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# Some Operator Ideals in Non-Commutative Functional Analysis

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**Abstract.** We study classes of linear maps between operator spaces $E$ and $F$ which factorize through maps arising in a natural manner by the Pisier vector-valued non-commutative $L^p$-spaces $S_p[E^*]$ based on the Schatten classes on the separable Hilbert space $\ell^2$. These classes of maps, firstly introduced in [28] and called *p-nuclear maps*, can be viewed as Banach operator ideals in the category of operator spaces, that is in non-commutative (quantized) functional analysis. We also discuss some applications to the split property for inclusions of $W^*$-algebras such as those describing the physical observables in Quantum Field Theory.

**Keywords:** *Linear spaces of operators; Operator algebras and ideals on Hilbert spaces; Classifications, factors*

**AMS subject classification:** Primary 47 D 15, 47 D 25, secondary 46 L 35

## 1. Introduction

The investigation of classes of operator ideals in Hilbert and Banach space theories has a long history. First, some interesting classes of maps between classical function spaces were considered, subsequently many classes of operators were intensively studied. Due to the vastness of the subject, we refer the reader to the monographies [24, 25, 27, 34, 35] and the references quoted therein. Many of these classes of maps can be considered as operator ideals in the category of Banach spaces, as it is well exposed in [35], where a wide class of operator ideals is treated from an axiomatic as well as a concrete viewpoint. On the other hand, if a Banach space $E$ is equipped with a sequence of compatible norms on all matrix spaces $\mathbb{M}_n(E)$, i.e. on matrices with entries in $E$, it can be viewed as a subspace of a $C^*$-algebra, that is a (concrete) operator space [41]. In this last context, the natural arrows between operator spaces are the completely bounded linear maps. All these ideas can be interpreted as a non-commutative (quantized) version of functional analysis as it is well explained in several papers (see, e.g., [2, 13 - 23, 33, 41, 44, 45]). Moreover, in the operator spaces context, interesting classes of completely bounded linear maps have been introduced and studied (see, e.g., [17 - 19, 21, 37, 38, 40]). These classes of maps can be naturally considered as operator ideals between operator spaces, i.e. in quantized functional analysis. This can be achieved by simply replacing the bounded maps (i.e. the natural arrows between Banach spaces) in the definition of an

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operator ideal given in [35], by the completely bounded maps which are the natural arrows in the category of operator spaces.

In this paper we study, for each $1 \leq p < +\infty$, classes $\mathfrak{N}_p(E, F)$ of linear maps between operator spaces $E$ and $F$ which factorize through maps arising in a natural manner by the Pisier vector-valued non-commutative $L^p$-spaces $S_p[E^*]$, based on the Schatten classes on the separable Hilbert space $\ell^2$. These maps, called *p-nuclear maps*, were firstly introduced in [28] in order to investigate the general theory of factorization in the context of operator spaces. These classes of maps can be viewed as operator ideals in operator space setting. Namely, we show that all spaces $\mathfrak{N}_p$ are normed complete (i.e. Banach) operator ideals. We also give a geometrical description for the image of the unit ball $T(E_1) \subset F$ under a $p$-nuclear injective map $T$. The description of the shape of $T(E_1)$ parallels those arising from Banach space context (see [5]) for nuclear maps, or from operator space context (see [21]) in the case of metrically nuclear operators. Finally we discuss some applications to the theory of inclusions of $W^*$-algebras. More precisely, we present some applications to the characterization of the split property of inclusions $N \subset M$ of $W^*$-algebras in terms of properties of the natural $L^2$-embedding (considered in [5]) of $N$ in the non-commutative measure space $L^2(M)$. The various characterizations of the split property have interesting applications in Quantum Field Theory (see, e.g., [6 - 8, 43]).

## 2. On spaces of operators

For the reader's convenience, we collect some preliminary results about operator spaces which we need in the following. Details and proofs can be found in the cited references. In this paper, all operator spaces are complete as normed spaces if it is not otherwise specified.

**2.1 Operator spaces.** For an arbitrary normed space $X$, $X_1$ denotes its (closed) unit ball. We consider a normed space $E$ together with a sequence of norms $\|\cdot\|_n$ on $\mathbb{M}_n(E)$, the space of the $(n \times n)$-matrices with entries in $E$. For $a, b \in \mathbb{M}_n$, $v, v_1 \in \mathbb{M}_n(E)$ and $v_2 \in \mathbb{M}_m(E)$, these norms satisfy

$$\|avb\|_n \leq \|a\|\|v\|_n\|b\| \quad \text{and} \quad \|v_1 \oplus v_2\|_{n+m} = \max\{\|v_1\|_n, \|v_2\|_m\} \quad . \tag{2.1}$$

where the above products are the usual row-column ones. Such a space, equipped with norms on the matrix spaces satisfying the above properties, is called an (abstract) *operator space*.

Let $T : E \to F$ and $T_n : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ are given by $T_n = T \otimes \mathrm{id}$ using the identification $\mathbb{M}_n(E) = E \otimes \mathbb{M}_n$. The linear map $T$ is said to be *completely bounded* if $\|T\|_{cb} := \sup_n \|T_n\| < +\infty$. Further, $\mathfrak{M}(E, F)$ denotes the set of all completely bounded maps between $E$ and $F$. *Complete contractions*, *complete isometries* and *complete quotient maps* have an obvious meaning. It is an important fact (see [41]) that a linear space $E$, together with norms on each $\mathbb{M}_n(E)$, has a realization as a concrete operator space, i.e. as a subspace of a $C^*$-algebra, if and only if these norms satisfy the properties in (2.1). We note that, if $\dim(E) > 1$, there would be a lot of non-isomorphic operator space structures on $E$ (see, e.g., [33]).

Given an operator space $E$ and $f = \mathbb{M}_n(E^*)$, the norms

$$\|f\|_n = \sup\left\{\|(f(v))_{(i,k)(j,l)}\| : v \in \mathbb{M}_m(E)_1 \text{ and } m \in \mathbb{N}\right\} \quad ((f(v))_{(i,k)(j,l)} := f_{ij}(v_{kl}) \in \mathbb{M}_{mn}) \tag{2.2}$$

determine a canonical operator space structure on $E^*$, which becomes itself an operator space, called in [2] the *standard dual* of $E$.

