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Approximation of Stochastic Differential Equations with Modified Fractional Brownian Motion

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Abstract. The modified fractional Brownian motion is a special semimartingale. This stochastic process is suitable for studying the phenomenon of long-range dependence in a wide range of fields. This paper introduces stochastic differential equations with respect to modified fractional Brownian motion. The solution of these equations is approximated by a splitting method whose convergence in probability is proved. An application of this method to determine ε-optimal controls for a stochastic control problem is also given.

Keywords: *Modified fractional Brownian motion, splitting method, stochastic integral, ε-optimal control*

AMS subject classification: 60 H 10, 93 C 35

1. Introduction

A key component in stochastic analysis and its applications is the Itô equation

$$\left.\begin{aligned} dX(t) &= a(t, X(t))dt + b(t, X(t))dW(t) \\ X(0) &= X_0 \end{aligned}\right\} \tag{1}$$

where $W(t)$ is a Brownian motion (see, for example, Kloeden and Platen [6]). Recent studies have found that data in a large number of fields display long-range dependence (see Beran [2] and Peters [10]). In order to have a useful tool for description and analysis of long-range dependence processes, attempts have been made to replace a Brownian motion by a fractional Brownian motion B_H $(\frac{1}{2} < H < 1)$ in (1) because fractional Brownian motion displays long-range dependence in that range and has fractal sample paths (Mandelbrot and Van Ness [8]). But B_H is not a semimartingale, hence it is not possible to apply the Itô calculus. A stochastic analysis with respect to fractional Brownian motion is faced with difficulties. Lin [7] and Kleptsyna et al. [6] have discussed special stochastic differential equations which contain fractional Brownian motion.

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In this paper, we shall take another approach based on the modified fractional Brownian motion introduced in Anh et al. [1]. The covariance function (in the sense of distributions) of fractional Brownian motion is the Riesz kernel while the covariance function of modified fractional Brownian motion is the Bessel kernel. The Riesz kernel is not integrable, leading to many difficulties as expected of fractional Brownian motion. On the other hand, the Bessel kernel is an L^2-function in some range and at the same time inherits many useful properties of the Riesz kernel. These desirable properties make the modified fractional Brownian motion a suitable tool for studying long-range dependence in a local sense. A semimartingale representation of modified fractional Brownian motion is established in Anh et al. [1], and the Itô calculus in the sense of Protter [11] can be used. Therefore we consider stochastic differential equations of type (1) where the Brownian motion $W(t)$ is substituted by the modified fractional Brownian motion $B(t)$ and (1) is then defined by

$$X(t) = X_0 + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dB(s) \tag{2}$$

(see Section 2).

It is typical for stochastic differential equations to be interpreted as stochastic integral equations which contain a Riemann or Lebesgue integral and a stochastic integral. Consequently, stochastic differential equations can be interpreted as operator equations which contain two types of operators. Splitting methods use this structure. For example, the Zakai equation of filtration theory is solved by a splitting method by Bensoussan [3], and a nonlinear parabolic Itô equation is solved by a splitting method of Rothe type by Grecksch and Tudor [4].

The basic idea of the splitting method consitsts in the construction of two sequences of equations with a time discretization. The first sequence contains equations which are defined with probability 1. The second sequence contains equations with stochastic integrals. Both equations are coupled by initial conditions. Consequently, the splitting method is an approximation method such that, on the one hand, the equations of the first sequence can be solved as deterministic equations and, on the other hand, the equations of the second sequence can be solved by the determination of a purely stochastic problem, for example, by the simulation of a stochastic integral.

Here we construct a splitting approximation for a stochastic differential equation which contains a modified fractional Brownian motion so that the equations of the first sequence are random initial value problems for deterministic ordinary differential equations and the equations of the second sequence define Itô integrals with respect to a modified fractional Brownian motion.

