estimate

$$
\pi \int_{-\cot \delta} t^2 + 1 \qquad \pi
$$
\nthe estimate\n
$$
\|\left(I_{\delta}g\right) - f\|_{1 + \frac{1}{p}, p}
$$
\n
$$
\leq \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f\left(x(\cos \delta + t \sin \delta)\right)(\cos \delta + t \sin \delta) R(x, t, \delta)
$$
\n
$$
- \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right\|_{1 + \frac{1}{p}, p}
$$
\n
$$
\leq \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| \left[f\left(x(\cos \delta + t \sin \delta)\right)(\cos \delta + t \sin \delta) - \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right] R(x, t, \delta) \right\|_{1 + \frac{1}{p}, p}
$$
\n
$$
+ \frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x) [R(x, t, \delta) - 1] \right\|_{1 + \frac{1}{p}, p} dt.
$$
\n(22)

f has a derivative $f' \in L^1_{loc}(\mathbb{R}_+)$ such that

$$
+\frac{1}{\pi-\delta} \int_{-\cot\delta} \frac{1}{t^2+1} ||f(x)[R(x,t,\delta)-1]||_{1+\frac{1}{p},p} dt.
$$

We estimate now each norm under the integrals in (22). For this let us assume that
as a derivative $f' \in L^1_{loc}(\mathbb{R}_+)$ such that

$$
f(x(\cos\delta + t\sin\delta))(\cos\delta + t\sin\delta) - \left(1-\frac{\delta}{\pi}\right)^{-1} f(x)
$$

$$
= \int_{1}^{\cos\delta + t\sin\delta} \frac{d}{dy} [y f(xy)] dy - \frac{\delta}{\pi-\delta} f(x) \qquad (23)
$$

$$
= \int_{1}^{\cos\delta + t\sin\delta} [f(xy) + xyf'(xy)] dy - \frac{\delta}{\pi-\delta} f(x).
$$

Further, observe that owing to the uniform inequality $xK_1(x) \leq 1$ for the Macdonald function from (21) we obtain immediately that $R(x, t, \delta) \leq 1$. Consequently, if $f' \in L_{2+\frac{1}{2},p}(\mathbb{R}_+)$, then using the generalized M function from (21) we obtain immediately that $R(x,t,\delta) \leq 1$. Consequently, if $f' \in L_{2+\frac{1}{2},p}(\mathbb{R}_+)$, then using the generalized Minkowski inequality we have

$$
\left\| \left[f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right] R(x, t, \delta) \right\|_{1 + \frac{1}{p}, p}
$$

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\n
$$
\leq \left\| f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - f(x) \right\|_{1 + \frac{1}{p}, p} + \frac{\delta}{\pi - \delta} \|f\|_{1 + \frac{1}{p}, p}
$$
\n
$$
\leq \left[\|f'\|_{2 + \frac{1}{p}, p} + \|f\|_{1 + \frac{1}{p}, p} \right] \left\| \int_{1}^{\cos \delta + t \sin \delta} y^{-1 - \frac{1}{p}} dy \right\| + \frac{\delta}{\pi - \delta} \|f\|_{1 + \frac{1}{p}, p} \qquad (24)
$$
\n
$$
= p \left[\|f'\|_{2 + \frac{1}{p}, p} + \|f\|_{1 + \frac{1}{p}, p} \right] \left| 1 - (\cos \delta + t \sin \delta)^{-1/p} \right| + \frac{\delta}{\pi - \delta} \|f\|_{1 + \frac{1}{p}, p}.
$$
\nor the first integral in (22) we obtain\n
$$
\frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| \left[f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right] R(x, t, \delta) \right\| dt dt
$$

Now for the first integral in (22) we obtain

$$
\begin{aligned}\n\mathcal{L} \left[\left\| f' \right\|_{2+\frac{1}{p},p} + \left\| f \right\|_{1+\frac{1}{p},p} \right] \left| 1 - (\cos \delta + t \sin \delta)^{-1/p} \right| &+ \frac{\delta}{\pi - \delta} \left\| f \right\|_{1+\frac{1}{p},p}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{the first integral in (22) we obtain} \\
\frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| \left[f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \left(1 - \frac{\delta}{\pi} \right)^{-1} f(x) \right] R(x, t, \delta) \right\|_{1+\frac{1}{p},p} dt \\
&\leq \frac{p}{\pi} \left[\left\| f' \right\|_{2+\frac{1}{p},p} + \left\| f \right\|_{1+\frac{1}{p},p} \right] \\
&\times \int_{-\cot \delta}^{\infty} \frac{|1 - (\cos \delta + t \sin \delta)^{-1/p}|}{t^2 + 1} dt + \frac{\delta}{\pi} \left\| f \right\|_{1+\frac{1}{p},p}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{with } \delta \text{ takes the form } (p > 1)\n\end{aligned}
$$
\n
$$
I_{\delta} = \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{|1 - (\cos \delta + t \sin \delta)^{-1/p}|}{t^2 + 1} dt = \frac{\sin \delta}{\pi} \int_{0}^{\infty} \frac{\sinh(v/p)}{\cosh v - \cos \delta} dv. \tag{26}
$$
\n
$$
\text{continue to estimate}
$$

The integral on the right-hand side of the inequality (25) after the substitution e^v = $\cos \delta + t \sin \delta$ takes the form $(p>1)$

$$
\times \int_{-\cot\delta}^{\pi} \frac{|1 - (\cos\delta + t\sin\delta)^{-1/p}|}{t^2 + 1} dt + \frac{\delta}{\pi} ||f||_{1 + \frac{1}{p}, p}.
$$

grad on the right-hand side of the inequality (25) after the substitution $e^v = \sin\delta$ takes the form $(p > 1)$

$$
I_{\delta} = \frac{1}{\pi} \int_{-\cot\delta}^{\infty} \frac{|1 - (\cos\delta + t\sin\delta)^{-1/p}|}{t^2 + 1} dt = \frac{\sin\delta}{\pi} \int_{0}^{\infty} \frac{\sinh(v/p)}{\cosh v - \cos\delta} dv.
$$
 (26)
g. continue to estimate

Hence we continue to estimate

$$
\int_{\cot \delta}^{\infty} \frac{|1 - (\cos \delta + t \sin \delta)^{-1/p}|}{t^2 + 1} dt = \frac{\sin \delta}{\pi} \int_{0}^{\infty} \frac{\sinh(v/p)}{\cosh v - \cos \delta} dv. \qquad (26)
$$

\nwe to estimate
\n
$$
I_{\delta} = \frac{\sin \delta}{\pi} \left(\int_{0}^{1} + \int_{1}^{\infty} \right) \frac{\sinh(v/p)}{\cosh v - \cos \delta} dv
$$

\n
$$
\leq \frac{\sin \delta}{\pi} \left(\log(\cosh v - \cos \delta) \Big|_{0}^{1} + \int_{1}^{\infty} \frac{\sinh(v/p)}{\cosh v - 1} dv \right) \qquad (27)
$$

