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Asymptotic Behavior of M-Band Scaling Functions of Daubechies Type

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Abstract. This paper deals with the asymptotic behavior of M-band scaling functions ${}_N^M\phi$ and M-band symbols ${}_N^M H$ as $M \to \infty$ for $N \geq 2$. This is followed by pointwise convergence, and L^p-convergence ($1 \leq p < \infty$) of ${}_N^M\phi$, and the limit function g of ${}_N^M\phi$ as $M \to \infty$.

Keywords: *Wavelets, scaling functions, refinable functions*

AMS subject classification: 42C15

1. Introduction

For any integer $M \geq 2$, a function f is called *M-refinable* (or simply *refinable*) if it satisfies the *refinement equation*

$$f(x) = \sum_{s \in \mathbf{Z}} c(s) f(Mx - s) \tag{1.1}$$

and $\int_{\mathbb{R}} f(x)\,dx = 1$, where $\{c(s)\}$, called the *mask* of the refinement equation, satisfies the condition $\sum_{s\in\mathbf{Z}} c(s) = M$ and is of finite length. A function f is said to be *orthonormal* if it satisfies

$$\int_{\mathbb{R}} f(x) f(x-k)\,dx = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \qquad (k \in \mathbb{Z}).$$

By a *scaling function* we mean an M-refinable and orthonormal function. For a given sequence $\{c(s)\}$, we define

$$H(\xi) = \frac{1}{M} \sum_{s \in \mathbf{Z}} c(s) \exp(is\xi). \tag{1.2}$$

Then H is called a *filter* of the refinement equation (1.1) or a filter corresponding to the scaling function f. For any integer $N \geq 1$, H is said to have N vanishing moments if there exists a Laurent polynomial $\tilde{H}$ such that

$$H(z) = \left[\frac{1 - z^M}{M(1-z)}\right]^N \tilde{H}(z). \tag{1.3}$$

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For a scaling function f, let a sequence of closed subspaces $Vj \cdot (j \in \mathbb{Z})$ of square integrable function space $L^2(\mathbb{R})$ spanned by the functions

$$f_{j,k}(x) = \left\{M^{j/2} f(M^j x - k) : k \in \mathbb{Z}\right\}. \tag{1.4}$$

Then $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* of $L^2(\mathbb{R})$ if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$, and $f \in V_j$ if and only if $f(Mx) \in V_{j+1}$ for all $j \in \mathbb{Z}$.

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iii) $\{f(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0 for some of $f \in V_0$.

We denote the wavelet space W_j $(j \in \mathbb{Z})$ by the orthonormal complement spaces of V_j in V_{j+1} so that the wavelet decomposition

$$L^2 = \bigcup_{l\in\mathbb{Z}} W_j = V_k + \bigcup_{j\geq k} W_j \tag{1.5}$$

holds. In fact, (1.5) suggests the decomposition

$$f = \sum_{j\in\mathbb{Z}} g_j = \sum_{j\geq k} g_j + f_k \tag{1.6}$$

of $f \in L^2(\mathbb{R})$ where $g_j \in W_j$ and $f_k \in V_k$.

The literature of wavelets is replete with analysis of 2-band ($M = 2$) scaling functions. The wavelet theory when $M = 2$ can be found in the literature of wavelets (see Daubechies [2]). When $M = 2$, W_j is spanned by $\{2^j\psi(2^j \cdot -k)\}_{k\in\mathbb{Z}}$ and the mother wavelet can be constructed from the 2-band scaling functions ϕ in the form

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_{1-k}(-1)^k \phi(2x - k), \tag{1.7}$$

where c_k are the coefficient of the 2-band scaling functions defined by (1.1).

In short, the theory of wavelets for $M = 2$ has received considerable attention. However, the wavelet theory for $M > 2$ received much less attention. Bi et al. [1] and Heller [3] independently considered the design of filter with N vanishing moment and finite length. Bi et al. [1] also considered M-band scaling functions, M-band wavelets and constructed compactly supported orthonormal M-band wavelets. The major objective of this paper is to investigate the asymptotic behavior of M-band scaling functions and M-band symbols as $M \to \infty$.

