Asymptotic Behavior of M-Band Scaling Functions of Daubechies Type

N. **Bi, L. Debnath and Q. Sun**

Abstract. This paper deals with the asymptotic behavior of M-band scaling functions $\frac{M}{N}\phi$ and M-band symbols $\frac{M}{N}H$ as $M \rightarrow \infty$ for $N \ge 2$. This is followed by pointwise convergence, and L^P-convergence $(1 \leq p < \infty)$ of $\bigwedge^M \phi$, and the limit function g of $\bigwedge^M \phi$ as $M \to \infty$. Abstract. This paper deals with the asymptotic behavior of *M*-band scaling functions $\frac{M}{N}\phi$ and *M*-band symbols $\frac{M}{N}H$ as $M \to \infty$ for $N \geq 2$. This is followed by pointwise convergence,

and *L^p*-convergence

Keywords: *Wavelets, scaling functions, refinable functions*

AMS subject classification: 42C15

1. Introduction

For any integer $M \geq 2$, a function f is called M-refinable (or simply *refinable*) if it satisfies the *refinement equation*

$$
f(x) = \sum_{s \in \mathbf{Z}} c(s) f(Mx - s) \tag{1.1}
$$

fies the condition $\sum_{s\in\mathbb{Z}} c(s) = M$ and is of finite length. A function *f* is said to be *orihonorrnal* if it satisfies **If** ≥ 2 , a function f is called M -refinable (or sincent equation
 $f(x) = \sum_{s \in \mathbb{Z}} c(s) f(Mx - s)$
 I, where $\{c(s)\}$, called the mask of the refineme
 $\sum_{s \in \mathbb{Z}} c(s) = M$ and is of finite length. A funct

tisfies
 $f(x) = \sum_{s \in \mathbb{Z}} c(s) f(Mx - s)$
 $c(s)$, called the *mask* of the refineme
 $= M$ and is of finite length. A funct
 $c(-k) dx = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$

($k \in \mathbb{Z}$).

In an *M*-refinable and orthonormal function
 H

$$
\int_{\mathbb{R}} f(x)f(x-k) dx = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \qquad (k \in \mathbb{Z}).
$$

By a *scaling function* we mean an M-refinable and orthonormal function. For a given sequence ${c(s)}$, we define

$$
H(\xi) = \frac{1}{M} \sum_{s \in \mathbb{Z}} c(s) \exp(is\xi).
$$
 (1.2)

Then *H* is called a *filter* of the refinement equation (1.1) or a filter corresponding to the scaling function f. For any integer $N \geq 1$, H is said to have N vanishing moments if there exists a Laurent polynomial \tilde{H} such that $H(k) = k \int d\mathbf{x} = \begin{cases} 1 & \text{if } k \neq 0 \\ 0 & \text{if } k \neq 0 \end{cases}$ ($k \in \mathbb{Z}$).
 In an *M***-refinable and orthonormal function.** For a given
 $H(\xi) = \frac{1}{M} \sum_{s \in \mathbb{Z}} c(s) \exp(is\xi))$. (1.2)

2 refinement equation (1.1) or a filter corr

$$
H(z) = \left[\frac{1-z^M}{M(1-z)}\right]^N \tilde{H}(z).
$$
 (1.3)

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For a scaling function *f*, let a sequence of closed subspaces $Vj \cdot (j \in \mathbb{Z})$ of square integrable function space $L^2(\mathbb{R})$ spanned by the functions

n and Q. Sun
\non f, let a sequence of closed subspaces
$$
Vj \cdot (j \in \mathbb{Z})
$$
 of square
\ne $L^2(\mathbb{R})$ spanned by the functions
\n
$$
f_{j,k}(x) = \left\{ M^{j/2} f(M^j x - k) : k \in \mathbb{Z} \right\}.
$$
\n(1.4)
\na *multiresolution analysis* of $L^2(\mathbb{R})$ if it satisfies the following

Then $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* of $L^2(\mathbb{R})$ if it satisfies the following conditions:

- (i) $V_i \subset V_{i+1}$, and $f \in V_j$ if and only if $f(Mx) \in V_{i+1}$ for all $j \in \mathbb{Z}$.
- (ii) $\bigcup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}}V_j=\{0\}.$
- (iii) $\{f(\cdot k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_0 for some of $f \in V_0$.

We denote the wavelet space W_j ($j \in \mathbb{Z}$) by the orthonormal complement spaces of V_i in V_{i+1} so that the wavelet decomposition

$$
(\text{1.4})
$$
\n
$$
\text{trresolution analysis of } L^{2}(\mathbb{R}) \text{ if it satisfies the following}
$$
\n
$$
j \text{ if and only if } f(Mx) \in V_{j+1} \text{ for all } j \in \mathbb{Z}.
$$
\n
$$
L^{2}(\mathbb{R}) \text{ and } \bigcap_{j \in \mathbb{Z}} V_{j} = \{0\}.
$$
\n
$$
\text{thonormal basis of } V_{0} \text{ for some of } f \in V_{0}.
$$
\n
$$
\text{ace } W_{j} \quad (j \in \mathbb{Z}) \text{ by the orthonormal complement spaces}
$$
\n
$$
L^{2} = \bigcup_{i \in \mathbb{Z}} W_{j} = V_{k} + \bigcup_{j \geq k} W_{j}
$$
\n
$$
\text{the decomposition}
$$
\n
$$
L^{2} = \bigcup_{i \in \mathbb{Z}} W_{i} = V_{i} + \bigcup_{j \geq k} W_{j}
$$
\n
$$
\text{the decomposition}
$$
\n
$$
L^{2} = \bigcup_{i \in \mathbb{Z}} W_{i} = V_{i} + \bigcup_{j \geq k} W_{j}
$$
\n
$$
\text{the decomposition}
$$

holds. In fact, (1.5) suggests the decomposition

R) and
$$
\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.
$$

\nnonormal basis of V_0 for some of $f \in V_0$.
\nwe W_j $(j \in \mathbb{Z})$ by the orthonormal complement spaces
\nat decomposition
\n
$$
f^2 = \bigcup_{i\in\mathbb{Z}} W_j = V_k + \bigcup_{j\geq k} W_j
$$
\n
$$
f = \sum_{j\in\mathbb{Z}} g_j = \sum_{j\geq k} g_j + f_k
$$
\n
$$
f_k \in V_k.
$$
\n(1.6)

of $f \in L^2(\mathbb{R})$ where $g_i \in W_i$ and $f_k \in V_k$.

The literature of wavelets is replete with analysis of 2-band $(M = 2)$ scaling functions. The wavelet theory when $M = 2$ can be found in the literature of wavelets (see Daubechies [2]). When $M = 2$, W_j is spanned by $\{2^j \psi(2^j \cdot -k)\}_{k \in \mathbb{Z}}$ and the mother wavelet can be constructed from the 2-band scaling functions ϕ in the form *the decomposition*
 $f = \sum_{j \in \mathbb{Z}} g_j = \sum_{j \geq k} g_j + f_k$ (1.6)
 t, and $f_k \in V_k$.
 t, and $f_k \in V_k$.
 t, and $f_k \in V_k$.
 t, and $f_k = 2$ can be found in the literature of wavelets (see
 $f_k = 2$, W_j is spanned by $\{2^$

$$
\psi(x) = \sum_{k \in \mathbb{Z}} c_{1-k} (-1)^k \phi(2x - k), \qquad (1.7)
$$

where c_k are the coefficient of the 2-band scaling functions defined by (1.1) .

In short, the theory of wavelets for $M = 2$ has received considerable attention. However, the wavelet theory for $M > 2$ received much less attention. Bi et al. [1] and Heller [3] independently considered the design of filter with *N* vanishing moment and finite length. Bi et al. $[1]$ also considered M-band scaling functions, M-band wavelets and constructed compactly supported orthonormal M-band wavelets. The major objective of this paper is to investigate the asymptotic behavior of M-band scaling functions and M-band symbols as $M \to \infty$. *N* for $M > 2$ received much le
 N considered the design of filter
 N [1] also considered *M*-band

ompactly supported orthonorn

per is to investigate the asym

also smooths as $M \to \infty$.

let
 $N H(\xi) = \frac{1}{2} \sum_{s=0}^{$

For any integer $N \geq 1$, let

$$
{N}H(\xi)=\frac{1}{2}\sum{s=0}^{2N-1}N a(s)\exp(is\xi)
$$

be a solution of the equation

Asymptotic Behavior of *M*-Band Scaling Functions
\nthe equation
\n
$$
|_{N}H(\xi)|^{2} = \cos^{2N}\left(\frac{\xi}{2}\right) \sum_{s=0}^{N-1} {2N-1+s \choose s} \sin^{2s}\left(\frac{\xi}{2}\right).
$$
\n(1.8)
\nwe solution of equation (1.8) in the form
\n
$$
{N}H(\xi) = \frac{1}{2} \sum{s=0}^{2N-1} {}_{N}a(s)e^{is\xi}
$$
\n(1.9)

We note that the solution of equation (1.8) in the form

$$
{N}H(\xi) = \frac{1}{2} \sum{s=0}^{2N-1} {}_{N}a(s)e^{is\xi}
$$
\n(1.9)
\n
$$
N \ge 2.
$$
\nscaling functions $_N \phi$ with symbol $_N H$ when $M = 2$, and
\n
$$
{N}H\left(-\frac{\xi}{2} + \pi\right) \exp\left(-\frac{i\xi}{2}\right) {}{N} \hat{\phi}\left(\frac{\xi}{2}\right),
$$
\n(1.10)
\n
$$
{N}F \left(\frac{\xi}{2} + \pi\right) \exp\left(-\frac{i\xi}{2}\right) {}{N} \hat{\phi}\left(\frac{\xi}{2}\right),
$$
\n(1.11)

is *not* unique, but finite when $N \geq 2$.

Daubechies [2] introduced scaling functions $N\phi$ with symbol $N\phi$ when $M = 2$, and wavelets $N\psi$ defined by

$$
N\hat{\psi}(\xi) = N H\left(-\frac{\xi}{2} + \pi\right) \exp\left(-\frac{i\xi}{2}\right) N \hat{\phi}\left(\frac{\xi}{2}\right),\tag{1.10}
$$

where \hat{f} is the Fourier transform of an integrable function f defined by

$$
\hat{f}(\xi) = \int_{\mathbf{R}} \exp(-ix\xi) f(x) dx.
$$

For these wavelets $_N\psi$, $\{2_N^{j/2}\psi(2^j-k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. The Hölder index of $_N\psi$ is about $\frac{\ln 3}{2\ln 2}N$ for large N, and it has N vanishing moments, that **IS,** $2^{j/2}_N \psi(2^j \cdot -k) \}_{j,k \in \mathbb{Z}}$
ut $\frac{\ln 3}{2 \ln 2} N$ for large N
 $x^k{}_N \psi(x) dx = 0$

$$
\int_{\mathbb{R}} x^k \, N \psi(x) \, dx = 0 \qquad (0 \leq k \leq N-1).
$$

Moreover, for any $N \geq 1$, the scaling function $N\phi$ has minimal support in the class of compactly supported scaling functions ϕ for which we may find a compactly supported orthonormal wavelet ψ in V_1 which has N vanishing moments and satisfies
 $\int_{\mathbb{R}} \psi(x)\phi(x-k) dx = 0$ $(k \in \mathbb{Z})$ orthonormal wavelet ψ in V_1 which has N vanishing moments and satisfies

$$
\int_{\mathbb{R}} \psi(x)\phi(x-k) dx = 0 \qquad (k \in \mathbb{Z})
$$

where V_1 is the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\sqrt{2}\phi(2\cdot-k)\}_{k\in\mathbb{Z}}$.