For any index set $I$, one can consider the linear space $\mathbb{M}_I(E)$ of the set of $(I \times I)$-matrices with entries in $E$ such that

$$\|v\|_{\mathbb{M}_I(E)} := \sup_{\Delta} \|v^{\Delta}\|_{\mathbb{M}_{\Delta}(E)} < +\infty$$

where $\Delta$ denotes any finite subset of $I$. For each index set $I$, $\mathbb{M}_I(E)$ is, in a natural way, an operator space via the inclusion $\mathbb{M}_I(E) \subset \mathfrak{B}(\mathcal{H}\otimes\ell^2(I))$ if $E$ is realized as a subspace of $\mathfrak{B}(\mathcal{H})$. Of interest is also the definition of $\mathbb{K}_I(E)$ as the set of those elements $v \in \mathbb{M}_I(E)$ with $v = \lim_{\Delta} v^{\Delta}$. Obviously, $\mathbb{M}_I(\mathbb{C}) \equiv \mathbb{M}_I = \mathfrak{B}(\ell^2(I))$ and $\mathbb{K}_I(\mathbb{C}) \equiv \mathbb{K}_I = \mathfrak{K}(\ell^2(I))$, the set of all compact operators on $\ell^2(I)$.

Given an index set $I$, we can define as usual a map $\mathcal{X} : \mathbb{M}(E^*) \to \mathfrak{M}(E, \mathbb{M}_I)$, which is a complete isomorphism, by

$$(\mathcal{X}(f)(v))_{ij} = f_{ij}(v). \tag{2.3}$$

Moreover, if $f \in \mathbb{K}(E^*)$, then $\mathcal{X}(f)$ is the norm limit of finite rank maps so $\mathcal{X}(\mathbb{K}_I(E^*)) \subset \mathfrak{K}(E, \mathbb{K}_I)$ and therefore $\mathcal{X}(f)(v) \in \mathbb{K}_I$ for each $v \in E$ (see [16: p. 172]).

Among the operator space structures on a Hilbert space $H$, we recall ([17]) the *row* and *column* Hilbert spaces $H_r$, $H_c$ respectively, together with the self-dual $OH$ structure introduced by Pisier [36, 39] by interpolation

$$OH(I) := (H_c, H_r)_{1/2}$$

where the cardinality of the index set $I$ is equal to the (Hilbert) dimension of $H$. Using the results contained in [39: Section 1], it is easy to show that (see [22]) the norm of an element $x \in \mathbb{M}_n(OH(I))$ can be computed as

$$\|x\|_{\mathbb{M}_n(OH(I))} = \|(x_{ij}, x_{kl})\|^{1/2}_{\mathbb{M}_{n^2}}. \tag{2.4}$$

Moreover, a non-commutative version of the Cauchy-Schwarz inequality can be also proved ([22] and [39: Section 1]). Namely, if $x \in \mathbb{M}_m(OH(I))$ and $y \in \mathbb{M}_n(OH(I))$, then

$$\|(x_{ij}, y_{kl})\|_{\mathbb{M}_{mn}} \leq \|x\|_{\mathbb{M}_m(OH(I))}\|y\|_{\mathbb{M}_n(OH(I))}. \tag{2.5}$$

In the Formulae (2.4) and (2.5) the entries of the numerical matrices are as those given in (2.2).

**2.2 Tensor products between operator spaces.** Let $E$, $F$ be operator spaces, one can form other tensor products between $E$ and $F$ [16, 17]. We recall the (operator) *projective* tensor product and the *spatial* tensor product denoted respectively by $E \otimes_\wedge F$, $E \otimes_{min} F$. The *Haagerup* tensor product $E \otimes_h F$ is also of particular interest in the general theory of operator spaces.

The projective tensor product allows one to describe the predual of a $W^*$-tensor product of von Neumann algebras in terms of their preduals. Namely, let $N$ and $M$ be $W^*$-algebras. Then the predual $(N \overline{\otimes} M)_*$ is completely isomorphic to the projective tensor product $N_* \otimes_\wedge M_*$. The detailed proof of the above results can be found in [16: Section 3].

**2.3 Metrically nuclear maps.** The class of *metrically nuclear maps* $\mathfrak{N}(E, F)$ between operator spaces $E$ and $F$ has been introduced and studied in [18]. It is defined as

$$\mathfrak{N}(E, F) = E^* \otimes_\wedge F / \ker \mathcal{X}$$

where $\mathcal{X}$ is the map (2.3) which is, in this case, a complete quotient map. The metrically nuclear norm is just the quotient one (see [21: Theorem 2.3]). Another (more concrete) description of metrically nuclear operators has been given in [21] at the same time and independently. Moreover, also a geometrical characterization (Definition 2.6) has been presented in [21]. All spaces $\mathfrak{N}(E, F)$ are themselves operator spaces which are complete if the range space $F$ is complete. Moreover, the metrically nuclear maps satisfy the ideal property (see [21: Proposition 2.4]).

**2.4 Non-commutative vector-valued $L^p$-spaces.** A quantized version of vector-valued spaces of functions has been introduced and studied by Pisier in [38, 40]. The vector-valued non-commutative $L^p$-spaces are defined by interpolation as

$$S_p[H, E] = \big(S_\infty(H) \otimes_{min} E, S_1(H) \otimes_\wedge E\big)_{1/p}$$

where $S_p(H)$ denotes the Schatten class of order $p$ on $H$. If the Hilbert space $H$ is kept fixed, we always write $S_p[E]$ instead of $S_p[H, E]$.

We conclude with a result, quite similar to that contained in [16: Proposition 3.1], which will be useful in the sequel. Let $H$ be a Hilbert space of (Hilbert) dimension given by the index set $I$. Making the identification $H \equiv \ell^2(I)$ we get the following

**Proposition 1.** *An element $u$ in $S_p[H, E] \subset \mathbb{M}_I(E)$ satisfies $\|u\|_{S_p[H,E]} < 1$ if and only if there exist elements $a, b \in S_{2p}(H) \subset \mathbb{M}_I$ with $\|a\|_{S_{2p}(H)} = \|b\|_{S_{2p}(H)} = 1$, and an element $v \in \mathbb{M}_I(E)$ with $\|v\|_{\mathbb{M}_I(E)} < 1$ such that*

$$u = avb.$$

*Furthermore, one can choose $v \in \mathbb{K}_I(E)$.*

**Proof.** By [38: Théorème 2], it is enough to show only the "if"- part of the statement. Suppose that $u \in \mathbb{M}_I(E)$ can be written as $u = avb$ as above, and let $\varepsilon > 0$ be fixed. Then there exists $F(\varepsilon) \subset I$ such that

$$\left.\begin{array}{l} \|a^{F_1} - a^{F_2}\|_{S_{2p}[E]} \\ \|b^{G_1} - b^{G_2}\|_{S_{2p}[E]} \end{array}\right\} \le \frac{\varepsilon}{3\|v\|_{\mathbb{M}_I(E)}}$$

whenever $F_1, F_2, G_1, G_2 \supset F(\varepsilon)$ are finite subsets of $I$. Considering finite subsets $F, G, \widehat{F}, \widehat{G} \subset I$, we get

$$\begin{aligned}\|a^F v b^G - a^{\widehat{F}} v b^{\widehat{G}}\|_{S_p[E]} \\ \leq \|(a^F - a^{F\wedge\widehat{F}}) v b^G\|_{S_p[E]} \\ + \|(a^{\widehat{F}} - a^{F\wedge\widehat{F}}) v b^{\widehat{G}}\|_{S_p[E]} + \|a^{F\wedge\widehat{F}} v (b^{G\vee\widehat{G}} - b^{G\wedge\widehat{G}})\|_{S_p[E]}.\end{aligned}$$

