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On Two-Point Right Focal Eigenvalue Problems

P. J. Y. Wong and R. P. Agarwal

Abstract. We consider the boundary value problem

$$\left.\begin{array}{ll} (-1)^{n-p}y^{(n)} = \lambda F(t, y, y', \ldots, y^{(p)}) & (n \geq 2,\ t \in (0,1)) \\ y^{(i)}(0) = 0 & (0 \leq i \leq p-1) \\ y^{(i)}(1) = 0 & (p \leq i \leq n-1) \end{array}\right\}.$$

where $\lambda > 0$ and $1 \leq p \leq n-1$ are fixed. The values of λ are characterized so that the boundary value problem has a positive solution. We also establish explicit intervals of λ. Examples are included to dwell upon the importance of the results obtained.

Keywords: *Eigenvalues, positive solutions, boundary value problems*

AMS subject classification: 34 B 15

1. Introduction

In this paper we shall consider the n-th order differential equation

$$(-1)^{n-p}y^{(n)} = \lambda F(t, y, y', \cdots, y^{(p)}) \qquad (t \in (0,1)) \tag{1.1}$$

together with the focal boundary conditions

$$\left.\begin{array}{ll} y^{(i)}(0) = 0 & (0 \leq i \leq p-1) \\ y^{(i)}(1) = 0 & (p \leq i \leq n-1) \end{array}\right\} \tag{1.2}$$

where $n \geq 2$, $\lambda > 0$ and p is a fixed integer satisfying $1 \leq p \leq n-1$. Throughout, it is assumed that there exist continuous functions $f : [0,\infty)^{p+1} \to (0,\infty)$ and $u, v : (0,1) \to \mathbb{R}$ such that the following conditions are fulfilled:

(A1) $f(x_0, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_p)$ is non-decreasing, for each fixed $(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$, $0 \leq j \leq p$.

Patricia J. Y. Wong: Nanyang Techn. Univ., Div. Math., 469 Bukit Timah Road, Singapore 259756, Singapore; e-mail: wongjyp@nievax.nie.ac.sg
Ravi P. Agarwal: Nat. Univ. Singapore, Dept. Math., 10 Kent Ridge Crescent, Singapore 119260, Singapore; e-mail: matravip@ilonis.nus.edu.sg

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(A2) For $(x_0, \ldots, x_p) \in [0, \infty)^{p+1}$,

$$u(t) \le \frac{F(t, x_0, \ldots, x_p)}{f(x_0, \ldots, x_p)} \le v(t).$$

(A3) $u = u(t)$ is non-negative and is not identically zero on any non-degenerate subinterval of $(0, 1)$.

(A4) $\int_0^1 t^{n-p-1} v(t)\, dt < \infty$.

By a *positive solution* y of problem (1.1)-(1.2) we mean a function $y \in C^{(n)}(0,1)$ satisfying equation (1.1) on $(0,1)$ and fulfilling conditions (1.2), and which is non-negative and not identically zero on $[0,1]$. If, for a particular λ, the boundary value problem (1.1)-(1.2) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of problem (1.1)-(1.2). We let

$$E = \left\{ \lambda > 0 \,\middle|\, \text{Problem (1.1)-(1.2) has a positive solution} \right\}$$

be the set of eigenvalues of the boundary value problem (1.1)-(1.2). Further, we introduce the notations

$$f_0 = \lim_{\substack{x_j \to 0^+ \\ 0 \le j \le p}} \frac{f(x_0, \ldots, x_p)}{x_0 + \ldots + x_p} \quad \text{and} \quad f_\infty = \lim_{\substack{x_j \to \infty \\ 0 \le j \le p}} \frac{f(x_0, \ldots, x_p)}{x_0 + \ldots + x_p}.$$

First, we shall characterize the values of λ for which the boundary value problem (1.1)-(1.2) has a positive solution. To be specific, we shall show that the set E is an interval and establish conditions under which E is a bounded or unbounded interval. Next, on relaxing the monotonicity condition (A1), explicit eigenvalue intervals are obtained in terms of f_0 and f_∞.

The motivation for the present work stems from many recent investigations. In fact, when $n = 2$, the boundary value problem (1.1)-(1.2) models a wide spectrum of nonlinear phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, where only positive solutions are meaningful (see, e.g., [4, 7, 9, 10, 16, 19, 24]). For the special case $\lambda = 1$, problem (1.1)-(1.2) and its particular and related cases have been the subject matter of many recent publications on singular boundary value problems, for this we refer to [2, 3 , 8, 18, 20, 21, 23, 29]. Further, in the case of second order boundary value problems, (1.1)-(1.2) occurs in applications involving nonlinear elliptic problems in annular regions (see, e.g., [5, 6, 17, 26]). Once again, in all these applications, it is frequent that only solutions that are positive are useful.

Recently, several eigenvalue problems related to problem (1.1)-(1.2) have been tackled. To cite a few examples, Fink, Gatica and Hernandez [15] have dealt with the boundary value problem

$$\left.\begin{aligned} y'' + \lambda q(t) f(y) = 0 \quad (t \in (0,1)) \\ y(0) = y(1) = 0. \end{aligned}\right\}$$

A more general problem, namely

$$\left.\begin{aligned} y^{(n)} + \lambda q(t) f(y) = 0 \qquad & (t \in (0,1)) \\ y^{(i)}(0) = y^{(n-2)}(1) = 0 \qquad & (0 \le i \le n-2) \end{aligned}\right\}$$

has been studied by Chyan and Henderson [8]. Further, Eloe and Henderson [11, 12] have considered the n-th order differential equation

$$y^{(n)} + q(t) f(y) = 0 \qquad (t \in (0,1))$$

subject to the two types of boundary conditions

$$\begin{aligned} y^{(i)}(0) = y^{(n-2)}(1) = 0 \qquad & (0 \le i \le n-2) \\ y^{(i)}(0) = y(1) = 0 \qquad & (0 \le i \le n-2). \end{aligned}$$

It is noted that in all these eigenvalue problems, the nonlinear term that appears in the differential equation concerned is always a function of y only, whereas in equation (1.1) the nonlinear term is a function of $y^{(j)}$ $(0 \le j \le p)$. Hence, the differential equation under consideration is more general. As such our results not only extend the work done on the above eigenvalue problems, but also complement those in [3, 13, 14, 25, 27, 28, 30 - 33], as well as include several other known criteria offered in [1].

The outline of the paper is as follows. In Section 2 we shall state a fixed point theorem due to Krasnosel'skii [22], and develop some properties of certain Green function which are needed later. By defining an appropriate Banach space and cone, the characterization of the set E is carried out in Section 3. Finally, in Section 4 we shall establish explicit eigenvalue intervals in terms of f_0 and f_∞.

2. Preliminaries

In this section, we shall state a fixed point theorem due to Krasnosel'skii [22] and present some inequalities of certain Green function which are vital in later sections.