We define

homormal wavelet
$$
\psi
$$
 in V_1 which has N vanishing moments and satisfies

\n
$$
\int_{\mathbb{R}} \psi(x)\phi(x-k) \, dx = 0 \quad (k \in \mathbb{Z})
$$
\nwhere V_1 is the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\sqrt{2}\phi(2-k)\}_{k \in \mathbb{Z}}$.

\nWe define

\n
$$
\int_{N}^{M} a(s) = \sum_{s_1 + \dots + s_{M-1} = s} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \left(2\sin\frac{j\pi}{M}\right)^{-2s_j} \quad (0 \leq s \leq N-1) \quad (1.11)
$$
\nand

\n
$$
P(t) = \sum_{s=0}^{N-1} \int_{N}^{M} a(s) \, t^s. \tag{1.12}
$$

and

$$
P(t) = \sum_{s=0}^{N-1} \, \frac{M}{N} a(s) \, t^s. \tag{1.12}
$$

By the Riesz lemma [2: P. 172/Lemma 6.1.3], there exists a unique solution *H* of the equation

nth and Q. Sun
\n2: p. 172/Lemma 6.1.3], there exists a unique solution *H* of the
\n
$$
|H(\xi)|^2 = \left(\frac{\sin \frac{M\xi}{2}}{M \sin \frac{\xi}{2}}\right)^{2N} P(2 - 2 \cos \xi), \qquad (1.13)
$$
\n
$$
\left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^{N} \sum_{s=0}^{N-1} \tilde{c}(s) e^{is\xi} = \frac{1}{M} \sum_{s=0}^{M N-1} c(s) e^{is\xi} \qquad (1.14)
$$
\nall roots in the open unit disk, where $P(z)$ is a polynomial in *z*.

such that

$$
(M \sin \frac{1}{2})
$$

$$
H(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^{N} \sum_{s=0}^{N-1} \tilde{c}(s) e^{is\xi} = \frac{1}{M} \sum_{s=0}^{M N-1} c(s) e^{is\xi} \qquad (1.14)
$$

and $\sum_{s=0}^{N-1} \tilde{c}(s) z^s$ has all roots in the open unit disk, where $P(z)$ is a polynomial in z. Denote the solution of equations (1.13) and (1.14) by $^{M}_{N}H$. Let $^{M}_{N}\phi$ be the solution of the refinement equation (1.1) with the symbol $_{N}^{M}H$.

Bi et al. [1] and Heller [3] independently proved that $\frac{M}{N}\phi$ is orthonormal, and represents a scaling function. Furthermore, $\frac{M}{N}\phi$ has minimal support in the class of compactly supported scaling functions ϕ for which we may find compactly supported orthonormal the refinement equation (1.1) with the symbol $_{NH}^{MH}$.
Bi et al. [1] and Heller [3] independently proved that $_{NH}^{M}\phi$ is orthonormal, and represents a scaling function. Furthermore, $_{NH}^{M}\phi$ has minimal support in the of L^2 spanned by ${\lbrace \sqrt{M} \phi(M \cdot -k) \rbrace_{k \in \mathbb{Z}}}$. For this reason, we call ${}_{N}^{M} \phi$ as M-band scaling *functions of Daubechies type.*

When $M = 2$, Daubechies [2] and Pollen [4] studied the 2-band scaling functions of Daubechies type. On the other hand, for M-band scaling functions of Daubechies type, Bi et al. [1] investigated the asymptotic behavior of the Hölder index of $\frac{M}{N}\phi$ as *functions of Daubechies type.*

When $M = 2$, Daubechies [2] and Pollen [4] studied the 2-band scaling functions

of Daubechies type. On the other hand, for *M*-band scaling functions of Daubechies

type, Bi et al. [1] in pointwise as $M \to \infty$ where g is given by r which we may find comp:
 $M - 1$) such that ψ_s has

orthogonal basis of V_1 , w
 k _k_k ϵ _Z. For this reason, w

2] and Pollen [4] studied

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the asymptotic behavior of

Zhang [5] pro

$$
g(x) = \begin{cases} x + \frac{\sqrt{6}}{6} & \text{if } 0 < x \le 1 \\ -x + 1 - \frac{\sqrt{6}}{6} & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

They have also shown that $\frac{M}{2}\phi$ is locally linear on an open set with full measure and locally linearly dependent when $M \geq 3$.

This paper deals with studying the asymptotic behavior of M-band scaling functions $M_N^M \phi$ and *M*-band symbols $^M_N H$ as $M \to \infty$, for any $N \geq 2$. More precisely, we investigate the local polynomial structure of $\frac{M}{N}\phi$ on an open set with full measure, the asymptotic behavior of $^{M}_{N}H$, and then the pointwise convergence and L^{p} convergence of $^{M}_{N}\phi$ as This paper deals with studying the asymptotic behavior of *M*-band scaling functions $M\phi$ and *M*-band symbols $M\theta$ as $M \to \infty$, for any $N \ge 2$. More precisely, we investigate the local polynomial structure of $M\phi$ on $M \to \infty$. In Section 2, we consider the local polynomial structure of $M\phi$ on an open set. Section 3 deals with the asymptotic behavior of M-band symbols $M\phi$. This is the local polynomial structure of $\mathbb{N}\phi$ on an open set with full measure, the asymptotic
behavior of $\mathbb{N}H$, and then the pointwise convergence and *LP*-convergence of $\mathbb{N}\phi$ as
 $M \to \infty$. In Section 2, we consid followed by pointwise convergence and L^p -convergence $(1 \leq p < \infty)$ of $\frac{M}{N}\phi$. Finally, some remarks on the limit function g of $\frac{M}{N}\phi$ as $M \to \infty$ are discussed.

2. Local polynomial functions

We say that a function supported in $[a, b]$ is *locally polynomial* on an open set $A \subset [a, b]$ if it is a polynomial on every connected component of *A.*

Theorem 2.1. Let $M > N$ and $\substack{M \\ N} \phi$ be the solution of the refinement equation (1.1) with symbol $_{N}^{M}H$. Then there exists an open set $A\subset (0,N+\frac{N-1}{M-1})$ with Lebesgue measure $N+\frac{N-1}{M-1}$ such that $\frac{M}{N}\phi$ is locally polynomial on A. *st* a function supported in $[a, b]$ is *locally* plynomial on every connected component $em 2.1$. Let $M > N$ and $\frac{M}{N}\phi$ be the solution $l \frac{M}{N}H$. Then there exists an open set $A \subset$ such that $\frac{M}{N}\phi$ is *locally p*

Moreover, the above assertion holds for a more general class of refinable functions. A proof of this theorem is given by Bi et al. [1], and is omitted.

Theorem 2.2. Let $M-1 > r \neq 0$ and ϕ be the solution of the refinement equation *(1.1) with symbol*

$$
H(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^N Q_r(\xi)
$$

where $Q_r(0) = 1$ and $Q_r(\xi)$ may be written as $Q_r(\xi) = \sum_{k=0}^r c(k)e^{ik\xi}$. Then there exists *an open set* $A \subset (0, N + \frac{r}{M-1})$ with Lebesgue measure $N + \frac{r}{M-1}$ such that ϕ is locally *polynomial on A.*

To prove Theorem 2.2, we need some lemmas.

Let ϕ be as in Theorem 2.2. We define

rem 2.2, we need some lemmas.
\nTheorem 2.2. We define
\n
$$
\Phi(x) = (\phi(x), ..., \phi(x+N-1))^{T}
$$
\n
$$
\tilde{\Phi}(x) = (\phi(x+1), ..., \phi(x+N))^{T}
$$
\n
$$
(x \in (0,1))
$$
\n
$$
m_{j} = \int_{\mathbb{R}} x^{j} \phi(x) dx \qquad (0 \le j \le N-1).
$$

and

$$
m_j = \int_{\mathbb{R}} x^j \phi(x) dx \qquad (0 \le j \le N-1).
$$

Let

$$
\Phi(x) = (\phi(x+1), \dots, \phi(x+N))
$$

$$
m_j = \int_{\mathbb{R}} x^j \phi(x) dx \qquad (0 \le j \le N-1).
$$

$$
A(x) = ((x+k)^j)_{0 \le j,k \le N-1} \qquad \text{and} \qquad \widetilde{A}(x) = ((x+k)^j)_{0 \le j \le N-1}
$$

Denote the transpose of a matrix (or a vector) A by A^T . Then we have the following

Lemma 2.1. Let $M - 1 > r$ and ϕ be as in Theorem 2.2. Then

Asymptotic Behavior of M-Band Scaling Functions
\nWe say that a function supported in [a, b] is locally polynomial on an open set
$$
A \subset [a, b]
$$

\nif it is a polynomial on every connected component of A.
\n**Theorem 2.1.** Let $M > N$ and $\frac{M}{N}\phi$ be the solution of the refinement equation (1.1)
\nwith symbol $\frac{M}{N}H$. Then there exists an open set $A \subset (0, N + \frac{N-1}{N-1})$ with Lebesgue measure
\n $N + \frac{N-1}{M-1}$ such that $\frac{M}{N}\phi$ is locally polynomial on A.
\nMoreover, the above assertion holds for a more general class of refinable functions.
\nA proof of this theorem is given by Bi et al. [1], and is omitted.
\nTherefore, the above assertion holds for a more general class of refinable functions.
\nA proof of this theorem 2.2. Let $M - 1 > r \neq 0$ and ϕ be the solution of the refinement equation
\n(1.1) with symbol
\n
$$
H(\xi) = \left(\frac{1 - e^{iK}}{M(1 - e^{i\xi})}\right)^N Q_r(\xi)
$$
\nwhere $Q_r(0) = 1$ and $Q_r(\xi)$ may be written as $Q_r(\xi) = \sum_{k=0}^r c_k(k)e^{ik\xi}$. Then there exists
\nan open set $A \subset (0, N + \frac{r}{M-1})$ with Lebesgue measure $N + \frac{r}{M-1}$ such that ϕ is locally
\npolynomial on A.
\nTo prove Theorem 2.2. We define
\n
$$
\Phi(x) = (\phi(x) \dots, \phi(x + N - 1))^T
$$
\n
$$
\tilde{\Phi}(x) = (\phi(x + 1), \dots, \phi(x + N))^T
$$
\n
$$
x = \int_{\mathbb{R}} x^j \phi(x) dx \qquad (0 \leq j \leq N - 1).
$$
\nLet
\n
$$
A(x) = ((x + k)^j)_{0 \leq j,k \leq N-1}
$$
 and $\tilde{A}(x) = ((x + k)^j)_{0 \leq j \leq N-1}$.
\nDenote the transpose of a matrix (or a vector) A by A^T . Then we have the following
\nLemma 2.1. Let $M - 1 > r$ and ϕ be as in Theorem 2.2

on (0, 1) and ϕ is polynomial on $\bigcup_{i=0}^{N-1}$ $\left(j + \left(\frac{r}{M-1}, 1\right)\right)$.

