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Finite Chainability, Locally Lipschitzian and Uniformly Continuous Functions

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Abstract. We present a notion of a finitely chainable subset of a metric space X. We show that Y is a finitely chainable subset of X if and only if $f(Y)$ is a bounded subset of $\mathbb{R}$ for any uniformly locally Lipschitzian or uniformly continuous real-valued function f on X. As a corollary we reprove the Atsuji theorem in a slightly stronger form.

Keywords: *Metric spaces, finite chainable subsets, uniformly continuous functions, uniformly locally Lipschitzian functions*

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0. Introduction

In infinite dimensional metric spaces not all continuous images of bounded sets are bounded. Indeed, in 1948 Hewitt [1: p. 69] showed that in a metric space X each continuous, real-valued function is bounded if and only if X is compact.

What happens for uniformly continuous functions? To explain better this problem we begin with

Example 0.1. Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of l_2 and let $\|\cdot\|$ denote the Euclidean norm. Let X_n be the segment joining e_n with e_{n+1}, i.e. $X_n = \{e_n + t(e_{n+1} - e_n) : 0 \le t \le 1\}$. Let $X = \bigcup_{n=1}^{\infty} X_n$. Equip X with two different metrics ρ and d defined by

$$d(x,y) = \|x-y\|$$

and

$$\rho(x,y) = \begin{cases} 2^{-n}d(x,y) & \text{if } x,y \in X_n \\ 2^{-n}d(x,e_{n+1}) + D_{n,m} + 2^{-m}d(e_m,y) & \text{if } x \in X_n, y \in X_m \ (n<m) \\ 2^{-n}d(y,e_{n+1}) + D_{n,m} + 2^{-m}d(e_m,x) & \text{if } y \in X_n, x \in X_m \ (n<m) \end{cases}$$

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where $D_{n,m} = \sum_{j=n+1}^{m-1} 2^{-j} d(e_j, e_{j+1})$. Finally, consider a function $f : X \to \mathbb{R}$ defined by

$$f(x) = n + t \qquad \text{if } x = e_n + t(e_{n+1} - e_n).$$

Note the following:

a) (X, d) and (X, ρ) are two bounded metric spaces.

b) d and ρ are equivalent but not uniformly equivalent metrics on X (i.e. for every $x \in X$ and $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$, depending not only on ε but also on x, such that $\rho(x, y) < \varepsilon$ whenever $d(x, y) < \delta_1$ and $d(x, y) < \varepsilon$ whenever $\rho(x, y) < \delta_2$.

c) f is a real-valued unbounded function on X.

d) f is a uniformly continuous function on the metric space (X, d).

e) f is a continuous but not uniformly continuous function on the metric space (X, ρ).

The situation pointed out in Example 0.1 is not unexpected. Indeed, in 1956 Atsuji [2: Theorem 2] showed that each uniformly continuous real-valued function on a metric space (X, d) is bounded if and only if X is a finite chainable space, i.e. for every $\varepsilon > 0$ there are finitely many points $p_1, ..., p_l$ and a positive integer m such that any point of X can be bound with some p_j by a finite sequence of $m + 1$ points $x = x_0, ..., x_m = p_j$ of X satisfying $d(x_{k-1}, x_k) < \varepsilon$ $(k = 1, ..., m)$.

In this note we introduce and study a notion of a finitely chainable subset of a metric space X. The main result of it is Theorem 2.1, which gives a characterization of finitely chainable subsets of X. Also, we reprove the Atsuji theorem [2: Theorem 2] in a slightly stronger form.

1. Finite chainability property

In the sequel, X denotes a metric space with a metric d, $B(x, r)$ the open ball of a centre x and radius r and $A^\varepsilon = \{y \in X : \text{dist}(y, A) < \varepsilon\}$ the ε-neighbourhood of a set $A \subset X$. Let $x, y \in X$ and $\varepsilon > 0$.

