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The Riemann Problem for a Two-Dimensional Hyperbolic System of Nonlinear Conservation Laws: Multiplication of Distribution Solutions

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Abstract. In the papers [6 - 8], the author has constructed the Riemann solutions to a two-dimensional hyperbolic system of nonlinear conservation laws for any piecewise constant initial data having two discontinuity rays with origin as vertex. It has been found that, for some initial data, the Riemann solutions no longer lie in $L^{\infty}_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$, and the non-classical waves (labelled as Dirac-contact waves) have arisen. But it remains open in [6 - 8] to verify that the non-classical solutions constructed satisfy the system considered. In the present paper, we borrow the new mathematical theory of generalized functions, chiefly initiated by J. F. Colombeau and Rosinger, to deal with the diffculty of the multiplication of distribution solutions. The non-classical Riemann solutions we constructed in [6 - 8] satisfy the system in the sense of association. The present paper provides a good example of applications for this new mathematical theory in powerfully handling the product of generalized functions.

Keywords: *Two-dimensional conservation laws, Riemann problems, non-classical waves, multiplication of distributions*

AMS subject classification: 35 L 65

1. Introduction

We are concerned with the two-dimensional hyperbolic system of nonlinear conservation laws

$$\left.\begin{aligned} u_t + (uv)_y &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}\right\} \qquad ((x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+) \tag{1.1}$$

with initial data

$$(u,v)|_{t=0} = \begin{cases} (u_2, v_2) & \text{for } x, y > 0 \\ (u_1, v_1) & \text{otherwise} \end{cases} \tag{1.2}$$

where (u_i, v_i) $(i = 1,2)$ are constant states. We call (1.1), (1.2) a *Riemann problem*.

System (1.1) is the special form of the mathematical simplification of the two-dimensional linearized model of the cochlea. We recall that in the absence of fluid

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viscosity the linearized equations of motion in the fluid-filled inner ear (cochlea) read

$$\rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial y}=0 \tag{1.3}$$

$$\rho\frac{\partial v}{\partial t}+\frac{\partial p}{\partial x}=0 \tag{1.4}$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \tag{1.5}$$

where (u,v) and p stand for the fluid velocity and pressure, respectively, while ρ is the (constant) density (see [9] and the references therein). Neglecting (1.5) and assuming $p=\rho uv$, one finds that (1.3), (1.4) are reduced to the system (1.1).

In the papers [6 - 8] we constructed the Riemann solutions to (1.1) for any piecewise constant initial data (1.2). In particular, some Riemann solutions contain the non-classical waves (labelled as the Dirac-contact waves). In other words, these Riemann solutions no longer lie in $L^\infty_{loc}(\mathbb{R}^2\times\mathbb{R}_+)$, the space of locally bounded functions on $\mathbb{R}^2\times\mathbb{R}_+$, although the initial data belong to $L^\infty(\mathbb{R}^2)$. This is distinctly different from that people investigated before (sec [14]). In [6 - 8], the non-classical Riemann solutions to (1.1), (1.2) were shown to possess high singularity on the Dirac-contact waves and can be viewed as the bounded functions in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ plus the Schwartz generalized functions supported on the Dirac-contact waves. At this time, there arises a concurrent problem of how to define the product of two Schwartz generalized functions. According to L. Schwartz's theory, it is impossible to define the product of two arbitrary Schwartz generalized functions since the space of all Schwartz generalized functions is not an algebraic one [13]. Because of this difficulty, we failed to verify in [6 - 8] that the non-classical Riemann solutions constructed satisfy the system (1.1). In the present paper, we apply a new mathematical theory of generalized functions, chiefly introduced by J. F. Colombeau and Rosinger [1, 2, 12], to dealing with the multiplication of the distributions appearing in the non-classsical Riemann solutions to (1.1), (1.2), and have fully solved the open problem left in [6 - 8]. This new theory of generalized functions is the extension of Schwartz generalized function theory and allows us to define the product of two arbitrary Schwartz generalized functions. It was later developed by Oberguggenberger [10, 11] and other people (see, e.c., [2 - 4, 5, 15]). We remark in passing here that in recent years, the non-classical waves have attracted great interest, and many people have paid attention to investigating them (see [16 - 21]).

The program of this paper is as follows:

In Section 2, for the reader's convenience, we shall give a glimpse of this new mathematical theory of generalized functions and then interpret in what sense the Riemann solutions we constructed in [6 - 8] satisfy (1.1), (1.2) in the framework of this new mathematical theory (cf. (2.3), (2.4)). In Section 3 we only pay our attention to a representative case that $u_1\ge 0\ge u_2$ and $v_1\ge 0\ge v_2$ $(u_1\ne u_2, v_1\ne v_2$ and $u_1v_1\ne u_2v_2)$ (at this time, both u and v are singular on the Dirac-contact waves and there arises the difficulty of the product of distributions). Then wc verify that the non-classical Riemann solutions satisfy (1.1) in the sense of association.

2. The new generalized function space $\mathcal{G}(\Omega)$

In this section we briefly describe the definition of the new generalized function space $\mathcal{G}(\Omega)$ introduced by Colombeau and Rosinger [1, 2, 12].

Let Ω be an open set in $\mathbb{R}^n$ and we denote by $\mathcal{E}_M[\Omega]$ the set of all the maps $R(\varepsilon, x) : (0,1] \times \Omega \to C$ such that:

(i) For any $\varepsilon > 0$, the map $R(\varepsilon, x)$ is a C^∞-function of the variable $x \in \Omega$.