\n
$$
\leq \sin \delta \log \left(2^{-1} \sin \frac{-2}{\rho} \frac{\delta}{2} \right) + \sin \delta A_{p}
$$

\n
$$
A_{p} = 1 + \int_{1}^{\infty} \frac{\sinh(v/p)}{\cosh v - 1} dv \qquad (p > 1).
$$
 (28)

where

$$
a \delta \log \left(2^{-1} \sin \frac{2}{\rho} \frac{\delta}{2} \right) + \sin \delta A_p
$$

$$
A_p = 1 + \int_{1}^{\infty} \frac{\sinh(v/p)}{\cosh v - 1} dv \qquad (p > 1).
$$
 (28)

 \bar{z}

Thus we can rewrite estimate (25) as

 \sim

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\nIn rewrite estimate (25) as
\n
$$
\frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| \left[f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right] R(x, t, \delta) \right\|_{1 + \frac{1}{p}, p} dt
$$
\n
$$
\leq p \sin \delta \left[\log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + A_p \right] \|f'\|_{2 + \frac{1}{p}, p} + \left[p \sin \delta \log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + p \sin \delta A_p + \frac{\delta}{\pi} \right] \|f\|_{1 + \frac{1}{p}, p}.
$$
\n(29)

To estimate the norm in the second integral in (22) we represent the kernel (21) by means of the identity (see [1: Vol. 2/Subsection 7.11, Formula (21)])

$$
\frac{d}{dx}[xK_1(x)] = -xK_0(x). \tag{30}
$$

Therefore, owing to the limit property $xK_1(x) \rightarrow 1$ as $x \rightarrow 0$ we obtain the representation

$$
\left[e^{\sin \theta \log \left(\frac{2}{\sin \theta}\right)} + e^{\sin \theta \log \theta}\right] + \frac{1}{\pi} \left[e^{\sin \theta \log \theta}\right]
$$
\n
$$
= \left[1: \text{Vol. } 2/\text{Subsection } 7.11, \text{ Formula (21)}\right]
$$
\n
$$
= \frac{d}{dx} [xK_1(x)] = -xK_0(x). \tag{30}
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\right]
$$
\n
$$
= \lim_{x \to \infty} \left[e^{\cos \theta \log \theta} + \frac{1}{2} \log \left(\frac{2}{x}\right)\
$$

Substituting it in the norm in (22) we apply the generalized Minkowski inequality and with a simple change of variables we derive

it in the norm in (22) we apply the generalized Minkowski inequality and
\ne change of variables we derive
\n
$$
||f(x)[R(x,t,\delta)-1]||_{1+\frac{1}{p},p}
$$
\n
$$
= \left(\int_{0}^{\infty} x^{p} \left(\int_{0}^{x \sin \delta(t^{2}+1)^{1/2}} yK_{0}(y) dy\right)^{p} |f(x)|^{p} dx\right)^{1/p}
$$
\n
$$
\leq \int_{0}^{\infty} yK_{0}(y) \left(\int_{\frac{1}{\sin \delta(t^{2}+1)^{1/2}}}^{\infty} x^{p} |f(x)|^{p}\right)^{1/p} dy.
$$
\n(32) of formula (32) we return to the respective integral in (22) and estimate it
\n
$$
\frac{1}{\pi-\delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^{2}+1} ||f(x)[R(x,t,\delta)-1]||_{1+\frac{1}{p},p} dt
$$

Making use of formula (32) we return to the respective integral in (22) and estimate i as follows: \overline{a}

$$
\begin{array}{c}\n\text{(32) we return to the respective integral in (}\n\\ \n\frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \|f(x)[R(x, t, \delta) - 1]\|_{1 + \frac{1}{p},p} dt\n\end{array}
$$

$$
\leq \frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \int_{0}^{\infty} yK_0(y) \left(\int_{\frac{\sin \theta(t)^2 + 1}{\sin \theta(t)^2 + 1}}^{\infty} x^p |f(x)|^p dx \right)^{1/p} dy dt
$$
\n
$$
= \frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \int_{0}^{\infty} uK_0(u\sqrt{t^2 + 1}) du \left(\int_{\frac{u}{\sin \theta}}^{\infty} x^p |f(x)|^p dx \right)^{1/p} dt.
$$
\n
$$
< 2 - p. Dividing the integral with respect to u in (33) as
$$
\n
$$
\left(\int_{0}^{\delta^{\alpha}} + \int_{\delta^{\alpha}}^{\infty} \right) uK_0(u\sqrt{t^2 + 1}) du
$$
\n
$$
= to estimate the norm (33) and we have
$$

Let $0 < \alpha < 2 - p$. Dividing the integral with respect to *u* in (33) as

$$
\left(\int\limits_{0}^{\delta^{\alpha}}+\int\limits_{\delta^{\alpha}}^{\infty}\right) u K_{0}(u\sqrt{t^{2}+1}) du
$$

we continue to estimate the norm (33) and we have

$$
\left(\int_{0}^{\delta^{a}} + \int_{\delta^{a}}^{\infty} \right) u K_{0}(u \sqrt{t^{2} + 1}) du
$$

continue to estimate the norm (33) and we have

$$
\frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \left(\int_{0}^{\delta^{a}} + \int_{\delta^{a}}^{\infty} \right) u K_{0}(u \sqrt{t^{2} + 1}) du \left(\int_{\frac{\epsilon^{a}}{\sin \delta}}^{\infty} x^{p} |f(x)|^{p} dx\right)^{1/p} dt
$$
(34)
$$
\leq \frac{||f||_{1 + \frac{1}{p}, p}}{\pi - \delta} \int_{0}^{\delta^{a}} u \int_{-\cot \delta}^{\infty} K_{0}(u|t|) dt du + \int_{0}^{\infty} u K_{0}(u) du \left(\int_{\frac{\epsilon^{a}}{\sin \delta}}^{\infty} x^{p} |f(x)|^{p} dx\right)^{1/p} .
$$
(34)
we clearly that according to the absolute continuity of the Lebesgue integral for an
itrary $\varepsilon > 0$ there exists a number $\delta \in (0, \frac{\pi}{2})$ such that

$$
\int_{\frac{\epsilon^{a}}{\sin \delta}}^{\infty} x^{p} |f(x)|^{p} dx \leq \varepsilon^{p}.
$$
(35)
ace invoking the values of the integrals 2.16.2.1-2 from [7: Vol.2] we obtain finally

Now clearly that according to the absolute continuity of the Lebesgue integral for an arbitrary $\varepsilon > 0$ there exists a number $\delta \in (0, \frac{\pi}{2})$ such that

$$
\int_{\frac{\delta^{\alpha}}{\sin \delta}}^{\infty} x^p |f(x)|^p dx \leq \varepsilon^p. \tag{35}
$$