Stochastic differential equations with respect to a modified fractional Brownian motion are introduced in Section 2. The splitting approximation is formulated in Section 3 (Theorem 3.1). We prove this approximation in the sense of convergence in probability in Section 4. In Section 5 we consider an application in control theory to construct ε-controls by the above splitting method.

2. A stochastic differential equation with respect to modified fractional Brownian motion

Let $B(t) = B_{\alpha,\gamma}(t)$ $(1 < \gamma < \frac{3}{2}, \alpha > 0)$ be a Gaussian process on a given complete probability space $(\Omega, \mathcal{F}, P)$ whose increments are stationary and have the spectral density

$$f_{\alpha,\gamma}(\lambda) = \frac{c}{(\alpha^2 + \lambda^2)^\gamma} \cdot \frac{\lambda^2}{1 + \lambda^2} \qquad (\lambda \in \mathbb{R}, c > 0).$$

$B(t)$ is called a *modified fractional Brownian motion* (see Anh et al. [1: Formula (21)]). This definition is suggested by the spectral density of the increments of a fractional Brownian motion $B_H(t)$:

$$f_H(\lambda) = \frac{1}{2\pi} \cdot \frac{1}{\lambda^{2H+1}} \cdot \frac{\lambda^2}{1 + \lambda^2} \qquad (0 < H < 1, \lambda \in \mathbb{R})$$

where H is the so-called Hurst index (see [1: Lemma 2.2]).

But $B_H(t)$ is not a semimartingale (see [7: Lemma 2.2]). On the other hand, $B(t)$ has a semimartingale representation. In fact, [1: Proposition 3.5] gives

$$B(t) - B(s) = \int_s^t Y(u)\,du + k(W(t) - W(s)), \tag{3}$$

where $Y(u)$ is a stationary process, $W(t)$ is a Wiener process and k is a positive constant which depends on α and γ. The processes are adapted with respect to the σ-algebras

$$\mathcal{G}_t = \sigma\Big\{B(s) - B(s - \tau),\ s \le t, \tau > 0\Big\} \tag{4}$$

and it holds that $\mathcal{F}_t = \mathcal{G}_t$, where

$$\mathcal{F}_t = \sigma\Big\{W(s) - W(s - \tau),\ s \le t, \tau > 0\Big\}. \tag{5}$$

$\mathcal{F}_t$ is a right-continuous increasing family of σ-algebras.

Here we consider the stochastic differential equation

$$\left.\begin{aligned} dX(t) &= a(t, X(t))dt + b(t, X(t))dB(t) \\ X(0) &= X_0 \end{aligned}\right\} \tag{6}$$

which is defined by

$$X(t) = X_0 + \int_0^t a(s, X(s))\,ds + \int_0^t b(s, X(s))\,dB(s) \tag{7}$$

where, in view of (3),

$$\int_0^t b(s, X(s))\, dB(s) = \int_0^t b(s, X(s)) Y(s)\, ds + k \int_0^t b(s, X(s))\, dW(s)$$

and the second integral of the last formula is the usual Itô integral with respect to a Wiener process $W(t)$.

We assume the following conditions:

$$\begin{aligned} & X_0 \text{ is a } \mathcal{G}_0 - \text{measurable variable with } P\{|X_0| < \infty\} = 1 \\ & a : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ and } b : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ are measurable functions} \\ & |a(t,x)| + |b(t,x)| \le K(1 + |x|) && (8) \\ & |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le L|x - y| && (9) \end{aligned}$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}$, where $K, L > 0$ are fixed constants.