(ii) If $D = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$ is any partial derivation operator and if K is any compact subset of Ω, then there exists an integer N and two constants $C > 0$ and $\eta > 0$ with $0 < \eta < 1$ such that $\sup_{x \in K} |DR(\varepsilon, x)| \leq \frac{C}{\varepsilon^N}$ if $0 < \varepsilon < \eta$.

Next let $\mathcal{N}[\Omega]$ be the set of all elements $R \in \mathcal{E}_M[\Omega]$ with the property that, for all D and K as above, we have an integer N such that for all $q \geq N$ there exist $C_q, \eta_q > 0$ with $\sup_{x \in K} |DR(\varepsilon, x)| \leq C_q \varepsilon^q$ if $0 < \varepsilon < \eta_q$. Obviously, $\mathcal{N}[\Omega]$ is an ideal of the algebra $\mathcal{E}_M[\Omega]$. The new generalized function space $\mathcal{G}(\Omega)$ is defined as the quotient algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega].$$

The operations in $\mathcal{G}(\Omega)$ such as differentiation, addition and multiplication are those naturally defined on representatives; in particular, multiplication is possible in $\mathcal{G}(\Omega)$ because $\mathcal{N}[\Omega]$ is an ideal of the algebra $\mathcal{E}_M[\Omega]$.

Two generalized functions $G_1, G_2 \in \mathcal{G}(\Omega)$ are said to be *associated* (in notation: $G_1 \approx G_2$) if there exist some representatives R_1 and R_2 of G_1 and G_2, respectively, such that for all $\psi \in \mathcal{D}(\Omega)$

$$\lim_{\varepsilon \to 0} \int_\Omega \big(R_1(\varepsilon, x) - R_2(\varepsilon, x)\big)\psi(x)\, dx = 0.$$

An element $G \in \mathcal{G}(\Omega)$ is said to have a distribution $T \in \mathcal{D}'(\Omega)$ as *macroscopic aspect*, if $G \approx T$, i.e. for all $\psi \in \mathcal{D}(\Omega)$

$$\lim_{\varepsilon \to 0} \int_\Omega R(\varepsilon, x)\psi(x)\, dx = \langle T(\lambda), \psi(\lambda - x)\rangle$$

for some representative R of G.

Now we are in a position to give in what sense the Riemann solutions we constructed in [6 - 8] satisfy (1.1), (1.2). First of all, we note that (1.1), (1.2) are invariant under the self-similar transformation

$$\left.\begin{array}{l} x \to \alpha x' \\ y \to \alpha y' \\ t \to \alpha t' \end{array}\right\} \qquad (\alpha > 0).$$

We should seek self-similar solutions of the form

$$\big(u(x,y,t), v(x,y,t)\big) = \big(u(\xi,\eta), v(\xi,\eta)\big) \qquad \big(\xi = \tfrac{x}{t}, \eta = \tfrac{y}{t}\big).$$

Thus (1.1) changes into

$$\left.\begin{aligned} -\xi u_\xi - \eta u_\eta + (uv)_\eta &= 0 \\ -\xi v_\xi - \eta v_\eta + (uv)_\xi &= 0 \end{aligned}\right\} \qquad ((\xi,\eta) \in \mathbb{R}^2) \tag{2.1}$$

and (1.2) into

$$(u(\xi,\eta), v(\xi,\eta)) \to \begin{cases} (u_2, v_2) & \text{for } \xi \to \infty, \eta \to \infty \\ (u_1, v_1) & \text{for } \xi \to \infty, \eta \to -\infty \ \text{ or } \ \xi \to -\infty, |\eta| \to \infty. \end{cases} \tag{2.2}$$

Definition 2.1. The Riemann solution $(U,V) \in \mathcal{G}(\mathbb{R}^2)$ is said to satisfy (1.1), (1.2) if

$$\left.\begin{aligned} -\xi U_\xi - \eta U_\eta + (UV)_\eta &\approx 0 \\ -\xi V_\xi - \eta V_\eta + (UV)_\xi &\approx 0. \end{aligned}\right\} \tag{2.3}$$

The sense in which $(U,V) \in \mathcal{G}(\mathbb{R})$ satisfies the initial data (1.2) should be investigated carefully. In the present paper, we simply give the following justification:

Let $\big(R_u(\varepsilon,\xi,\eta), R_v(\varepsilon,\xi,\eta)\big)$ be any representative of (U,V). If

$$\big(R_u(\varepsilon,\xi,\eta), R_v(\varepsilon,\xi,\eta)\big)$$

converges to some function pair

$$\big(u(\xi,\eta), v(\xi,\eta)\big) = \big(u(\tfrac{x}{t}), v(\tfrac{y}{t})\big) \in L^\infty_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$$

in $(\mathcal{D}_1(\mathbb{R}^2))'$ as $\varepsilon \to 0$, i.e.

$$\begin{aligned} \lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)\varphi(\xi,\eta)\,d\xi d\eta &= \iint_{\mathbb{R}^2} u(\xi,\eta)\varphi(\xi,\eta)\,d\xi d\eta \\ \lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} R_v(\varepsilon,\xi,\eta)\varphi(\xi,\eta)\,d\xi d\eta &= \iint_{\mathbb{R}^2} v(\xi,\eta)\varphi(\xi,\eta)\,d\xi d\eta \end{aligned} \tag{2.4$_1$}$$

for each $\varphi \in \mathcal{D}_1(\mathbb{R}^2)$, where

$$\mathcal{D}_1(\mathbb{R}^2) = \Big\{\varphi \in C_0^\infty(\mathbb{R}^2) : \varphi(\xi,0) = 0 \text{ for all } \xi \in \mathbb{R} \text{ and } \varphi(0,\eta) = 0 \text{ for all } \eta \in \mathbb{R}\Big\},$$

and if

$$u\Big(\frac{x}{t},\frac{y}{t}\Big) \to u_0(x,y) \quad \text{and} \quad v\Big(\frac{x}{t},\frac{y}{t}\Big) \to v_0(x,y) \qquad \text{in } L^1_{loc}(\mathbb{R}^2) \tag{2.4$_2$}$$

as $t \to 0+$, we say that $(U,V) \in \mathcal{G}(\mathbb{R}^2)$ satisfies (1.2).