Proof. We first note that ϕ is supported on $\left[0, N + \frac{r}{M-1}\right]$ and

$$
\det A(x) = \prod_{0 \le i < j \le N-1} (j - i) \neq 0.
$$

Therefore, from the first formula in (2.1) we get

First formula in (2.1) we get
\n
$$
\Phi(x) = (\det A(x))^{-1} A^*(x) (m_0, ..., m_{N-1})^T
$$

on $\left(\frac{r}{M-1}, 1\right)$, where $A^*(x)$ denotes the adjoint matrix of $A(x)$. Then the second assertion follows from (2.1).

Now we prove (2.1). By taking the Fourier transform of both sides of the refinement equation (1.1), we obtain

$$
\hat{\phi}(\xi) = H\left(\frac{\xi}{M}\right)\hat{\phi}\left(\frac{\xi}{M}\right). \tag{2.2}
$$

Therefore, $D^j \hat{\phi}(2k\pi) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $0 \leq j$ **818** *N. Bi, L. Debnath and Q. Sun*
 Therefore, from the first formula in (2.1) we get
 $\Phi(x) = (\det A(x))^{-1} A^*(x)(m_0, \ldots, m_{N-1})^T$

on $\left(\frac{r}{M-1}, 1\right)$, where $A^*(x)$ denotes the adjoint matrix of $A(x)$. Then the second differential operator, and furthermore

prove (2.1). By taking the Fourier transform of both sides of the refinement
\n1), we obtain\n
$$
\hat{\phi}(\xi) = H\left(\frac{\xi}{M}\right)\hat{\phi}\left(\frac{\xi}{M}\right).
$$
\n(2.2)\n
$$
D^j \hat{\phi}(2k\pi) = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0\} \text{ and } 0 \le j \le N - 1, \text{ where } D = \frac{\partial}{\partial \xi} \text{ is a}
$$
\noperator, and furthermore\n
$$
\sum_{k \in \mathbb{Z}} (x + k)^j \phi(x + k) = \int_{\mathbb{R}} x^j \phi(x) dx = m_j \qquad (0 \le j \le N - 1) \qquad (2.3)
$$

by the Poisson summation formula. Then the first assertion in (2.1) follows from (2.3)

Lemma 2.2. Let ϕ be the same as in Theorem 2.2. Then there exist real numbers $a(0), \ldots, a(r)$ and polynomials P_1, \ldots, P_r with degree at most $N-1$ such that

$$
\hat{\phi}(2k\pi) = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0\} \text{ and } 0 \leq j \leq N - 1, \text{ where } D = \frac{\partial}{\partial \xi} \text{ is a}
$$
\n
\n
$$
\sum_{\mathbf{z}} (x + k)^j \phi(x + k) = \int_{\mathbb{R}} x^j \phi(x) dx = m_j \qquad (0 \leq j \leq N - 1) \qquad (2.3)
$$
\n
\nsummation formula. Then the first assertion in (2.1) follows from (2.3)
\n2. Let ϕ be the same as in Theorem 2.2. Then there exist real numbers
\nand polynomials P_1, \ldots, P_r with degree at most $N - 1$ such that
\n
$$
\phi\left(\frac{x + j}{M}\right) = a(j)\phi(x) + P_j(x) \qquad (0 \leq j \leq r, x \in (0, 1)) \qquad (2.4)
$$

and

$$
\sum_{k \in \mathbb{Z}} (x+k)^j \phi(x+k) = \int_{\mathbb{R}} x^j \phi(x) dx = m_j \qquad (0 \le j \le N-1) \qquad (2.3)
$$

the Poisson summation formula. Then the first assertion in (2.1) follows from (2.3) **E**
Lemma 2.2. Let ϕ be the same as in Theorem 2.2. Then there exist real numbers
),..., $a(r)$ and polynomials $P_1,...,P_r$ with degree at most $N-1$ such that

$$
\phi\left(\frac{x+j}{M}\right) = a(j)\phi(x) + P_j(x) \qquad (0 \le j \le r, x \in (0,1)) \qquad (2.4)
$$

$$
\phi\left(\sum_{j=1}^k \frac{\varepsilon_j}{M^j} + \frac{x}{M^k}\right) = \prod_{j=1}^k a(\varepsilon_j)\phi(x) + P_{\varepsilon_k}\left(\sum_{j=2}^k \frac{\varepsilon_j}{M^{j-1}} + \frac{x}{M^{k-1}}\right)
$$

$$
+ \sum_{i=0}^{k-2} \prod_{l=k-i}^k a(\varepsilon_l)P_{\varepsilon_{k-1-i}}\left(\sum_{j=i+3}^k \frac{\varepsilon_j}{M^{j-i-2}} + \frac{x}{M^{k-i-2}}\right)
$$

ere $\varepsilon_j \in \{0,1,...,r\}$ and $x \in (0,1)$.
Proof. By the refinement equation (1.1), we obtain

$$
\phi\left(\frac{x+j}{M}\right) = \sum_{l=0}^{(M-1)N+r} c_l \phi(x+j-l) = \sum_{l=0}^j c_{j-l} \phi(x+l) \qquad (2.6)
$$

(0,1). From Lemma 2.1, there exist polynomials $Q_j \in \Pi_{N-1}$ and numbers d_j $(1 \le N)$ such that

where $\varepsilon_j \in \{0, 1, \ldots, r\}$ *and* $x \in (0, 1)$ *.*

Proof. By the refinement equation (1.1), we obtain

$$
\phi\left(\frac{x+j}{M}\right) = \sum_{l=0}^{(M-1)N+r} c_l \phi(x+j-l) = \sum_{l=0}^{j} c_{j-l} \phi(x+l) \tag{2.6}
$$

on (0,1). From Lemma 2.1, there exist polynomials $Q_j \in \Pi_{N-1}$ and numbers d_j (1 \leq *j* \leq *N*) such that $\phi(x + j) = d_j \phi(x) + Q_j(x)$ (2.7) $j \leq N$) such that

$$
\phi(x+j) = d_j \phi(x) + Q_j(x) \tag{2.7}
$$

where Π_{N-1} denotes the class of polynomials with degrees at most $N-1$. Then (2.4) follows from (2.6) and (2.7) , and (2.5) follows by using formula (2.4) *k* times

For any $\varepsilon_i \in \{0, 1, \ldots, r\}$ and $1 \leq i \leq k$, define

$$
A(\varepsilon_1,\ldots,\varepsilon_k)=\left(\sum_{j=1}^k\frac{\varepsilon_j}{M^j}+\frac{r}{(M-1)M^k},\sum_{j=1}^k\frac{\varepsilon_j}{M^j}+\frac{1}{M^k}\right).
$$

Then $A(\varepsilon_1, \ldots, \varepsilon_k) \subset (0, \frac{r}{M-1})$ when $\varepsilon_k \neq r$. Furthermore, we have the following

Lemma 2.3. Let $A(\varepsilon_1, \ldots, \varepsilon_k)$ be defined as above. Then

$$
A(\varepsilon_1,\ldots,\varepsilon_k)\cap A(\varepsilon'_1,\ldots,\varepsilon'_{k'})=\emptyset
$$

when $\varepsilon_k, \varepsilon'_{k'} \neq r$ *except* $k = k'$ *and* $(\varepsilon_1, \ldots, \varepsilon_k) = (\varepsilon'_1, \ldots, \varepsilon'_{k'})$.

Proof. Define

$$
\sqrt{1-\epsilon} \quad \text{where } \quad j=1 \quad \text{where } j=1 \
$$

Then it suffices to prove that

$$
a(\varepsilon'_1,\ldots,\varepsilon'_{k'})>a(\varepsilon_1,\ldots,\varepsilon_k)\quad\Longrightarrow\quad a(\varepsilon'_1,\ldots,\varepsilon'_{k'})\geq b(\varepsilon_1,\ldots,\varepsilon_k)
$$

We note that

$$
\begin{aligned}\n\varepsilon_1, \ldots, \varepsilon_k &= \sum_{j=1} \frac{\varepsilon_j}{M^j} + \frac{r}{(M-1)M^k} \quad \text{and} \quad b(\varepsilon_1, \ldots, \varepsilon_k) = \sum_{j=1} \frac{\varepsilon_j}{M^j} + \frac{1}{M^i} \\
\text{t suffices to prove that} \\
a(\varepsilon'_1, \ldots, \varepsilon'_{k'}) > a(\varepsilon_1, \ldots, \varepsilon_k) \quad \Longrightarrow \quad a(\varepsilon'_1, \ldots, \varepsilon'_{k'}) \ge b(\varepsilon_1, \ldots, \varepsilon_k) \\
\text{te that} \\
M^j a(\varepsilon_1, \ldots, \varepsilon_k) &= M^j \sum_{j=1}^j \frac{\varepsilon_j}{M^i} + M^j a(\varepsilon_{j+1}, \ldots, \varepsilon_k) \subset M^j \sum_{j=1}^j \frac{\varepsilon_j}{M^i} + (0, 1) \\
M^j b(\varepsilon_1, \ldots, \varepsilon_k) &= M^j \sum_{j=1}^j \frac{\varepsilon_j}{M^j} + M^j b(\varepsilon_{i+1}, \ldots, \varepsilon_k) \subset M^j \sum_{j=1}^j \frac{\varepsilon_j}{M^j} + (0, 1).\n\end{aligned}
$$

and

$$
u(\epsilon_1, \ldots, \epsilon_k) > u(\epsilon_1, \ldots, \epsilon_k) \qquad \longrightarrow \qquad u(\epsilon_1, \ldots, \epsilon_k) \leq v(\epsilon_1, \ldots, \epsilon_k)
$$
\nwe that

\n
$$
M^j a(\epsilon_1, \ldots, \epsilon_k) = M^j \sum_{j=1}^j \frac{\epsilon_i}{M^i} + M^j a(\epsilon_{j+1}, \ldots, \epsilon_k) \subset M^j \sum_{j=1}^j \frac{\epsilon_i}{M^i} + (0, 1)
$$
\n
$$
M^j b(\epsilon_1, \ldots, \epsilon_k) = M^j \sum_{j=1}^j \frac{\epsilon_i}{M^i} + M^j b(\epsilon_{j+1}, \ldots, \epsilon_k) \subset M^j \sum_{j=1}^j \frac{\epsilon_i}{M^i} + (0, 1].
$$
\nare the problem reduces to prove

\n
$$
b(\epsilon_1, \ldots, \epsilon_k) \leq a(\epsilon'_1, \ldots, \epsilon'_k)
$$

Therefore the problem reduces to prove

$$
b(\varepsilon_1,\ldots,\varepsilon_k)\leq a(\varepsilon'_1,\ldots,\varepsilon'_{k'})
$$

Therefore the problem reduces to prove
 $b(\varepsilon_1, \ldots, \varepsilon_k) \le a(\varepsilon'_1, \ldots, \varepsilon'_k)$

for the following two cases: (i) $\varepsilon'_1 \ne \varepsilon_1$ and (ii) $\varepsilon_1 = \varepsilon'_1$ and $k = 1$ or $k' = 1$.