Hence invoking the values of the integrals $2.16.2.1-2$ from $[7:$ Vol.2 we obtain finally

ly that according to the absolute continuity of the Lebesgue integral for an
$$
\varepsilon > 0
$$
 there exists a number $\delta \in (0, \frac{\pi}{2})$ such that\n
$$
\int_{\frac{\delta^{\alpha}}{\sin \delta}}^{\infty} x^{p} |f(x)|^{p} dx \leq \varepsilon^{p}.
$$
\n(35)\n\nUsing the values of the integrals 2.16.2.1-2 from [7: Vol.2] we obtain finally\n
$$
\frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^{2} + 1} \|f(x)[R(x, t, \delta) - 1]\|_{1 + \frac{1}{p}, p} dt < 2\delta^{\alpha} \|f\|_{1 + \frac{1}{p}, p} + \varepsilon.
$$
\n(36)\n\nisining estimates (36) and (29), from (22) we get

Now combining estimates (36) and (29), from (22) we get

$$
\int_{\frac{\epsilon}{\sin\delta}}^{\infty} x^{p} |f(x)|^{p} dx \leq \epsilon^{p}.
$$
 (35)
Hence involving the values of the integrals 2.16.2.1-2 from [7: Vol.2] we obtain finally

$$
\frac{1}{\pi - \delta} \int_{-\cot\delta}^{\infty} \frac{1}{t^{2} + 1} ||f(x)| R(x, t, \delta) - 1||_{1 + \frac{1}{p}, p} dt < 2\delta^{\alpha} ||f||_{1 + \frac{1}{p}, p} + \epsilon.
$$
 (36)
Now combining estimates (36) and (29), from (22) we get

$$
||(I_{\delta}g) - f||_{1 + \frac{1}{p}, p}
$$

$$
< p \sin \delta \left[\log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + A_{p} \right] ||f'||_{2 + \frac{1}{p}, p} + \epsilon
$$
 (37)
$$
+ \left[p \sin \delta \log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + p \sin \delta A_{p} + \frac{\delta}{\pi} + 2\delta^{\alpha} \right] ||f||_{1 + \frac{1}{p}, p}.
$$

Invoking inequality (35) one can let $\delta = c_{\alpha, p} \epsilon^{p(1 - \alpha)^{-1}}$, where $c_{\alpha, p}$ is a constant. Assume
also that $||f||_{1 + \frac{1}{p}, p} \leq E_{1}$ and $||f'||_{2 + \frac{1}{p}, p} \leq E_{2}$ ($E_{1}, E_{2} > 0$). Then, taking into account

 $\begin{aligned} E_1^{\text{S}} &\geq 2 \end{aligned}$
 $E_1^{\text{S}} \text{ and } \begin{aligned} \|f'\|_{2+\frac{1}{p},p} &\leq 1 \end{aligned}$ E_2 $(E_1, E_2 > 0)$. Then, taking into account our previous estimates, from (19) we obtain the desired estimate of the difference (18) as

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\nOur previous estimates, from (19) we obtain the desired estimate of the difference (18)
\nas
\n
$$
||f_{\epsilon} - f||_{1+\frac{1}{p},p} < \frac{2^{1/p-1}}{\pi^2 p^{1/p} (c_{\alpha,p} q)^{1/q}} \frac{[\Gamma(1/2p)]^2}{\Gamma(1/p)} \epsilon^{\frac{2-p-\alpha}{1-\alpha}}
$$
\n
$$
+ p c_{\alpha,p} (E_1 + E_2) \epsilon^{p(1-\alpha)^{-1}} \log \left(2^{-1} \sin^{-2} \left(\frac{\epsilon^{p(1-\alpha)^{-1}} c_{\alpha,p}}{2}\right)\right)
$$
\n(38)
\n
$$
+ \epsilon + 2E_1 c_{\alpha,p}^{\alpha} \epsilon^{p\alpha(1-\alpha)^{-1}} + (p A_p (E_1 + E_2) + 1) c_{\alpha,p} \epsilon^{p(1-\alpha)^{-1}}.
$$
\nWe summarize our results in this section by the following
\nTheorem 1. Let $f \in L_p([0,1]) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+) \quad (1 < p < 2)$ and let f possess
\na derivative $f' \in L_{loc}^1(\mathbb{R}_+) \cap L_{2+\frac{1}{p},p}(\mathbb{R}_+).$ Then, assuming that $||f||_{1+\frac{1}{p},p} \le E_1$ and
\n
$$
||f'||_{2+\frac{1}{p},p} \le E_2 \quad (E_1, E_2 > 0), \text{ the function (10), i.e.}
$$
\n
$$
f_{\epsilon}(x) = \frac{2}{\pi^2 x} \int_0^{\infty} \frac{\tau \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} K_{ir}(x)h(\tau) d\tau
$$
\n(39)
\nwith

We summarize our results in this section by the following

Theorem 1. Let $f \in L_p([0,1]) \cap L_{1+\frac{1}{2},p}(\mathbb{R}_+)$ $(1 < p < 2)$ and let f possess *a derivative* $f' \in L^1_{loc}(\mathbb{R}_+) \cap L_{2+\frac{1}{2},p}(\mathbb{R}_+).$ Then, assuming that $||f||_{1+\frac{1}{2},p} \leq E_1$ and *y* the following
 (\mathbb{R}_{+}) $(1 < p < 2)$ and let f possess
 ien, assuming that $||f||_{1+\frac{1}{p},p} \leq E_1$ and
 (i, i.e.
 $\frac{\delta}{\delta}$)
 *K*_{ir}(*x*)*h*(*r*) *dr* (39)
 $(0 < \alpha < 2-p)$ (40)
 iat $||f_{\epsilon} - f||_{1+\frac{1}{p},p}$ satis

$$
F = L_{loc}(\mathbf{R}_{+}) + L_{2 + \frac{1}{p},p}(\mathbf{R}_{+}). \quad \text{Then, assuming that } ||j||_{1 + \frac{1}{p},p} \le L_{1} \text{ and}
$$
\n
$$
E_{2} \quad (E_{1}, E_{2} > 0), \text{ the function (10), i.e.}
$$
\n
$$
f_{\epsilon}(x) = \frac{2}{\pi^{2}x} \int_{0}^{\infty} \frac{\tau \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} K_{i\tau}(x)h(\tau) d\tau \qquad (39)
$$
\n
$$
\delta = c_{\alpha, p} \epsilon^{p(1-\alpha)^{-1}} \qquad (0 < \alpha < 2 - p) \qquad (40)
$$
\n
$$
E_{2} \quad \text{and} \quad \delta = \frac{2}{\pi^{2}x} \int_{0}^{\infty} \frac{\tau \sinh((\pi - \delta)\tau)}{\cosh(\pi \tau)} K_{i\tau}(x)h(\tau) d\tau \qquad (39)
$$

with

$$
\delta = c_{\alpha, p} \varepsilon^{p(1-\alpha)^{-1}} \qquad (0 < \alpha < 2-p) \tag{40}
$$

is a regularized solution of equation (3) *such that* $||f_{\epsilon} - f||_{1+\frac{1}{2},p}$ *satisfies estimate* (38) *for all* $1 < p < 2$.