Then it follows from [11: p. 194/Theorem 6] that

Theorem 1. *Problem* (6) *has a unique solution* X *which is a semimartingale.*

3. A splitting scheme

We construct the following splitting method:

Consider partitions

$$0 = t_0^{(r)} < t_1^{(r)} < \cdots < t_{N_r}^{(r)} = t \in [0,T]$$

for $r = 1, 2, \ldots, N_r$ $(N_r \ge 1)$ where $t_{j+1}^{(r)} - t_j^{(r)} = h_r \to 0$ as $r \to \infty$. For brevity of notation, we shall write $t_j^{(r)} = t_j$. We introduce the problems

$$X_{1r}(s) = X_{1r}(t_j) + \int_{t_j}^{s} a(u, X_{1r}(u))\, du \qquad (s \in [t_j, t_{j+1})) \tag{10}$$

$$X_{2r}(s) = X_{2r}(t_j) + \int_{t_j}^{s} b(u, X_{1r}(u))\, dB(u) \qquad (s \in [t_j, t_{j+1})) \tag{11}$$

with

$$X_{1r}(t_j) = X_{2r}(t_j - 0) \tag{12}$$

$$X_{2r}(t_j) = X_{1r}(t_{j+1} - 0) \tag{13}$$

$$X_{1r}(0) = X_0 \tag{14}$$

and we define

$$X_{1r}(t) = X_{1r}(t-0) \tag{15}$$

and

$$X_{2r}(t) = X_{2r}(t-0). \tag{16}$$

The method contains some advantages. Equation (10) is, for fixed $\omega \in \Omega$, a deterministic problem and can be solved by numerical methods of the deterministic theory. Equation (11) can be solved by the simulation of a stochastic integral with respect to a modified fractional Brownian Motion.

Theorem 2. *It holds for all $\delta > 0$ that*

$$\lim_{r\to\infty} P\{|X_{1r}(t) - X(t)| > \delta\} = 0 \qquad \textit{and} \qquad \lim_{r\to\infty} P\{|X_{2r}(t) - X(t)| > \delta\} = 0.$$

Theorem 2 shows that $X_{1r}(t)$ and $X_{2r}(t)$ approximate the solution $X(t)$ for fixed $t \in [0,T]$.

4. Proof of the main result

We introduce over $[0,t]$ the processes Y_l with

$$Y_l(\omega,s) = \begin{cases} Y(\omega,s) & \text{if } \int_0^s |Y(\omega,u)|^2 du \le l^2 \\ 0 & \text{otherwise} \end{cases} \qquad (l \in \mathbb{N}).$$

Then we define $B_l(s) = \int_0^s Y_l(u)\,du + kW(s)$. If $V(t) = \int_0^t Y(u)\,du$, then we can define a σ-finite measure β on $\mathcal{B}[0,T] \times \mathcal{F}$ by

$$\beta(A) = \int_\Omega \int_A \Big(k^2 dt dP(\omega) + |V_{[0,T]}|(\omega)|V|(\omega, dt)\, dP(\omega)\Big)$$

for $A \in \mathcal{B}[0,T]$ so that, for $\mathcal{F}_t$-measurable functions $F(t)$ with $\int_{\Omega\times[0,T]} |F(t)|^2 d\beta < \infty$, it holds that

$$E\left|\int_0^T F(t)\, dB(t)\right|^2 \le 2\int_{\Omega\times[0,T]} |F(t)|^2 d\beta$$

(see [1: Formulae (45) and (46)]). Here, $|V|$ is a positive measure defined by

$$|V|(A) = \sup_\pi \sum_{i=1}^{N_r} \big|V(\omega, t_{i+1}) - V(\omega, t_i)\big|,$$

π being a partition $\{t_1 < t_2 < \dots < t_{N_r+1} : t_k \in A\}$ for intervals A and $|V|(\omega,t) \le |V_{[0,T]}|(\omega)$ a.s.

If we substitute Y by Y_l, then the measure β is majorized by

$$\beta_l(A) = \int_\Omega \int_A (k^2 + l^2)\, dt dP \tag{17}$$

($A \in \mathcal{B}[0,T]$) and consequently

$$E\left|\int_0^t b(s,X(s))\,dB_l(s)\right|^2 \le 2(k^2+l^2)E\int_0^t b^2(s,X(s))\,ds. \tag{18}$$

Further, we introduce $X_0^l = \max\{-l, \min\{X_0, l\}\}$.