For the case (i), we get $\varepsilon'_1 > \varepsilon_1$,

For the case (i), we get $\varepsilon_1' > \varepsilon_1$, otherwise

$$
e^{i\theta}
$$

reduces to prove

$$
b(\varepsilon_1, \ldots, \varepsilon_k) \le a(\varepsilon'_1, \ldots, \varepsilon'_k)
$$

is: (i) $\varepsilon'_1 \neq \varepsilon_1$ and (ii) $\varepsilon_1 = \varepsilon'_1$ and $k =$
get $\varepsilon'_1 > \varepsilon_1$, otherwise

$$
a(\varepsilon'_1, \ldots, \varepsilon'_k) < \frac{\varepsilon'_1 + 1}{M} \le a(\varepsilon_1, \ldots, \varepsilon_k)
$$

Therefore, we have

which is a contradiction. Therefore, we have

$$
b(\varepsilon_1,\ldots,\varepsilon_k) \leq \frac{\varepsilon_1+1}{M} \leq a(\varepsilon'_1,\ldots,\varepsilon'_{k'}).
$$

For the case (ii), *k'* must be one, otherwise

$$
b(\varepsilon_1, \dots, \varepsilon_k) \le \frac{\varepsilon_1 + 1}{M} \le a(\varepsilon'_1, \dots, \varepsilon'_{k'}).
$$

the case (ii), k' must be one, otherwise

$$
a(\varepsilon'_1, \dots, \varepsilon'_k) < \frac{\varepsilon_1}{M} + \sum_{j=2}^k \frac{r}{M^j} + \frac{r}{(M-1)M^k} = \frac{\varepsilon_1}{M} + \frac{r}{(M-1)M} = a(\varepsilon_1)
$$

is a contradiction. Therefore, we have

$$
b(\varepsilon_1, \dots, \varepsilon_k) \le \frac{\varepsilon_1}{M} + \sum_{j=2}^{k-1} \frac{r}{M^j} + \frac{r-1}{M^k} + \frac{1}{M^k} \le a(\varepsilon'_1)
$$

which is a contradiction. Therefore, we have

dition. Therefore, we have

\n
$$
b(\varepsilon_1 \ldots, \varepsilon_k) \leq \frac{\varepsilon_1}{M} + \sum_{j=2}^{k-1} \frac{r}{M^j} + \frac{r-1}{M^k} + \frac{1}{M^k} \leq a(\varepsilon_1')
$$

and the lemma is proved \blacksquare

Proof of Theorem 2.2. We define

branch and Q. Sun

\n
$$
\mathbf{rem\ 2.2. We define}
$$
\n
$$
O = \bigcup_{k=1}^{\infty} \bigcup_{\substack{(\epsilon_1, \ldots, \epsilon_{k-1}) \in \{0, 1, \ldots, r\}^{k-1} \\ \epsilon_k \in \{0, 1, \ldots, r-1\}}} A(\epsilon_1, \ldots, \epsilon_k)
$$

and

$$
A = \left(\bigcup_{i=0}^{N} (O + i)\right) \bigcup \left(\bigcup_{i=0}^{N-1} \left(i + \left(\frac{r}{M-1}, 1\right)\right)\right).
$$

Then ϕ is local polynomial on *A* by Lemmas 2.1 and 2.2. By Lemma 2.3, we obtain

IAI=(N_l)(l_Mrl)+N k=1 (' ' k-1)€(0,1 ,.... ,) kE{0,1,1} =(N - 1)(1 - M— 1) +N (i — (r+1 k-1 =N+ M— l' H()= (3.1) = aq(s)(e' - W. (3.2)

This proves the theorem \blacksquare

3. Asymptotic behavior of *M-band* symbol

We write

$$
{}_{N}^{M}H(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^{N} {}_{N}^{M}\widetilde{H}(\xi)
$$
\n(3.1)

and

$$
{}_{N}^{M}\widetilde{H}(\xi) = \sum_{s=0}^{N-1} a_{M}(s)(e^{i\xi} - 1)^{s}.
$$
 (3.2)

Then we have the following

Theorem 3.1. Let $_{N}^{\mathcal{M}}\tilde{H}(\xi)$ be defined by (3.2). Then $a_{\mathcal{M}}(0) = 1$ and the limit of

Theorem 3.1. Let
$$
\mathcal{R}(k)
$$
 be defined by (3.2). Then $a_M(0)$
\n $a_M(s)M^{-s}$ exists for $1 \le s \le N - 1$ and
\n
$$
\lim_{M \to \infty} {}^M_H H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^N \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s
$$

$$
\lim_{M \to \infty} M H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^{N} \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s
$$
\nwhere $\alpha(s) = \lim_{M \to \infty} a_M(s)M^{-s}$ $(0 \le s \le N - 1)$. Furthermore,
\n
$$
|M^{-(N-1)}a_M(N-1)| \le 2^{-N+1} \left(1 - \left|\frac{2}{\pi}\right|^{2N}\right)^{1/2}
$$

To prove Theorem 3.1, we need some lemmas.

Define

ve Theorem 3.1, we need some lemmas.
\n
$$
A(k, s) = \sum_{l=0}^{s} {2N - 1 + l \choose l} (2k\pi)^{-2l} A(k - 1, s - l) \qquad (k \ge 2)
$$

and

$$
A(1,s) = {2N-1+s \choose s} (2\pi)^{-2s}.
$$

Then $A(k, s) \ge A(k - 1, s)$ and $|A(k, s) - A(k - 1, s)| \le C_s k^{-2}$ holds for all $k \ge 2$, where C_s is a constant depending on *s* only. Therefore, $\lim_{k\to\infty} A(k, s)$ exists for all $0 \leq s \leq N - 1$. Denote its limit by A, $(0 \leq s \leq N - 1)$ (the explicit computation of *A, will* be given in Section 5). Then we have the following: $A(1, s) = {2N - 1 + s \choose s} (2\pi)^{-2s}.$
 $-1, s)$ and $|A(k, s) - A(k - 1, s)| \le C_s k^{-2}$ holds for all $k \ge 2$,
 t depending on s only. Therefore, $\lim_{k\to\infty} A(k, s)$ exists for all
 e its limit by A_s $(0 \le s \le N - 1)$ (the explicit comp

Lemma 3.1. Let $\frac{M}{N}a(s)$ be defined by (1.6). Then

$$
\lim_{M \to \infty} \frac{M}{N} a(s) M^{-2s} = A_s \qquad (0 \le s \le N - 1).
$$
 (3.3)

Proof. First we prove the assertion when *M* is odd. Denote $M' = \frac{M-1}{2}$. Then we may write

- 1. Denote its limit by A_s
$$
(0 \le s \le N - 1)
$$
 (the explicit co-
iven in Section 5). Then we have the following:
\n**3.1.** Let ${}^M_M a(s)$ be defined by (1.6). Then
\n
$$
\lim_{M \to \infty} {}^M_n a(s) M^{-2s} = A_s \qquad (0 \le s \le N - 1).
$$
\nFirst we prove the assertion when M is odd. Denote $M' = \frac{M}{2}$
\n
$$
{}^M_n a(s) = M^{2s} \sum_{s_1 + ... + s_{M'} = s} \left(\prod_{j=1}^{M'} \left(2N - 1 + s_j \right) (2j\pi)^{-2s_j} \right)
$$
\n
$$
\times \left(\prod_{j=1}^{M'} \left(1 + O\left(\frac{s_j j}{M}\right)^2 \right) \right)
$$
\n
$$
= M^{2s} \sum_{l=1}^{M'} \sum_{s_1 + ... + s_{M'} = s} \left(\prod_{j=1}^{M'} \left(2N - 1 + s_j \right) (2j\pi)^{-2s_j} \right)
$$
\n
$$
\times \left(O\left(\frac{s_l l}{M}\right)^2 \prod_{j=1}^{2^{l-1}} \left(1 + O\left(\frac{s_j j}{M}\right)^2 \right) \right)
$$
\n
$$
+ M^{2s} \sum_{s_1 + ... + s_{M'} = s} \prod_{j=1}^{M'} \left(2N - 1 + s_j \right) (2j\pi)^{-2s_j}
$$
\n
$$
= M^{2s} \left(\sum_{l=1}^{M'} I_{l,M}(s) + A(M', s) \right)
$$

where $O(\frac{s_j j}{M})^2$ denotes a term bounded by $C(\frac{s_j j}{M})^2$ for some constant *C* independent of

M. Obviously, we have

N. Bi, L. Debnath and Q. Sun
\nviously, we have
\n
$$
0 \le I_{l,M}(s)
$$
\n
$$
\le C \sum_{s_1=1}^s {2N-1+s_1 \choose s_1} (2l\pi)^{-2(s_1-1)} M^{-2}
$$
\n×
$$
\sum_{s_1+...+s_{l-1}+s_{l+1}+...+s_{M'}=s-s_l} \left(\prod_{j\neq l, 1 \le j \le M'} {2N-1+s_j \choose s_j} (2j\pi)^{-2s_j} \right)
$$
\n≤ $CA(M',s) \sum_{s_1=1}^s (2l\pi)^{-2(s_1-1)} M^{-2}$
\n≤ $CM^{-2}A(M',s)$.

Hereafter the letter *C* would denote a constant independent of *M* which may be different at different instances. Therefore, we get $M^{-2s}M_{(s)}(s) \to A_s$ as $M \to \infty$ when *M* is odd.