Theorem 2.1 (see [22]). *Let B be a Banach space, and let $C \subset B$ a cone. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\bar{\Omega}_2 \setminus \Omega_1) \to C$$

be a completely continuous operator such that, either

(a) $\|Sy\| \le \|y\|$ $(y \in C \cap \partial\Omega_1)$ *and* $\|Sy\| \ge \|y\|$ $(y \in C \cap \partial\Omega_2)$

or

(b) $\|Sy\| \ge \|y\|$ $(y \in C \cap \partial\Omega_1)$ *and* $\|Sy\| \le \|y\|$ $(y \in C \cap \partial\Omega_2)$.

Then S has a fixed point in $C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

To obtain a solution for problem (1.1)-(1.2), we require a mapping whose kernel $G(t,s)$ is the Green function of the boundary value problem

$$\left.\begin{array}{ll} y^{(n)}=0 & \\ y^{(i)}(0)=0 & (0\le i\le p-1) \\ y^{(i)}(1)=0 & (p\le i\le n-1) \end{array}\right\}$$

where $1\le p\le n-1$ is fixed. The Green function $G(t,s)$ can be explicitly expressed as (see [1])

$$G(t,s)=\frac{1}{(n-1)!}\begin{cases} \sum_{i=0}^{p-1}\binom{n-1}{i}t^i(-s)^{n-i-1} & \text{if } 0\le s\le t\le 1 \\ -\sum_{i=p}^{n-1}\binom{n-1}{i}t^i(-s)^{n-i-1} & \text{if } 0\le t\le s\le 1. \end{cases} \tag{2.1}$$

Further, the signs of the derivatives of $G(t,s)$ with respect to t are known (see [1]). In fact, for $(t,s)\in[0,1]\times[0,1]$,

$$\begin{array}{ll} (-1)^{n-p}G^{(i)}(t,s)\ge 0 & (0\le i\le p-1) \\ (-1)^{n-i}G^{(i)}(t,s)\ge 0 & (p\le i\le n-1). \end{array} \tag{2.2}$$

Remark 2.1. From (2.2), we have $(-1)^{n-p}G^{(i)}(t,s)\ge 0$ $(0\le i\le p)$ and $(-1)^{n-p}G^{(p+1)}(t,s)\le 0$. Therefore, it follows that $(-1)^{n-p}G^{(i)}(t,s)$ is non-decreasing in t $(0\le i\le p-1)$ and $(-1)^{n-p}G^{(p)}(t,s)$ is non-increasing in t.

Lemma 2.1. *Let $\delta\in(0,\frac{1}{2})$ be given. Then for each $0\le j\le p$ and $(t,s)\in[\delta,1-\delta]\times[0,1]$ we have*

$$(-1)^{n-p}G^{(j)}(t,s)\ge k_\delta^j(-1)^{n-p}G^{(j)}(s,s) \tag{2.3}$$

where $0<k_\delta^j\le 1$ is a constant given by

$$k_\delta^j=\begin{cases} \min_{s\in[\delta,1]}\dfrac{G^{(j)}(\delta,s)}{G^{(j)}(1,s)} & \text{if } 0\le j\le p-1 \\ 1 & \text{if } j=p. \end{cases} \tag{2.4}$$

Proof. First, we shall consider the case $0\le j\le p-1$. For $s\le t$, by the monotonicity of the function $(-1)^{n-p}G^{(j)}(t,s)$ (see Remark 2.1), inequality (2.3) holds for

$$k_\delta^j=1. \tag{2.5}$$

For $t\le s$, inequality (2.3) is satisfied provided that

$$k_\delta^j\le\min_{\substack{t\in[\delta,1-\delta]\\ s\in[\delta,1]}}\frac{(-1)^{n-p}G^{(j)}(t,s)}{(-1)^{n-p}G^{(j)}(s,s)}.$$

Since

$$\min_{\substack{t\in[\delta,1-\delta]\\ s\in[\delta,1]}} \frac{(-1)^{n-p}G^{(j)}(t,s)}{(-1)^{n-p}G^{(j)}(s,s)} \geq \min_{s\in[\delta,1]} \frac{(-1)^{n-p}G^{(j)}(\delta,s)}{(-1)^{n-p}G^{(j)}(1,s)} = \min_{s\in[\delta,1]} \frac{G^{(j)}(\delta,s)}{G^{(j)}(1,s)},$$

inequality (2.3) holds if

$$k_\delta^j \leq \min_{s\in[\delta,1]} \frac{G^{(j)}(\delta,s)}{G^{(j)}(1,s)} \quad (\in (0,1]). \tag{2.6}$$

Coupling (2.5) and (2.6), we take k_δ^j to be the right side of (2.6).

Next, we shall prove for the case $j = p$. For $t \leq s$, in view of Remark 2.1, it is obvious that inequality (2.3) holds for

$$k_\delta^p = 1. \tag{2.7}$$

Further, from (2.1) we find

$$G^{(p)}(t,s) = \frac{1}{(n-p-1)!} \begin{cases} 0 & \text{if } 0 \leq s \leq t \leq 1 \\ -(t-s)^{n-p-1} & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.8}$$

Hence, for $s \leq t$, inequality (2.3) is actually $0 \geq k_\delta^p \cdot 0$, which is of course true for any constant k_δ^p. In view of (2.7), we take $k_\delta^p = 1$ ∎

Lemma 2.2. *For each $0 \leq j \leq p$ and $(t,s) \in [0,1] \times [0,1]$, we have*

$$(-1)^{n-p}G^{(j)}(t,s) \leq \phi_j(s) \tag{2.9}$$

where

$$\phi_j(s) = \begin{cases} \dfrac{1}{(n-j-1)!} \sum_{i=0}^{p-j-1} \dbinom{n-j-1}{i} s^{n-j-i-1} & \textit{if } 0 \leq j \leq p-1 \\ \dfrac{s^{n-p-1}}{(n-p-1)!} & \textit{if } j = p. \end{cases} \tag{2.10}$$

Proof. First, we shall prove for the case $0 \leq j \leq p-1$. On differentiating expression (2.1) with respect to t, j-times, we get

$$G^{(j)}(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum_{i=j}^{p-1} \dbinom{n-1}{i} i^{(j)} t^{i-j}(-s)^{n-i-1} & \text{if } 0 \leq s \leq t \leq 1 \\ -\sum_{i=p}^{n-1} \dbinom{n-1}{i} i^{(j)} t^{i-j}(-s)^{n-i-1} & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.11}$$

Subsequently, in view of Remark 2.1 and (2.11), we find

$$\begin{aligned} (-1)^{n-p}G^{(j)}(t,s) &\leq (-1)^{n-p}G^{(j)}(1,s) \\ &= \frac{(-1)^{n-p}}{(n-1)!} \sum_{i=j}^{p-1} \binom{n-1}{i} i^{(j)}(-s)^{n-i-1} \\ &\leq \frac{1}{(n-1)!} \sum_{i=j}^{p-1} \binom{n-1}{i} i^{(j)} s^{n-i-1} \\ &= \phi_j(s). \end{aligned}$$