3. The Lebedev-Skalskaya transform

In this section we will consider a regularized solution of the Lebedev-Skalskaya integral equation (see $[6, 11, 13]$)

$$
< 2.
$$
\n
\nebedev-Skalskaya transform\n
\non we will consider a regularized solution of the Lebedev-Skalskaya integral\n
\n
$$
Re[f](x) = \cosh(\pi x) \int_{0}^{\infty} ReK_{\frac{1}{2}+ix}(y)f(y) dy = g(x) \qquad (x \ge 0)
$$
\n(41)\n
\n
$$
i + ix(y)
$$
 is the real part of the Macdonald function with the index $\frac{1}{2} + ix$,\n
\n
$$
ReK_{\frac{1}{2}+ix}(y) = \frac{K_{\frac{1}{2}+ix}(y) + K_{\frac{1}{2}-ix}(y)}{2}.
$$
\n
$$
E_{\nu}(x) = \int_{0}^{\infty} e^{-x \cosh u} \cosh(\nu u) du \qquad (43)
$$

where Re $K_{\frac{1}{2}+i\tau}(y)$ is the real part of the Macdonald function with the index $\frac{1}{2}+ix$, namely

$$
Re K_{\frac{1}{2}+i\,}(y) = \frac{K_{\frac{1}{2}+i\,}(y) + K_{\frac{1}{2}-i\,}(y)}{2}.
$$
 (42)

The integral representation of the Macdonald function (see [1: Vol. *2,* Subsection *7.12,* Formula *(21)1)*

$$
K_{\nu}(x) = \int_{0}^{\infty} e^{-x \cosh u} \cosh(\nu u) du \qquad (43)
$$

immediately yields to the following formula for the kernel (42):

e and S. B. Yakubovich
\nto the following formula for the kernel (42):
\n
$$
ReK_{\frac{1}{2}+iz}(y) = \int_{0}^{\infty} e^{-y \cosh u} \cosh \left(\frac{u}{2}\right) \cos(ux) du \qquad (44)
$$
\n
$$
imate
$$
\n
$$
|ReK_{\frac{1}{2}+iz}(y)| \le K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2}} \frac{e^{-y}}{\sqrt{y}}.
$$
\n(45)
\nnate (similar to (17)) is given, for example, in [16: Formula (6.10,11)],

and the uniform estimate

$$
+iz(y) = \int_{0}^{y \cosh u} \cosh\left(\frac{u}{2}\right) \cos(ux) du \qquad (44)
$$

$$
|\text{Re}K_{\frac{1}{2}+iz}(y)| \le K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2}} \frac{e^{-y}}{\sqrt{y}}.
$$
 (45)

A more precise estimate (similar to (17)) is given, for example, in [16: Formula (6.10,11)] namely

uniform estimate
\n
$$
|\text{Re}K_{\frac{1}{2}+i\mathbf{z}}(y)| \le K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2}} \frac{e^{-y}}{\sqrt{y}}.
$$
\n
$$
\text{forceise estimate (similar to (17)) is given, for example, in [16: Formula (6.10,11)],}
$$
\n
$$
|\text{Re}K_{\frac{1}{2}+i\mathbf{z}}(y)| \le e^{-\mu x} K_{\frac{1}{2}}(y \cos \mu) = \sqrt{\frac{\pi}{2}} \frac{e^{-y \cos \mu - \mu x}}{\sqrt{y \cos \mu}} \qquad (\mu \in [0, \frac{\pi}{2}))
$$
\n
$$
\text{that } f \in L_{\frac{1}{2}-\frac{1}{2},p}([0,1]) \cap L_p(\mathbb{R}_+) \quad (1 < p < 2).
$$
\n
$$
\text{Then owing to (45) and the}
$$

Assume that $f \in L_{\frac{1}{2}-\frac{1}{2},p}([0,1]) \cap L_p(\mathbb{R}_+)$ $(1 < p < 2)$. Then owing to (45) and the Holder inequality it is not difficult to observe that the integral *(41)* converges. Indeed, we have

$$
\int_{0}^{\infty} \frac{1}{(x+1)^{g/2}} e^{-\int_{-\frac{1}{2},y}^{y} f(x,y) dx} \qquad (\mu \in [0, \frac{1}{2})
$$
\n
$$
= \int_{0}^{\infty} \frac{1}{(x+1)^{g/2}} e^{-\int_{-\frac{1}{2},y}^{y} f(x,y) dx} \qquad (1 < y < 2).
$$
\n
$$
\int_{0}^{\infty} |\text{Re}K_{\frac{1}{2}+ix}(y) f(y)| dy
$$
\n
$$
\leq \int_{0}^{1} \sqrt{y} K_{\frac{1}{2}}(y) \frac{|f(y)|}{\sqrt{y}} dy + \int_{1}^{\infty} K_{\frac{1}{2}}(y) |f(y)| dy
$$
\n
$$
\leq \int_{0}^{1} \int_{0}^{\infty} \sqrt{y} K_{\frac{1}{2}}(y) \frac{|f(y)|}{\sqrt{y}} dy + \int_{1}^{\infty} K_{\frac{1}{2}}(y) |f(y)| dy
$$
\n
$$
\leq \int_{0}^{1} \left[\sqrt{y} K_{\frac{1}{2}}(y) \right]^{q} dy + \int_{1}^{\infty} K_{\frac{1}{2}}(y) |f(y)| dy + \int_{1}^{\infty} \left[K_{\frac{1}{2}}(y) \right]^{q} dy \right)^{1/q} ||f||_{p}
$$
\n
$$
< \infty.
$$
\n
$$
\text{us introduce a regularization operator}
$$
\n
$$
(\hat{I}_{\delta}h)(x) = \frac{4}{\pi^{2}} \int_{0}^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\cosh(\pi \tau)} \text{Re}K_{\frac{1}{2}+i\tau}(x) h(\tau) d\tau \qquad (48)
$$
\n
$$
\text{re } x > 0 \text{ and } \delta \in (0, \frac{\pi}{2}). \text{ Now we estimate the norm of the operator (48) under the
$$

Let us introduce a regularization operator

a regularization operator
\n
$$
(\hat{I}_{\delta}h)(x) = \frac{4}{\pi^2} \int_{0}^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\cosh(\pi \tau)} \operatorname{Re}K_{\frac{1}{2}+i\tau}(x)h(\tau) d\tau
$$
\n(48)

where $x > 0$ and $\delta \in (0, \frac{\pi}{2})$. Now we estimate the norm of the operator (48) under the condition $h \in L_p(\mathbb{R}_+)$ (1 < p < 2). In the same manner as in (11) we obtain

$$
(\hat{I}_{\delta}h)(x) = \frac{4}{\pi^2} \int_0^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\cosh(\pi \tau)} \operatorname{Re}K_{\frac{1}{2} + i\tau}(x)h(\tau) d\tau
$$
(48)
> 0 and $\delta \in (0, \frac{\pi}{2})$. Now we estimate the norm of the operator (48) under the
in $h \in L_p(\mathbb{R}_+)$ (1 < $p < 2$). In the same manner as in (11) we obtain

$$
\|(\hat{I}_{\delta}h)\|_p \le \frac{4}{\pi^2} \int_0^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\cosh(\pi \tau)} |h(\tau)| \left(\int_0^{\infty} |\operatorname{Re}K_{\frac{1}{2} + i\tau}(x)|^p dx\right)^{1/p} d\tau.
$$
(49)