We note that $(2.4)_1$ is feasible for the Riemann solution to the system (1.1) since the non-classical wave appearing in the Riemann solution develops only from the discontinuity line $x = 0, y \geq 0$ or $y = 0, x \geq 0$ of the initial data as time evolves.

3. Verification of (2.3), (2.4) for non-classical Riemann solutions constructed in [6 - 8]

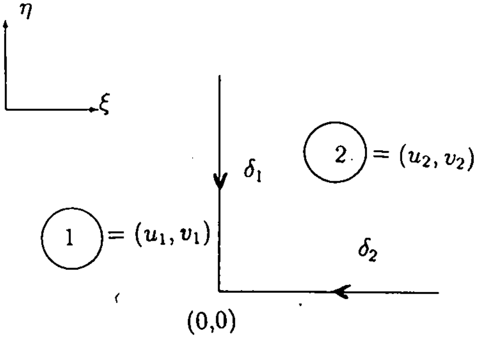
In this section we shall verify that the non-classical Riemann solutions we constructed in [6 - 8] satisfy (2.3), (2.4). We only pay our attention to the case that $u_1 \geq 0 \geq u_2$ and $v_1 \geq 0 \geq v_2$ $(u_1 \neq u_2, v_1 \neq v_2$ and $u_1 v_1 \neq u_2 v_2)$. Other cases can be treated similarly.

At this time, the Riemann solution to (1.1), (1.2) is constructed as follows [6 - 8]:

(i) By the initial condition (2.2), there exists a sufficiently large circle in the (ξ, η)-plane, outside which the solution to (1.1), (1.2) looks like that of the Riemann problem for corresponding one-dimensional system of conservation laws, and what we should do is to determine the interaction in the circle of waves coming from infinity.

(ii) There exists a Dirac-contact wave (denoted by δ_1), which comes from infinity and is given by $\xi = 0, \eta > 0$, since $u_1 \geq 0 \geq u_2$ $(u_1 > u_2)$ (see [6 - 8]). Also, there is another Dirac-contact wave (denoted by δ_2), which comes from infinity and is given by $\eta = 0, \xi > 0$, since $v_1 \geq 0 \geq v_2$ $(v_1 > v_2)$.

(iii) δ_1 and δ_2 hit at the point (0,0) (see the following figure).



Next we give the expressions of the elements R_u and R_v of the Riemann solution $(U, V) \in \mathcal{G}(\mathbb{R}^2)$ and verify that for $\psi \in C_0^\infty(\mathbb{R}^2)$

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}^2} \Big[R_u(\varepsilon, \xi, \eta)((\xi\psi)_\xi + (\eta\psi)_\eta) - R_u(\varepsilon, \xi, \eta) R_v(\varepsilon, \xi, \eta)\psi_\eta \Big] d\xi d\eta = 0 \tag{3.1}$$

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}^2} \Big[R_v(\varepsilon, \xi, \eta)((\xi\psi)_\xi + (\eta\psi)_\eta) - R_u(\varepsilon, \xi, \eta) R_v(\varepsilon, \xi, \eta)\psi_\xi \Big] d\xi d\eta = 0. \tag{3.2}$$

We first consider $u_1 v_1 = 0$ and then discuss the case that $u_1 v_1 \neq 0$.

3.1 The case $u_1 v_1 = 0$. Without loss of generality, we assume that $u_1 = 0$ and let

$$\left.\begin{aligned} \overline{u}(\xi, \eta) &= u_2 H(\xi, \eta) \\ \overline{v}(\xi, \eta) &= v_1 + (v_2 - v_1) H(\xi, \eta) \end{aligned}\right\} \tag{3.3}$$

where $H(\xi, \eta) = 1$ for $\xi, \eta > 0$ and $H(\xi, \eta) = 0$ otherwise. From the construction above, we know that the Riemann solution to (1.1), (1.2) equals

$$(\overline{u}(\xi, \eta), \overline{v}(\xi, \eta)) = \begin{cases} (u_2, v_2) & \text{for } \xi > 0, \eta > 0 \\ (u_1, v_1) = (0, v_1) & \text{for } \xi < 0, -\infty < \eta < \infty \ \text{ or } \ \xi > 0, \eta < 0. \end{cases}$$

But $(\overline{u}(\xi,\eta),\overline{v}(\xi,\eta))$ is not a solution to (1.1), (1.2) in the sense of distributions (or association). In fact, (2.1) never holds true for $(\overline{u}(\xi,\eta),\overline{v}(\xi,\eta))$ in the sense of distributions (or association) since

$$\left.\begin{aligned}\iint_{\mathbb{R}^2}\Big[((\xi\psi)_\xi+(\eta\psi)_\eta)\overline{u}(\xi,\eta)-\psi_\eta\overline{u}(\xi,\eta)\overline{v}(\xi,\eta)\Big]d\xi d\eta&=u_2v_2\int_0^\infty\psi(\xi,0)\,d\xi\\ \iint_{\mathbb{R}^2}\Big[((\xi\psi)_\xi+(\eta\psi)_\eta)\overline{v}(\xi,\eta)-\psi_\xi\overline{u}(\xi,\eta)\overline{v}(\xi,\eta)\Big]d\xi d\eta&=u_2v_2\int_0^\infty\psi(0,\eta)\,d\eta\end{aligned}\right\}\tag{3.4}$$

and $u_2v_2\neq u_1v_1=0$. Thus the Riemann solution to (1.1), (1.2) is much more than $(\overline{u}(\xi,\eta),\overline{v}(\xi,\eta))$. The right sides of (3.4) carry enough information for us to give the expressions of the elements R_u and R_v of the Riemann solution $(U,V)\in\mathcal{G}(\mathbb{R}^2)$ to (1.1), (1.2).