We first prove Theorem 2 with Y and X_0 being substituted by Y_l and X_0^l.

Lemma 1. *Let X_l be the solution of equation (6) for $Y = Y_l$ and $X_0 = X_0^l$. Then there exists a positive constant C_l with $E|X_l(s)|^2 \le C_l$ for all $s \in [0,t]$.*

Proof. It is clear that

$$X_l^2(s) \le 2|X_0^l|^2 + 4\left|\int_0^s a(u,X_l(u))\,du\right|^2 + 4\left|\int_0^s b(u,X_l(u))\,dB_l(u)\right|^2.$$

In view of (18), we get

$$E|X_l(s)|^2 \le 2E|X_0^l|^2 + 4E\left|\int_0^s a(u,X_l(u))\,du\right|^2 + 8(k^2+l^2)E\int_0^s b^2(u,X_l(u))\,du.$$

It follows from (8) and the Cauchy-Schwarz inequality that

$$E|X_l(s)|^2 \le 2E|X_0^l|^2 + 8TK^2\int_0^s E|X_l(u)|^2du + 8T^2K^2$$
$$+ 16K^2(k^2+l^2)\int_0^s E|X_l(u)|^2du + 16K^2(k^2+l^2)T.$$

The result now follows from the Gronwall lemma ∎

Next we show some a priori estimates for X_{1r} and X_{2r} if $Y = Y_l$ and $X_0 = X_0^l$. Let X_{1r}^l and X_{2r}^l be the solutions of problems (10) and (11), respectively, for $Y = Y_l$ and $X_0 = X_0^l$. Conditions (12) and (13) give $X_{2r}^l(t_{j+1}-0) = X_{1r}^l(t_{j+1})$ and $X_{2r}^l(t_j) = X_{1r}^l(t_{j+1}-0)$, and we obtain from (10) and (11) for $s = t_{j+1}-0$ that

$$X_{1r}^l(t_{j+1}) - X_{1r}^l(t_j) = \int_{t_j}^{t_{j+1}} a(u,X_{1r}^l(u))\,du + \int_{t_j}^{t_{j+1}} b(u,X_{1r}^l(u))\,dB_l(u).$$

Consequently,

$$X_{1r}^l(t) - X_{1r}^l(0) = \int_0^t a(u,X_{1r}^l(u))\,du + \int_0^{t_{N_r-1}} b(u,X_{1r}^l(u))\,dB_l(u)$$

and then

$$E|X_{1r}^l(t)|^2 \le 2E|X_0^l|^2 + 4E\left|\int_0^t a(u, X_{1r}^l(u))\,du\right|^2 + 4E\left|\int_0^{t_{N_r-1}} b(u, X_{1r}^l(u))\,dB_l(u)\right|^2.$$

The Cauchy-Schwarz inequality, condition (8) and formula (18) give

$$\begin{aligned} E|X_{1r}^l(t)|^2 &\le 2E|X_0^l|^2 + 8TK^2(T + 2(k^2 + l^2)) \\ &\quad + 8K^2(T + 2(k^2 + l^2)) \int_0^t E|X_{1r}^l(u)|^2 du. \end{aligned} \tag{19}$$

The Gronwall lemma then implies

$$E|X_{1r}^l(t)|^2 \le \left(2E|X_0^l|^2 + 8TK^2(T + 2(k^2 + l^2))\right) \exp\left\{8TK^2(T + 2(k^2 + l^2))\right\}. \tag{20}$$

Obviously, we have $X_{2r}^l(t) = X_{1r}^l(t) + (X_{2r}^l(t) - X_{1r}^l(t))$. Then we have

$$\begin{aligned} E|X_{2r}^l(t)|^2 &\le 2E|X_{1r}^l(t)|^2 + 2E\left|\int_{t-h_r}^t b(u, X_{1r}^l(u))\,dB_l(u)\right|^2 \\ &\le 2E|X_{1r}^l(t)|^2 + 8K^2(k^2 + l^2) \int_{t-h_r}^t (1 + E|X_{1r}^l(u)|^2)\,du \\ &\le 2E|X_{1r}^l(t)|^2 + 8K^2(k^2 + l^2)T\left(1 + \sup_{t\in[0,T]} E|X_{1r}^l(t)|^2\right) \\ &< \infty. \end{aligned} \tag{21}$$