Hence by the Holder inequality and estimate (45) we have

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\nHence by the Hölder inequality and estimate (45) we have
\n
$$
\|(\hat{I}_{\delta}h)\|_{p} \leq \frac{4}{\pi^{2}} \left(\int_{0}^{\infty} K_{\frac{1}{2}}^{p}(x) dx\right)^{1/p} \|h\|_{p} \left(\int_{0}^{\infty} \frac{\cosh^{q}((\pi - \delta)\tau)}{\cosh^{q}(\pi \tau)} d\tau\right)^{1/q}.
$$
\n(50)
\nEstimating the integrals in (50) as
\n
$$
\left(\int_{0}^{\infty} |K_{\frac{1}{2}}(x)|^{p} dx\right)^{1/p} = \sqrt{\frac{\pi}{2}} \left(\int_{0}^{\infty} e^{-px} x^{-p/2} dx\right)^{1/p}
$$
\n(51)

Estimating the integrals in (50) as

$$
\rho \leq \frac{4}{\pi^2} \left(\int_0^\infty K^p_2(x) dx \right)^{1/p} ||h||_p \left(\int_0^\infty \frac{\cosh^q((\pi - \delta)\tau)}{\cosh^q(\pi \tau)} d\tau \right)^{1/q}
$$
(50)
ntegrals in (50) as

$$
\left(\int_0^\infty |K_{\frac{1}{2}}(x)|^p dx \right)^{1/p} = \sqrt{\frac{\pi}{2}} \left(\int_0^\infty e^{-px} x^{-p/2} dx \right)^{1/p}
$$
(51)
$$
= \sqrt{\frac{\pi}{2}} p^{1/2 - 1/p} \left[\Gamma \left(1 - \frac{p}{2} \right) \right]^{1/p}
$$

$$
\int_0^\infty \frac{\cosh^q((\pi - \delta)\tau)}{\cosh^q(\pi \tau)} d\tau \right)^{1/q} \leq 2 \left(\int_0^\infty e^{-q\delta\tau} d\tau \right)^{1/q} = \frac{2}{(q\delta)^{1/q}}
$$
(52)
$$
||(\hat{t}, h)||_p \leq \frac{4}{(q\delta)^{1/p}} \left[\Gamma \left(1 - \frac{p}{2} \right) \right]^{1/p} ||h||
$$

and

$$
\sqrt{\frac{3}{6}} \int \sqrt{\frac{1}{2}} \sqrt
$$

we have finally

$$
\frac{\cosh^{q}(\pi\tau)}{\cosh^{q}(\pi\tau)} d\tau \Bigg\} \le 2 \left(\int_{0}^{e^{-q} \pi\tau} d\tau \right) = \frac{\overline{q\delta}^{1/q}}{\overline{q\delta}^{1/q}} \tag{52}
$$
\n
$$
\| (\hat{I}_{\delta}h) \|_{p} \le \frac{4}{\pi\sqrt{\pi}(\overline{q\delta})^{1/q}} p^{1/2 - 1/p} \left[\Gamma\left(1 - \frac{p}{2}\right) \right]^{1/p} \|h\|_{p}.
$$

As a consequence of the calculations above the following lemma concerning the composition $(\hat{I}_{\delta} \text{Re}[f])$ is valid. $\|(\hat{I}_{\delta}h)\|_{p} \leq \frac{4}{\pi\sqrt{\pi}(q\delta)^{1/q}} p^{1/2-1/p} \left[\Gamma\left(1-\frac{p}{2}\right)\right]$
As a consequence of the calculations above the followin
position $(\hat{I}_{\delta}\mathbf{Re}[f])$ is valid.
Lemma 2. Let $f \in L_{\frac{1}{p}-\frac{1}{2},p}([0,1]) \cap L_{p}(\mathbb{R}_{+}) \quad$

As a consequence of the calculations above the following lemma concerning the position
$$
(\hat{I}_{\delta} \mathbf{Re}[f])
$$
 is valid.
\nLemma 2. Let $f \in L_{\frac{1}{p}-\frac{1}{2},p}([0,1]) \cap L_p(\mathbb{R}_+)$ $(1 \le p \le \infty)$. Then
\n $(\hat{I}_{\delta} \mathbf{Re}[f]) = \frac{\sin(\delta/2)}{\pi} \left[\int_{0}^{\infty} K_0 \left(\sqrt{x^2 + y^2 - 2xy \cos \delta} \right) f(y) dy + \int_{0}^{\infty} \frac{(x+y)K_1 \left(\sqrt{x^2 + y^2 - 2xy \cos \delta} \right)}{\sqrt{x^2 + y^2 - 2xy \cos \delta}} f(y) dy \right]$ \n(7.20). (54)
\nProof. Substituting the value of $\mathbf{Re}[f]$ by formula (41) into (48), changing the

Proof. Substituting the value of $\mathbf{Re}[f]$ by formula (41) into (48), changing the order of integration in the absolute convergent iterated integral for any $\delta \in (0, \frac{\pi}{2})$ as it can be shown by using the Hölder inequality in the inner integral and the estimate (46), we obtain after the calculation of the integral with respect to τ by means of formula $(2.16.55.2)$ in [7: Vol. 2] immediately representation (54) $\int_{0}^{+\infty} \frac{\sqrt{x^2 + y^2 - 2xy \cos \delta}}{\sqrt{x^2 + y^2 - 2xy \cos \delta}}$
Substituting the value of **Re**[*f*] by formula (4
egration in the absolute convergent iterated inte
vn by using the Hölder inequality in the inner inte
after the calculat

As above denote by f_e the left-hand side of (54) and write the equality

mediately representation (54)
$$
\blacksquare
$$

the left-hand side of (54) and write the equality

$$
f_{\epsilon} - f = (\hat{I}_{\delta}(h - g)) + (\hat{I}_{\delta}g) - f
$$
(55)

 ε ($\varepsilon > 0$) and invoking estimate (53) we have

As above denote by
$$
f_{\epsilon}
$$
 the left-hand side of (54) and write the equality
\n
$$
f_{\epsilon} - f = (\hat{I}_{\delta}(h - g)) + (\hat{I}_{\delta}g) - f
$$
\n(55)
\nwhere $g = \text{Re}[f]$ is the exact right-hand side of equation (41). Assuming that $||h - g||_p < \epsilon$
\n $(\epsilon > 0)$ and invoking estimate (53) we have
\n
$$
||f_{\epsilon} - f||_p \leq \frac{4\epsilon}{\pi\sqrt{\pi}(q\delta)^{1/q}} p^{1/2 - 1/p} \left[\Gamma\left(1 - \frac{p}{2}\right) \right]^{1/p} + ||(\hat{I}_{\delta}g) - f||_p.
$$
\n(56)