Let T_1,T_2 be two bounded linear functionals given by

$$\langle T_1,\psi\rangle=\int_0^\infty r(\xi)\,\psi(\xi,0)\,d\xi\tag{3.5}$$

$$\langle T_2,\psi\rangle=\int_0^\infty s(\eta)\,\psi(0,\eta)\,d\eta\tag{3.6}$$

for $\psi\in C_0^\infty(\mathbb{R}^2)$, where r and s are some continuous functions on $[0,\infty)$ yet to be determined below. It is easily seen that T_2 is supported on the open half η-axis $\xi=0,\eta>0$ and T_1 is supported on the open half ξ-axis $\eta=0,\xi>0$. Now we define

$$\begin{aligned}R_u(\varepsilon,\xi,\eta)&=(\overline{u}*w_{1\varepsilon})(\xi,\eta)+u_2v_2(T_1*w_{1\varepsilon})(\xi,\eta)\\ R_v(\varepsilon,\xi,\eta)&=(\overline{v}*w_{2\varepsilon})(\xi,\eta)+u_2v_2(T_2*w_{2\varepsilon})(\xi,\eta)\end{aligned}\tag{3.7}$$

where $w_{i\varepsilon}(\xi,\eta)=\frac{1}{\varepsilon^2}\theta_i(\frac{\xi}{\varepsilon})\phi_i(\frac{\eta}{\varepsilon})$ and $\theta_i,\phi_i\in\mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}}\theta_i(\xi)\,d\xi=\int_{\mathbb{R}}\phi_i(\xi)\,d\xi=1$ $(i=1,2)$. We rewrite (3.7) as

$$\begin{aligned}R_u(\varepsilon,\xi,\eta)&=u_2\int_{-\xi/\varepsilon}^\infty\theta_1(x)\,dx\int_{-\eta/\varepsilon}^\infty\phi_1(y)\,dy\\ &\quad+u_2v_2\frac{1}{\varepsilon}\phi_1\left(-\frac{\eta}{\varepsilon}\right)\int_{-\xi/\varepsilon}^\infty r(\xi+\varepsilon x)\theta_1(x)\,dx\end{aligned}\tag{3.8}$$

$$\begin{aligned}R_v(\varepsilon,\xi,\eta)&=v_1+(v_2-v_1)\int_{-\xi/\varepsilon}^\infty\theta_2(x)\,dx\int_{-\eta/\varepsilon}^\infty\phi_2(y)\,dy\\ &\quad+u_2v_2\frac{1}{\varepsilon}\theta_2\left(-\frac{\xi}{\varepsilon}\right)\int_{-\eta/\varepsilon}^\infty s(\eta+\varepsilon y)\phi_2(y)\,dy.\end{aligned}\tag{3.9}$$

From (3.1), (3.2) we can determine r and s. To do this, we compute that for $\psi \in C_0^\infty(\mathbb{R}^2)$, (3.8) and (3.9) give

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta = u_2\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta + u_2v_2\int_0^\infty r(\xi)\psi(\xi,0)\,d\xi \tag{3.10}$$

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2} R_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta = v_1\iint_{\mathbb{R}^2}\psi(\xi,\eta)\,d\xi d\eta + (v_2-v_1)\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta + u_2v_2\int_0^\infty s(\eta)\psi(0,\eta)\,d\eta \tag{3.11}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta\\
&= u_2v_2\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta + u_2^2v_2\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\frac{1}{\varepsilon}\theta_2\left(-\frac{\xi}{\varepsilon}\right)\\
&\quad\times\int_{-\xi/\varepsilon}^{\infty}\theta_1(x)\,dx\int_{-\eta/\varepsilon}^{\infty}\phi_1(y)\,dy\cdot\int_{-\eta/\varepsilon}^{\infty} s(\eta+\varepsilon y)\phi(y)\,dyd\xi d\eta\\
&\quad+u_2v_2\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\frac{1}{\varepsilon}\phi_1\left(-\frac{\eta}{\varepsilon}\right)\int_{-\xi/\varepsilon}^{\infty} r(\xi+\varepsilon x)\theta_1(x)\,dx\\
&\quad\times\left(v_1+(v_2-v_1)\int_{-\xi/\varepsilon}^{\infty}\theta_2(x)\,dx\cdot\int_{-\eta/\varepsilon}^{\infty}\phi_2(y)\,dy\right)d\xi d\eta\\
&\quad+u_2^2v_2^2\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\frac{1}{\varepsilon}\phi_1\left(-\frac{\eta}{\varepsilon}\right)\frac{1}{\varepsilon}\theta_2\left(-\frac{\xi}{\varepsilon}\right)\int_{-\xi/\varepsilon}^{\infty} r(\xi+\varepsilon x)\theta_1(x)\,dx\\
&\quad\times\int_{-\eta/\varepsilon}^{\infty} s(\eta+\varepsilon y)\phi_2(y)\,dyd\xi d\eta\\
&= u_2v_2\int_0^\infty\int_0^\infty \psi(\xi,\eta)d\xi d\eta + u_2^2v_2\cdot A\int_0^\infty s(\eta)\psi(0,\eta)\,d\eta
\end{aligned}\tag{3.12}$$