Hereafter the letter C would denote a constant independent of M which may be different
at different instances. Therefore, we get
$$
M^{-2s}Ma(s) \to A_s
$$
 as $M \to \infty$ when M is odd
When M is even, we may write

$$
M^{-2s}Ma(s)
$$

$$
= \sum_{s_{M/2}=0}^{s} {N-1+s_{M/2} \choose s_{M/2}} (2M)^{-2s_{M/2}}
$$

$$
\times \left(M^{-2(s-s_{M/2})} \sum_{s_1+\ldots+s_{M/2-1}=s-s_{M/2}} \prod_{j=1}^{M/2-1} {2N-1+s_j \choose s_j} (2\sin \frac{j\pi}{M})^{-2s_j} \right)
$$

$$
= \sum_{s_{M/2}=0}^{s} {N-1+s_{M/2} \choose s_{M/2}} 4^{-s_{M/2}} M^{-2s_{M/2}} I(s-s_{M/2}).
$$

Using the same procedure to prove the assertion when *M* is odd, we may prove that $I(t) \to A_t$ as $M \to \infty$ for all $0 \le t \le N - 1$. Thus also $M^{-2s}M_a(s) \to A_s$ as $M \to \infty$ when *M* is even **^I**

Lemma 3.2. *Let* $_{N}^{M}a(N - 1)$ *be defined by* (1.11). *Then*

$$
\infty \text{ for all } 0 \le t \le N - 1. \text{ Thus also } M^{-2s}M_{\alpha}(s)
$$

Let
$$
M_{\alpha}(N-1) \text{ be defined by (1.11). Then}
$$

$$
M^{-2N+2}M_{\alpha}(N-1) \le \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right)2^{-2(N-1)}.
$$

oved by Bi et al. in [1] that

$$
\sum_{s=0}^{M-1} \left|M_{\alpha}(s) + \frac{2\pi s}{M}\right|^2 = 1.
$$

Proof. It is proved by Bi et al. in [1] that

$$
\int 4^{-M/2} M^{-(2M/2)} (s - s_{M/2}).
$$

\nprove the assertion when *M* is odd, we may prove that
\n $0 \le t \le N - 1$. Thus also $M^{-2s} N a(s) \to A_s$ as $M \to \infty$
\n-1) be defined by (1.11). Then
\n
$$
a(N - 1) \le \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right) 2^{-2(N-1)}.
$$
\net al. in [1] that
\n
$$
\sum_{s=0}^{M-1} \left| \frac{M}{N} H\left(\xi + \frac{2\pi s}{M}\right) \right|^2 = 1.
$$
\n(3.4)
\n
$$
\frac{e^{iM\xi}}{i(\xi + 2s\pi/M)} \left| \frac{2N}{M} \le \left(\frac{\sin \frac{M\xi}{2}}{M \sin \frac{\xi}{2}}\right)^{2N} \ge \left(\frac{2}{\pi}\right)^{2N}
$$

We note that

$$
\text{proved by Bi et al. in [1] that}
$$
\n
$$
\sum_{s=0}^{M-1} \left| \frac{M}{N} H\left(\xi + \frac{2\pi s}{M}\right) \right|^2 = 1.
$$
\n
$$
\sum_{s=0}^{M-1} \left| \frac{1 - e^{iM\xi}}{M(1 - e^{i(\xi + 2s\pi/M)})} \right|^{2N} \ge \left| \frac{\sin \frac{M\xi}{2}}{M \sin \frac{\xi}{2}} \right|^{2N} \ge \left(\frac{2}{\pi}\right)^{2N}
$$

when $|\xi| \leq \frac{\pi}{M}$. Therefore, we get

Asymptotic Behavior of *M*-Band Scaling Functions
\nherefore, we get
\n
$$
\sum_{s=0}^{M-1} \left| \frac{1 - e^{iM\xi}}{M(1 - e^{i(\xi + 2s\pi/M)})} \right|^{2N} \ge \left(\frac{2}{\pi} \right)^{2N} \qquad (\xi \in \mathbb{R}).
$$
\n(3.5)

We recall that

$$
\begin{aligned}\n\text{erefore, we get} \\
\sum_{n=0}^{n-1} \left| \frac{1 - e^{iM\xi}}{M(1 - e^{i(\xi + 2s\pi/M)})} \right|^{2N} &\geq \left(\frac{2}{\pi}\right)^{2N} \qquad (\xi \in \mathbb{R}) \\
|\binom{M}{N}H(\xi)|^2 &= \left(\frac{\sin\frac{M\xi}{2}}{M\sin\frac{\xi}{2}}\right)^{2N} \sum_{s=0}^{N-1} 2^{2s} \binom{M}{N} a(s) \left(\sin\frac{\xi}{2}\right)^{2s} \\
\text{where it follows from (3.4) and (3.5) that}\n\end{aligned}
$$

and
$$
{}_{N}^{M}a(0) = 1.
$$
 Then it follows from (3.4) and (3.5) that
\n
$$
1 - \left(\frac{2}{\pi}\right)^{2N} \geq M^{-2N+2} {}_{N}^{M}a(N-1) \left(\sin \frac{M\xi}{2}\right)^{2(N-1)}
$$
\n
$$
\times 2^{2(N-1)} \sum_{s=0}^{M-1} \frac{\sin^2 \frac{M\xi}{2}}{M^2 \sin^2(\frac{\xi}{2} + \frac{s\pi}{M})}
$$
\n
$$
= M^{-2N+2} {}_{N}^{M}a(N-1)2^{2(N-1)} \left(\sin \frac{M\xi}{2}\right)^{2(N-1)}
$$
\nSubstituting $\xi = \frac{\pi}{M}$ in (3.6) gives the lemma
\nProof of Theorem 3.1. Let
\n
$$
Q(t) = \sum_{s=0}^{N-1} A_{s}t^{s}.
$$
\n(3.7)
\nBy Lemma 3.1, we can write

Substituting $\xi = \frac{\pi}{M}$ in (3.6) gives the lemma **I**

Proof of Theorem 3.1. Let

$$
Q(t) = \sum_{s=0}^{N-1} A_s t^s.
$$
 (3.7)

By Lemma 3.1, we can write

(3.6) gives the lemma **R**
\nn 3.1. Let
\n
$$
Q(t) = \sum_{s=0}^{N-1} A_s t^s.
$$
\nwrite
\n
$$
\widetilde{P}(t) = \sum_{s=0}^{N-1} \underset{M}{M} a(s) t^s = \sum_{s=0}^{N-1} \beta_M(s) (M^2 t)^s
$$

where $\beta_M(s) \to A_s$ as $M \to \infty$. We then set

$$
Q(t) = \prod_{j=1}^{N-1} \left(\frac{t-t_j}{-t_j} \right).
$$

Then there exists a sequence $\{t_{j,M}\}_{j=1}^{N-1}$ such that

$$
Q(t) = \prod_{j=1} \left(\frac{t - t_j}{-t_j} \right).
$$

$$
\{t_{j,M}\}_{j=1}^{N-1} \text{ such that}
$$

$$
\widetilde{P}(t) = \prod_{j=1}^{N-1} \left(\frac{M^2 t - t_{j,M}}{-t_{j,M}} \right)
$$

and $t_{j,M} \to t_j$ as $M \to \infty$.

Recall that $\tilde{P}(t) > 0$ when $t > 0$. Therefore, $t_j \notin (0, \infty)$, and $t_{j,M} \notin (0, \infty)$ as M is 824 N. Bi, L. Debnath and Q. Sun
Recall that $\tilde{P}(t) > 0$ when $t > 0$
sufficiently large. When $t = 2 - e^{i\xi}$ $-e^{-i\xi}$, we may write *M*₂ *t* $\tilde{P}(t) > 0$ when $t > 0$. Therefore, $t_j \notin (0, \infty)$, and $t_{j,M} \notin (0, \infty)$
arge. When $t = 2 - e^{i\xi} - e^{-i\xi}$, we may write
 $M^2t - t_{j,M} = [M(e^{i\xi} - 1) - \theta_{j,M}] [M(e^{-i\xi} - 1) - \theta_{j,M}] \times \beta_{j,M}$

$$
M^{2}t-t_{j,M}=[M(e^{i\xi}-1)-\theta_{j,M}][M(e^{-i\xi}-1)-\theta_{j,M}]\times\beta_{j,M}
$$

where

Bi, L. Debnath and Q. Sun
\nthat
$$
\tilde{P}(t) > 0
$$
 when $t > 0$. Therefore, $t_j \notin (0, \infty)$, and $t_{j,M} \notin (0, \infty)$
\n t large. When $t = 2 - e^{i\xi} - e^{-i\xi}$, we may write
\n
$$
M^2t - t_{j,M} = [M(e^{i\xi} - 1) - \theta_{j,M}][M(e^{-i\xi} - 1) - \theta_{j,M}] \times \beta_{j,M}
$$
\n
$$
\theta_{j,M} = \frac{-t_{j,M} - \sqrt{t_{j,M}^2 - 4t_{j,M}M^2}}{2M} \to -\sqrt{-t_j}
$$
\nas $M \to \infty$.

Furthermore, the real part of $\theta_{j,M}$ is always less than zero when *M* is large enough. Therefore, the root of $M(z-1) - \theta_{j,M}$ is contained in the open unit disk and

$$
\prod_{j=1}^N \frac{M(e^{i\xi}-1)-\theta_{j,M}}{-\theta_{j,M}}
$$

is a trigonometrical polynomial with real coefficients. By the Riesz lemma [2: p. 172/Lemma 6.1.3] we obtain

$$
{N}^{M}\widetilde{H}(\xi)=\prod{j=1}^{N}\left[\frac{M(e^{i\xi}-1)-\theta_{j,M}}{-\theta_{j,M}}\right].
$$

Hence

$$
M(z-1) = \sigma_{j,M} \text{ is contained in the open}
$$
\n
$$
\prod_{j=1}^{N} \frac{M(e^{i\xi} - 1) - \theta_{j,M}}{-\theta_{j,M}}
$$
\n
$$
\text{polynomial with real coefficients. By the obtain}
$$
\n
$$
\begin{aligned}\nM\widetilde{H}(\xi) &= \prod_{j=1}^{N} \left[\frac{M(e^{i\xi} - 1) - \theta_{j,M}}{-\theta_{j,M}} \right]. \\
\lim_{M \to \infty} \sum_{s=0}^{N-1} M^{-s} a_M(s) t^s &= \prod_{j=1}^{N-1} \left(\frac{t + \sqrt{-t_j}}{\sqrt{-t_j}} \right) \\
\int_{0}^{M^{-s} a_M(s)} \text{ exists for all } 0 \le s \le N - 1.\n\end{aligned}
$$
\n
$$
\left(\frac{1 - e^{i\xi}}{M(1 - e^{i\xi/M})} \right)^N \sum_{s=0}^{N-1} (a_M(s)M^{-s}) \times \left(M(s)M^{-s} \right)^{N} \left(\frac{M(s)}{M(s)M^{-s}} \right)^{N} \left(\frac{M(s)}{M(s)M^{-s
$$

and the limit $\lim_{M\to\infty} M^{-s} a_M(s)$ exists for all $0 \le s \le N - 1$.