To estimate the norm $\|(\hat{I}_{\delta}g) - f\|_p$ by the same substitution as in (20) we obtain the representation

ske and S. B. Yakubovich
\n
$$
\text{form } ||(\hat{I}_{\delta}g) - f||_{p} \text{ by the same substitution as in (20) we obtain the}
$$
\n
$$
(\hat{I}_{\delta}g)(x) = \frac{1}{\pi} \int_{-\cot\delta}^{\infty} \frac{\hat{R}(x, t, \delta)}{t^{2} + 1} f(x(\cos\delta + t\sin\delta)) dt
$$
\n(57)

where

$$
\hat{R}(x, t, \delta) = x \sin \frac{\delta}{2} \left[\sin \delta (t^2 + 1) K_0 \left(x \sin \delta (t^2 + 1)^{1/2} \right) + (1 + \cos \delta + t \sin \delta)(t^2 + 1)^{1/2} K_1 \left(x \sin \delta (t^2 + 1)^{1/2} \right) \right].
$$
\n(58)

Hence similar as in the case of the Kontorovich-Lebedev transform we have the estimate

ere
\n
$$
\hat{R}(x, t, \delta) = x \sin \frac{\delta}{2} \left[\sin \delta (t^2 + 1) K_0 (x \sin \delta (t^2 + 1)^{1/2}) + (1 + \cos \delta + t \sin \delta)(t^2 + 1)^{1/2} K_1 (x \sin \delta (t^2 + 1)^{1/2}) \right].
$$
\n(58)
\n
$$
\text{since similar as in the case of the Kontorovich-Lebedev transform we have the estimate}
$$
\n
$$
\| (\hat{I}_{\delta}g) - f \|_p
$$
\n
$$
\leq \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta)) \hat{R}(x, t, \delta) - \left(1 - \frac{\delta}{\pi}\right)^{-1} f(x) \right\|_p dt
$$
\n
$$
\leq \frac{\sin \frac{\delta}{2} \sin \delta}{\pi} \int_{-\cot \delta}^{\infty} \left\| f(x(\cos \delta + t \sin \delta)) x K_0 (x \sin \delta (t^2 + 1)^{1/2}) \right\|_p dt
$$
\n
$$
+ \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta)) x \sin \frac{\delta}{2} (t^2 + 1)^{1/2} \right\|_p dt
$$
\n
$$
+ \frac{1}{\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta)) (\cos \delta + \sin \delta) x \sin \frac{\delta}{2} (t^2 + 1)^{1/2} \right\|_p
$$
\n
$$
\times K_1 (x \sin \delta (t^2 + 1)^{1/2}) - \frac{\pi}{2(\pi - \delta)} f(x) \right\|_p dt
$$
\n
$$
= I_1(\delta) + I_2(\delta) + I_3(\delta).
$$
\n(59)

Let us estimate now each of the integrals $I_i(\delta)$ $(1 \leq i \leq 3)$. Since for all $x > 0$ the

Let us estimate how each of the integrals
$$
I_i(\theta) \ (1 \le i \le 3)
$$
. Since for all $x > 0$ the inequality $xK_0(x) < 1$ holds, then for the integral $I_1(\delta)$ we obtain\n
$$
I_1(\delta) = \frac{\sin \frac{\delta}{2} \sin \delta}{\pi} \int_{-\cot \delta}^{\infty} \left\| f(x(\cos \delta + t \sin \delta)) x K_0(x \sin \delta (t^2 + 1)^{1/2}) \right\|_p dt
$$
\n
$$
< \frac{\sin \frac{\delta}{2} ||f||_p}{\pi} \int_{-\cot \delta}^{\infty} \frac{dt}{\sqrt{t^2 + 1} (\cos \delta + t \sin \delta)^{1/p}}
$$
\n
$$
= \frac{\sqrt{2} \sin \frac{\delta}{2}}{\pi} ||f||_p \int_0^{\infty} \frac{\cosh((\frac{1}{p} - \frac{1}{2})v)}{\sqrt{\cosh v - \cos \delta}} dv.
$$
\n(60)

l,

 $\mathcal{F}(\mathcal{F})$ and $\mathcal{F}(\mathcal{F})$

Note that the last integral is obtained by the substitution $e^v = \cos \delta + t \sin \delta$. Hence we write $(1 < p < 2)$

Once that the last integral is obtained by the substitution
$$
e^v = \cos \delta + t \sin \delta
$$
. Hence we write $(1 < p < 2)$

\n
$$
\left(\int_{0}^{1} + \int_{1}^{\infty} \int \frac{\cosh(\left(\frac{1}{p} - \frac{1}{2}\right)v)}{\sqrt{\cosh v - \cos \delta}} dv\right)
$$
\n
$$
\leq 2\sqrt{2} \cosh\left(\frac{1}{p} - \frac{1}{2}\right) \int_{0}^{1} \frac{dv}{\sqrt{v^2 + 16 \sin^2(\delta/2)}} + \frac{1}{\sqrt{2}} \int_{1}^{\infty} \frac{\cosh(\left(\frac{1}{p} - \frac{1}{2}\right)v)}{\sinh(v/2)} dv \qquad (61)
$$
\n
$$
= B_p \log\left(4 \sin\left(\frac{\delta}{2}\right) + \sqrt{1 + 16 \sin^2\left(\frac{\delta}{2}\right)}\right) + C_p
$$
\nwhere the constants B_p and C_p are defined as

\n
$$
B_p = 2\sqrt{2} \cosh\left(\frac{1}{p} - \frac{1}{2}\right) \qquad \text{and} \qquad C_p = \frac{1}{\sqrt{2}} \int_{1}^{\infty} \frac{\cosh(\left(\frac{1}{p} - \frac{1}{2}\right)v)}{\sinh(v/2)} dv. \qquad (62)
$$
\nThus we obtain finally

\n
$$
I_1(\delta) < \frac{\sqrt{2} \sin \frac{\delta}{2}}{\pi} \left[B_p \log\left(4 \sin\left(\frac{\delta}{2}\right) + \sqrt{1 + 16 \sin^2\left(\frac{\delta}{2}\right)}\right) + C_p\right] ||f||_p. \qquad (63)
$$
\nConcerning the sum of integrals $I_2(\delta) + I_3(\delta)$ similar to (22) and (24) we derive

where the constants B_p and C_p are defined as

$$
B_p \log \left(4 \sin \left(\frac{\pi}{2} \right) + \sqrt{1 + 16 \sin^2 \left(\frac{\pi}{2} \right)} \right) + C_p
$$

ne constants B_p and C_p are defined as

$$
B_p = 2\sqrt{2} \cosh \left(\frac{1}{p} - \frac{1}{2} \right) \quad \text{and} \quad C_p = \frac{1}{\sqrt{2}} \int_{1}^{\infty} \frac{\cosh \left(\left(\frac{1}{p} - \frac{1}{2} \right) v \right)}{\sinh \left(v/2 \right)} dv. \tag{62}
$$