$$+u_2v_2(v_1+(v_2-v_1)B)\int_0^\infty r(\xi)\psi(\xi,0)\,d\xi + u_2^2v_2^2AB\,r(0)s(0)\psi(0,0)$$

where $A=\int_R \theta_2(\xi)\big(\int_\xi^\infty \theta_1(x)dx\big)d\xi$ and $B=\int_R \phi_1(\eta)\big(\int_\eta^\infty \phi_2(y)dy\big)d\eta$. Appropriately choosing θ_i and ϕ_i $(i=1,2)$ such that $A=0$ and $B=\frac{v_1}{v_1-v_2}$, one gets at once from (3.12) that

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta = u_2v_2\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta. \tag{3.13}$$

Using (3.10) and (3.13), it follows from (3.1) that $\int_0^\infty \big(r(\xi)-\xi r'(\xi)+1\big)\psi(\xi,0)\,d\xi=0$ which implies that $r(\xi)-\xi r'(\xi)+1=0$ $(\xi\ge 0)$, i.e., $r(\xi)=a\xi-1$ $(\xi\ge 0)$. Here a is any constant.

Similarly, from (3.11) and (3.13) we deduce from (3.2) that $s(\eta)=b\eta-1$ $(\eta\ge 0)$, with b arbitrary constant, for the θ_i and ϕ_i $(i=1,2)$ chosen above. Therefore, $(U,V)\in\mathcal{G}(\mathbb{R}^2)$ satisfies (2.3). And (2.4) is trivial. Thus we have proved that $(U,V)\in\mathcal{G}(\mathbb{R}^2)$, with $\overline{u}(\xi,\eta)+u_2v_2T_1$ and $\overline{v}(\xi,\eta)+u_2v_2T_2$ as their macroscopic aspects, respectively, satisfies (1.1) and (1.2) in the sense of association when $u_1=0>u_2$ and $v_1\ge 0>v_2$.

Remarks 1. We have infinitely many different Schwartz generalized functions T_1 and T_2 defined above since $r(\xi)=a\xi-1$ and $s(\eta)=b\eta-1$, and so the macroscopic aspects of U and V are infinitely numerous. In this sense the Riemann solution to (1.1), (1.2) is not unique.

2. The elements of the Riemann solution $(U,V)\in\mathcal{G}(\mathbb{R}^2)$ strongly depend on the choices of regularization process, which eventually depend on the system and the initial datum. Thus the ambiguity is removed in defining the multiplication of two distributions $U,V\in\mathcal{G}(\mathbb{R}^2)$ (see (3.13)).

3.2 The case $u_1v_1\neq 0$. We set

$$\begin{aligned}\overline{u}(\xi,\eta)&=u_1+(u_2-u_1)\,H(\xi,\eta)\\ \overline{v}(\xi,\eta)&=v_1+(v_2-v_1)\,H(\xi,\eta)\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\langle T_1,\psi\rangle&=\int_0^\infty (a\xi-1)\psi(\xi,0)\,d\xi\\ \langle T_2,\psi\rangle&=\int_0^\infty (b\eta-1)\psi(0,\eta)\,d\eta\end{aligned}\tag{3.15}$$

where H is the same as (3.3) and a,b are arbitrary constants. As above, we can define $\overline{U},\overline{V}\in\mathcal{G}(\mathbb{R}^2)$ with $\overline{u}(\xi,\eta)+\rho T_1$ and $\overline{v}(\xi,\eta)+\rho T_2$ as their macroscopic aspects, $\rho=$

$u_2v_2 - u_1v_1 \neq 0$. Let R_u and R_v be representatives of $\overline{U}$ and $\overline{V}$, respectively, where

$$\begin{aligned}
\overline{R}_u(\varepsilon,\xi,\eta) &= (\overline{u} * w_{1\varepsilon})(\xi,\eta) + \rho(T_1 * w_{1\varepsilon})(\xi,\eta) \\
&= u_1 + (u_2 - u_1) \int_{-\xi/\varepsilon}^{\infty} \theta_1(x)\,dx \int_{-\eta/\varepsilon}^{\infty} \phi_1(y)\,dy \\
&\quad + \rho\frac{1}{\varepsilon}\phi_1\left(-\frac{\eta}{\varepsilon}\right) \int_{-\xi/\varepsilon}^{\infty} (a\xi + a\varepsilon x - 1)\theta_1(x)\,dx \\
\overline{R}_v(\varepsilon,\xi,\eta) &= (\overline{v} * w_{2\varepsilon})(\xi,\eta) + \rho(T_2 * w_{2\varepsilon})(\xi,\eta) \\
&= v_1 + (v_2 - v_1) \int_{-\xi/\varepsilon}^{\infty} \theta_2(x)\,dx \int_{-\eta/\varepsilon}^{\infty} \phi_2(y)\,dy \\
&\quad + \rho\frac{1}{\varepsilon}\theta_2\left(-\frac{\xi}{\varepsilon}\right) \int_{-\eta/\varepsilon}^{\infty} (b\eta + \varepsilon b y - 1)\phi_2(y)\,dy.
\end{aligned} \tag{3.16}$$