For any integer $N \geq 1$, let

$$_N H(\xi) = \frac{1}{2} \sum_{s=0}^{2N-1} {_N a(s)} \exp(is\xi)$$

be a solution of the equation

$$|{}_N H(\xi)|^2 = \cos^{2N}\left(\frac{\xi}{2}\right) \sum_{s=0}^{N-1} \binom{2N-1+s}{s} \sin^{2s}\left(\frac{\xi}{2}\right). \tag{1.8}$$

We note that the solution of equation (1.8) in the form

$${}_N H(\xi) = \frac{1}{2} \sum_{s=0}^{2N-1} {}_N a(s) e^{is\xi} \tag{1.9}$$

is *not* unique, but finite when $N \geq 2$.

Daubechies [2] introduced scaling functions ${}_N\phi$ with symbol ${}_N H$ when $M = 2$, and wavelets ${}_N\psi$ defined by

$${}_N\hat{\psi}(\xi) = {}_N H\left(-\frac{\xi}{2} + \pi\right) \exp\left(-\frac{i\xi}{2}\right) {}_N\hat{\phi}\left(\frac{\xi}{2}\right), \tag{1.10}$$

where $\hat{f}$ is the Fourier transform of an integrable function f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi) f(x)\, dx.$$

For these wavelets ${}_N\psi$, $\{2^{j/2}_N \psi(2^j \cdot -k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. The Hölder index of ${}_N\psi$ is about $\frac{\ln 3}{2 \ln 2} N$ for large N, and it has N vanishing moments, that is,

$$\int_{\mathbb{R}} x^k\, {}_N\psi(x)\, dx = 0 \qquad (0 \leq k \leq N-1).$$

Moreover, for any $N \geq 1$, the scaling function ${}_N\phi$ has minimal support in the class of compactly supported scaling functions ϕ for which we may find a compactly supported orthonormal wavelet ψ in V_1 which has N vanishing moments and satisfies

$$\int_{\mathbb{R}} \psi(x)\phi(x-k)\, dx = 0 \qquad (k \in \mathbb{Z})$$

where V_1 is the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\sqrt{2}\,\phi(2 \cdot -k)\}_{k\in\mathbb{Z}}$.

We define

$${}^M_N a(s) = \sum_{s_1+\ldots+s_{M-1}=s} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \left(2 \sin \frac{j\pi}{M}\right)^{-2s_j} \qquad (0 \leq s \leq N-1) \tag{1.11}$$

and

$$P(t) = \sum_{s=0}^{N-1} {}^M_N a(s)\, t^s. \tag{1.12}$$

By the Riesz lemma [2: p. 172/Lemma 6.1.3], there exists a unique solution H of the equation

$$|H(\xi)|^2 = \left(\frac{\sin\frac{M\xi}{2}}{M\sin\frac{\xi}{2}}\right)^{2N} P(2-2\cos\xi), \tag{1.13}$$

such that

$$H(\xi) = \left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)^{N} \sum_{s=0}^{N-1} \tilde{c}(s)\, e^{is\xi} = \frac{1}{M} \sum_{s=0}^{MN-1} c(s)\, e^{is\xi} \tag{1.14}$$

and $\sum_{s=0}^{N-1} \tilde{c}(s)\, z^s$ has all roots in the open unit disk, where $P(z)$ is a polynomial in z. Denote the solution of equations (1.13) and (1.14) by ${}_N^M H$. Let ${}_N^M\phi$ be the solution of the refinement equation (1.1) with the symbol ${}_N^M H$.