Next, for the case $j = p$, it is clear from Remark 2.1 and (2.8) that

$$(-1)^{n-p}G^{(p)}(t,s) \le (-1)^{n-p}G^{(p)}(0,s) = \phi_p(s).$$

The proof of the lemma is complete ∎

Let $y \in C^{(p)}[0,1]$ be such function that $y^{(j)}$ is non-negative on $[0,1]$ for each $0 \le j \le p$. We shall denote, for each $0 \le j \le p$,

$$M_j = \int_0^1 \phi_j(s)v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\,ds \tag{2.12}$$

and

$$N_j = \int_0^1 (-1)^{n-p}G^{(j)}(s,s)u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\,ds. \tag{2.13}$$

In view of Lemma 2.2 and conditions (A2) and (A3), it is clear that $M_j \ge N_j > 0$ $(0 \le j \le p)$. Further, we define the constants

$$\theta_j = k_\delta^j N_j \left(\max_{0\le \ell\le p} M_\ell\right)^{-1} \qquad (0 \le j \le p) \tag{2.14}$$

where k_δ^j $(0 \le j \le p)$ are given in (2.4). It is noted that $0 < \theta_j \le 1$ $(0 \le j \le p)$.

3. Characterization of eigenvalues

Let the Banach space $B = C^{(p)}[0,1]$ be equipped with the norm

$$\|y\| = \max_{0\le j\le p} \|y^{(j)}\|_\infty = \max_{0\le j\le p}\ \sup_{t\in[0,1]} |y^{(j)}(t)|.$$

For a given $\delta \in (0, \frac{1}{2})$, let

$$C_\delta = \left\{ y \in B \,\middle|\, y^{(j)}(t) \ge 0 \ \ (t \in [0,1]) \text{ and } \min_{t\in[\delta,1-\delta]} y^{(j)}(t) \ge \theta_j\|y\| \ \ (0 \le j \le p) \right\}.$$

We note that C_δ is a cone in B. Further, let

$$C_\delta(L) = \big\{ y \in C_\delta \,\big|\, \|y\| \le L \big\}.$$

We define the operator $S : C_\delta \to B$ by

$$Sy(t) = \int_0^1 (-1)^{n-p}G(t,s)\, F\big(s, y(s), y'(s), \ldots, y^{(p)}(s)\big)\,ds \qquad (t \in [0,1]). \tag{3.1}$$

To obtain a positive solution of problem (1.1)-(1.2), we shall seek a fixed point of the operator λS in the cone C_δ. It is clear that, for each $0 \le j \le p$,

$$(Sy)^{(j)}(t) = \int_0^1 (-1)^{n-p} G^{(j)}(t,s)\, F\big(s, y(s), y'(s), \dots, y^{(p)}(s)\big)\, ds \qquad (t \in [0,1]).$$

Thus, on using condition (A2) and the fact that $(-1)^{n-p}G^{(j)}(t,s) \ge 0$ $(0 \le j \le p)$ (see (2.2)), we find

$$(Uy)^{(j)}(t) \le (Sy)^{(j)}(t) \le (Vy)^{(j)}(t) \qquad (t \in [0,1],\ 0 \le j \le p) \tag{3.2}$$

where

$$Uy(t) = \int_0^1 (-1)^{n-p} G(t,s) u(s)\, f\big(y(s), y'(s), \dots, y^{(p)}(s)\big)\, ds \tag{3.3}$$

and

$$Vy(t) = \int_0^1 (-1)^{n-p} G(t,s) v(s)\, f\big(y(s), y'(s), \dots, y^{(p)}(s)\big)\, ds. \tag{3.4}$$

We shall now show that the operator S is compact on the cone C_δ. Let us consider the case when u is unbounded in a deleted right neighborhood of 0 and also in a deleted left neighborhood of 1. Clearly, v is also unbounded near 0 and 1. For $m \in \mathbb{N}$, define $u_m, v_m : [0,1] \to \mathbb{R}$ by

$$u_m(t) = \begin{cases} u\left(\dfrac{1}{m+1}\right) & \text{if } 0 \le t \le \dfrac{1}{m+1} \\ u(t) & \text{if } \dfrac{1}{m+1} \le t \le \dfrac{m}{m+1} \\ u\left(\dfrac{m}{m+1}\right) & \text{if } \dfrac{m}{m+1} \le t \le 1 \end{cases} \tag{3.5}$$

$$v_m(t) = \begin{cases} v\left(\dfrac{1}{m+1}\right) & \text{if } 0 \le t \le \dfrac{1}{m+1} \\ v(t) & \text{if } \dfrac{1}{m+1} \le t \le \dfrac{m}{m+1} \\ v\left(\dfrac{m}{m+1}\right) & \text{if } \dfrac{m}{m+1} \le t \le 1 \end{cases} \tag{3.6}$$

and the operators $U_m, V_m : C_\delta \to B$ by

$$U_m y(t) = \int_0^1 (-1)^{n-p} G(t,s) u_m(s)\, f\big(y(s), y'(s), \dots, y^{(p)}(s)\big)\, ds \tag{3.7}$$

$$V_m y(t) = \int_0^1 (-1)^{n-p} G(t,s) v_m(s)\, f\big(y(s), y'(s), \dots, y^{(p)}(s)\big)\, ds. \tag{3.8}$$

It is standard that for each m, both U_m and V_m are compact operators on C_δ. Let $L > 0$ and $y \in C_\delta(L)$. Then, in view of condition (A1) and Lemma 2.2 we find, for each $0 \le j \le p$,

$$\begin{aligned}
&\left|(V_m y)^{(j)}(t) - (Vy)^{(j)}(t)\right| \\
&\le \int_0^1 (-1)^{n-p} G^{(j)}(t,s)|v_m(s) - v(s)|\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\
&= \int_0^{\frac{1}{m+1}} (-1)^{n-p} G^{(j)}(t,s)|v_m(s) - v(s)|\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\
&\quad + \int_{\frac{m}{m+1}}^1 (-1)^{n-p} G^{(j)}(t,s)|v_m(s) - v(s)|\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\
&\le f(L, \ldots, L) \left[\int_0^{\frac{1}{m+1}} \phi_j(s) \left| v\left(\frac{1}{m+1}\right) - v(s) \right| ds \right. \\
&\quad \left. + \int_{\frac{m}{m+1}}^1 \phi_j(s) \left| v\left(\frac{m}{m+1}\right) - v(s) \right| ds \right].
\end{aligned}$$

The integrability of $\phi_j v$ $(0 \le j \le p)$ (ensured by condition (A4)) implies that V_m converges uniformly to V on $C_\delta(L)$. Hence, V is compact on C_δ. Similarly, we can verify that U_m converges uniformly to U on $C_\delta(L)$ and therefore U is compact on C_δ. It follows from inequality (3.2) that the operator S is compact on C_δ ∎