Now, if $F, G, \widehat{F}, \widehat{G} \supset F(\varepsilon)$, we obtain

$$\|a^F v b^G - a^{\widehat{F}} v b^{\widehat{G}}\|_{S_p[E]} \leq \varepsilon.$$

Then $\{a^F v b^G\}$, with $F, G \subset I$ finite subsets, is a Cauchy net in $S_p[E]$ which converges to an element of $S_p[E]$ and must coincide with $u$ ∎

## 3. $p$-nuclear maps

In this section we study the basic properties of classes $\mathfrak{N}_p(E,F) \subset \mathfrak{K}(E,F)$, $1 \leq p < +\infty$, of linear maps between operator spaces $E$ and $F$ which are limits of finite rank maps. These maps, called *p-nuclear maps*, were firstly considered in [28] in connection with the general theory of factorization and were defined through operators arising by the Pisier non-commutative vector-valued $L^p$-spaces $S_p[H,E]$. Although the case $p = +\infty$ seems to present no complications, for simplicity we deal only with the cases $1 \leq p < +\infty$.

In the sequel we indicate with $\underline{x}$ any element of $\mathbb{C}^n$ ($n$ any integer). If $\{a_i\}_{i=1}^n \subset E$, we denote the numerical sequence $\{\|a_i\|\}_{i=1}^n$ simply by $\underline{a}$. For any $1 \leq p \leq +\infty$, as usual, $q = \frac{p}{p-1}$ is the conjugate exponent of $p$ (where $q = +\infty$ is the conjugate exponent of $p = 1$).

We start with an elementary lemma whose proof is left to the reader [1].

**Lemma 1.** *If $1 < p < +\infty$ and $r$, $s$ are positive real numbers, then*

$$rs \leq \frac{1}{p} r^p + \frac{1}{q} s^q.$$

Now we consider the situation where $1 \leq p < +\infty$ and $q$ is the conjugate exponent of $p$.

Let $A_i \in \mathfrak{M}(E, S_p^*)$ $(i = 1,2)$ be completely bounded maps and consider the linear map between $E$ and $S_p^*$ given by $Ax = A_1 x \oplus A_2 x$ (where we have kept fixed any identification $H \equiv \ell^2 \cong H \oplus H$). At the same way, let $b_i \in S_p[E]$ $(i = 1,2)$ and consider the element $b \in \mathrm{M}_\infty(E)$ given by $b = b_1 \oplus b_2$.

[1] We are indebted to M.Junge for a suggestion relative to this point.

**Lemma 2.** *In the above situation we get:*

**(i)** $A \in \mathfrak{M}(E, S_p^*)$ *with* $\|A\|_{cb} \leq \|\underline{A}\|_q$.

**(ii)** $b \in S_p[E]$ *with* $\|b\|_{S_p[E]} = \|\underline{b}\|_p$.

**Proof.** The case with $p = 1$ in statement $(i)$ is easy and is left to the reader. For the other cases in statement $(i)$, taking into account [38: Théorème 2] and [40: Corollary 1.3], we compute, for $u, v \in (\mathbb{M}_n)_1$, $\|x\|_{\mathbb{M}_n(E)} < 1$ and $n$ integer,

$$\begin{aligned}\|\widetilde{M}_{u,v}Ax\|^q_{S_q((H\oplus H)\otimes\mathbb{C}^n)} &= \|\widetilde{M}_{u,v}A_1x\|^q_{S_q(H\otimes\mathbb{C}^n)} + \|\widetilde{M}_{u,v}A_2x\|^q_{S_q(H\otimes\mathbb{C}^n)} \\ &\leq \|A_1x\|^q_{\mathbb{M}_n(S_q(H))} + \|A_2x\|^q_{\mathbb{M}_n(S_q(H))} \\ &\leq \|A_1\|^q_{cb} + \|A_2\|^q_{cb}.\end{aligned}$$

Taking the supremun on the left, first on the unit balls $(\mathbb{M}_n)_1$, $E_1$ and then on $n \in \mathbb{N}$, we obtain the assertion again by [38: Théorème 2]. The proof of part (ii) follows at the same way ∎

Let any identification $\ell^2 \equiv H \cong \oplus_{i=1}^N H$ be fixed and $1 \leq p < +\infty$. Suppose that we have a sequence $\{A_i\}_{i=1}^N \subset \mathfrak{M}(S_p(H), E)$. We define a linear operator $A : S_p(\oplus_{i=1}^N H) \to E$ as follows. Let $x \in S_p(\oplus_{i=1}^N H)$. First we cut the off-diagonal part of $x$. Then we define

$$Ax = \sum_{i=1}^N A_i P_i x P_i \tag{3.1}$$

where $P_j$ is the orthogonal projection corresponding to the $j$-subspace in the direct sum $\oplus_{i=1}^N H$.

We have the following

**Lemma 3.** *The map* $A : S_p(\oplus_{i=1}^N H) \to E$ *defined as above is completely bounded and*

$$\|A\|_{cb} \leq \|\underline{A}\|_q$$

*where* $q$ *is the conjugate exponent of* $p$.

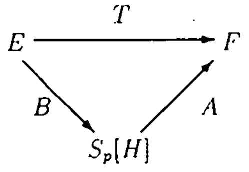
**Proof.** It is easy to note that $A$ is bounded as an operator between $S_p(\oplus_{i=1}^N H)$ and $E$. Hence, in order to compute its completely bounded norm, it is enough to pass to the transpose map $A^* : E^* \to S_p(\oplus_{i=1}^N H)^*$ which is of the same type as those described in Lemma 2. So, after a similar calculation, we get

$$\|A\|_{cb} \equiv \|A^*\|_{cb} \leq \|\underline{A}^*\|_q \equiv \|\underline{A}\|_q$$

which is the assertion ∎

Now we are ready to define the classes of $p$-nuclear maps between operator spaces.

**Definition 1** [28: Definition 3.1]. Let $E$ and $F$ be operator spaces and $1 \leq p < +\infty$. A linear map $T : E \to F$ will be called *p-nuclear* if there exists a Hilbert space $H$ and elements $b \in S_p[H, E^*]$ and $A \in \mathfrak{M}(S_p(H), F)$ such that $T$ factorizes according to




where $B = \mathcal{X}(b) \in \mathfrak{M}(E, S_p(H))$ (see [40: Lemma 3.15]). We also define, for a $p$-nuclear map $T$,

$$\nu_p(T) = \inf \left\{ \|A\|_{cb} \|b\|_{S_p[H,E^*]} \right\}$$

where the infimum is taken over all factorizations for $T$ as above. The class of all $p$-nuclear maps between $E$ and $F$ will be denoted by $\mathfrak{N}_p(E, F)$.