Definition 1.1. An ε-chain of length m joining x with y is a finite sequence of $m + 1$ points (not necessarily distinct) of X, $x_0 = x, ..., x_m = y$ satisfying $d(x_k, x_{k-1}) < \varepsilon$ $(k = 1, ..., m)$.

Definition 1.2 (Compare with [2: Definition 3], where the case $Y = X$ has been considered). A subset Y of X is said to be X-*finitely chainable* if for each $\varepsilon > 0$ there are a finite set $q_1, ..., q_{l(\varepsilon)}$ of points of X and a positive integer $m_Y = m_Y(\varepsilon)$ such that any point of Y can be bound with some q_j $(1 \le j \le l(\varepsilon))$ by an ε-chain with length $m_Y(\varepsilon)$. The function $m_Y : [0, \infty] \to \mathbb{N}$, $\varepsilon \to m_Y(\varepsilon)$ is said *link's number function*. It is a non-increasing function.

Example 1.3. We can equip $\mathbb{R}$ with many metrics. For example, the functions

$$d_1(x, y) = |x - y| \tag{1.1}$$

$$d_2(x,y) = \frac{|x-y|}{1+|x-y|} \tag{1.2}$$

$$d_3(x,y) = |\arctan(x) - \arctan(y)| \tag{1.3}$$

are three equivalent metrics on $\mathbb{R}$ but only d_1 and d_2 are uniformly equivalent. The following is easy to see:

a) $(\mathbb{R}, d_1)$ is an unbounded and not finite chainable space.

b) $(\mathbb{R}, d_2)$ is a bounded but not finite chainable space.

c) $(\mathbb{R}, d_3)$ is a bounded finite chainable space.

Now we summarize a few properties of X-finite chainable subsets.

Proposition 1.4. *Let (X,d) be a metric space. Then:*

1) *The property to be X-finite chainable subset is an immersion property, i.e. if Y is X-finitely chainable, then Y is Z-finitely chainable for every metric space Z which contains metrically X.*

2) *The property to be X-finitely chainable is hereditary, i.e. if Y is X-finitely chainable, then each subset Z of Y is X-finitely chainable.*

3) *Let $\{(X_j, d_j), j = 1, ..., n\}$ be a finite family of metric spaces. Then a subset $A = A_1 \times ... \times A_n$ in the metric product space $X = \prod_{j=1}^{n} X_j$ is X-finitely chainable if and only if A_j is X_j-finitely chainable for $j = 1, ..., n$.*

4) *Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be a sequence of metric spaces and let $X = \prod_{n=1}^{\infty} X_n$ be the Cartesian product of X_n endowed with the metric*

$$d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}.$$

For $A_n \subset X_n$ $(n \in \mathbb{N})$ consider the set $A = \prod_{n=1}^{\infty} A_n$. Then A is X-finite chainable if and only if A_n is X_n-finitely chainable for every $n \in \mathbb{N}$. (This is a version of the Tichonoff Theorem for finite chainability.)

5) *The property to be X-finitely chainable is a metric property but not a topological one, i.e. equivalent but not uniformly equivalent metrics can induce different X-chainable subsets. For uniformly equivalent metrics the classes of X-finitely chainable subsets with respect to them are the same.*

6) *The family of X-finitely chainable subsets of X contains the family of bounded metrically convex subsets of X, whenever X is a complete metric space.*

7) *The family of X-finitely chainable subsets of X is contained (properly in general) in the family of the bounded subsets of X.*

8) *If E is a normed space, then a subset Y of E is E-finitely chainable if and only if Y is bounded.*

9) *Let Y be a subset of a complete metric space X. Then Y is relatively compact if and only if Y is X-finite chainable and the link's number function admits a maximum.*

10) *Let (X, d_X) and (Z, d_Z) be two metric spaces. Let $f : X \to Z$ be a uniformly continuous function. Then f maps X-finitely chainable subsets of X into Z-finitely chainable subsets of Z.*

Proof. We only prove statements 4 - 10.