We observe that

$$
{NH}^M\left(\frac{\xi}{M}\right)=\left(\frac{1-e^{i\xi}}{M(1-e^{i\xi/M})}\right)^N\sum{s=0}^{N-1}(a_M(s)M^{-s})\times\left(M(e^{i\xi/M}-1)\right)^s.
$$

Then we find that

$$
\lim_{M\to\infty} {}_{N}^{M}H\left(\frac{\xi}{M}\right)=\left(\frac{1-e^{i\xi}}{-i\xi}\right)^{N}\sum_{s=0}^{N-1}\alpha(s)(i\xi)^{s}.
$$

We recall from Lemma 3.2 that

$$
I\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{M(1 - e^{i\xi/M})}\right)^{N} \sum_{s=0}^{N-1} (a_M(s)M^{-s}) \times \left(M(e^{i\xi/M} - 1)\right)
$$

and that

$$
\lim_{M \to \infty} \frac{M}{N} H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^{N} \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s.
$$

from Lemma 3.2 that

$$
\prod_{j=1}^{N-1} \frac{1}{-t_{j,M}} = M^{-2(N-1)} \frac{M}{N} \alpha(N-1) \le 2^{-2(N-1)} \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right).
$$

obtain

Then we obtain

Find that

\n
$$
\lim_{M \to \infty} \frac{M}{N} H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^{N} \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s.
$$
\nfrom Lemma 3.2 that

\n
$$
\prod_{j=1}^{N-1} \frac{1}{-t_{j,M}} = M^{-2(N-1)} \frac{M}{N} \alpha(N-1) \leq 2^{-2(N-1)} \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right).
$$
\nobtain

\n
$$
|M^{-N+1} \alpha_M(N-1)| = \left|\prod_{j=1}^{N} \frac{1}{\sqrt{-t_{j,M}}} \right| \leq 2^{-N+1} \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right)^{1/2}.
$$
\n(3.8)

\n3.1 is thus proved

Theorem 3.1 is thus proved \blacksquare

4. Pointwise convergence and LP-convergence

The B-spline B_N of degree $N-1$ is defined with the help of the Fourier transform by

$$
\widehat{B_N}(\xi)=\Big(\frac{1-e^{i\xi}}{-i\xi}\Big)^N.
$$

Theorem 4.1. Let $_{N}^{M}\phi$ be the solution of the refinement equation (1.1) with symbol $\widehat{B_N}(\xi) = \left(\frac{1-e^{i\xi}}{-i\xi}\right)^N$.
Theorem 4.1. Let $\stackrel{M}{N}\phi$ be the solution of the refinement equation (1
H and $1 \leq p < \infty$. Then $\stackrel{M}{N}\phi$ converges pointwisely and in *LP*-norm to *First* **convergence and**
 B_N of degree $N - 1$ is defin
 $\widehat{B_N}(\xi) =$
 n 4.1. Let $\stackrel{M}{N}\phi$ be the solution
 $p < \infty$. Then $\stackrel{M}{N}\phi$ converges
 $g(x) = B_N(x)$
 $\lim_{M \to \infty} M^{-s} a_M(s)$ and $B^{\{0\}}$

Figure 2.2.2.2.3.3.3.3.4.4.4.5. The equation is given by:

\n
$$
\widehat{B_N}(\xi) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^N.
$$
\nwhere $B_N(\xi) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^N$.

\nwhere $B_N(\xi) = \frac{1 - e^{i\xi}}{i\xi}$ is the solution of the *refinement equation* (1.1) with symbol $\frac{M\phi}{N\phi}$ converges pointwise by *and in L^p-norm to* and $B_N(\xi) = B_N(x) + \sum_{s=1}^{N-1} \alpha(s)B_N^{(s)}(x)$ (4.1) is a solution of B_N . The following conditions are:

\n
$$
B_N(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{N-1} \alpha(s)B_N^{(s)}(x)
$$
\nwhere B_N is the solution of B_N . The function B_N is the solution of B_N . The function B_N is the solution of B_N . The function B_N is the solution of B_N . The function B_N is the solution of B_N . The function B_N is the solution of B_N . The function B_N is the solution of B_N is the solution of B_N . The function B_N is the solution of B_N is the solution of B_N . The function B_N is the solution of B_N is the solution of B_N . The function B_N is the solution of B_N is the solution of B_N .

where $\alpha(s) = \lim_{M \to \infty} M^{-s} a_M(s)$ and $B_N^{(s)}$ is the s-th derivative of B_N . Furthermore, *g is orthonormal.*

We need the following lemmas to prove this theorem whose proof will be given later. For a compactly supported integrable function *f,* the k-moment of *f is* defined by

$$
\lim_{k \to \infty} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k} \log \frac{1}{k}
$$
\n
$$
\text{where } \log \frac{1}{k} \log \frac{1
$$

Then we have the following

Lemma 4.1. *Let NO be the solution of the refinement equation (1.1) with symbol* ^f*H. Then*

have the following
\n
$$
\begin{aligned}\n\text{na } 4.1. \quad \text{Let } \substack{M \\ N} \phi \text{ be the solution of the refinement equation (1.1) with symbol} \\
\text{and} \\
\lim_{M \to \infty} m_k \binom{M}{N} \phi &= m_k(B_N) + \sum_{j=1}^{N-1} \alpha(j) m_k(B_N^{(j)}) \qquad (0 \le k \le N-1) \qquad (4.2)\n\end{aligned}
$$

holds.

Proof. Let $D = i \frac{d}{d\xi}$ be a differential operator. Then for any compactly supported integrable function f, we have $m_k(f) = (D^k \hat{f})(0)$. Define

$$
h_M(\xi)=\prod_{j=1}^{\infty}\underset{N}{M}\widetilde{H}\left(\frac{\xi}{M^j}\right).
$$

Then we have

$$
h_M(\xi) = \stackrel{M}{N}\widetilde{H}\left(\frac{\xi}{M}\right)h_M\left(\frac{\xi}{M}\right)
$$

and

function
$$
f
$$
, we have $m_k(f) = (D^k \hat{f})(0)$. Define

\n
$$
h_M(\xi) = \prod_{j=1}^{\infty} \frac{M}{N} \tilde{H}\left(\frac{\xi}{M^j}\right).
$$
\ne

\n
$$
h_M(\xi) = \frac{M}{N} \tilde{H}\left(\frac{\xi}{M}\right) h_M\left(\frac{\xi}{M}\right)
$$
\n
$$
D^k h_M(\xi) = M^{-k} \sum_{j=0}^k \binom{k}{j} \left(D^j \frac{M}{N} \tilde{H}\right) \left(\frac{\xi}{M}\right) \left(D^{k-j} h_M\right) \left(\frac{\xi}{M}\right).
$$

Hence, we find

$$
(1 - M^{-k})(D^k h_M)(0) = M^{-k} \sum_{j=1}^k {k \choose j} (D^j{}_N^M \widetilde{H})(0) (D^{k-j} h_M)(0).
$$

From Theorem 3.1, $M^{-j}(D^j {}_N^M \widetilde{H})(0) \rightarrow (-1)^j j! \alpha(j)$ as $M \rightarrow \infty$ $(0 \le j \le N - 1)$.
Hence $D^k h_M(0) \rightarrow (-1)^k k! \alpha(k)$ as $M \rightarrow \infty$ $(0 \le k \le N - 1)$.

We recall that

$$
D^j N^j \widetilde{H} (0) \to (-1)^j j! \alpha(j) \text{ as } M
$$

\n
$$
D^j N^j \widetilde{H} (0) \to (-1)^j j! \alpha(j) \text{ as } M
$$

\n
$$
N^j \alpha(k) \text{ as } M \to \infty \quad (0 \le k \le N - 1)
$$

\n
$$
N^j \widetilde{H} (\xi) = \left(\frac{1 - e^{iM \xi}}{M (1 - e^{i \xi})} \right)^N N^j \widetilde{H} (\xi).
$$

\n
$$
h_M(\xi), \text{ and}
$$

\n
$$
0(0) = \sum_{j=0}^k {k \choose j} (D^j \widehat{B_N})(0) (D^{k-j})
$$

Therefore, $\widehat{N}\phi(\xi) = \widehat{B_N}(\xi)h_M(\xi)$, and

$$
(D^k \widehat{M\phi})(0)=\sum_{j=0}^k \binom{k}{j} (D^j \widehat{B_N})(0) (D^{k-j}h_M)(0).
$$

Hence

 $\ddot{}$

e
$$
D^*h_M(0) \to (-1)^*k! \alpha(k)
$$
 as $M \to \infty$ $(0 \le k \le N - 1)$.
\nWe recall that
\n
$$
\frac{M}{N}H(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^N \frac{M}{N} \widetilde{H}(\xi).
$$
\nefore $\sqrt{M}\phi(\xi) = \widehat{B_N}(\xi)h_M(\xi)$, and
\n $(D^k \widehat{M}\phi)(0) = \sum_{j=0}^k {k \choose j} (D^j \widehat{B_N})(0)(D^{k-j}h_M)(0).$
\ne
\n $(D^k \widehat{M}\phi)(0) \to \sum_{j=0}^k \frac{k!}{(k-j)!} (-1)^j \alpha(j)D^{k-j} \widehat{B_N}(0) = \sum_{j=0}^{N-1} \alpha(j) \int_{\mathbb{R}} x^k B_N^{(j)}(x) dx$
\nthe lemma is proved
\nLemma 4.2. Let $d_{j,M}$ $(0 \le j \le N - 1)$ be numbers such that
\n
$$
\frac{M}{N}\phi\left(\frac{x+j}{M}\right) = d_{j,M}\frac{M}{N}\phi(x) + Q_{j,M}(x) \qquad (x \in (0,1), 0 \le j \le N - 1) \qquad (4.3)
$$

and the lemma is proved \blacksquare

Lemma 4.2. Let $d_{j,M}$ $(0 \leq j \leq N-1)$ be numbers such that

$$
M \phi \left(\frac{x+j}{M} \right) = d_{j,M} M \phi(x) + Q_{j,M}(x) \qquad (x \in (0,1), \, 0 \le j \le N-1) \tag{4.3}
$$

holds for a polynomial $Q_{j,M} \in \Pi_{N-1}$ *. Then*

$$
\beta_j = \lim_{M \to \infty} d_{j,M} = (-1)^{N-1} \alpha (N-1) \sum_{s=0}^j \frac{(-1)^s N!}{s!(N-s)!}
$$

$$
2^{N-1} |\alpha (N-1)| \le \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right)^{1/2} \qquad (0 \le j \le N)
$$

and

$$
|\beta_j| \le 2^{N-1} |\alpha(N-1)| \le \left(1 - \left(\frac{2}{\pi}\right)^{2N}\right)^{1/2} \qquad (0 \le j \le N-1).
$$