**Remark 1.** As the linear map $\mathcal{X}(b)$ is norm limit of finite rank maps, hence has separable range, without loss of generality we can reduce ourselves in Definition 1 to consider $H \equiv \ell^2$ and omit the dependance on $H$ in the sequel if it is not otherwise specified.

**Remark 2.** We have $\mathfrak{N}_p(E, F) \subset \mathfrak{K}(E, F)$ as $\mathcal{X}(\mathbb{K}_I(E^*)) \subset \mathfrak{K}(E, \mathbb{K}_I)$ where, as usual, $\mathcal{X}$ is the map defined in (2.3).

Now we show that the classes $\mathfrak{N}_p(E, F)$ are all normed linear spaces.

**Proposition 2.** *$(\mathfrak{N}_p(E, F), \nu_p)$ is a normed space for each $1 \leq p < +\infty$.*

**Proof.** If $T = A\mathcal{X}(b) \in \mathfrak{N}_p(E, F)$, one has $\|T\| \leq \|A\|_{cb} \|b\|_{S_p[E^*]}$ and, taking the infimum on the right, one obtains $\|T\| \leq \nu_p(T)$. So $\nu_p(T)$ is non-degenerate. Now we have only to verify the triangle inequality for $\nu_p$. We start treating the cases $p > 1$. Let $T_i \in \mathfrak{N}_p(E, F)$ $(i = 1, 2)$ and $\varepsilon > 0$ be fixed and choose $A_i$ and $b_i$ such that

$$\|A_i\|_{cb} = (\nu_p(T_i)(1 + \varepsilon))^{\frac{1}{q}},$$
$$\|b_i\|_{S_p[E^*]} = (\nu_p(T_i)(1 + \varepsilon))^{\frac{1}{p}}.$$

We consider the linear map $A : S_p \to F$ defined as in (3.1) where any decomposition of $\ell^2 \equiv H \cong H \oplus H$ is kept fixed. We also put, under this decomposition of $H$, $b := b_1 \oplus b_2$. Applying Lemmas 2 and 3, we obtain $\|A\|_{cb} \leq \|\underline{A}\|_q$ and $\|b\|_{S_p[E^*]} = \|\underline{b}\|_p$, hence $T_1 + T_2 = A\mathcal{X}(b)$. By Lemma 1 we get

$$\begin{aligned} \nu_p(T_1 + T_2) &\leq \frac{1}{p}\left(\|b_1\|^p_{S_p[E^*]} + \|b_2\|^p_{S_p[E^*]}\right) + \frac{1}{q}\left(\|A_1\|^q_{cb} + \|A_2\|^q_{cb}\right) \\ &\leq (1 + \varepsilon)\big(\nu_p(T_1) + \nu_p(T_2)\big) \end{aligned}$$

and the statement follows as $\varepsilon$ is arbitrary. Now we consider the case $p = 1$ and suppose that $\nu_1(T_1) \leq \nu_1(T_2)$. For each $s > 1$ we choose $A_i^{(s)}$ and $b_i^{(s)}$ such that

$$\|A_i^{(s)}\|_{cb} = (\nu_1(T_i)(1 + \varepsilon))^{\frac{1}{r}}$$
$$\|b_i^{(s)}\|_{S_1[E^*]} = (\nu_1(T_i)(1 + \varepsilon))^{\frac{1}{s}}$$

where $\frac{1}{s} + \frac{1}{t} = 1$. Again by Lemma 1, we get

$$\nu_1(T_1 + T_2) \leq \frac{1}{s}(\nu_1(T_1)^{\frac{1}{s}} + \nu_1(T_2)^{\frac{1}{s}})^s(1+\varepsilon) + \frac{1}{t}\nu_1(T_2)(1+\varepsilon).$$

Noticing that the above inequality holds for each $s > 1$, we can take the limit as $s$ goes to 1 (so $t'$ automatically goes to $+\infty$) and obtain

$$\nu_1(T_1 + T_2) \leq (\nu_1(T_1) + \nu_1(T_2))(1+\varepsilon)$$

which is the assertion ∎

Actually, $(\mathfrak{N}_p(E,F), \nu_2)$ are all Banach spaces of linear maps between $E$ and $F$ (see below). Furthermore, one has for $T \in \mathfrak{N}_p(E,F)$ a summation

$$T = \sum_{i\in\mathbf{N}} f_i(\cdot)y_i$$

where $\{f_i\} \subset E^*$ and $\{y_i\} \subset F$. It is easy to see that such a summation is unconditionally convergent in the norm topology of $\mathfrak{B}(E,F)$. Moreover, if $T \in \mathfrak{N}_p(E,F)$, then $T$ is completely bounded (see Remark 2).

## 4. Ideals between operator spaces

As we have already mentioned, one can point out the properties which characterize classes of operator ideals also in non-commutative functional analysis, that is in operator spaces setting. Examples of such operator ideals have been studied in [17 - 19, 21, 37, 38, 40] where it has been shown that some of these spaces of maps have also a natural operator space structure themselves. In this section we start with these definitions and show that the $p$-nuclear maps $\mathfrak{N}_p$ are examples of Banach operator ideals in the setting of non-commutative functional analysis. In order to do this, we follow the strategy of the celebrated monograph [35] of Pietsch.

We indicate with $\mathfrak{M}$ the classes of all completely bounded maps. Namely, $\mathfrak{M}(E,F)$ is just the space of completely bounded maps between the operator spaces $E$ and $F$.

The following definition is our startpoint.

**Definition 2.** A subclass $\mathfrak{J} \subset \mathfrak{M}$ will be said an *operator ideal* if

**(i)** $I_1 \in \mathfrak{J}$ where 1 is the 1-dimensional space,

**(ii)** $\mathfrak{J}(E,F)$ is a linear space for each couple of operator spaces $E$ and $F$,

**(iii)** $\mathfrak{M}\mathfrak{J}\mathfrak{M} \subset \mathfrak{J}$ (ideal property).

Moreover, let there exist quasi-norms $\varphi$ [30: Subsection 15.10] which satisfy the following conditions:

**(a)** $\varphi(I_1) = 1$.

**(b)** $\varphi(S+T) \leq \kappa(\varphi(S) + \varphi(T)) \quad (\kappa \geq 1)$.

(c) If $T \in \mathfrak{M}(E_0, E), S \in \mathfrak{I}(E, F), R \in \mathfrak{M}(F, F_0)$, then

$$\varphi(RST) \le \|R\|_{cb}\varphi(S)\|T\|_{cb}$$

with each $(\mathfrak{I}(E, F), \varphi)$ complete as topological vector space.