Statement 4: *Necessity.* Let $A = \prod_{n=1}^{\infty} A_n$ be X-finitely chainable. We show that A_n is X_n-finitely chainable for every n. Fix $\varepsilon > 0$ and consider $\eta = \frac{\varepsilon}{2^n(1+\varepsilon)}$. By the X-finite chainability of A, there exists a number $j(\eta)$ of elements $p^1, ..., p^{j(\eta)} \in X$ and $m = m(\eta) \in \mathbb{N}$ such that any $x = \{x_n\} \in A$ can be bound with some p^i $(1 \leq i \leq j(\eta))$ by an η-chain in X $x^0 = x, ..., x^m = p^i$ satisfying $d(x^{l-1}, x^l) < \varepsilon$ $(l = 1, ..., m)$. Then the n-th coordinate x_n of x can be bound with the n-th coordinate p_n^i for some $i \in \{1, ..., j(\eta)\}$ with an ε-chain in X_n of length $m(\eta)$ since

$$2^{-n}\frac{d_n(x_n^{l-1}, y_n^l)}{1 + d_n(x_n^{l-1}, y_n^l)} \leq d(x^{l-1}, x^l) < \eta \quad \text{implies} \quad d_n(x_n^{l-1}, x_n^l) < 2^n \frac{\eta}{1 - 2^n\eta} = \varepsilon.$$

Sufficiency. Let A_n be X_n-finitely chainable for every n. Take $\varepsilon > 0$ and fix n such that $\sum_{k=n+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. Then the thesis follows from property 3) applied to $A_1 \times ... \times A_n$ and from the fact that

$$d(\{x_k\}, \{y_k\}) = \sum_{k=}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} < \sum_{k=1}^{n} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} + \frac{\varepsilon}{2}.$$

Statement 5: Examples 0.1 and 1.3 show that the property to be X-finitely chainable is not a topological one. Now let d_1 and d_2 be two uniformly equivalent metrics on X. Let A be a subset (X, d_1)-finitely chainable and let $\varepsilon > 0$ be fixed. Then there exists $\eta > 0$ such that $d_1(x, y) < \eta$ implies $d_2(x, y) < \varepsilon$. On the other hand, there are $p_1, ..., p_{l(\eta)} \in X$ and $m(\eta) \in \mathbb{N}$ such that every $x \in X$ can be bound with some p_j by an η-chain $x = x_0, ..., x_{m(\eta)} = p_j$ such that $d_1(x_l, x_{l+1}) < \eta$ $(l = 0, ..., m(\eta) - 1)$. Note that $d_2(x_l, x_{l+1}) < \varepsilon$, and consequently A is (X, d_2)-finitely chainable.

Statement 6: First of all, a bounded set $A = [0, 1] \cup \{2\}$ is $\mathbb{R}$-finitely chainable but not metrically convex. Now, let A be a bounded, metrically convex subset of X, i.e. for any $x, y \in A$ there is a point $z \in A$ such that $d(x, y) = d(x, z) + d(y, z)$. A theorem of Menger [3: p. 41] states that a convex and complete metric space contain together with x and y a metric segment whose extremities are x and y, that is a subset isometric to an interval of length $d(x, y)$. Hence we see that if $x, y \in A$, there exist $x = x_0, ..., x_m = y$ such that

$$d(x, y) = \sum_{i=1}^{m} d(x_{i-1}, x_i) \qquad \text{and} \qquad d(x_{i-1}, x_i) < \varepsilon. \tag{1.4}$$

In addition, we can assume that (1.4) holds with

$$d(x_{i+1}, x_i) + d(x_{i+2}, x_{i+1}) \geq \varepsilon.$$

Indeed, since

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_i, x_{i+2})$$

if

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) < \varepsilon$$

we can exclude x_{i+1} from the chain. Hence by (1.4) it follows that $\frac{n\varepsilon}{2} \leq d(x, y) < (n+1)\varepsilon$. Hence, any pair can be bound with an ε-chain of length $m(\varepsilon) < 2\frac{\text{diam}(Y)}{\varepsilon}$.