Together with (20), we get from (21) that $E|X_{2r}^l(t)|^2$ is also bounded. Hence we have proved

Lemma 2. *There is a positive constant D_l with*

$$E|X_{1r}^l(t)|^2,\ E|X_{2r}^l(t)|^2 \le D_l$$

for all r.

Remark 1. In a similar manner we can prove the boundednes properties

$$E|X_{1r}^l(t_{j+1} - 0)|^2,\ E|X_{2r}^l(t_{j+1} - 0)|^2 \le D_l \qquad (j = 0, \ldots, N_r - 2)$$

and

$$E|X_{1r}^l(s)|^2,\ E|\bar{X}_{2r}^l(s)|^2 \le D_l \qquad (s \in (t_j, t_{j+1}),\ j = 0, \ldots, N_r - 1).$$

Next we show

Lemma 3. *There is a positive constant D_l' with*

$$E\left|X_{1r}^l(s) - X_{2r}^l(s)\right|^2 \leq D_l' h_r$$

for $s \in [t_j, t_{j+1})$ and $j = 0, \dots, N_r - 1$.

Proof. Integrating (10) backward over $[s, t_{j+1})$ with $s \in [t_j, t_{j+1})$ gives

$$X_{1r}^l(t_{j+1} - 0) = X_{1r}^l(s) + \int_s^{t_{j+1}} a(u, X_{1r}^l(u))\, du. \tag{22}$$

It then follows from (11), (13) and (22) that

$$X_{2r}^l(s) - X_{1r}^l(s) = \int_s^{t_{j+1}} a(u, X_{1r}^l(u))\, du + \int_{t_j}^{s} b(u, X_{1r}^l(u))\, dB_l(u)$$

and we obtain from (8), the definition of Y_l, the Cauchy-Schwarz inequality and (18) that

$$\begin{aligned} E|X_{2r}^l(s) - X_{1r}^l(s)|^2 &\leq 2E\left|\int_s^{t_{j+1}} a(u, X_{1r}^l(u))\, du\right|^2 + 2E\left|\int_{t_j}^{s} b(u, X_{1r}^l(u))\, dB_l(u)\right|^2 \\ &\leq 4h_r^2 K^2 + 4K^2 \int_{t_j}^{t_{j+1}} E|X_{1r}^l(u)|^2 du \\ &\quad + 4K^2(l^2 + k^2)h_r + 4K^2(l^2 + k^2) \int_{t_j}^{t_{j+1}} E|X_{1r}^l(u)|^2 du. \end{aligned}$$

Remark 1 then gives the statement ∎

Remark 2. In a similar manner we also get

$$\begin{aligned} E\left|X_{1r}^l(t_{j+1} - 0) - X_{1r}^l(t_j)\right|^2 &= E\left|X_{1r}^l(t_{j+1} - 0) - X_{2r}^l(t_j - 0)\right|^2 \to 0 \\ E\left|X_{2r}^l(t_{j+1} - 0) - X_{2r}^l(t_j)\right|^2 &= E\left|X_{2r}^l(t_{j+1} - 0) - X_{1r}^l(t_{j+1} - 0)\right|^2 \to 0 \\ E\left|X_{2r}^l(t_j) - X_{1r}^l(t_j)\right|^2 &\to 0 \end{aligned}$$

for $r \to \infty$.