Then we call $(\mathfrak{I}, \varphi)$ a *quasi normed* or *Banach* operator ideal according to $\kappa > 1$ or $\kappa = 1$, respectively.

Now we show that the classes $(\mathfrak{N}_p, \nu_p)$ of $p$-nuclear maps are Banach operator ideals. For the reader's convenience we split up the proof in two propositions.

**Proposition 3.** *Let $E_0$, $E$, $F$ and $F_0$ be operator spaces, $T: E_0 \to E$, $S: E \to F$ and $R: F \to F_0$ linear maps. If $T \in \mathfrak{M}(E_0, E)$, $S \in \mathfrak{N}_p(E, F)$ and $R \in \mathfrak{M}(F, F_0)$, then $RST \in \mathfrak{N}_p(E_0, F_0)$ and*

$$\nu_p(RST) \le \|R\|_{cb}\nu_p(S)\|T\|_{cb}.$$

**Proof.** If $S \in \mathfrak{N}_p(E, F)$ and $\varepsilon > 0$, by Proposition 1 there exist $a, b \in (S_{2p})_1$, $f \in \mathbb{M}_\infty(E^*)$ and $A \in \mathfrak{M}(S_p, F)_1$ such that $\|f\|_{cb} \le \nu_p(S) + \frac{\varepsilon}{\|R\|_{cb}\|T\|_{cb}}$ and $S = A\mathcal{X}(af(\cdot)b)$, ($\mathcal{X}$ is the map given in (2.3)). Now

$$RSTx = RAa(f(Tx)b$$

where $f \circ T \in \mathbb{M}_\infty(E_0^*)$ and $\|f \circ T\|_{cb} \le \|f\|_{cb}\|T\|_{cb}$. Then we get $RST \in \mathfrak{N}_p(E_0, F_0)$ and

$$\nu_p(RST) \le \|R\|_{cb}\|f \circ T\|_{cb} \le \|R\|_{cb}\|f\|_{cb}\|T\|_{cb} \le \|R\|_{cb}\nu_p(S)\|T\|_{cb} + \varepsilon$$

and the proof follows ∎

**Proposition 4.** *$(\mathfrak{N}_p(E, F)), \nu_p)$ is complete as normed space.*

**Proof.** We have already proved in Proposition 2 that $(\mathfrak{N}_p(E, F)), \nu_p)$ is a normed space for each $p$; so it is enough to show that any absolutely summable sequence $\{T_i\} \subset (\mathfrak{N}_p(E, F), \nu_p)$ is summable in $(\mathfrak{N}_p(E, F), \nu_p)$. We begin with the cases $p > 1$. Let $\{T_i\}$ be an absolutely summable sequence (i.e. $\sum_{i=1}^{+\infty} \nu_p(T_i) < +\infty$) where $T_i = A_i\mathcal{X}(b_i)$ for some sequences $\{A_i\} \subset \mathfrak{M}(S_p, F)$ and $\{b_i\} \subset S_p[E^*]$ such that

$$\|A_i\|_{cb} \le \left(\nu_p(T_i) + \frac{\varepsilon}{2^{i+1}}\right)^{\frac{p-1}{p}} \qquad \text{and} \qquad \|b_i\|_{S_p[E^*]} \le \left(\nu_p(T_i) + \frac{\varepsilon}{2^{i+1}}\right)^{\frac{1}{p}}$$

and $H \cong \ell^2$ as usual. Now we fix any identification $H \equiv \ell^2 \cong \oplus_{i=1}^{+\infty} H$ and define

$$b = \bigoplus_{i=1}^{+\infty} b_i \in \mathbb{M}_\infty(E^*).$$

It is easy to show that $b \in S_p[\oplus_{i=1}^{+\infty} H, E^*]$ as norm limit of the sequence

$$\sigma_N = b_1 \oplus \ldots \oplus b_N \oplus 0 \oplus \ldots$$

thanks to

$$\|\sigma_N\|^p_{S_p[\oplus_{i=1}^{+\infty} H, E^*]} = \sum_{i=1}^{N} \|b_i\|^p_{S_p[H,E^*]} \leq \sum_{i=1}^{N} \nu_p(T_i) + \frac{\varepsilon}{2} \leq \sum_{i=1}^{+\infty} \nu_p(T_i) + \frac{\varepsilon}{2}$$

(see [40: Corollary 1.3]). Moreover, we can define linear maps $A_N : S_p(\oplus_{i=1}^N H) \to F$ as in (3.1) which are completely bounded and satisfy, by Lemma 3,

$$\|A_N\|^q_{cb} \leq \sum_{i=1}^{+\infty} \|A_i\|^q_{cb} \leq \sum_{i=1}^{+\infty} \nu_p(T_i) + \frac{\varepsilon}{2}.$$

By these considerations, one can easily show that the direct limit $\varinjlim A_N$ defines a bounded map on $\cup_N S_p(\oplus_{i=1}^N H)$ which uniquely extends to a completely bounded map $A \in \mathfrak{M}(S_p(\oplus_{i=1}^{+\infty} H), F)$ as $\cup_N S_p(\oplus_{i=1}^N H)$ is dense in $S_p(\oplus_{i=1}^{+\infty} H)$. Then we have

$$b \in S_p[\oplus_{i=1}^{+\infty} H, E^*] \cong S_p[H, E^*]$$
$$A \in \mathfrak{M}(S_p(\oplus_{i=1}^{+\infty} H), F) \cong \mathfrak{M}(S_p(H), F).$$

Finally, if one defines $T = A\mathcal{X}(b)$, then $T \in \mathfrak{N}_p(E, F)$ and

$$\nu_p(T - T_N) \leq \left( \sum_{i=N+1}^{+\infty} \nu_p(T_i) + \frac{\varepsilon}{2} \right)^{\frac{p-1}{p}} \left( \sum_{i=N+1}^{+\infty} \nu_p(T_i) + \frac{\varepsilon}{2} \right)^{\frac{1}{p}} < \varepsilon$$

if $N$ is big enough; that is $\{T_n\}$ is summable in $\mathfrak{N}_p(E, F)$.