Statement 7: Note that Examples 0.1 and 1.3 furnish bounded but not X-finitely chainable subsets. The boundedness of an X-finite chainable subset A follows from the fact that $A \subset \cup_{j=1}^{m(\varepsilon)} B(p_j, m_A(\varepsilon)\varepsilon)$ for fixed $\varepsilon > 0$.

Statement 8: Every element x of a bounded set $A \subset E$ can be bound with zero by an ε-chain with knots on the segment $[0, x]$ of length $m(\varepsilon) < \frac{\sup\{\|x\|: x \in A\}}{\varepsilon}$.

Statement 9: For any $\varepsilon > 0$ we have $m_Y(\varepsilon) = 1$. On the other side, let $M = \max\{m_Y(\varepsilon) : \varepsilon > 0\}$. Fix $\varepsilon > 0$. Then there are finite number of points $p_1, ..., p_{l(\varepsilon/M)} \in X$ such that every point of Y can be bound with some p_j by an $\frac{\varepsilon}{M}$-chain with length M. Thus $Y \subset \cup_{n=1}^{l(\varepsilon/M)} B(p_j, \varepsilon)$.

Statement 10: Fix $\varepsilon > 0$. Let $\delta = \delta(\varepsilon)$ be such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. Let Y be X-finitely chainable subset of X. Then there are finite number of points $p_1, ..., p_{l(\delta)} \in X$ such that any $y \in Y$ can be bound with some p_j by a δ-chain of length $m_Y(\delta)$. Then any point of $f(Y)$ can be bound with some $f(p_j)$ by an ε-chain of length $m_Y(\delta)$ ∎

Now we want to examine some properties (frame, amount, length and so on) of the chains with start knots fixed. In this way we will be able to define a non finite chainability measure that will be useful to prove the connexion between X-finite chainability, uniform continuity and uniformly local Lipschitz continuity of functions.

Let (X, d) be a metric space and let $\varepsilon > 0$ be fixed. We denote by $P(x, \varepsilon, n)$ the set of all points in X which can be bound with x by an ε-chain of length n, i.e.

$$P(x,\varepsilon,n) = \left\{ y \in X \;\middle|\; \begin{array}{l} \text{There exist } \{z_1, ..., z_{n-1}\} \subset X \text{ such that} \\ d(x, z_1) < \varepsilon, d(z_1, z_2) < \varepsilon, ..., d(z_{n-1}, y) < \varepsilon \end{array} \right\}. \tag{1.5}$$

Moreover, we denote by $P(x, \varepsilon)$ the set of all points in X which can be bound with x by an ε-chain with an arbitrary finite length, i.e.

$$P(x, \varepsilon) = \bigcup_{n \in \mathbb{N}} P(x, \varepsilon, n). \tag{1.6}$$

With this notation, step by step, it is easy to verify the following

Proposition 1.5.

a) $P(x, \varepsilon, 1) = B(x, \varepsilon)$.

b) $P(x, \varepsilon, n+1) = (P(x, \varepsilon, n))^{\varepsilon}$ *(so any* $P(x, \varepsilon, n)$ *is an open set).*

c) $P(x, \varepsilon, n+1) = P(x, \varepsilon, n)$ *for some* n *implies* $P(x, \varepsilon, m) = P(x, \varepsilon, n)$ *for any* $m \geq n$.

d) $(P(x, \varepsilon))^{\varepsilon} = P(x, \varepsilon)$, *i.e.* $P(x, \varepsilon)$ *is an isolated set, so if* X *is a connected metric space, then* $P(x, \varepsilon) = X$ *for any* $x \in X$ *and* $\varepsilon > 0$.

e) *A relation* R *on* $X \times X$ *defined by* $(x, y) \in R$ *if and only if* $x \in P(y, \varepsilon)$ *is an equivalence relation on* $X \times X$.