Theorem 3.1. *There exists a constant $c > 0$ such that $(0, c] \subseteq E$.*

Proof. Let $L > 0$ be given. Define

$$c = \frac{L}{f(L, \ldots, L)} \left[\max_{0 \le j \le p} \int_0^1 \phi_j(s) v(s)\, ds \right]^{-1}. \tag{3.9}$$

Let $\lambda \in (0, c]$. We shall prove that $(\lambda S)(C_\delta(L)) \subseteq C_\delta(L)$. For this, let $y \in C_\delta(L)$. First, we shall show that $\lambda S y \in C_\delta$. From (3.2) and condition (A3) it is clear that, for each $0 \le j \le p$,

$$(\lambda S y)^{(j)}(t) \ge \lambda \int_0^1 (-1)^{n-p} G^{(j)}(t,s) u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \ge 0 \tag{3.10}$$

for all $t \in [0,1]$. Further, it follows from (3.2) and Lemma 2.2 that

$$\begin{aligned}(Sy)^{(j)}(t) &\le \int_0^1 (-1)^{n-p} G^{(j)}(t,s) v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\le \int_0^1 \phi_j(s) v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &= M_j\end{aligned}$$

for all $t \in [0,1]$ and $0 \le j \le p$. Thus, $\|(Sy)^{(j)}\|_\infty \le M_j$ $(0 \le j \le p)$ which readily leads to

$$\|Sy\| \le \max_{0 \le \ell \le p} M_\ell. \tag{3.11}$$

Now, on using (3.2), Lemma 2.1, (3.11) and (2.14) we find, for $t \in [\delta, 1-\delta]$ and $0 \le j \le p$,

$$\begin{aligned}(\lambda Sy)^{(j)}(t) &\ge \lambda \int_0^1 (-1)^{n-p} G^{(j)}(t,s) u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\ge \lambda \int_0^1 k_\delta^j (-1)^{n-p} G^{(j)}(s,s) u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &= \lambda k_\delta^j N_j \\ &\ge \lambda k_\delta^j N_j \|Sy\| \left(\max_{0 \le \ell \le p} M_\ell\right)^{-1} \\ &= \lambda \theta_j \|Sy\| \\ &= \theta_j \|\lambda Sy\|.\end{aligned}$$

Therefore,

$$\min_{t \in [\delta, 1-\delta]} (\lambda Sy)^{(j)}(t) \ge \theta_j \|\lambda Sy\| \qquad (0 \le j \le p). \tag{3.12}$$

Inequalities (3.10) and (3.12) imply that $\lambda Sy \in C_\delta$.

Next, we shall show that $\|\lambda Sy\| \le L$. For this, on using (3.2), Lemma 2.2, condition (A1) and (3.9) successively we get, for each $0 \le j \le p$ and $t \in [0,1]$,

$$\begin{aligned}(\lambda Sy)^{(j)}(t) &\le \lambda \int_0^1 (-1)^{n-p} G^{(j)}(t,s) v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\le \lambda \int_0^1 \phi_j(s) v(s) f(L, \ldots, L)\, ds\end{aligned}$$

$$\leq c\int_0^1 \phi_j(s)v(s)f(L,\ldots,L)ds$$

$$\leq c\,f(L,\ldots,L)\max_{0\leq \ell\leq p}\int_0^1 \phi_\ell(s)v(s)\,ds$$

$$= L.$$

Hence,

$$\|\lambda Sy\| \leq L.$$

We have shown that $(\lambda S)(C_\delta(L)) \subseteq C_\delta(L)$. Also, standard arguments yield that λS is completely continuous. By the Schauder fixed point theorem, λS has a fixed point in $C_\delta(L)$. Clearly, this fixed point is a positive solution of problem (1.1)-(1.2) and therefore λ is an eigenvalue of problem (1.1)-(1.2). Since $\lambda \in (0,c]$ is arbitrary, it follows immediately that $(0,c] \subseteq E$ ∎

The next theorem makes use of the monotonicity and compactness of the operator S on the cone C_δ. We refer to [15: Theorem 3.2] for its proof.

Theorem 3.2 (see [15: Theorem 3.2]). *Suppose that $\lambda_0 \in E$. Then, for each $0 < \lambda < \lambda_0$, $\lambda \in E$.*

The following corollary is immediate from Theorem 3.2.

Corollary 3.1. *E is an interval.*

We shall establish conditions under which E is a bounded or unbounded interval. For this, we need the following results.

Theorem 3.3. *Let λ be an eigenvalue of problem (1.1) – (1.2) and $y \in C_\delta$ be a corresponding eigenfunction. Suppose that $(n-p)$ is odd and $y^{(i)}(0) = q_i$ $(p \leq i \leq n-1)$ where $q_i \geq 0, p \leq i \leq n-2$ and $q_{n-1} > 0$. Then λ satisfies*

$$\begin{aligned}
&\max_{p\leq \ell\leq n-1}\left(\sum_{i=0}^{n-\ell-1}\frac{q_{i+\ell}}{i!}\right)\left[f(D_0,\ldots,D_p)\int_0^1\frac{(1-s)^{n-\ell-1}}{(n-\ell-1)!}v(s)\,ds\right]^{-1}\\
&\quad\leq \lambda \leq \min_{p\leq \ell\leq n-1}\left(\sum_{i=0}^{n-\ell-1}\frac{q_{i+\ell}}{i!}\right)\left[f(0,\ldots,0)\int_0^1\frac{(1-s)^{n-\ell-1}}{(n-\ell-1)!}u(s)\,ds\right]^{-1}
\end{aligned} \tag{3.13}$$

where

$$D_j = \sum_{i=0}^{n-p-1}\frac{q_{p+i}}{(p+i-j)!} \qquad (0 \leq j \leq p). \tag{3.14}$$

Proof. For $m \in \mathbb{N}$ we define $f_m = f * \psi_m$ where ψ_m is a standard mollifier [8, 15] such that f_m is Lipschitz and converges uniformly to f. For a fixed m, let λ_m be an eigenvalue and y_m with $y_m^{(i)}(0) = q_i$ $(p \leq i \leq n-1)$ be a corresponding eigenfunction of the boundary value problem

$$(-1)^{n-p}y_m^{(n)} = \lambda_m F_m\big(t, y_m, y_m', \ldots, y_m^{(p)}\big) \qquad (t \in [0,1]) \tag{3.15}$$

$$\left.\begin{array}{ll} y_m^{(i)}(0) = 0 & (0 \le i \le p-1) \\ y_m^{(i)}(1) = 0 & (p \le i \le n-1) \end{array}\right\} \tag{3.16}$$

where F_m converges uniformly to F and

$$u_m(t) \le \frac{F_m(t, x_0, \ldots, x_p)}{f_m(x_0, \ldots, x_p)} \le v_m(t) \tag{3.17}$$