We recall that

$$
\sum_{k \in \mathbb{Z}} (x+k)^j \frac{M}{N} \phi(x+k) = m_j \left(\frac{M}{N}\phi\right) \qquad (0 \le j \le N-1).
$$

Proof. We recall that

$$
\sum_{k \in \mathbb{Z}} (x+k)^j \, M_N \phi(x+k) = m_j \, M_N \phi) \qquad (0 \le j \le N-1).
$$
\n
$$
\sum_{k \in \mathbb{Z}} k^j \, M_N \phi(\cdot + k) = Q_j \in \Pi_{N-1} \qquad (0 \le j \le N-1)
$$

Then we obtain

$$
\sum_{k\in\mathbb{Z}}k^{j}{}_{N}^{M}\phi(\cdot+k)=Q_{j}\in\Pi_{N-1}\qquad(0\leq j\leq N-1)
$$

or, in matrix form,

Asymptotic Behavior of *M*-Band Scaling Functions
\nmatrix form,
\n
$$
\begin{pmatrix}\n1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & N \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2^{N-1} & \cdots & N^{N-1}\n\end{pmatrix}\n\begin{pmatrix}\nM\phi(x+1) \\
N\phi(x+2) \\
\vdots \\
M\phi(x+N)\n\end{pmatrix} = -\n\begin{pmatrix}\n1 \\
0 \\
\vdots \\
0\n\end{pmatrix}\n\begin{pmatrix}\nM\phi(x) \\
N\phi(x) + \begin{pmatrix}\nQ_0(x) \\
Q_1(x) \\
\vdots \\
Q_{N-1}(x)\n\end{pmatrix}
$$
\n, 1). Therefore, we get
\n
$$
\begin{pmatrix}\nM\phi(x+j) = \frac{(-1)^j N!}{j!(N-j)!} M\phi(x) + \tilde{Q}_j(x) \\
\vdots \\
Q_j \in \Pi_{N-1}.\n\end{pmatrix}
$$
\n
$$
\begin{pmatrix}\n2M\xi & 1 \end{pmatrix} \times N^{N-1}
$$
\n(4.4)

on $(0, 1)$. Therefore, we get

$$
N^{N-1} / \sqrt{M}\phi(x+N) / \sqrt{0} / \sqrt{Q_{N-1}(x) / Q}
$$

get

$$
M\phi(x+j) = \frac{(-1)^j N!}{j!(N-j)!} M\phi(x) + \tilde{Q}_j(x)
$$
 (4.4)

where $\tilde{Q}_j \in \Pi_{N-1}$. It follows from Theorem 3.1 that $M_{\mathbf{H}(\xi)} = \int e^{iM\xi} - 1 \int_{N} \int_{N-1}^{N-1} f(x) dx$

^NMH() (M(e' - 1)) *aM(s)(e(eiM MN-1* ⁼*ci(N -* 1) (1_ e') *(eM -* i)N_i + *o(1)e* **^M** *1:* k=0

where $o(1)$ means a number tending to zero as $M \to \infty$. From (2.4) and (4.4), there $\text{exist polynomials } Q_{\boldsymbol{j}} \in \Pi_{N-1} \text{ such that }$

here
$$
o(1)
$$
 means a number tending to zero as $M \to \infty$. From (2.4) and (4.4), the
\n
$$
\frac{M}{N}\phi\left(\frac{x+j}{M}\right) = \sum_{l=0}^{j} c_{j-l}\frac{M}{N}\phi(x+l)
$$
\n
$$
= (-1)^{N-1}\alpha(N-1)\sum_{l=0}^{j} (1+o(1))\frac{M}{N}\phi(x+l)
$$
\n
$$
= \left((-1)^{N-1}\alpha(N-1)\sum_{l=0}^{j} (-1)^{l} \binom{N}{l}\right) \frac{M}{N}\phi(x) + o(1)\frac{M}{N}\phi(x) + Q_{j}(x).
$$
\nThis proves the first assertion.
\nObserve that $\sum_{s=0}^{N} (-1)^{s} \binom{N}{s} = 0$. Then we have\n
$$
\left|\sum_{s=0}^{j} (-1)^{s} \binom{N}{s}\right| = \left|\sum_{s=0}^{N-j-1} (-1)^{s} \binom{N}{s}\right|.
$$
\nis easy to see that\n
$$
\left|\sum_{s=0}^{j} (-1)^{s} \binom{N}{s}\right| \leq \binom{N}{j}
$$
\nthen $j < \frac{N}{2}$. Then from the identity $2^{N} = (1+1)^{N} = \sum_{s=0}^{N} \binom{N}{s}$ we get $\binom{N}{(N-1)/2}$
\n
$$
= \frac{1}{N} \sum_{s=0}^{N} (-1)^{s} \binom{N}{s} \leq 2^{N-1}
$$
. Observe that

This proves the first assertion.

Observe that $\sum_{s=0}^{N} (-1)^{s} {N \choose s} = 0$. Then we have

Use the that
$$
\sum_{s=0}^{n-1} (-1)^s \binom{N}{s} = \left| \sum_{s=0}^{N-j-1} (-1)^s \binom{N}{s} \right|
$$
.

\nIt is easy to see that

\n
$$
\left| \sum_{s=0}^{j} (-1)^s \binom{N}{s} \right| \leq \binom{N}{j}
$$
\nwhen $j < \frac{N}{2}$. Then from the identity $2^N = (1+1)^N = \sum_{s=0}^{N} \binom{N}{s}$.

It is easy to see that

$$
\left|\sum_{s=0}^j (-1)^s \binom{N}{s}\right| \le \binom{N}{j}
$$

when $j < \frac{N}{2}$. Then from the identity $2^N = (1+1)^N = \sum_{s=0}^N {N \choose s}$ we get ${N \choose (N-1)/2} \le 2^{N-1}$ when *N* is odd and ${N \choose N/2-1} \le 2^{N-1}$. Observe that

$$
\left|\sum_{s=0}^{j}(-1)^{s} {N \choose s}\right| \le {N \choose j}
$$

Then from the identity $2^{N} = (1+1)^{N} = \sum_{s=0}^{N} {N \choose s}$ we get
l is odd and ${N \choose N/2-1} \le 2^{N-1}$. Observe that

$$
{N \choose j} \le {N \choose (N-1)/2} \text{ when } N \text{ is odd}
$$

$$
{0 \le j < \frac{N}{2}}.
$$

and assertion follows from Theorem 3.1

Thus, the second assertion follows from Theorem **3.11**

Proof of Theorem 4.1. It follows from (2.1) that

$$
A(x)_N^M \Phi(x) = (m_0 \begin{pmatrix} M \phi \\ N \phi \end{pmatrix}, \dots, m_{N-1} \begin{pmatrix} M \phi \\ N \phi \end{pmatrix})^T \text{ on } \left(\frac{N-1}{M-1}, 1 \right)
$$

where
$$
N \Phi(x) = \begin{pmatrix} M \phi(x), \dots, M \phi(x+N-1) \end{pmatrix}.
$$
 Recall that

$$
M_{\phi}(x + N - 1)).
$$
 Recall that

$$
m_k(M_{\phi}) \to \sum_{j=0}^{N-1} \alpha(j) m_k(B_N^{(j)})
$$

by Lemma 4.1. Therefore,

$$
m_k(\substack{M \ N} \phi) \to \sum_{j=0}^N \alpha(j) m_k(B_N^{(j)})
$$

Therefore,

$$
\substack{M \ \Phi(x) \to \sum_{j=0}^{N-1} \alpha(j) A^{-1}(x) \left(m_0(B_N^{(j)}), \dots, m_{N-1}(B_N^{(j)}) \right)^T}
$$

pointwisely on (0, 1). This proves that

$$
{}_N^M \phi(x) \to g(x) = \sum_{j=0}^{N-1} \alpha(j) B_N^{(j)}(x).
$$

 $M_N^M \phi(x)$ reduces to prove that $M_N^M \phi(x)$ is uniformly bounded. Recall that

$$
\underset{N}{M}\phi(x) \to g(x) = \sum_{j=0} \alpha(j)B_{N}^{(j)}(x).
$$
\nObviously, by the dominated convergence theorem, the *L^p*-convergence (1 $\leq p < \infty$) of $\underset{N}{M}\phi(x)$ reduces to prove that $\underset{N}{N}\phi(x)$ is uniformly bounded. Recall that

\n
$$
\underset{N}{M}\phi\left(\frac{x+j}{M}\right) = d_{j,M}\underset{N}{M}\phi(x) + Q_{j,M}(x) \quad \text{and} \quad |d_{j,M}| \leq \left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{1/2}
$$
\nwhen *M* is sufficiently large and *Q_{j,M}* is uniformly bounded by *C*. Thus we get

when M is sufficiently large and $Q_{j,M}$ is uniformly bounded by $C.$ Thus we get

$$
\begin{pmatrix}\nM \\
M\n\end{pmatrix}\n\begin{pmatrix}\n\sqrt{\pi} \\
\sqrt{\pi}\n\end{pmatrix}
$$
\n
$$
I \text{ is sufficiently large and } Q_{j,M} \text{ is uniformly bounded by } C. \text{ Thus we get}
$$
\n
$$
\sup_{x \in A(\epsilon_1, \ldots, \epsilon_k)} |M \phi(x)| \le C \left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{\frac{1}{2}k} + C \sum_{j=0}^{k-1} \left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{\frac{1}{2}j}
$$
\n
$$
\le C \left(\frac{\pi}{2}\right)^{2N}.
$$
\n
$$
\text{vs from the proof of Theorem 2.1 that}
$$
\n
$$
\bigcup_{k=1}^{\infty} \bigcup_{\substack{\epsilon_i \in \{0, 1, \ldots, N-1\}, 1 \le i \le k-1 \\ \epsilon_k \in \{0, 1, \ldots, N-2\}}}\nA(\epsilon_1, \ldots, \epsilon_k) \cup \left(\frac{N-1}{M-1}, 1\right)
$$

It follows from the proof of Theorem 2.1 that

$$
\leq C \left(\frac{\pi}{2}\right)^{2N}.
$$

ne proof of Theorem 2.1 that

$$
\bigcup_{k=1}^{\infty} \bigcup_{\substack{\epsilon_i \in \{0, 1, \ldots, N-1\}, 1 \leq i \leq k-1 \\ \epsilon_k \in \{0, 1, \ldots, N-2\}}} A(\epsilon_1, \ldots, \epsilon_k) \cup \left(\frac{N-1}{M-1}, 1\right)
$$

has Lebesgue measure 1. This proves that $\frac{M}{N}\phi$ is uniformly bounded. Recall that $\frac{M}{N}\phi$ is orthonormal. Then the limit g of $\frac{M}{N}\phi$ in the L^2 -norm is also orthonormal \blacksquare

5. The limit function

In this section we will give a method to construct the limit function g in Theorem 4.1. Let

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\n**function**
\n
$$
= \text{will give a method to construct the limit function } g \text{ in Theorem 4.1.}
$$
\n
$$
G(z) = \sum_{s=0}^{N-1} \alpha(s) z^s \quad \text{and} \quad Q(z) = \sum_{s=0}^{N-1} A_s z^{2s}.
$$
\n
$$
= G(iz)G(-iz) = Q(z). \tag{5.1}
$$
\n
$$
= \text{for } \frac{M}{N}\phi \text{ in Theorem 4.1. Then } g \text{ is unique determined by } G.
$$
\n
$$
= \text{Out } A_s \quad (0 \le s \le N-1) \text{ explicitly. Observe that } g \text{ is orthonormal by}
$$
\n
$$
= \text{Out } A_s \text{ and } G(z) = \text{Out }
$$

Then, by the proof of Theorem 3. 1, we get

$$
G(iz)G(-iz) = Q(z). \tag{5.1}
$$

Let g be the limit of $\frac{M}{N}\phi$ in Theorem 4.1. Then g is unique determined by G.