However, $(\overline{U},\overline{V}) \in \mathcal{G}(\mathbb{R}^2)$ does not satisfy (1.1) in the sense of association. As a matter of fact, from (3.16) we compute that for $\psi \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \big(u_1 + (u_2 - u_1)H(\xi,\eta)\big)\psi(\xi,\eta)\,d\xi d\eta \\
&\quad + \lim_{\varepsilon\to 0}\frac{\rho}{\varepsilon}\iint_{\mathbb{R}^2} \psi(\xi,\eta)\phi_1\left(-\frac{\eta}{\varepsilon}\right)\int_{-\xi/\varepsilon}^{\infty} (a\xi + a\varepsilon x - 1)\theta_1(x)\,dx \\
&= u_1 \iint_{\mathbb{R}^2} \psi(\xi,\eta)\,d\xi d\eta \\
&\quad + (u_2 - u_1)\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta + \rho\int_0^\infty (a\xi - 1)\psi(\xi,0)\,d\xi
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta \\
&= u_1v_1 \iint_{\mathbb{R}^2} \psi(\xi,\eta)\,d\xi d\eta + (u_2v_2 - u_1v_1)\int_0^\infty\int_0^\infty \psi(\xi,\eta)\,d\xi d\eta \\
&\quad + \rho\big(u_1 + (u_2 - u_1)A\big)\int_0^\infty (b\eta - 1)\psi(0,\eta)\,d\eta
\end{aligned} \tag{3.18}$$

$$+\rho(v_1+(v_2-v_1)B)\int_0^\infty (a\xi-1)\psi(\xi,0)\,d\xi+\rho^2 AB\psi(0,0)$$

where $A=\int_{\mathbb{R}}\theta_2(\xi)\big(\int_\xi^\infty\theta_1(x)\,dx\big)d\xi$ and $B=\int_{\mathbb{R}}\phi_1(\eta)\big(\int_\eta^\infty\phi_2(y)\,dy\big)d\eta$. After choosing θ_i and ϕ_i $(i=1,2)$ such that $A=\frac{u_1}{u_1-u_2}$ and $B=\frac{v_1}{v_1-v_2}$, from (3.17) and (3.18) one gets at once that for $\psi\in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}&\lim_{\varepsilon\to0}\iint_{\mathbb{R}^2}\Big[\overline{R}_u(\varepsilon,\xi,\eta)((\xi\psi)_\xi+(\eta\psi)_\eta)-\overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi_\eta(\xi,\eta)\Big]d\xi d\eta\\&\quad=-\rho^2AB\psi_\eta(0,0).\end{aligned}\tag{3.19}$$

Similarly,

$$\begin{aligned}&\lim_{\varepsilon\to0}\iint_{\mathbb{R}^2}\Big[\overline{R}_v(\varepsilon,\xi,\eta)((\xi\psi)_\xi+(\eta\psi)_\eta)-\overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi_\xi(\xi,\eta)\Big]d\xi d\eta\\&\quad=-\rho^2AB\psi_\xi(0,0).\end{aligned}\tag{3.20}$$

Thus, $(\overline{U},\overline{V})\in\mathcal{G}(\mathbb{R}^2)$, with $\overline{R}_u$ and $\overline{R}_v$ as their respective elements, is not a solution to (1.1),(1.2).

Indeed, at this time, not only is the solution to (1.1),(1.2) singular on the half lines $\xi=0,\eta>0$ and $\eta=0,\xi>0$, but also singular at the point (0,0). The right-handed sides of (3.19) and (3.20) contain information enough to make us give the expressions of the solution to (1.1), (1.2) at the point (0,0). In fact, we define two distributions μ_1 and μ_2 supported on the origin

$$\langle\mu_1,\psi\rangle=c_1\,\psi_\eta(0,0)\qquad\text{and}\qquad\langle\mu_2,\psi\rangle=c_2\,\psi_\xi(0,0)\tag{3.21}$$

for $\psi\in C_0^\infty(\mathbb{R}^2)$, where c_1 and c_2 are constants to be determined below. We set $U,V\in\mathcal{G}(\mathbb{R}^2)$ with R_u and R_v as their representatives, respectively. Here R_u and R_v are determined by

$$\begin{aligned}R_u(\varepsilon,\xi,\eta)&=\overline{R}_u(\varepsilon,\xi,\eta)+(\mu_1*w_{3\varepsilon})(\xi,\eta)\\&=\overline{R}_u(\varepsilon,\xi,\eta)+c_1\cdot\frac{1}{\varepsilon^3}\theta_3\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)\\R_v(\varepsilon,\xi,\eta)&=\overline{R}_v(\varepsilon,\xi,\eta)+(\mu_2*w_{4\varepsilon})(\xi,\eta)\\&=\overline{R}_v(\varepsilon,\xi,\eta)+c_2\cdot\frac{1}{\varepsilon^3}\theta_4'\left(-\frac{\xi}{\varepsilon}\right)\phi_4\left(-\frac{\eta}{\varepsilon}\right)\end{aligned}\tag{3.22}$$

where $w_{i\varepsilon}(\xi,\eta)=\frac{1}{\varepsilon^3}\theta_i(\frac{\xi}{\varepsilon})\phi_i(\frac{\eta}{\varepsilon})$ and $\theta_i,\phi_i\in C_0^\infty(\mathbb{R})$ with $\int_{\mathbb{R}}\theta_i(\xi)\,d\xi=\int_{\mathbb{R}}\phi_i(\xi)\,d\xi=1$ $(i=3,4)$.