(see (3.5) and (3.6) for the definitions of u_m and v_m). It is clear that y_m is the unique solution of the initial value problem (3.15)-(3.18), where

$$\left.\begin{array}{ll} y_m^{(i)}(0) = 0 & (0 \le i \le p-1) \\ y_m^{(i)}(0) = q_i & (p \le i \le n-1). \end{array}\right\} \tag{3.18}$$

Since

$$\begin{aligned} (-1)^{n-p} y_m^{(n)}(t) &= \lambda_m F_m(t, y_m, y_m', \ldots, y_m^{(p)}) \\ &\ge \lambda_m u_m(t) f_m\left(y_m(t), y_m'(t), \ldots, y_m^{(p)}(t)\right) \\ &\ge 0 \end{aligned} \tag{3.19}$$

we have $y_m^{(n-1)}$ is non-increasing and hence

$$y_m^{(n-1)}(t) \le y_m^{(n-1)}(0) = q_{n-1} \qquad (t \in [0,1]). \tag{3.20}$$

Noting that

$$y_m^{(i)}(t) = q_i + \int_0^t y_m^{(i+1)}(s)\,ds \qquad (p \le i \le n-2,\ t \in [0,1]) \tag{3.21}$$

we obtain, on using (3.20),

$$y_m^{(n-2)}(t) = q_{n-2} + \int_0^t y_m^{(n-1)}(s)\,ds \le q_{n-2} + q_{n-1} t \qquad (t \in [0,1]).$$

Applying the above inequality and continuing integrating, we find

$$y_m^{(p)}(t) \le \sum_{i=0}^{n-p-1} q_{p+i} \frac{t^i}{i!} \qquad (t \in [0,1]). \tag{3.22}$$

Next, using the relation

$$y_m^{(i)}(t) = \int_0^t y_m^{(i+1)}(s)\,ds \qquad (0 \le i \le p-1,\ t \in [0,1]) \tag{3.23}$$

successive integration of (3.22) yields

$$y_m^{(j)}(t) \le \sum_{i=0}^{n-p-1} q_{p+i} \frac{t^{p+i-j}}{(p+i-j)!} \le D_j \qquad (t \in [0,1],\ 0 \le j \le p). \tag{3.24}$$

Now, it follows from (3.15), (3.17) and (3.24) that, for $t \in [0,1]$,

$$\lambda_m u_m(t) f_m(0,\ldots,0) \le (-1)^{n-p} y_m^{(n)}(t) \le \lambda_m v_m(t) f_m(D_0,\ldots,D_p). \tag{3.25}$$

In view of the initial conditions $y_m^{(i)}(0) = q_i \ \ (p \le i \le n-1)$ repeated integration of (3.25) from 0 to t provides

$$a_\ell(t) \le y_m^{(\ell)}(t) \le b_\ell(t) \qquad (t \in [0,1],\ p \le \ell \le n-1) \tag{3.26}$$

where

$$a_\ell(t) = \sum_{i=0}^{n-\ell-1} q_{i+\ell} \frac{t^i}{i!} - \lambda_m f_m(D_0,\ldots,D_p) \int_0^t \frac{(t-s)^{n-\ell-1}}{(n-\ell-1)!} v_m(s)\, ds$$

$$b_\ell(t) = \sum_{i=0}^{n-\ell-1} q_{i+\ell} \frac{t^i}{i!} - \lambda_m f_m(0,\ldots,0) \int_0^t \frac{(t-s)^{n-\ell-1}}{(n-\ell-1)!} u_m(s)\, ds.$$

In order to have $y_m^{(\ell)}(1) = 0 \ \ (p \le \ell \le n-1)$ (see (3.16)), from inequality (3.26) it is necessary that $a_\ell(1) \le 0$ and $b_\ell(1) \ge 0 \ \ (p \le \ell \le n-1)$ or, equivalently,

$$\lambda_m \ge \left(\sum_{i=0}^{n-\ell-1} \frac{q_{i+\ell}}{i!} \right) \left[f_m(D_0,\ldots,D_p) \int_0^1 \frac{(1-s)^{n-\ell-1}}{(n-\ell-1)!} v_m(s)\, ds \right]^{-1} = \alpha_\ell \tag{3.27}$$

$$(p \le \ell \le n-1)$$

and

$$\lambda_m \le \left(\sum_{i=0}^{n-\ell-1} \frac{q_{i+\ell}}{i!} \right) \left[f_m(0,\ldots,0) \int_0^1 \frac{(1-s)^{n-\ell-1}}{(n-\ell-1)!} u_m(s)\, ds \right]^{-1} = \beta_\ell \tag{3.28}$$

$$(p \le \ell \le n-1).$$

Coupling (3.27) and (3.28), we find

$$\max_{p \le \ell \le n-1} \alpha_\ell \le \lambda_m \le \min_{p \le \ell \le n-1} \beta_\ell. \tag{3.29}$$

From (3.26) it is seen that $\{y_m^{(n-1)}\}_{m \ge 1}$ is a uniformly bounded sequence on $[0,1]$. Using initial conditions (3.18) and repeated integrations we find that $\{y_m^{(i)}\}_{m \ge 1}$ $(0 \le$

$i \le n-2$) is a uniformly bounded sequence. Thus, there exists a subsequence, which can be relabeled as $\{y_m\}_{m\ge 1}$ that converges uniformly (in fact, in $C^{(n-1)}$-norm) to some y on $[0,1]$. We note that each y_m can be expressed as

$$y_m(t) = \lambda_m \int_0^1 (-1)^{n-p} G(t,s)\, F_m\big(s, y_m(s), y_m'(s), \ldots, y_m^{(p)}(s)\big) ds \tag{3.30}$$

for $t \in [0,1]$. Since $\{\lambda_m\}_{m\ge 1}$ is a bounded sequence (from (3.29)), there is a subsequence, which can be relabeled as $\{\lambda_m\}_{m\ge 1}$ that converges to some λ. Then, letting $m \to \infty$ in (3.30) yields

$$y(t) = \lambda \int_0^1 (-1)^{n-p} G(t,s)\, F\big(s, y(s), y'(s), \ldots, y^{(p)}(s)\big) ds \tag{3.31}$$

for $t \in [0,1]$. This means that y is an eigenfunction of problem (1.1)-(1.2) corresponding to the eigenvalue λ. Further, $y^{(i)}(0) = q_i$ $(p \le i \le n-1)$ and inequality (3.13) follows immediately from (3.29) ∎

Theorem 3.4. *Let λ be an eigenvalue of problem* (1.1) − (1.2) *and $y \in C_\delta$ be a corresponding eigenfunction. Further, let $d = \|y\|$. Then*

$$\lambda \ge \frac{d}{f(d,\ldots,d)} \left[\max_{0\le j\le p} \int_0^1 \phi_j(s) v(s)\, ds \right]^{-1} \tag{3.32}$$

and

$$\lambda \le \frac{d}{f(\theta_0 d,\ldots,\theta_p d)} \left[\max_{0\le j\le p} \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}\left(\frac{1}{2}, s\right) u(s)\, ds \right]^{-1}. \tag{3.33}$$

Proof. First, we shall prove (3.32). For this, let $t_0 \in [0,1]$ and $J \in \{0,1,\ldots,p\}$ be such that $d = \|y\| = y^{(J)}(t_0)$. Then, applying (3.2), Lemma 2.2 and condition (A1) we find

$$\begin{aligned} d &= y^{(J)}(t_0) \\ &= (\lambda S y)^{(J)}(t_0) \\ &\le \lambda \int_0^1 (-1)^{n-p} G^{(J)}(t_0,s) v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big) ds \\ &\le \lambda \int_0^1 \phi_J(s) v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big) ds \\ &\le \lambda \int_0^1 \phi_J(s) v(s)\, f(d,\ldots,d)\, ds \\ &\le \lambda f(d,\ldots,d) \max_{0\le j\le p} \int_0^1 \phi_j(s) v(s)\, ds \end{aligned}$$

from which (3.32) is immediate.