f) *The family* $\{P(x, \varepsilon) : x \in X\}$ *is an uniformly isolated partition, i.e.* $(P(x, \varepsilon))^{\varepsilon} \cap (P(y, \varepsilon))^{\varepsilon} = \emptyset$ *if* $P(x, \varepsilon) \neq P(y, \varepsilon)$.

g) *$(\cup_{i\in I}P(x_i,\varepsilon))^\varepsilon = \cup_{i\in I}P(x_i,\varepsilon)$ for any index set I.*

h) *If there is infinite number of distinct sets $P(x_n,\varepsilon)$ $(n\in\mathbb{N})$ and (Z,d) is an unbounded metric space, then a function $f : X \to Z$ defined by*

$$f(x)=\begin{cases}0 & \text{if } x\notin \cup_{n\in\mathbb{N}}P(x_n,\varepsilon)\\ z_n & \text{if } x\in P(x_n,\varepsilon), n \text{ even}\\ w_n & \text{if } x\in P(x_n,\varepsilon), n \text{ odd}\end{cases}$$

where $w_n, z_n \in Z$ are fixed points such that $d(w_n,z_n) > n$ is an unbounded uniformly locally Lipschitz function on X.

Now, let Y be a bounded subset of X. Denote by $N(Y)$ the set of all numbers $\varepsilon > 0$ for which Y is chainable by ε-chains with fixed finite length, i.e

$$N(Y)=\left\{\varepsilon>0 \,\middle|\, \begin{array}{l}\text{There exist } p_1,\dots,p_{i(\varepsilon)}\in X, m_Y(\varepsilon)\in\mathbb{N}\\ \text{such that } Y\subset\cup_{j=1}^{i(\varepsilon)}P(p_j,\varepsilon,m_Y(\varepsilon))\end{array}\right\}. \tag{1.7}$$

Of course, if $\varepsilon\in N(Y)$, then the real interval $[\varepsilon,\infty)$ is contained in $N(Y)$. Put

$$c(Y)=\inf N(Y). \tag{1.8}$$

This is a measure of non finite chainability of Y and Y is X-finitely chainable if and only if $c(Y)=0$.

Moreover, the following is easy to see:

a) $c(Y)\le \operatorname{diam}(Y)$.

b) $c(A\cup B)\le\max\{c(A),c(B)\}$.

c) $c(\bar{A})=c(A)$.

d) $A\subset B$ implies $c(A)\le c(B)$.

e) X is complete if for any decreasing sequence of X-finitely chainable closed subsets $\{F_n\}$ one has $\cap_{n\in\mathbb{N}}F_n\neq\emptyset$.

2. The main results

In the sequel $\mathbb{R}$ is endowed with the Euclidean metric. For sake of completeness, we recall that a function $f : X\to\mathbb{R}$ is said to be *uniformly locally Lipschitzian* if there are $\rho>0$ and $L>0$ such that

$$\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} \,\middle|\, x,y\in X \text{ with } 0<d(x,y)<\rho\right\}\le L. \tag{2.1}$$

Theorem 2.1. *Let (X,d) be a metric space and let $Y\subset X$. Then the following conditions are equivalent:*

(i) *Y is X-finitely chainable.*

(ii) *For any uniformly continuous function $f : X \to \mathbb{R}$, $f(Y)$ is a bounded subset of $\mathbb{R}$.*

(iii) *For any uniformly locally Lipschitzian function $f : X \to \mathbb{R}$, $f(Y)$ is a bounded subset of $\mathbb{R}$.*

Proof. By Proposition 1.4/8) and 10), (i) implies (ii). Of course, (ii) implies (iii). So we show that (iii) implies (i). Suppose, on the contrary, that $Y \subset X$ is not X-finitely chainable. We will construct a real-valued, uniformly locally Lipschitzian function on X, unbounded on Y. Fix a positive number $\varepsilon_0 < c(Y)$. Then for any finite set of points $p_1, \dots, p_l$ and for any $n \in \mathbb{N}$,

$$Y \setminus \bigcup_{j=1}^{l} P(p_j, \varepsilon_0, n) \neq \emptyset. \tag{2.2}$$

It can happen or not that there are finitely many points $p_1, \dots, p_l \in X$ such that

$$Y \subset \bigcup_{j=1}^{l} P(p_j, \varepsilon_0).$$

We examine both cases separately.