For the case $p = 1$ we choose

$$\|A_i\|_{cb} \leq 1 \qquad \text{and} \qquad \|b_i\|_{S_1[E^*]} \leq \nu_1(T_i) + \frac{\varepsilon}{2^{i+1}}.$$

As in the previous case, we can construct $\sigma_N$, $b \in S_1[H, E^*]$ and $A_N$, $A \in \mathfrak{M}(S_1(H), F)$ such that $T_N := A_N \mathcal{X}(\sigma_N)$ and $T := A\mathcal{X}(b)$ are all 1-nuclear maps. Furthermore

$$\nu_1(T - T_N) \leq \sum_{i=N+1}^{+\infty} \nu_1(T_i) + \frac{\varepsilon}{2} < \varepsilon$$

if $N$ is big enough. The proof is now complete ∎

Summarizing we have the following

**Theorem 1.** $(\mathfrak{N}_p, \nu_p)$ $(1 \leq p < +\infty)$ *are Banach operator ideals.*

**Proof.** The proof immediately follows collecting the results contained in Section 3 and in Propositions 3 and 4 ∎

## 5. A geometrical description

Analogously to metrically nuclear operators (see [21]), we give a suitable geometrical description for the range of a $p$-nuclear injective map.

We start with an absolutely convex set $Q$ in an operator space $E$ and indicate with $V$ its algebraic span. Consider a sequence $\mathcal{Q} \equiv \{Q_n\}$ of sets with the following properties:

**(i)** $Q_1 \equiv Q$ and every $Q_n$ is an absolutely convex absorbing set of $\mathbb{M}_n(V)$ with $Q_n \subset \mathbb{M}_n(Q)$,

**(ii)** $Q_{m+n} \cap (\mathbb{M}_m(V) \oplus \mathbb{M}_n(V)) = Q_m \oplus Q_n$,

**(iii)** For $\lambda > 0$ and $x \in Q_n$, $x \in \lambda Q_n$ implies $bx \in \lambda Q_n$ and $xb \in \lambda Q_n$ for each numerical matrix $b \in (\mathbb{M}_n)_1$.

We say that a (possibly) infinite matrix $f$ with entries in the algebraic dual $V'$ of $V$ has *finite $\mathcal{Q}$-norm* if

$$\|f\|_{\mathcal{Q}} \equiv \sup\left\{\|f^{\Delta}(q)\| : q \in Q_n,\ n \in \mathbb{N},\ \Delta\right\} < +\infty$$

where $f^{\Delta}$ indicates an arbitrary finite truncation relative to the finite set $\Delta$; the enumeration of the entries of the numerical matrix $f^{\Delta}(q)$ is as that given in (2.2).

**Definition 3.** An absolutely convex set $Q \subset E$ is said to be $(p, \mathcal{Q})$-*nuclear* $(1 \leq p < +\infty$ and $\mathcal{Q}$ a fixed sequence as above with $Q_1 = Q)$ if there exist matrices $\alpha, \beta \in S_{2p}$ and a (possible infinite) matrix of linear functionals $f \in \mathbb{M}_\infty(V')$ with $\|f\|_{\mathcal{Q}} < +\infty$ such that, if $x \in Q_n$, one has

$$\|x\|_{\mathbb{M}_n(E)} \leq \|\alpha f(x)\beta\|_{\mathbb{M}_n(S_p)}. \tag{5.1}$$

We omit the dependence on $\mathcal{Q}$ if it causes no confusion.

One can easily see that a $(p, \mathcal{Q})$-nuclear set is relatively compact, hence bounded in the norm topology of $E$ and therefore $V$, together with the Minkowski norms determinated by the $Q_n$'s on $\mathbb{M}_n(V)$, is a (not necessarily complete) operator space. As in the metrically nuclear case contained in [21: Section 2] and the case of completely summing maps described in [40: Remark 3.7] one can reinterpret the above definition as a factorization condition.

**Proposition 5.** *Let $E \subset \mathfrak{B}(\mathcal{H})$ be a (concrete) operator space, $Q \subset E$ a $(p, \mathcal{Q})$-nuclear set and $V$ its algebraic span. Then the canonical immersion $V \overset{i}{\hookrightarrow} E$ is a $p$-nuclear map when $V$ is equipped with the operator space structure determined by the sequence $\mathcal{Q}$*

**(i)** *if there exists a completely bounded projection $P : \mathfrak{B}(\mathcal{H}) \to E$ when $p \neq 2$*

**(ii)** *without any other condition on $E$ if $p = 2$.*

**Proof.** According to Definition 3, there exist matrices $\alpha, \beta \in S_{2p}$ and a matrix $f$ of linear functionals with $\|f\|_{\mathcal{Q}} < +\infty$ satisfying the property described above. We define $b = \alpha f \beta$, so $b \in S_2[V^*]$. We consider $W := \overline{\alpha f(V)\beta}^{S_p}$ and define on $W$ a linear map $A : W \to E$ as $Ax = v$ if $x = \alpha f(v)\beta$. This map extends firstly to all of $W$

by continuity, and then to a completely bounded map between $S_p$ and $\mathfrak{B}(\mathcal{H})$ by the celebrated Arveson-Wittstock-Hahn-Banach Theorem [45]. So we obtain $i = PA\mathcal{X}(b)$ which is $p$-nuclear as $PA$ is a completely bounded map between $S_2$ and $E$ (see (5.1)). In the case $p = 2$ we can extend $A$ to all of $S_2$ if one define $Ax = 0$ on $W^{\perp}$. Being $S_2 \cong OH$ a homogeneous Hilbertian operator space (see [39: Proposition 1.5]), $A$ is completely bounded and the proof is now complete ∎

We now consider an injective completely bounded operator $T : E \to F$ and the sequence $\mathcal{Q}_T$ given by

$$\mathcal{Q}_T = \{T_n(\mathbb{M}_n(E)_1)\}_{n\in\mathbb{N}}.$$

For such sequences the properties (i) - (iii) in the beginning are automatically satisfied, and, if $T(E_1)$ is a $(p, \mathcal{Q}_T)$-nuclear set, we call it simply $(p, T)$-nuclear and indicate the $\mathcal{Q}_T$-norm of a matrix of functional $f$ by $\|f\|_T$.

As it happens in some interesting well-known cases (compact operators, nuclear and metrically nuclear maps), also for the class of $p$-nuclear maps we have a description in terms of geometrical properties (i.e. the shape) of the range of such maps.