Next, using (3.2) and the fact that $\min_{t\in[\delta,1-\delta]} y^{(j)}(t) \geq \theta_j d \quad (0 \leq j \leq p)$ we get for any $0 \leq j \leq p$

$$\begin{aligned} d &\geq y^{(j)}\left(\frac{1}{2}\right) \\ &\geq \lambda \int_0^1 (-1)^{n-p} G^{(j)}\left(\frac{1}{2}, s\right) u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\geq \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}\left(\frac{1}{2}, s\right) u(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\geq \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}\left(\frac{1}{2}, s\right) u(s)\, f(\theta_0 d, \ldots, \theta_p d)\, ds. \end{aligned}$$

The above inequality readily leads to

$$d \geq \lambda f(\theta_0 d, \ldots, \theta_p d) \max_{0\leq j\leq p} \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}\left(\frac{1}{2}, s\right) u(s)\, ds$$

which is the same as (3.33) ∎

Theorem 3.5. *Let*

$$\begin{aligned} F_B &= \left\{ f \,\middle|\, \frac{x}{f(x, \ldots, x)} \text{ is bounded for } x \in [0, \infty) \right\} \\ F_0 &= \left\{ f \,\middle|\, \lim_{x\to\infty} \frac{x}{f(x, \ldots, x)} = 0 \right\} \\ F_\infty &= \left\{ f \,\middle|\, \lim_{x\to\infty} \frac{x}{f(x, \ldots, x)} = \infty \right\}. \end{aligned}$$

Then the following statements hold:

(a) *If $f \in F_B$, then $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$.*

(b) *If $f \in F_0$, then $E = (0, c]$ for some $c \in (0, \infty)$.*

(c) *If $f \in F_\infty$, then $E = (0, \infty)$.*

Proof. Case (a) is immediate from (3.33).

Case (b): Since $F_0 \subseteq F_B$, it follows from Case (a) that $E = (0, c)$ or $E = (0, c]$ for some $c \in (0, \infty)$. In particular,

$$c = \sup E. \tag{3.34}$$

Let $\{\lambda_m\}_{m\geq 1}$ be a monotonically increasing sequence in E which converges to c, and let $\{y_m\}_{m\geq 1}$ in C_δ be a corresponding sequence of eigenfunctions. Further, let $d_m = \|y_m\|$.

Then (3.33) implies that no subsequence of $\{d_m\}_{m\geq 1}$ can diverge to infinity. Thus, there exists $L > 0$ such that $d_m \leq L$ for all m. So y_m is uniformly bounded. Hence, there is a subsequence of $\{y_m\}_{m\geq 1}$ relabeled as the original sequence, which converges uniformly to some $y \in C_\delta$. Noting that $\lambda_m S y_m = y_m$, we have

$$cSy_m = \frac{c}{\lambda_m} y_m. \tag{3.35}$$

Since $\{cSy_m\}_{m\geq 1}$ is relatively compact, y_m converges to y and λ_m converges to c, letting $m \to \infty$ in (3.35) gives $cSy = y$, i.e. $c \in E$. This completes the proof for Case (b).

Case (c) follows from Theorem 3.2 and (3.32) ∎

Example 3.1. Consider the boundary value problem

$$\left.\begin{aligned} &y^{(4)} = \lambda \frac{1}{(t^4+6t^2-12t+19)^a}(y+y'+y''+7)^a \quad (t \in (0,1)) \\ &y(0) = y'(0) = y''(1) = y^{(3)}(1) = 0 \end{aligned}\right\}$$

where $\lambda > 0$ and $a \geq 0$. Here, $n = 4$ and $p = 2$. Taking $f(y, y', y'') = (y + y' + y'' + 7)^a$, we find

$$\frac{F(t, y, y', y'')}{f(y, y', y'')} = \frac{1}{(t^4+6t^2-12t+19)^a}.$$

Hence, we may take

$$u(t) = v(t) = \frac{1}{(t^4+6t^2-12t+19)^a}.$$

All the hypotheses (A1) - (A4) are satisfied.

Case 1: $0 \leq a < 1$. We have

$$\frac{x}{f(x,x,x)} = \frac{x}{(x+x+x+7)^a} = \frac{x}{(3x+7)^a}.$$

Since $f \in F_\infty$, by Theorem 3.5/(c), $E = (0, \infty)$. For instance, when $\lambda = 24$, the boundary value problem has a positive solution given by $y(t) = t^2(t^2 - 4t + 6)$.

Case 2: $a = 1$. Since $f \in F_B$, by Theorem 3.5/(a) the set E is an open or half-closed interval. Further, from Case 1 and Theorem 3.2 we note that E contains the interval $(0, 24]$.

Case 3: $a > 1$. Since $f \in F_0$, by Theorem 3.5/(b) the set E is a half-closed interval. Again, as in Case 2 it is noted that $(0, 24] \subseteq E$.

4. Intervals of eigenvalues

For the rest of the paper, we shall not require conditions (A1) and (A4). However, we need the functions u and v to be continuous on the closed interval $[0,1]$. For a given $\delta \in (0, \frac{1}{2})$ and $0 \le j \le p$, we shall define the number $t_j^* \in [0,1]$ as

$$\int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s)\, ds = \sup_{t\in[0,1]} \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t, s) u(s)\, ds.$$

In view of Remark 2.1, it is clear that

$$t_j^* = \begin{cases} 1 & \text{if } 0 \le j \le p-1 \\ 0 & \text{if } j = p. \end{cases} \tag{4.1}$$