Lemma 4. *Let X_l be the solution of equation (6) for $Y = Y_l$ and $X_0 = X_0^l$, and let X_{1r}^l and X_{2r}^l the solutions of equations (10) and (11) for $Y = Y_l$ and $X_0 = X_0^l$. Then*

$$\lim_{r \to \infty} E\left|X_{2r}^l(t) - X_l(t)\right|^2 = 0 \tag{23}$$

and

$$\lim_{r\to\infty} E\left|X_{1r}^l(t) - X_l(t)\right|^2 = 0. \tag{24}$$

Proof. Consider the problems (10) and (11) for $s = t_{j+1} - 0$. It follows from these equations and (12) and (13) that

$$X_{2r}^l(t) - X_{2r}^l(0) = \int_0^t a(u, X_{1r}^l(u))\,du + \int_0^t b(u, X_{1r}^l(u))\,dB_l(u).$$

Consequently,

$$\begin{aligned} X_{2r}^l(t) - X_l(t) = X_{2r}^l(0) - X_0^l &+ \int_0^t \left[a(u, X_{1r}^l(u)) - a(u, X_{2r}^l(u))\right] du \\ &+ \int_0^t \left[a(u, X_{2r}^l(u)) - a(u, X_l(u))\right] du \\ &+ \int_0^t \left[b(u, X_{1r}^l(u)) - b(u, X_{2r}^l(u))\right] dB_l(u) \\ &+ \int_0^t \left[b(u, X_{2r}^l(u)) - b(u, X_l(u))\right] dB_l(u). \end{aligned}$$

Then we can estimate using the Cauchy-Schwarz inequality, the definition of Y_l, formulas (18) and (9) that

$$\begin{aligned} E\left|X_{2r}^l(t) - X_l(t)\right|^2 \le 2E\left|X_{2r}^l(0) - X_0^l\right|^2 &+ D_l'' \int_0^t E\left|X_{2r}^l(u) - X_l(u)\right|^2 du \\ &+ D_l'' \int_0^t E\left|X_{1r}^l(u) - X_{2r}^l(u)\right|^2 du \end{aligned} \tag{25}$$

where D_l'' is a positive constant.

If we consider

$$E\left|X_{2r}^l(0) - X_0^l\right|^2 = E\left|X_{1r}^l(t_1 - 0) - X_0^l\right|^2 \le 2TK^2 h_r \exp\{2K^2T\}$$

and Lemma 3, then an application of the Gronwall lemma to (25) gives

$$E\left|X_{2r}^l(t) - X_l(t)\right|^2 \le D_l''' h_r \tag{26}$$

where D_l''' is a positive constant. The last relation proves (23). In conjunction with Lemma 3 and (26), the inequality

$$E\left|X_{1r}^l(t) - X_l(t)\right|^2 \le 2E\left|X_{1r}^l(t) - X_{2r}^l(t)\right|^2 + 2E\left|X_{2r}^l(t) - X_l(t)\right|^2$$

gives assertion (24) ∎

We can now prove Theorem 2.

Let $\delta > 0$ be chosen arbitrarily. Introduce the events

$$A_{i\delta r} = \{\omega : |X(\omega, t) - X_{ir}(\omega, t)| > \delta\} \qquad (i = 1, 2)$$

and

$$B_l^t = \left\{\omega : \int_0^t |Y(\omega, u)|^2 du \le l^2 \text{ and } |X_0(\omega)| < l\right\}.$$

Obviously, $P(B_l^t) \to 1$ for $l \to \infty$. Let $\varepsilon > 0$ be chosen arbitrarily. Then there exists l_0 so that $P(\Omega \setminus B_l^t) < \varepsilon$ for $l \ge l_0$. Thus,

$$P\{A_{i\delta r}\} = P\{A_{i\delta r} \cap B_l^t\} + P\{A_{i\delta r} \cap (\Omega \setminus B_l^t)\}$$

and

$$P\{A_{i\delta r}\} \le P\{|X_l(\omega, t) - X_{ir}^l(\omega, t)| > \delta\} + \varepsilon.$$

Consequently, with the Markov inequality and Lemma 4, we get $\lim_{r\to\infty} P\{A_{i\delta r}\} \le \varepsilon$, that is, $\lim_{r\to\infty} P\{A_{i\delta r}\} = 0$ for $i = 1, 2$.