**Proposition 6.** *Let $E$ and $F$ be operator spaces with $F \subset \mathfrak{B}(\mathcal{H})$ and $T : E \to F$ a completely bounded injective operator. Then $T \in \mathfrak{N}_p(E, F)$ if and only if $T(E_1)$ is a $(p, T)$-nuclear set in $F$ when*

**(i)** *there exists a completely bounded projection $P : \mathfrak{B}(\mathcal{H}) \to F$ when $p \neq 2$,*

**(ii)** *without any other condition on $F$ if $p = 2$.*

**Proof.** It is easy to verify that, if $T \in \mathfrak{N}_p(E, F)$ is injective, one can write for $T$ a decomposition

$$T = A\mathcal{X}(\alpha f(\cdot)\beta).$$

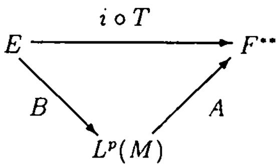
Then $\alpha$, $f \circ T^{-1}$ and $\beta$ allow us to say that $T(E_1)$ is $(p, T)$-nuclear (here we have chosen $A$ such that $\|A\|_{cb} = 1$). Conversely, if $T(E_1)$ is a $(p, T)$-nuclear set in $F$ with $\alpha$, $\beta$ and $f$ as in Definition 3, we can consider the matrix $f \circ T \in \mathbb{M}_\infty(E^*)$ and define $A = S_p \to F$ as in the proof of Proposition 5. Then we obtain for $T$ the factorization $T = A\mathcal{X}(b)$ where $b := \alpha(f \circ T)\beta \in S_p[E^*]$ ∎

As in the case relative to metrically nuclear maps [21], the definition of a $(p, \mathcal{Q})$-nuclear set may appear rather involved. This is due to the well-known fact that the inclusion $\mathbb{M}_n(V)_1 \subset \mathbb{M}_n(V_1)$ is strict in general. But, for an injective completely bounded operator $T$ as above, a $(p, T)$-nuclear set $T(V_1)$ is intrinsically defined in terms of $T$.

**Remark 3.** An interesting example in the cases with $p \neq 2$ for which Proposition 6 can be applied is when the range space is an injective $C^*$-algebra.

A characterization of completely $p$-summing maps in terms of a factorization condition is given in [40: Remark 3.7] and involve ultrapowers [26]. As ultrapowers of $W^*$-algebras of the same type may produce $W^*$-algebras of other type in general (see [9: Section II], for similar questions, or [35: Section 19] for the commutative case), one can argue that in general $\mathfrak{N}_p(E, F) \neq \Pi_p(E, F)$, the completely summing maps considered by Pisier in [40], even if we always have $\mathfrak{N}_p(E, F) \subset \Pi_p(E, F)$. It would be of interest to understand if (and when) the above inclusion is in fact an equality.

Other cases which could involve ultrapowers are the definition of factorable maps in the setting of quantized functional analysis, in a similar manner as that considered in [32]. Such a kind of factorable maps might be the true quantized counterpart of factorable maps of the Banach spaces case (see [32, 34, 35]; see also [19] for some related questions relative to the quantized case). Following [32, 35], one could consider those maps which factor according to the commutative diagram




where $i$ is the canonical completely isometric immersion of $F$ in $F^{**}$ [2] and $M$ is a $W^*$-algebra whose non-commutative measure spaces $L^p(M)$ [31] should be equipped with a suitable operator space structure. The maps $A$ and $B$ in the above commutative diagram should be completely bounded. Indeed, it has been shown in [23] that there exists only one canonical (i.e. obtained by interpolation between $M$ and its predual $M_*$) operator space structure on the non-commutative $L^p$ space $L^p(M)$ when $M$ and $p$ are kept fixed.

A complete analysis of the above framework, specially the study of quantized counterparts of results of Banach space theory, would be desirable. We refer the reader to the monoghraph [28] for a complete exposition of the whole matter and for very interesting results concerning the general theory of factorization.

## 6. The split property for inclusions of $W^*$-algebras. An application to Quantum Field Theory

This final section is devoted to some applications of $p$-nuclear maps to the theory of the split inclusions of $W^*$-algebras. Some natural applications to Quantum Field Theory will be also discussed.

We suppose that all $W^*$-algebras considered here have a separable predual. For standard results about the theory of $W^*$-algebras see, e.g. [42].

An inclusion $N \subset M$ of $W^*$-algebras is said to be *split* if there exists a type $I$ interpolating $W^*$-factor $F$, that is $N \subset F \subset M$. The split property has been intensively studied [5, 11, 12] in the last years for natural applications to Quantum Field Theory [6 - 8]. In [5], canonical non-commutative embeddings $\Phi_i : M \to L^i(M)$ $(i = 1,2)$ are considered. These embeddings are constructed, by a cyclic and separating (i.e. standard) vector $\Omega \in L^2(M)$ for $M$, in the following way:

$$\begin{aligned} \Phi_1 &: a \in M \to (\,\cdot\,\Omega, Ja\Omega) \in L^1(M) \\ \Phi_2 &: a \in M \to \Delta^{1/4}a\Omega \in L^2(M). \end{aligned} \tag{6.1}$$

The $W^*$-algebra $M$ is supposed to act standardly on the Hilbert space $L^2(M)$ and, in the above formulae, $\Delta$ and $J$ are the Tomita operators of $M$ relative to the standard vector $\Omega$. In this way the split property is analyzed considering the nuclear properties of the restrictions of $\Phi_i$ $(i = 1,2)$ to the subalgebra $N$. The nuclear property and its connection with the split property has interesting applications in Quantum Field Theory (see [6 - 8]). Following this approach, in [21] the split property has been exactly characterized in terms of the $L^1$-embedding $\Phi_1$ constructed by a fixed standard vector for $M$ as above. The characterization of the split property in terms of the other canonical $L^2$-embedding $\Phi_2$ is also an interesting problem. Here we give some partial results (involving the theory of $p$-nuclear maps) which go towards a complete characterization of the split property in terms of the $L^2$-embedding. The results relative to the $L^1$-embedding contained in [21] are also reported for the sake of completeness.

**Theorem 2.** *Let $N \subset M$ be an inclusion of $W^*$-factors with separable preduals and $\omega \in M_*$ a faithful state. Let $\Phi_i : M \to L^i(M)$ $(i = 1,2)$ be the embeddings associated to the state $\omega$ and given in* (6.1). *Consider the following statements:*

**(i)** $\Phi_{2\restriction N} \in \mathfrak{N}_1(N, (L^2(M)))$.

**(ii)** $\Phi_{2\restriction N} \in \mathfrak{N}(N, (L^2(M)))$.

**(iii)** $\Phi_{1\restriction N} \in \mathfrak{N}(N, (L^1(M)))$.

**(iv)** *$N \subset M$ is a split inclusion.*

**(v)** $\Phi_{2\restriction N} \in \mathfrak{N}_2(N, (L^2(M)))$.

*We have the following chain of implications:*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \iff (iv) \Rightarrow (v).$$

In the above Theorem 2 we suppose that $L^i(M)$ $(i = 1,2)$ are endowed with the following operator space structures:

(a) as the predual of $M^\circ$, the opposite algebra of $M$, for $L^1(M)$,

(b) the Pisier self-dual $OH$-structure for $L^2(M)$.

**Proof of Theorem 2.** Some of the above statements are contained in [21]. So we only deal with the remaining ones.

$(i) \Rightarrow (ii)$: If $\Phi_{2\restriction N}$ is 1-nuclear, then it has the form $A\mathcal{X}(b)$ where $b \in S_1[N^*]$ [2] and $A : S_1 \to S_2$ is completely bounded. As $S_1[N^*] \cong N^* \otimes_\wedge S_1$ [40], the assertion now follows by the ideal property of the metrically nuclear operators.