Theorem 4.1. *Suppose that conditions* (A2) *and* (A3) *hold. Let* $\delta \in (0, \frac{1}{2})$ *be given. Then, for each* λ *satisfying*

$$L < \lambda < R \tag{4.2}$$

where

$$L = \left\{ f_\infty \left(\sum_{j=0}^{p} k_\delta^j \right) \left[\min_{0\le j\le p} \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s)\, ds \right] \right\}^{-1}$$

and

$$R = \left\{ f_0 (p+1) \left[\max_{0\le j\le p} \int_0^1 \phi_j(s) v(s)\, ds \right] \right\}^{-1}$$

the boundary value problem (1.1)-(1.2) *has a positive solution.*

Proof. Let λ satisfy condition (4.2). Noting that $\theta_j \le k_\delta^j$ $(0 \le j \le p)$, we let $\varepsilon > 0$ be such that

$$\begin{aligned} &\left\{ (f_\infty - \varepsilon) \left(\sum_{j=0}^{p} \theta_j \right) \left[\min_{0\le j\le p} \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s)\, ds \right] \right\}^{-1} \\ &\quad \le \lambda \le \left\{ (f_0 + \varepsilon)(p+1) \left[\max_{0\le j\le p} \int_0^1 \phi_j(s) v(s)\, ds \right] \right\}^{-1}. \end{aligned} \tag{4.3}$$

Next, we choose $w > 0$ so that

$$f(x_0, \ldots, x_p) \le (f_0 + \varepsilon) \sum_{j=0}^{p} x_j \qquad (0 < x_j \le w,\ 0 \le j \le p). \tag{4.4}$$

Let $y \in C_\delta$ be such that $\|y\| = w$. Then, applying (3.2), Lemma 2.2, (4.4) and (4.3) successively we find, for each $0 \le j \le p$ and $t \in [0,1]$,

$$\begin{aligned}(\lambda S y)^{(j)}(t) &\le \lambda \int_0^1 (-1)^{n-p} G^{(j)}(t,s) v(s) f\big(y(s), y'(s), \dots, y^{(p)}(s)\big) ds \\ &\le \lambda \int_0^1 \phi_j(s) v(s) f\big(y(s), y'(s), \dots, y^{(p)}(s)\big) ds \\ &\le \lambda \int_0^1 \phi_j(s) v(s) (f_0 + \varepsilon) \sum_{\ell=0}^{p} y^{(\ell)}(s)\, ds \\ &\le \lambda \int_0^1 \phi_j(s) v(s) (f_0 + \varepsilon)(p+1)\|y\|\, ds \\ &\le \lambda (f_0 + \varepsilon)(p+1)\|y\| \max_{0 \le \ell \le p} \int_0^1 \phi_\ell(s) v(s)\, ds \\ &\le \|y\|.\end{aligned}$$

Hence,

$$\|\lambda S y\| \le \|y\|. \tag{4.5}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (4.5) holds for $y \in C_\delta \cap \partial\Omega_1$. Further, let $T > 0$ be such that

$$f(x_0, x_1, \dots, x_p) \ge (f_\infty - \varepsilon) \sum_{j=0}^{p} x_j \qquad (x_j \ge T,\ 0 \le j \le p). \tag{4.6}$$

Let $y \in C_\delta$ be such that $\|y\| = T' \equiv \max\{2w, T(\min_{0 \le \ell \le p} \theta_\ell)^{-1}\}$. Then, for $t \in [\delta, 1-\delta]$ and $0 \le j \le p$,

$$y^{(j)}(t) \ge \theta_j \|y\| \ge \theta_j \cdot \frac{T}{\min_{0 \le \ell \le p} \theta_\ell} \ge T$$

which in view of (4.6) leads to

$$f\big(y(t), y'(t), \dots, y^{(p)}(t)\big) \ge (f_\infty - \varepsilon) \sum_{j=0}^{p} y^{(j)}(t) \qquad (t \in [\delta, 1-\delta]). \tag{4.7}$$

Using (3.2), (4.7) and (4.3) we find, for each $0 \le j \le p$,

$$\begin{aligned}(\lambda S y)^{(j)}(t_j^*) &\ge \lambda \int_0^1 (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) f\big(y(s), y'(s), \dots, y^{(p)}(s)\big) ds \\ &\ge \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) f\big(y(s), y'(s), \dots, y^{(p)}(s)\big) ds\end{aligned}$$

$$\geq \lambda \int_{\delta}^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) (f_\infty - \varepsilon) \sum_{\ell=0}^{p} y^{(\ell)}(s)\, ds$$

$$\geq \lambda \int_{\delta}^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) (f_\infty - \varepsilon) \sum_{\ell=0}^{p} \theta_\ell \|y\|\, ds$$

$$\geq \lambda (f_\infty - \varepsilon) \left(\sum_{\ell=0}^{p} \theta_\ell \right) \|y\| \min_{0 \leq \ell \leq p} \int_{\delta}^{1-\delta} (-1)^{n-p} G^{(\ell)}(t_\ell^*, s) u(s)\, ds$$

$$\geq \|y\|.$$

Therefore,

$$\|\lambda S y\| \geq \|y\|. \tag{4.8}$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < T'\}$, then (4.8) holds for $y \in C_\delta \cap \partial\Omega_2$.

Now that we have obtained (4.5) and (4.8), it follows from Theorem 2.1 that λS has a fixed point $y \in C_\delta \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $w \leq \|y\| \leq T'$. Obviously, this y is a positive solution of problem (1.1)-(1.2) ∎

The following corollary is immediate from Theorem 4.1.

Corollary 4.1. *Suppose that conditions* (A2) *and* (A3) *hold. Let* $\delta \in (0, \frac{1}{2})$*be given. Then*

$$(L, R) \subseteq E$$

where L *and* R *are defined in Theorem 4.1.*

Theorem 4.2. *Suppose that conditions* (A2) *and* (A3) *hold. Let* $\delta \in (0, \frac{1}{2})$ *be given. Then, for each* λ *satisfying*

$$L' < \lambda < R' \tag{4.9}$$

where

$$L' = \left\{ f_0 \left(\sum_{j=0}^{p} k_\delta^j \right) \left[\min_{0 \leq j \leq p} \int_{\delta}^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s)\, ds \right] \right\}^{-1}$$

and

$$R' = \left\{ f_\infty (p+1) \left[\max_{0 \leq j \leq p} \int_0^1 \phi_j(s) v(s)\, ds \right] \right\}^{-1}$$

the boundary value problem (1.1)-(1.2) *has a positive solution.*

Proof. Let λ satisfy condition (4.9). Again, in view of the inequality $\theta_j \leq k_\delta^j$ $(0 \leq j \leq p)$ let $\varepsilon > 0$ be such that

$$\begin{aligned} &\left\{ (f_0 - \varepsilon) \left(\sum_{j=0}^{p} \theta_j \right) \left[\min_{0 \leq j \leq p} \int_{\delta}^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s)\, ds \right] \right\}^{-1} \\ &\leq \lambda \leq \left\{ (f_\infty + \varepsilon)(p+1) \left[\max_{0 \leq j \leq p} \int_0^1 \phi_j(s) v(s)\, ds \right] \right\}^{-1}. \end{aligned} \tag{4.10}$$

Let $\bar{w} > 0$ be such that

$$f(x_0, \ldots, x_p) \geq (f_0 - \varepsilon) \sum_{j=0}^{p} x_j \qquad (0 < x_j \leq \bar{w},\ 0 \leq j \leq p). \tag{4.11}$$