First case: There exist $p_1, \dots, p_l$ such that $Y \subset \bigcup_{j=1}^{l} P(p_j, \varepsilon_0)$. Then by (2.2) for some p_j we have $P(p_j, \varepsilon_0, n) \neq P(p_j, \varepsilon_0)$ for any $n \in \mathbb{N}$. Hence, by Proposition 1.5/c), $P(p_j, \varepsilon_0, n) \neq P(p_j, \varepsilon_0, m)$ for any $n \neq m$. We define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin P(p_j, \varepsilon_0) \\ d(x, p_j) & \text{if } x \in P(p_j, \varepsilon_0, 1) \\ (n-1)\varepsilon_0 + \operatorname{dist}(x, P(p_j, \varepsilon_0, n-1)) & \text{if } x \in P(p_j, \varepsilon_0, n) \setminus P(p_j, \varepsilon_0, n-1) \end{cases} \tag{2.3}$$

The function f is unbounded on Y and uniformly locally Lipschitzian on $X \setminus P(p_j, \varepsilon_0)$. We show that f is uniformly locally Lipschitzian on X. Put $\rho = \varepsilon_0$ and fix $x_1, x_2 \in X$ which satisfy

$$d(x_1, x_2) < \rho. \tag{2.4}$$

We show that

$$|f(x_1) - f(x_2)| < 2d(x_1, x_2). \tag{2.5}$$

Since p_j and ε_0 are fixed, to shorten notation, we will write P instead of $P(p_j, \varepsilon_0)$, P_n instead of $P(p_j, \varepsilon_0, n)$ and $P_0 = \{p_j\}$. Note that if there is $l \in \{1, 2\}$ such that $x_l \in P$, then $x_l \in P_n$ for some $n \in \mathbb{N}$. Put

$$n_0 = \min\{n \in \mathbb{N} : \{x_1, x_2\} \cap P_n \neq \emptyset\}. \tag{2.6}$$

Without loss, we can assume that $x_1 \in P_{n_0}$. By (2.4),

$$d(x_2, P_{n_0}) \leq d(x_1, x_2) < \varepsilon_0.$$

Hence $\{x_1, x_2\} \subset P_{n_0+1}$ by Proposition 1.5/b). Moreover, by (2.6), $\{x_1, x_2\} \subset P_{n_0+1} \setminus P_{n_0-1}$ (if $n_0 = 0$, $\{x_1, x_2\} \subset P_1$). Note that, if $x_1, x_2 \in P_{n_0} \setminus P_{n_0-1}$ then, by the definition of f,

$$|f(x_1) - f(x_2)| \le |d(x_1, P_{n_0-1}) - d(x_2, P_{n_0-1})| \le d(x_1, x_2). \tag{2.7}$$

Now suppose $x_2 \in P_{n_0+1} \setminus P_{n_0}$ and $x_1 \in P_{n_0} \setminus P_{n_0-1}$ (hence $n_0 \ge 1$). By Proposition 1.5/b)

$$d(x_2, P_{n_0-1}) \ge \varepsilon_0. \tag{2.8}$$

We show that

$$\varepsilon_0 - d(x_1, x_2) \le d(x_1, P_{n_0-1}) \le \varepsilon_0. \tag{2.9}$$

Since $x_1 \in P_{n_0}$, $d(x_1, P_{n_0-1}) < \varepsilon_0$.