$(ii) \Rightarrow (iii)$: We start considering the linear map $T : OH \to L^1(M)$ given by

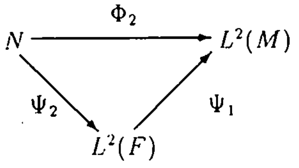
$$(Tx)(b) := (x, \Delta^{1/4} b^* \Omega).$$

We remark that the canonical $L^1$ embedding $\Phi_{1\restriction N}$ of $N$ in $L^1(M)$ is precisely $T\Phi_{2\restriction N}$. As $OH \cong \overline{OH}^*$ [39] and the antilinear identification $b \in M^\circ \to b^* \in M$ is completely bounded, it is easy to show that $T$ is also a completely bounded map by (2.3) and (2.4).

[2] Indeed $b$ can be chosen in $S_1[N_*]$ as $\Phi_2$ is a normal map [21, 22].

The assertion now follows again by the ideal property of the metrically nuclear maps as $OH \cong S_2$ [40].

$(iv) \Rightarrow (v)$: If there exists a type $I$ interpolating factor $F$ between $N$ and $M$, then $\Phi_2$ factors according to




where $\Psi_2$ arises from $S_2[N_*]$ and $\Psi_1$ is bounded (see [5]). As $\Psi_1$ is also completely bounded (see [39: Proposition 1.5]), the proof is now complete ∎

We note that the results summarized in the above Theorem 2 are obtained for inclusions of $W^*$-factors. However, in some interesting cases such those arising from Quantum Field Theory (where the von Neumann algebras of local observables might have a non-trivial centre, see [4]), the above results also holds.

We suppose that the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of von Neumann algebras of local observables of a quantum theory acts on the Hilbert space $\mathcal{H}$ of the vacuum representation and satisfies all typical assumptions (without a priori the split property) of Quantum Field Theory. The vector $\Omega \in \mathcal{H}$ will be any standard vector for the net such as the vacuum vector which is cyclic and separating for each element $\mathfrak{A}(\mathcal{O})$ of $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ (see, e.g., [43]). The following corollary is nothing but the application of the results contained in Theorem 2 to Quantum Field Theory.

**Corollary 1.** *Let $\mathcal{O} \subset \operatorname{int}(\widehat{\mathcal{O}})$ be double cones in the Physical Space-Time and $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ the corresponding inclusion of von Neumann algebras of local observables. Let $\Delta$ be the Modular Operator of $\mathfrak{A}(\widehat{\mathcal{O}})$ relative to the vector $\Omega$. Consider the following statements:*

**(i)** *The set $\{\Delta^{1/4}a\Omega : a \in \mathfrak{A}(\mathcal{O})_1\} \subset \mathcal{H}$ is $\big(1, \Phi_{2\restriction\mathfrak{A}(\mathcal{O})}\big)$-nuclear.*

**(ii)** *The set $\{\Delta^{1/4}a\Omega : a \in \mathfrak{A}(\mathcal{O})_1\} \subset \mathcal{H}$ is $\Phi_{2\restriction\mathfrak{A}(\mathcal{O})}$-decomposable (see [21 : Definition 2.6]).*

**(iii)** *The set $\{(\cdot\, a\Omega, \Omega) : a \in \mathfrak{A}(\mathcal{O})_1\} \subset (\mathfrak{A}(\widehat{\mathcal{O}})')_*$ is $\Phi_{1\restriction\mathfrak{A}(\mathcal{O})}$-decomposable.*

**(iv)** *$\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ is a split inclusion.*

**(v)** *The set $\{\Delta^{1/4}a\Omega : a \in \mathfrak{A}(\mathcal{O})_1\} \subset \mathcal{H}$ is $\big(2, \Phi_{2\restriction\mathfrak{A}(\mathcal{O})}\big)$-nuclear.*

*We have the following chain of implications:*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \iff (iv) \Rightarrow (v).$$

**Proof.** It is enough to show only $(iii) \Rightarrow (iv)$ whose proof is outlined in [5, 21]. If (iii) holds, then $\Phi_{1\restriction\mathfrak{A}(\mathcal{O})}$ is extendible. Hence the map

$$\eta : a \otimes b \in \mathfrak{A}(\mathcal{O}) \otimes \mathfrak{A}(\widehat{\mathcal{O}})' \to ab \in \mathfrak{A}(\mathcal{O}) \vee \mathfrak{A}(\widehat{\mathcal{O}})'$$

extends to a normal homomorphism of $\mathfrak{A}(\mathcal{O})\bar{\otimes}\mathfrak{A}(\hat{\mathcal{O}})'$ onto $\mathfrak{A}(\mathcal{O}) \vee \mathfrak{A}(\hat{\mathcal{O}})'$. But this homomorphism is in fact an isomorphism by an argument exposed in [4: pp. 129 - 130]. Moreover, as $\mathfrak{A}(\mathcal{O})' \wedge \mathfrak{A}(\hat{\mathcal{O}})$ is properly infinite [29], the assertion now follows by [11: Corollary 1] ∎

**Remark 4.** The conditions contained in Theorem 2 and Corollary 1, which assure the split property, are weaker than similar conditions considered in [5].

We conclude with a characterization which is analogue to one contained in [5].

**Proposition 7.** *A factor $M$ with separable predual is a type $I$ factor if and only if, for some (or equivalently for every) faithful state $\omega \in M_*$, we have*

$$\Phi_{2,\omega} \in \mathfrak{N}_2(M, L^2(M)).$$

**Proof.** Suppose that there exists a faithful state $\omega \in M_*$ such that $\Phi_{2,\omega}$ is 2-nuclear. Then $\Phi_{2,\omega}$ is automatically compact as norm limit of finite rank maps (see Remark 2) so, by [5: Corollary 2.9], $M$ is a type $I$ factor. Conversely, by Theorem 2, if $M$ is a type $I$ factor, $\Phi_{2,\omega}$ is 2-nuclear for each faithful normal positive functional $\omega \in M_*$ ∎

The results presented in this section are related to the description of the split property for an inclusion $N \subset M$ of von Neumann algebras in terms of properties of classes of linear maps such as canonical embeddings of $N$ in non-commutative $L^p$-spaces $L^p(M)$ of $M$ (see [5, 6, 21]). The complete characterization of the split property in terms of the $L^1$-embedding is contained in [21] whereas an analogous characterization in terms of the other $L^2$ canonical embedding seems to be a very hard problem which is still open [23]. We hope to return to this question in the near future.

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**Note Added.** After submitting this paper, we have been informed by M. Junge that the $p$-nuclear operators were firstly introduced in [28]. In a first version of the present paper, we independently defined the $p$-nuclear maps adopting a different terminology.

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