It remains to verify that R_u and R_v given by (3.22) satisfy (3.1) and (3.2) for

$\psi \in C_0^\infty(\mathbb{R}^2)$. Actually, from (3.22) we have that for $\psi \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}
&\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)\,\psi(\xi,\eta)d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta + c_1\cdot\frac{1}{\varepsilon^3}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\theta_3\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta + c_1\cdot\frac{1}{\varepsilon}\iint_{\mathbb{R}^2}\psi(-\varepsilon\xi,-\varepsilon\eta)\theta_3(\xi)\phi_3'(\eta)\,d\xi d\eta \\
&= \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta + c_1\psi_\eta(0,0) + O(\varepsilon)
\end{aligned} \tag{3.23}$$

since $\psi(-\varepsilon\xi,-\varepsilon\eta) = \psi(0,0) - \varepsilon\xi\,\psi_\xi(0,0) - \varepsilon\eta\,\psi_\eta(0,0) + O(\varepsilon^2)$, while

$$\begin{aligned}
&\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta \\
&\quad= \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta \\
&\quad\quad+ c_1\cdot\frac{1}{\varepsilon^3}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\theta_3\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)\overline{R}_v(\varepsilon,\xi,\eta)\,d\xi d\eta \\
&\quad\quad+ c_2\cdot\frac{1}{\varepsilon^3}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\theta_4'\left(-\frac{\xi}{\varepsilon}\right)\phi_4\left(-\frac{\eta}{\varepsilon}\right)\overline{R}_u(\varepsilon,\xi,\eta)\,d\xi d\eta \\
&\quad\quad+ c_1c_2\frac{1}{\varepsilon^6}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\theta_3\left(-\frac{\xi}{\varepsilon}\right)\theta_4'\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)\phi_4\left(-\frac{\eta}{\varepsilon}\right)d\xi d\eta \\
&\quad\sim \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta + I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.
\end{aligned} \tag{3.24}$$

Now we choose θ_i and ϕ_i $(i = 2,3)$ such that

$$\left.\begin{aligned}
\int_{\mathbb{R}}\theta_2(\xi)\theta_3(\xi)\,d\xi = \int_{\mathbb{R}}\xi\theta_2(\xi)\theta_3(\xi)\,d\xi = 0 \\
\int_{\mathbb{R}}\phi_2(\xi)\phi_3(\xi)\,d\xi = 0 \\
\int_{\mathbb{R}}\theta_3(\xi)\left(\int_\xi^\infty\theta_2(x)\,dx\right)d\xi\cdot\int_{\mathbb{R}}\eta\phi_3'(\eta)\left(\int_\eta^\infty\phi_2(y)dy\right)d\eta = \frac{v_1}{v_2-v_1}.
\end{aligned}\right\} \tag{3.25}$$

This can be done, e.g., by taking $\theta_2, \theta_3 \in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{supp}\theta_2 \subset [0,1]$ and $\operatorname{supp}\theta_3 \subset [-1,0]$, and $\rho_i \in C_0^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}}\rho_i(\eta)\,d\eta = 1$ $(i = 1,2)$, $\operatorname{supp}\rho_1 \subset [-2,-1]$ and $\operatorname{supp}\rho_2 \subset [1,2]$. Set $\phi_3 = \theta_2$ and $\phi_2 = (1-B)\rho_1 + B\rho_2$, with $B = \frac{v_1}{v_1-v_2}$. From (3.16) and (3.25) it follows that

$$I_1^\varepsilon = c_1\cdot\frac{1}{\varepsilon^3}\iint_{\mathbb{R}^2}\psi(\xi,\eta)\theta_3\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)\overline{R}_v(\varepsilon,\xi,\eta)\,d\xi d\eta$$

$$
\begin{aligned}
&= \frac{c_1}{\varepsilon} \iint_{\mathbb{R}^2} \psi(-\varepsilon\xi, -\varepsilon\eta)\theta_3(\xi)\phi_3'(\eta)\overline{R}_v(\varepsilon, -\varepsilon\xi, -\varepsilon\eta)\, d\xi d\eta \\
&= \frac{c_1}{\varepsilon} \iint_{\mathbb{R}^2} \psi(-\varepsilon\xi, -\varepsilon\eta)\theta_3(\xi)\phi_3'(\eta)\Bigg[v_1 + (v_2 - v_1)\int_\xi^\infty \theta_2(x)\, dx \int_\eta^\infty \phi_2(y)\, dy \\
&\quad + \rho b\theta_2(\xi)\int_\eta^\infty (y-\eta)\phi_2(y)\, dy - \frac{\rho}{\varepsilon}\theta_2(\xi)\int_\eta^\infty \phi_2(y)\, dy\Bigg] d\xi d\eta \\
&= \frac{c_1}{\varepsilon} \iint_{\mathbb{R}^2} \Big(\psi(0,0) - \varepsilon\xi\psi_\xi(0,0) - \varepsilon\eta\psi_\eta(0,0) + O(\varepsilon^2)\Big)\theta_3(\xi)\phi_3'(\eta) \\
&\quad \times \Bigg\{v_1 + (v_2 - v_1)\int_\xi^\infty \theta_2(x)\, dx \int_\eta^\infty \phi_2(y)\, dy \\
&\quad + b\rho\theta_2(\xi)\int_\eta^\infty (y-\eta)\phi_2(y)\, dy\Bigg\} d\xi d\eta \\
&\quad - \frac{\rho c_1}{\varepsilon^2} \iint_{\mathbb{R}^2} \theta_2(\xi)\theta_3(\xi)\phi_3'(\eta)\int_\eta^\infty \phi_2(y)\, dy \cdot \Big\{\psi(0,0) - \varepsilon\xi\psi_\xi(0,0) \\
&\quad - \varepsilon\eta\psi_\eta(0,0) + \frac{1}{2}\varepsilon^2\xi^2\psi_{\xi\xi}(0,0) + \varepsilon^2\xi\eta\psi_{\xi\eta}(0,0) \\
&\quad + \frac{1}{2}\varepsilon^2\eta^2\psi_{\eta\eta}(0,0) + O(\varepsilon^3)\Big\} d\xi d\eta \\
&= O(\varepsilon).
\end{aligned}
\tag{3.26}
$$