Now we compute A_s $(0 \le s \le N - 1)$ explicitly. Observe that g is orthonormal by Theorem 4.1. Then we have Theorem 4.1. Then we have
 $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1$ ($\xi \in \mathbb{R}$).

By Theorem 4.1 and by the orthonormality of $\frac{M}{N}\phi$, we get $\hat{g}(\xi)$
 $\sum_{k \in \mathbb{Z}} O(\xi + 2k\pi)^{1/2} \cdot (1 + 2k\pi)^{1/2} = 1$ for all

$$
G(iz)G(-iz) = Q(z).
$$

1 Theorem 4.1. Then g is unique
 $0 \le s \le N - 1$ explicitly. Observe

$$
\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1 \qquad (\xi \in \mathbb{R}).
$$

 $)=G(i\xi)\widehat{B_N}(\xi)$ and

$$
G(z) = \sum_{s=0}^{N-1} \alpha(s)z^s \quad \text{and} \quad Q(z) = \sum_{s=0}^{N-1} A_s z^{2s}.
$$

of of Theorem 3.1, we get

$$
G(iz)G(-iz) = Q(z).
$$
(5.1)
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oute A_s $(0 \le s \le N-1)$ explicitly. Observe that g is orthonormal by
en we have

$$
\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1 \quad (\xi \in \mathbb{R}).
$$

and by the orthonormality of $\stackrel{M}{N}\phi$, we get $\hat{g}(\xi) = G(i\xi)\widehat{B_N}(\xi)$ and

$$
\sum_{k \in \mathbb{Z}} Q(\xi + 2k\pi) |\widehat{B_N}(\xi + 2k\pi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}. \qquad (5.2)
$$

$$
\sum_{k \in \mathbb{Z}} Q(i\xi + N).
$$
 Then $\widehat{B_{2N}}(\xi) = |\widehat{B_N}(\xi)|^2$ and the function \tilde{g} defined by

Let $\widetilde{B}_{2N}(x) = B_{2N}(x + N)$. Then $\widehat{\widetilde{B}_{2N}}(\xi) = |\widehat{B}_{N}(\xi)|^{2}$ and the function \tilde{g} defined by 1 nen, by the proof of Theorem 3.1, we get
 $G(iz)G(-iz) = Q(z)$. (5.1)

Let g be the limit of $\frac{M}{N}\phi$ in Theorem 4.1. Then g is unique determined by G.

Now we compute A_s , $(0 \le s \le N - 1)$ explicitly. Observe that g is orth value zero at integer lattice except $\tilde{g}(0) = 1$. Hence A_s satisfies the equation by the orthonormality of $\frac{M}{N}\phi$, w
 $\sum_{k \in \mathbb{Z}} Q(\xi + 2k\pi) |\widehat{B_N}(\xi + 2k\pi)|^2 = 1$
 $\kappa(\xi) = B_{2N}(\xi + N)$. Then $\widehat{B_{2N}}(\xi) = |\widehat{B_N}(\xi)|$
 $\sum_{s=0}^{N-1} A_s \widehat{B_{2N}^{(2s)}}(x)$ satisfies the interpolation of

ro at integer l

$$
\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1 \quad (\xi \in \mathbb{R}).
$$

By Theorem 4.1 and by the orthonormality of $\frac{M}{N}\phi$, we get $\hat{g}(\xi) = G(i\xi)\widehat{B_N}(\xi)$ and

$$
\sum_{k \in \mathbb{Z}} Q(\xi + 2k\pi)|\widehat{B_N}(\xi + 2k\pi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}. \tag{5.2}
$$

Let $\widetilde{B}_{2N}(x) = B_{2N}(x + N)$. Then $\widetilde{B}_{2N}(\xi) = |\widehat{B_N}(\xi)|^2$ and the function \tilde{g} defined by
 $\tilde{g}(x) = \sum_{s=0}^{N-1} A_s \widehat{B_{2N}^{(2s)}}(x)$ satisfies the interpolation condition, that means \tilde{g} takes the
value zero at integer lattice except $\tilde{g}(0) = 1$. Hence A_s satisfies the equation

$$
\begin{pmatrix}\n\tilde{B}_{2N}(0) & \tilde{B}_{2N}^N(0) & \dots & \tilde{B}_{2N}^{(2N-2)}(0) \\
\tilde{B}_{2N}(1) & \tilde{B}_{2N}^N(1) & \dots & \tilde{B}_{2N}^{(2N-2)}(1) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{B}_{2N}(N-1) & \tilde{B}_{2N}^N(N-1) & \dots & \tilde{B}_{2N}^{(2N-2)}(N-1)\n\end{pmatrix}\n\begin{pmatrix}\nA_0 \\
A_1 \\
\vdots \\
A_{N-1}\n\end{pmatrix} = \begin{pmatrix}\n1 \\
0 \\
\vdots \\
0\n\end{pmatrix}.
$$
 (5.3)
The above equation can be solved by the following iterative algorithm.
ALGORITHM.

ALGORITHM:

The above equation can be so
ALGORITHM:
Step 1. Define $B(s,\xi) = \sum$
Step 2. Define $A_s = G_s(0)$. $(k)e^{ik\xi}$ and $G_0(\xi) = 1$. **Step 2.** Define $A_3 = G_3(0)$. $\begin{cases}\n\vdots & \vdots \\
\tilde{B}_{2N}(N-1) & \tilde{B}_{2N}''(N-1) & \dots & \tilde{B}_{2N}^{(2N-2)}(N-1)\n\end{cases}\n\begin{cases}\n\vdots \\
A_{N-1}\n\end{cases}$ The above equation can be solved by the following iterative algori
 ALGORITHM:
 Step 1. Define $B(s,\xi) = \sum_{k \in \mathbb{Z}} \tilde$ **Step 4.** Return to Step 2 if $s \leq N-2$ and stop if $s=N-1$. $\widetilde{B}_{2N-2s}(k)e^{ik\xi}$ and ζ
 $e^{-i\xi} - e^{i\xi}$)⁻¹($G_s(\xi)$)
 $N-2$ and stop if *s*
 s : see that the solution

explicit description of

struct the coefficient
 $\sum_{s=0}^{N-1} A_s z^{2s} = \prod_{j=0}^{N-1} \left(\frac{z}{z}\right)$

From the above equation, we see that the solution of equation (5.2) is unique, and it is just equal to A_s . This gives explicit description of A_s where $0 \le s \le N - 1$.

Now we can show how to construct the coefficient $\alpha(s)$. First, we write

$$
0 = (2 - e^{-i\xi} - e^{i\xi})^{-1} (G_s(\xi) - A_s B(s))
$$

\n2 if $s \leq N - 2$ and stop if $s = N - 1$
\nation, we see that the solution of equa
\nhis gives explicit description of A_s wh
\nwe to construct the coefficient $\alpha(s)$. F
\n
$$
Q(z) = \sum_{s=0}^{N-1} A_s z^{2s} = \prod_{j=0}^{N-1} \left(\frac{z^2 - t_j}{-t_j} \right).
$$

\n
$$
\sum_{s=0}^{N-1} \alpha(s) z^s = \prod_{j=0}^{N-1} \left(\frac{z + \sqrt{-t_j}}{\sqrt{-t_j}} \right).
$$

Then $\alpha(s)$ satisfies

on, we see that the solution of e
\n
$$
gives explicit description of As to construct the coefficient $\alpha(s)$.
\n
$$
g(z) = \sum_{s=0}^{N-1} A_s z^{2s} = \prod_{j=0}^{N-1} \left(\frac{z^2 - t_j}{-t_j} \right)
$$
\n
$$
\sum_{s=0}^{N-1} \alpha(s) z^s = \prod_{j=0}^{N-1} \left(\frac{z + \sqrt{-t_j}}{\sqrt{-t_j}} \right)
$$
\n
$$
u(t) = 0 \text{ if } u \text{ is the constant.}
$$
$$

This give a explicit construction of g in Theorem 4.1.

Remark 1. From Theorem 3.1, we see that

$$
\lim_{M \to \infty} \frac{M}{N} H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi}\right)^N \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s = \hat{g}(\xi).
$$

Therefore, $\widehat{M}\phi(\xi) \to \hat{g}(\xi)$ uniformly on any bounded set.

Remark 2. Observe that the solution of equation (5.1) is not unique. In particular, the polynomial

$$
\tilde{Q}(z) = \prod_{j=0}^{N-1} \left(\frac{z \pm \sqrt{-t_j}}{\pm \sqrt{-t_j}} \right) \tag{5.4}
$$

also satisfies equation (5.1). After careful choice of positive or negative sign in (5.4), we can make \tilde{Q} to be a polynomial with real coefficients. Using the method of Theorem $\tilde{Q}(z) = \prod_{j=0} \left(\frac{z \pm \sqrt{-t_j}}{\pm \sqrt{-t_j}} \right)$
also satisfies equation (5.1). After careful choice of positive or ne
can make \tilde{Q} to be a polynomial with real coefficients. Using th
4.1, we may find a class of scaling $\left(\begin{array}{c} \frac{1}{j=0} \end{array}\right)$ + \pm ,
careful choice
with real coefficients $\frac{1}{N}\phi$
 $\left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)$

$$
{}_N^M \tilde{H}(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^N \sum_{s=0}^{N-1} \tilde{a}_M(s) e^{i\xi}
$$

satisfying equation (1.8) such that its limit function is $\sum_{s=0}^{N-1} \tilde{\alpha}(s) B_N^{(s)}(x)$ where $\tilde{Q}(z) =$ $\sum_{s=0}^{N-1} \tilde{\alpha}(s) z^{s}$.

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