Remark 3. Obviously, the above considerations are also true for a and b being $\mathcal{F}_t$-measurable functions so that conditions (8) and (9) hold for every $\omega \in \Omega$.

5. An application in control theory

Let $b(t, X(t)) \equiv b(t)$ and $a : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that $a(t, \cdot, \cdot)$ fulfils conditions (8) and (9) over $\mathbb{R} \times \mathbb{R}$, that is:

$$|a(t, x, u)| \le K(1 + |x| + |u|)$$

and

$$|a(t, x, u) - a(t, y, v)| \le L(|x - y| + |u - v|)$$

for all $t \in [0, T]$ and $x, y, u, v \in \mathbb{R}$. Let $\mathcal{V}$ be the set of all $\mathcal{F}_t$-measurable processes $U(t)$ with values in a compact set $M \subset \mathbb{R}$. Let X_0 be a $\mathcal{G}_0$-measurable variable with $E|X_0|^2 < \infty$. Further, let $F : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz-continous function. We now consider the optimal control problem

$$\left.\begin{aligned} dX(t) &= a(t, X(t), U(t))\, dt + b(t)\, dB(t) \\ X(0) &= X_0 \end{aligned}\right\} \tag{27}$$

$$\min\{\Phi(U) : U \in \mathcal{V}\} = \min\{EF(X(T)) : U \in \mathcal{V}\}. \tag{28}$$

We assume that problem (27)-(28) has a solution; that is, there is $U^* \in \mathcal{V}$ such that

$$\Phi(U^*) = \min\{\Phi(U) : U \in \mathcal{V}\}.$$

We denote by X^U the solution of (27) for a control U.

We first show that Φ depends continuously on U in the following sense:

Lemma 5. *For all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ so that, for $V_1, V_2 \in \mathcal{V}$ with*

$$E \sup_{0 \le t \le T} |V_1(t) - V_2(t)| \le \delta(\varepsilon),$$

it holds that

$$|\Phi(V_1) - \Phi(V_2)| \le \varepsilon.$$

Proof. We have

$$X^{V_1}(t) - X^{V_2}(t) = \int_0^t \Big(a(s, X^{V_1}(s), V_1(s)) - a(s, X^{V_2}(s), V_2(s))\Big) ds.$$

Hence,

$$\left|X^{V_1}(t) - X^{V_2}(t)\right| \le L \int_0^t \left[\left|X^{V_1}(s) - X^{V_2}(s)\right| + \left|V_1(s) - V_2(s)\right|\right] ds$$

and consequently the Gronwall lemma implies

$$E\left|X^{V_1}(T) - X^{V_1}(T)\right| \le TE \sup_{s \in [0,T]} \left|V_1(s) - V_2(s)\right| \exp\{LT\}. \tag{29}$$

The statement follows with $\delta(\varepsilon) = \varepsilon / L_F T \exp\{LT\}$ where L_F is the Lipschitz constant of F. ∎

Remark 4. If $\mathcal{U} \subset \mathcal{V}$ is compact in $L^\infty(\Omega \times [0,T])$, then the optimal controls belong to $\mathcal{U}$. Formula (29) shows that X depends continuously on U.

Definition 1. Let ε be a positive real number. A control $U_\varepsilon \in \mathcal{V}$ is called *ε-optimal* if for $\varepsilon > 0$ it holds that $\Phi(U_\varepsilon) - \Phi(U^*) \le \varepsilon$.

We consider the partitions of $[0,T]$ defined in Section 3 and introduce the step processes

$$U^{N_r}(t) = \sum_{j=0}^{N_r - 1} I_{[t_j, t_{j+1})}(t) U^{N_r j} \qquad (t \in [0,T])$$

where $U^{N_r j}$ are $\mathcal{F}_{t_j}$-measurable functions from Ω into M. Obviously, $U^{N_r} \in \mathcal{V}$. Because these step processes are dense in $\mathcal{V}$, there exist step processes which are ε-optimal. We can assert the approximation statement of the last section for the special equation (27) in the mean square sense.