Thus we obtain finally

 \sim $^{\circ}$

$$
I_1(\delta) < \frac{\sqrt{2}\sin\frac{\delta}{2}}{\pi} \left[B_p \log \left(4\sin\left(\frac{\delta}{2}\right) + \sqrt{1 + 16\sin^2\left(\frac{\delta}{2}\right)} \right) + C_p \right] ||f||_p. \tag{63}
$$

Concerning the sum of integrals $I_2(\delta) + I_3(\delta)$ similar to (22) and (24) we derive

ncerning the sum of integrals
$$
I_2(\delta) + I_3(\delta)
$$
 similar to (22) and (24) we derive
\n $I_2(\delta) + I_3(\delta)$
\n $\leq \frac{1}{2\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta)) - \frac{\pi}{\pi - \delta} f(x) \right\|_p dt$
\n $+ \frac{1}{2\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \frac{\pi}{\pi - \delta} f(x) \right\|_p dt$
\n $+ \frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1}$
\n $\times \left\| f(x) \left[2x \sin \frac{\delta}{2} (t^2 + 1)^{1/2} K_1(x \sin \delta (t^2 + 1)^{1/2}) - \cos^{-1} \frac{\delta}{2} \right] \right\|_p dt$
\n $+ \frac{2 \sin^2(\delta/4)}{\pi \cos(\delta/2)} \|f\|_p.$

Hence, appealing to (36) for the third integral of the right-hand side of inequality (64)

l,

we immediately obtain the estimate

-J. Glaeske and S. B. Yakubovich
\niately obtain the estimate
\n
$$
\frac{1}{\pi - \delta} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1}
$$
\n
$$
\times \left\| f(x) \left[2x \sin \frac{\delta}{2} (t^2 + 1)^{1/2} K_1 (x \sin \delta (t^2 + 1)^{1/2}) - \cos^{-1} \frac{\delta}{2} \right] \right\|_p dt
$$
\n
$$
< 2\delta^{\alpha} \cos^{-1} \frac{\delta}{2} \| f \|_p + \varepsilon
$$
\nan arbitrary $\varepsilon > 0$ there exists such a $\delta \in (0, \frac{\pi}{2})$ that
\n
$$
\int_{\frac{\delta}{\sin \delta}}^{\infty} |f(x)|^p dx \leq \varepsilon^p \qquad (0 < \alpha < 2 - p).
$$
\n(66)

where for an arbitrary $\epsilon > 0$ there exists such a $\delta \in (0, \frac{\pi}{2})$ that

$$
\int_{\frac{e^a}{\ln b}}^{\infty} |f(x)|^p dx \le \varepsilon^p \qquad (0 < \alpha < 2 - p). \tag{66}
$$

The second integral in the right-hand side of (64) is estimated by means of reprehave

where for an arbitrary
$$
\varepsilon > 0
$$
 there exists such a $\delta \in (0, \frac{\pi}{2})$ that
\n
$$
\int_{\frac{\delta^{\alpha}}{2\pi i}}^{\infty} |f(x)|^p dx \leq \varepsilon^p \qquad (0 < \alpha < 2 - p).
$$
\n(66)
\n
$$
\int_{\frac{\delta^{\alpha}}{2\pi i}}^{\infty} |f(x)|^p dx \leq \varepsilon^p \qquad (0 < \alpha < 2 - p).
$$
\n(66)
\nThe second integral in the right-hand side of (64) is estimated by means of representation (23) under the condition that f has a derivative $f' \in L_{1+\frac{1}{p},p}(\mathbb{R}_+).$ At first we have
\nhave
\n
$$
\frac{1}{2\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta))(\cos \delta + t \sin \delta) - \frac{\pi}{\pi - \delta} f(x) \right\|_p dt
$$
\n(67)
\n
$$
\leq \frac{q}{2\pi} \left[\|f'\|_{1+\frac{1}{p},p} + \|f\|_p \right] \int_{-\cot \delta}^{\infty} \frac{|1 - (\cos \delta + t \sin \delta)^{1/q}|}{t^2 + 1} dt + \frac{\delta}{\pi} \|f\|_p.
$$
\nThe integral on the right-hand side of inequality (67) can be treated in the same manner
\nThe integral on the right-hand side of inequality (67) can be treated in the same manner

The integral on the right-hand side of inequality (67) can be treated in the same manner as in (26) - (28). Thus we obtain an analogy of inequality (29), precisely

$$
\frac{1}{2\pi} \int_{-\cot\delta} \frac{1}{t^2+1} \left\| f(x(\cos\delta + t\sin\delta))(\cos\delta + t\sin\delta) - \frac{1}{\pi-\delta} f(x) \right\|_{p} dt
$$
\n
$$
\leq \frac{q}{2\pi} \left[\|f'\|_{1+\frac{1}{p},p} + \|f\|_{p} \right] \int_{-\cot\delta}^{\infty} \frac{|1 - (\cos\delta + t\sin\delta)^{1/q}|}{t^2+1} dt + \frac{\delta}{\pi} \|f\|_{p}.
$$
\nThe integral on the right-hand side of inequality (67) can be treated in the same manner as in (26) - (28). Thus we obtain an analogy of inequality (29), precisely\n
$$
\frac{1}{2\pi} \int_{-\cot\delta}^{\infty} \frac{1}{t^2+1} \left\| f(x(\cos\delta + t\sin\delta))(\cos\delta + t\sin\delta) - \frac{\pi}{\pi-\delta} f(x) \right\|_{p} dt
$$
\n
$$
\leq \frac{q}{2} \sin\delta \left[\log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + A_{q} \right] \|f'\|_{1+\frac{1}{p},p} \tag{68}
$$
\n
$$
+ \left[\frac{q}{2} \sin\delta \log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + \frac{q}{2} \sin\delta A_{q} + \frac{\delta}{2\pi} \right] \|f\|_{p}
$$
\nwhere the constant A_{q} is defined by equality (28) and q as usually equals $\frac{p}{p-1}$. Let us estimate the first integral in the right-hand side of inequality (64). By using the equality

estimate the first integral in the right-hand side of inequality (64). By using the equality

The constant
$$
A_q
$$
 is defined by equality (28) and q as usually equals $\frac{p}{p-1}$. Let us
the first integral in the right-hand side of inequality (64). By using the equality

$$
f(x(\cos \delta + t \sin \delta)) - \frac{\pi}{\pi - \delta} f(x) = x \int_{1}^{\cos \delta + t \sin \delta} f'(xy) dy - \frac{\delta}{\pi - \delta} f(x)
$$
(69)

we have

On Some Integral Transforms
\n
$$
\left\| f(x(\cos \delta + t \sin \delta)) - \frac{\pi}{\pi - \delta} f(x) \right\|_p
$$
\n
$$
\leq p \left[\|f'\|_{1 + \frac{1}{p}, p} + \|f\|_p \right] \left| 1 - (\cos \delta + t \sin \delta)^{-1/p} \right| + \frac{\delta}{\pi - \delta} \|f\|_p
$$
\n(70)