Now, suppose on the contrary that

$$d(x_1, P_{n_0-1}) < \varepsilon_0 - d(x_1, x_2).$$

Take $y \in P_{n_0-1}$ such that $d(x_1, y) < \varepsilon_0 - d(x_1, x_2)$. Then

$$d(x_2, P_{n_0-1}) \le d(x_2, y) \le d(x_1, y) + d(x_2, x_1) < d(x_2, x_1) + \varepsilon_0 - d(x_2, x_1) = \varepsilon_0,$$

which is a contradiction with (2.8). Note that in our case

$$\begin{aligned} |f(x_2) - f(x_1)| &= |\varepsilon_0 + d(x_2, P_{n_0}) - d(x_1, P_{n_0-1})| \\ &= \varepsilon_0 + d(x_2, P_{n_0}) - d(x_1, P_{n_0-1}) \\ &\le \varepsilon_0 + d(x_2, P_{n_0}) - (\varepsilon_0 - d(x_1, x_2)) \\ &= d(x_2, P_{n_0}) + d(x_1, x_2) \\ &\le 2\, d(x_1, x_2) \end{aligned} \tag{2.10}$$

which proves (2.5) if $\{x_1, x_2\} \cap P \neq \emptyset$. Since f is constant on $X \setminus P$, the result is proved.

Second case: For every $p_1, ..., p_l \in X$, $Y \setminus \cup_{j=1}^{l} P(p_j, \varepsilon_0) \neq \emptyset$. By Proposition 1.5/f), there is a sequence $\{y_k\} \subset Y$ such that $P(y_k, \varepsilon_0) \neq P(y_h, \varepsilon_0)$ for $k \neq h$. Let us define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \cup_{n\in\mathbb{N}} P(y_n, \varepsilon_0) \\ n & \text{if } x \in P(y_n, \varepsilon_0). \end{cases} \tag{2.11}$$

Reasoning as in the case of the previous function, we can show that or $\{x_1, x_2\}$ satisfying (2.4) is contained in $X \setminus \cup_{n\in\mathbb{N}} P(y_n, \varepsilon_0)$ or there exists a fixed $n \in \mathbb{N}$ such that $\{x_1, x_2\} \subset P(y_n, \varepsilon_0)$. Since f is constant on each $P(y_n, \varepsilon_0)$ and on $X \setminus \cup_{n\in\mathbb{N}} P(y_n, \varepsilon_0)$, (2.5) holds true. The proof is complete ∎

Remark 2.2. We want to give two examples which show that it is necessary to consider two cases examined in the proof of Theorem 2.1:

a) The space (X, d) from Example 0.1 satisfies the first case.

b) $X = \{\lambda e_n : n \in \mathbb{N} \text{ and } \lambda \in [1, 2]\}$, where e_n is the canonical basis of l_2 and $d(x, y) = \|x - y\|_2$, satisfies the second case.

It is worth saying that in Theorem 2.1 we can replace $\mathbb{R}$ with the Euclidean norm by any normed space $(E, \|\cdot\|)$, $E \neq \{0\}$. Indeed, we can define $g : X \to E$ by $g(x) = f(x)y$ where $y \neq 0$ is a fixed element from E and f is as in the proof of Theorem 2.1.

Corollary 2.3. *Let (X, d) be a metric space. Then X is compact if and only if X is finite chainable and each continuous, real-valued function on X is uniformly continuous.*

Proof. By Theorem 2.1, any real-valued continuous function is bounded on X. Thus the result follows by Hewitt's theorem ∎

Remark 2.4. Metric spaces (X, d) for which any real-valued, continuous function on X is uniformly continuous are widely studied in literature and are known as *UC* spaces (for references see [4]).

From Theorem 2.1 it is easy to reprove the Atsuji theorem [2: Theorem 2] in a slightly stronger form.

Corollary 2.5 (compare with [2: Theorem 2]). *Let X be a metric space. Then X is finitely chainable if and only if $f(X)$ is a bounded subset of $\mathbb{R}$ for any uniformly continuous or uniformly locally Lipschitzian function f on X.*

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