Lemma 6. *There is a constant $D > 0$ so that*

$$E\left|X_{2r}(t) - X(t)\right|^2, \; E\left|X_{1r}(t) - X(t)\right|^2 \le Dh_r$$

for all $t \in [0,T]$.

Here it is not necessary to introduce the processes X_l, X_{1r}^l and X_{2r}^l since the noise is only additive. Lemma 6 can be proved in the same manner as Lemma 4.

We now consider a step control U^{N_r} and apply this control to (27) and the corresponding equations (10) and (11), and define for (10) and (11) an optimal control problem

$$\min\{EF(X^{U^{N_r}}(T) : U^{N_r}\}. \tag{30}$$

Theorem 3. *Let U^{N_r*} be an optimal step control for (10), (11), (30). Then U^{N_r*} is ε-optimal for the original problem (27) – (28) for sufficiently large r.*

Proof. We get

$$\begin{aligned}\left|EF(X^{U^{N_r*}}(T)) - EF(X^{U^*}(T))\right| &\le \left|EF(X^{U^{N_r*}}(T)) - EF(X_{2r}^{U^{N_r*}}(T))\right| \\ &\quad + \left|EF(X_{2r}^{U^{N_r*}}(T)) - EF(X^{U^*}(T))\right|.\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}\left|EF(X^{U^{N_r*}}(T)) - EF(X_{2r}^{U^{N_r*}}(T))\right| &\le L_F\left(E|X^{U^{N_r*}}(T) - X_{2r}^{U^{N_r*}}(T)|^2\right)^{1/2} \\ &\le (Dh_r)^{1/2} \\ &\le \frac{\varepsilon}{2}\end{aligned} \tag{32}$$

for sufficiently large r, where the Lipschitz continuity of F and Lemma 6 for the control U^{N_r*} are applied. If $EF(X^{U^*}(T)) - EF(X_{2r}^{U^{N_r*}}(T)) > 0$, then we obtain, in a similar fashion,

$$EF(X^{U^*}(T)) - EF(X_{2r}^{U^{N_r*}}(T)) \le EF(X^{U^{N_r*}}(T)) - EF(X_{2r}^{U^{N_r*}}(T)) \le \frac{\varepsilon}{2} \tag{33}$$

for sufficiently large r. If $EF(X^{U^*}(T)) - EF(X_{2r}^{U^{N_r*}}(T)) < 0$, then

$$0 > EF(X^{U^*}(T)) - EF(X_{2r}^{U^{N_r*}}(T)) > EF(X^{U^*}(T)) - EF(X_{2r}^{U^*}(T)) > -\frac{\varepsilon}{2} \tag{34}$$

for sufficiently large r, where the Lipschitz continuity and Lemma 6 for the control U^* are applied. The results (32) – (34) give the statement in view of (31) ∎

Remark 5. Problem (10), (11), (30) can be interpreted as a discrete-time optimal control problem with entry X_0, steps

$$X_{2r}(t_1 - 0),\ X_{2r}(t_2 - 0),\ \ldots\ , X_{2r}(t_{N_r-1} - 0)$$

and exit $X_{2r}(t_{N_r} - 0)$, where the disturbances

$$\int_0^{t_1} b(u)\,dB(u), \qquad \int_{t_1}^{t_2} b(u)\,dB(u),\ \ldots\ , \int_{t_{N_r-1}}^{t_{N_r}} b(u)\,dB(u)$$

and the controls

$$U^{N_r0}, \qquad U^{N_r1},\ \cdots\ , U^{N_rN_r-1}$$

work on the steps $1, 2, \ldots, N_r$. The methods of optimal control of discrete-time random systems then can be applied to solve $EF(X_{2r}(T)) = \min$ (see, for example, Müller and Nollau [9]).

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