\nequently the final estimate will take the form

and consequently the final estimate will take the form

we have
\n
$$
\left\| f(x(\cos \delta + t \sin \delta)) - \frac{\pi}{\pi - \delta} f(x) \right\|_p
$$
\n
$$
\leq p \left[\|f'\|_{1 + \frac{1}{p}, p} + \|f\|_p \right] \left| 1 - (\cos \delta + t \sin \delta)^{-1/p} \right| + \frac{\delta}{\pi - \delta} \|f\|_p
$$
\nand consequently the final estimate will take the form
\n
$$
\frac{1}{2\pi} \int_{-\cot \delta}^{\infty} \frac{1}{t^2 + 1} \left\| f(x(\cos \delta + t \sin \delta)) - \frac{\pi}{\pi - \delta} f(x) \right\|_p dt
$$
\n
$$
\leq \frac{p}{2} \sin \delta \left[\log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + A_p \right] \|f'\|_{1 + \frac{1}{p}, p}
$$
\n
$$
+ \left[\frac{p}{2} \sin \delta \log \left(2^{-1} \sin^{-2} \frac{\delta}{2} \right) + \frac{p}{2} \sin \delta A_p + \frac{\delta}{2\pi} \right] \|f\|_p.
$$
\nNow meaning $\delta = \hat{c}_{\alpha, p} \epsilon^{p(1 - \alpha)^{-1}}$, where $\hat{c}_{\alpha, p}$ is a constant, and assuming that $||f||_p \leq E_1$ and $||f'||_{1 + \frac{1}{p}, p} \leq E_2$ ($E_1, E_2 > 0$) from (56), (59) taking into account estimates (63) -
\n(65), (68) and (71) we obtain
\n
$$
\|f_{\epsilon} - f\|_p
$$

(65), (68) and (71) we obtain

$$
+\left[\frac{p}{2}\sin\delta\log\left(2^{-1}\sin^{-2}\frac{\delta}{2}\right)+\frac{p}{2}\sin\delta A_p+\frac{\delta}{2\pi}\right]||f||_p.
$$
\n
$$
\text{w meaning } \delta = \hat{c}_{\alpha,p} \epsilon^{p(1-\alpha)^{-1}}, \text{ where } \hat{c}_{\alpha,p} \text{ is a constant, and assuming that } ||f||_p \leq E_1
$$
\n
$$
||f'||_r + \frac{1}{p}, p \leq E_2 \ (E_1, E_2 > 0) \text{ from (56), (59) taking into account estimates (63)-}, (68) and (71) we obtain\n
$$
||f_{\epsilon}-f||_p
$$
\n
$$
< \frac{4\epsilon^{(2-p-\alpha)/(1-\alpha)}}{\pi\sqrt{\pi}(q\hat{c}_{\alpha,p})^{1/p}} p^{1/2-1/p} \left[\Gamma\left(1-\frac{p}{2}\right)\right]^{1/p}
$$
\n
$$
+\frac{1}{2}(E_1 + E_2)(p+q)\hat{c}_{\alpha,p} \epsilon^{p(1-\alpha)^{-1}} \log\left(2^{-1}\sin^{-2}\left(\frac{\hat{c}_{\alpha,p}}{2}\epsilon^{p(1-\alpha)^{-1}}\right)\right)
$$
\n
$$
+\frac{\sqrt{2}}{\pi}E_1B_p\sin\left(\frac{\hat{c}_{\alpha,p}}{2}\epsilon^{p(1-\alpha)^{-1}}\right) + \sqrt{1+16\sin^2\left(\frac{\hat{c}_{\alpha,p}}{2}\epsilon^{p(1-\alpha)^{-1}}\right)}
$$
\n
$$
+ (E_1(C_p+1) + \frac{1}{2}(pA_p + qA_q)(E_1 + E_2))\hat{c}_{\alpha,p} \epsilon^{p(1-\alpha)^{-1}} + \epsilon
$$
\n
$$
+ E_1\left(\frac{2\sin^2(\hat{c}_{\alpha,p}\epsilon^{p(1-\alpha)^{-1}}/4)}{\pi\cos(\hat{c}_{\alpha,p}\epsilon^{p(1-\alpha)^{-1}}/2} + 2\hat{c}_{\alpha,p}^{\alpha}\epsilon^{p\alpha(1-\alpha)^{-1}}\cos^{-1}\left(\frac{\hat{c}_{\alpha,p}}{2}\epsilon^{p(1-\alpha)^{-1}}\right)\right).
$$
\n
$$
\text{Therefore we have proved the following}
$$
\n
$$
\text{Theorem 2. Let } f \in L_{\frac{1}{2}-\frac{1}{
$$
$$

Therefore we have proved the following

Theorem 2. Let $f \in L_{\frac{1}{n} - \frac{1}{2},p}([0,1]) \cap L_{1+\frac{1}{2},p}(\mathbb{R}_+)$ $(1 < p < 2)$ and let f possess a Therefore we have proved the following
 Theorem 2. Let $f \in L_{\frac{1}{p}-\frac{1}{2},p}([0,1]) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+)$ $(1 < p < 2)$

derivative $f' \in L_{loc}^1(\mathbb{R}_+) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+)$. Then assuming that $||f||_p \le E_2$ $(E_1, E_2 > 0)$ the Let $f \in L_{\frac{1}{p}-\frac{1}{2},p}([0,1]) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+)$ $(1 < p < 2)$ and let f possess
 $(\mathbb{R}_+) \cap L_{1+\frac{1}{p},p}(\mathbb{R}_+)$. Then assuming that $||f||_p \leq E_1$ and $||f'||_{1+\frac{1}{p},p}$
 i.e.
 $f_e(x) = \frac{4}{\pi^2} \int_0^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\$ *E*₂ $(E_1, E_2 > 0)$ *the function* (48), *i.e.*

$$
f_{\epsilon}(x) = \frac{4}{\pi^2} \int_{0}^{\infty} \frac{\cosh((\pi - \delta)\tau)}{\cosh(\pi \tau)} \operatorname{Re} K_{\frac{1}{2} + i\tau}(x) h(\tau) d\tau \tag{73}
$$

with

$$
\delta = \hat{c}_{\alpha, p} \, \varepsilon^{p(1-\alpha)^{-1}} \qquad (0 < \alpha < 2-p) \tag{74}
$$

 $(0 < \alpha < 2-p)$ *(74)*
 hat $||f_{\epsilon} - f||_{p}$ satisfies estimate (72) for *is a regularized solution of equation* (41) such that $||f_{\epsilon} - f||_{p}$ satisfies estimate (72) for *all* $1 < p < 2$.

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