Further, let $y \in C_\delta$ be such that $\|y\| = \bar{w}$. Then, on using (3.2), (4.11) and (4.10) successively we get, for each $0 \leq j \leq p$,

$$\begin{aligned}
(\lambda S y)^{(j)}(t_j^*) &\geq \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\
&\geq \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) (f_0 - \varepsilon) \sum_{\ell=0}^{p} y^{(\ell)}(s)\, ds \\
&\geq \lambda \int_\delta^{1-\delta} (-1)^{n-p} G^{(j)}(t_j^*, s) u(s) (f_0 - \varepsilon) \sum_{\ell=0}^{p} \theta_\ell \|y\|\, ds \\
&\geq \lambda (f_0 - \varepsilon) \left(\sum_{\ell=0}^{p} \theta_\ell \right) \|y\| \min_{0 \leq \ell \leq p} \int_\delta^{1-\delta} (-1)^{n-p} G^{(\ell)}(t_\ell^*, s) u(s)\, ds \\
&\geq \|y\|.
\end{aligned}$$

Therefore, inequality (4.8) follows immediately. If we set $\Omega_1 = \{y \in B \,|\, \|y\| < \bar{w}\}$, then (4.8) holds for $y \in C_\delta \cap \partial\Omega_1$.

Next, we may choose $\bar{T} > 0$ such that

$$f(x_0, \ldots, x_p) \leq (f_\infty + \varepsilon) \sum_{j=0}^{p} x_j \qquad (x_j \geq \bar{T},\ 0 \leq j \leq p). \tag{4.12}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

Case 1: Suppose that f is bounded, i.e. there exists some $M > 0$ such that

$$f(x_0, \ldots, x_p) \leq M \qquad (x_j \in [0, \infty),\ 0 \leq j \leq p). \tag{4.13}$$

We define

$$T_1 = \max\left\{ 2\bar{w},\ \lambda M \max_{0 \leq j \leq p} \int_0^1 \phi_j(s) v(s)\, ds \right\}.$$

Let $y \in C_\delta$ be such that $\|y\| = T_1$. From (3.2), Lemma 2.2 and (4.13) we find, for each

$0 \le j \le p$ and $t \in [0,1]$,

$$\begin{aligned}(\lambda S y)^{(j)}(t) &\le \lambda \int_0^1 \phi_j(s)v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\le \lambda \int_0^1 \phi_j(s)v(s) M\, ds \\ &\le \lambda M \max_{0\le \ell\le p} \int_0^1 \phi_\ell(s)v(s)\, ds \\ &\le T_1 \\ &= \|y\|.\end{aligned}$$

Hence, (4.5) holds.

Case 2: Suppose that f is unbounded. Then, there exists $T_1 > \max\{2\bar{w}, \bar{T}\}$ such that

$$f(x_0, \ldots, x_p) \le f(T_1, \ldots, T_1) \qquad (0 < x_j \le T_1,\, 0 \le j \le p). \tag{4.14}$$

Let $y \in C_\delta$ be such that $\|y\| = T_1$. Then, applying (3.2), Lemma 2.2, (4.14), (4.12) and (4.10) successively gives, for each $0 \le j \le p$ and $t \in [0,1]$,

$$\begin{aligned}(\lambda S y)^{(j)}(t) &\le \lambda \int_0^1 \phi_j(s)v(s)\, f\big(y(s), y'(s), \ldots, y^{(p)}(s)\big)\, ds \\ &\le \lambda \int_0^1 \phi_j(s)v(s) f(T_1, \ldots, T_1)\, ds \\ &\le \lambda \int_0^1 \phi_j(s)v(s)(f_\infty + \varepsilon) \sum_{\ell=0}^{p} T_1\, ds \\ &= \lambda \int_0^1 \phi_j(s)v(s)(f_\infty + \varepsilon)(p+1)\|y\|\, ds \\ &\le \lambda (f_\infty + \varepsilon)(p+1)\|y\| \max_{0\le \ell\le p} \int_0^1 \phi_\ell(s)v(s)\, ds \\ &\le \|y\|\end{aligned}$$

from which (4.5) follows immediately.

In both Cases 1 and 2, if we set $\Omega_2 = \{y \in B \mid \|y\| < T_1\}$, then (4.5) holds for $y \in C_\delta \cap \partial\Omega_2$.

Now that we have obtained (4.8) and (4.5), it follows from Theorem 2.1 that λS has a fixed point $y \in C_\delta \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $\bar{w} \le \|y\| \le T_1$. It is clear that this y is a positive solution of problem (1.1)-(1.2) ∎

Theorem 4.2 leads to the following corollary.

Corollary 4.2. *Suppose that conditions* (A2) *and* (A3) *hold. Let* $\delta \in (0, \frac{1}{2})$ *be given. Then*

$$(L', R') \subseteq E$$

where L' *and* R' *are defined in Theorem 4.2.*

Remark 4.1. If f is superlinear (i.e. $f_0 = 0$ and $f_\infty = \infty$) or sublinear (i.e. $f_0 = \infty$ and $f_\infty = 0$), then we conclude from Corollaries 4.1 and 4.2 that $E = (0, \infty)$. i.e. the boundary value problem (1.1)-(1.2) has a positive solution for any $\lambda > 0$.

Example 4.1. Consider the boundary value problem

$$\left.\begin{array}{l} y^{(3)} = \lambda \dfrac{1}{(2t^3 - 6t + 18)^a} (2y + 2y' + 12)^a \quad (t \in (0,1)) \\ y(0) = y'(1) = y''(1) = 0 \end{array}\right\}$$

where $\lambda > 0$ and $a \le 1$. In this example, $n = 3$ and $p = 1$. Choosing $f(y, y') = (2y + 2y' + 12)^a$, we may take

$$u(t) = v(t) = \frac{1}{(2t^3 - 6t + 18)^a}.$$

Hypotheses (A2) and (A3) are satisfied.

Case 1: $a < 1$. It is clear that f is sublinear. Hence, in view of Remark 4.1, for any $\lambda > 0$ the boundary value problem has a positive solution. In fact, we note that when $\lambda = 6$, the corresponding eigenfunction is given by $y(t) = t(t^2 - 3t + 3)$.

Case 2: $a = 1$. Here, $f_0 = \infty$ and $f_\infty = 2$. Further, from Lemma 2.2 we find that $\phi_0(s) = \frac{s^2}{2}$ and $\phi_1(s) = s$. Subsequently,

$$\max_{0 \le j \le 1} \int_0^1 \phi_j(s) v(s)\, ds = \int_0^1 s\, v(s)\, ds = 0.0339.$$

Hence, it follows from Theorem 4.2 that

$$E \supseteq \left(0, [2(1+1)(0.0339)]^{-1}\right) = (0, 7.37).$$

As an example, when $\lambda = 6 \in (0, 7.37)$, the corresponding eigenfunction is given by $y(t) = t(t^2 - 3t + 3)$.

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