Similarly, we have that

$$
I_2^\varepsilon = c_2 \cdot \frac{1}{\varepsilon^3} \iint_{\mathbb{R}^2} \psi(\xi,\eta)\theta_4'\left(-\frac{\xi}{\varepsilon}\right)\phi_4\left(-\frac{\eta}{\varepsilon}\right)\overline{R}_u(\varepsilon,\xi,\eta)\, d\xi d\eta = O(\varepsilon) \tag{3.27}
$$

if we set

$$
\left.
\begin{aligned}
\int_{\mathbb{R}} \phi_1(\eta)\phi_4(\eta)\, d\eta = \int_{\mathbb{R}} \eta\phi_1(\eta)\phi_4(\eta)\, d\eta = 0 \\
\int_{\mathbb{R}} \theta_1(\xi)\theta_4(\xi)\, d\xi = 0 \\
\int_{\mathbb{R}} \phi_4(\eta)\left(\int_\eta^\infty \phi_1(y)\, dy\right) d\eta \cdot \int_{\mathbb{R}} \xi\theta_4'(\xi)\left(\int_\xi^\infty \theta_1(x)\, dx\right) d\xi = \frac{u_1}{u_2 - u_1}
\end{aligned}
\right\}
$$

and

$$
\begin{aligned}
I_3^\varepsilon &= c_1 c_2 \frac{1}{\varepsilon^6} \iint_{\mathbb{R}^2} \psi(\xi,\eta)\theta_3\left(-\frac{\xi}{\varepsilon}\right)\theta_4'\left(-\frac{\xi}{\varepsilon}\right)\phi_3'\left(-\frac{\eta}{\varepsilon}\right)\phi_4\left(-\frac{\eta}{\varepsilon}\right) d\xi d\eta \\
&= O(\varepsilon)
\end{aligned}
\tag{3.28}
$$

if we set

$$\int_{\mathbb{R}} \xi^k \theta_3(\xi)\theta_4'(\xi)\,d\xi = 0 \quad \text{or} \quad \int_{\mathbb{R}} \eta^k \phi_3'(\eta)\phi_4(\eta)\,d\eta = 0 \qquad (k = 0,1,2,3,4).$$

Combining (3.26) - (3.28), we deduce from (3.24) that

$$\begin{aligned}&\iint_{\mathbb{R}^2} R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta\\ &\qquad = \iint_{\mathbb{R}^2} \overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi(\xi,\eta)\,d\xi d\eta + O(\varepsilon).\end{aligned} \tag{3.29}$$

Therefore, (3.23), (3.29) and (3.19) yield that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\Big[R_u(\varepsilon,\xi,\eta)\big((\xi\psi)_\xi + (\eta\psi)_\eta\big) - R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi_\eta\Big]\,d\xi d\eta\\ &= \lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\Big[\overline{R}_u(\varepsilon,\xi,\eta)\big((\xi\psi)_\xi + (\eta\psi)_\eta\big)\\ &\quad - \overline{R}_u(\varepsilon,\xi,\eta)\overline{R}_v(\varepsilon,\xi,\eta)\psi_\eta(\xi,\eta)\Big]\,d\xi d\eta + 3c_1\psi_\eta(0,0)\\ &= (3c_1 - \rho^2 AB)\psi_\eta(0,0)\\ &= 0\end{aligned}$$

for $\psi \in C_0^\infty(\mathbb{R}^2)$ if $c_1 = \frac{1}{3}\rho^2 AB$. In the same way, we have that $\psi \in C_0^\infty(\mathbb{R}^2)$

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^2}\Big[Rv(\varepsilon,\xi,\eta)\big((\xi\psi)_\xi + (\eta\psi)_\eta\big) - R_u(\varepsilon,\xi,\eta)R_v(\varepsilon,\xi,\eta)\psi_\xi\Big]\,d\xi d\eta = 0$$

for the same θ_i and ϕ_i $(i = 2,3,4)$ as above if $c_2 = \frac{1}{3}\rho^2 AB$. This verifies that $U, V \in \mathcal{G}(\mathbb{R}^2)$, with R_u and R_v as their representatives, respectively, satisfy (1.1) in the sense of association. The initial condition (2.4) is easily seen. We omit the details.

Remarks. 3. The approximation process above guarantees that $T_i\mu_j = 0$ $(i,j = 1,2)$, but $T_1T_2 \neq 0$ (when $u_1v_1 \neq 0$). One can choose other approximations to define the product of $\overline{U}$ and $\overline{V}$ so that not only $T_i\mu_j = 0$ $(i,j = 1,2)$ but also $T_1T_2 = 0$. However, it is more reasonable to define $T_1T_2 \neq 0$ when $u_1v_1 \neq 0$ since the intersection of the supports of T_1 and T_2 is non-void. The behavior of the solution to (1.1),(1.2) at the origin should be considered.

4. It is easily seen that $U, V \in \mathcal{G}(\mathbb{R}^2)$ have $\overline{u}(\xi,\eta)+\rho T_1+c_1\mu_1$ and $\overline{v}(\xi,\eta)+\rho T_2+c_2\mu_2$ as their respective macroscopic aspects.

We conclude this paper with the following

Theorem. *The non-classical Riemann solutions constructed in* [6 − 8] *satisfy* (1.1) *in the sense of associaton.*

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Added in proof:

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