Bi et al. [1] and Heller [3] independently proved that ${}_N^M\phi$ is orthonormal, and represents a scaling function. Furthermore; ${}_N^M\phi$ has minimal support in the class of compactly supported scaling functions ϕ for which we may find compactly supported orthonormal wavelets $\psi_s \in V_1$ $(1 \le s \le M-1)$ such that ψ_s has N-vanishing moments and $\{\phi(\cdot\,, -k), \psi_s(\cdot\,, -k)\}_{s;\,k\in\mathbf{Z}}$ is an orthogonal basis of V_1, where V_1 is a closed subspace of L^2 spanned by $\{\sqrt{M}\,\phi(M\cdot -k\}_{k\in\mathbf{Z}}$. For this reason, we call ${}_N^M\phi$ as *M-band scaling functions of Daubechies type.*

When $M = 2$, Daubechies [2] and Pollen [4] studied the 2-band scaling functions of Daubechies type. On the other hand, for M-band scaling functions of Daubechies type, Bi et al. [1] investigated the asymptotic behavior of the Hölder index of ${}_N^M\phi$ as $N \to \infty$. For $N = 2$, Sun and Zhang [5] proved that the exact Hölder index of ${}_2^M\phi$ is $1 - \frac{\ln(1+\theta)}{\ln M}$ where $\theta = \left\{\frac{1}{3}(2M^2+1)\right\}^{1/2}$. The function ${}_2^M\phi$ tends to a function g pointwise as $M \to \infty$ where g is given by

$$g(x) = \begin{cases} x + \frac{\sqrt{6}}{6} & \text{if } 0 < x \le 1 \\ -x + 1 - \frac{\sqrt{6}}{6} & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases} .$$

They have also shown that ${}_2^M\phi$ is locally linear on an open set with full measure and locally linearly dependent when $M \ge 3$.

This paper deals with studying the asymptotic behavior of M-band scaling functions ${}_N^M\phi$ and M-band symbols ${}_N^M H$ as $M \to \infty$, for any $N \ge 2$. More precisely, we investigate the local polynomial structure of ${}_N^M\phi$ on an open set with full measure, the asymptotic behavior of ${}_N^M H$, and then the pointwise convergence and L^p-convergence of ${}_N^M\phi$ as $M \to \infty$. In Section 2, we consider the local polynomial structure of ${}_N^M\phi$ on an open set. Section 3 deals with the asymptotic behavior of M-band symbols ${}_N^M H$. This is followed by pointwise convergence and L^p-convergence $(1 \le p < \infty)$ of ${}_N^M\phi$. Finally, some remarks on the limit function g of ${}_N^M\phi$ as $M \to \infty$ are discussed.

2. Local polynomial functions

We say that a function supported in $[a,b]$ is *locally polynomial* on an open set $A \subset [a,b]$ if it is a polynomial on every connected component of A.

Theorem 2.1. *Let $M > N$ and ${}_N^M\phi$ be the solution of the refinement equation* (1.1) *with symbol ${}_N^MH$. Then there exists an open set $A \subset (0, N+\frac{N-1}{M-1})$ with Lebesgue measure $N+\frac{N-1}{M-1}$ such that ${}_N^M\phi$ is locally polynomial on A.*

Moreover, the above assertion holds for a more general class of refinable functions. A proof of this theorem is given by Bi et al. [1], and is omitted.

Theorem 2.2. *Let $M-1 > r \neq 0$ and ϕ be the solution of the refinement equation* (1.1) *with symbol*

$$H(\xi) = \left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)^N Q_r(\xi)$$

where $Q_r(0) = 1$ and $Q_r(\xi)$ may be written as $Q_r(\xi) = \sum_{k=0}^r c(k)e^{ik\xi}$. Then there exists an open set $A \subset (0, N+\frac{r}{M-1})$ with Lebesgue measure $N + \frac{r}{M-1}$ such that ϕ is locally polynomial on A.

To prove Theorem 2.2, we need some lemmas.

Let ϕ be as in Theorem 2.2. We define

$$\begin{aligned}\Phi(x) &= \big(\phi(x),\ldots,\phi(x+N-1)\big)^T \\ \widetilde{\Phi}(x) &= \big(\phi(x+1),\ldots,\phi(x+N)\big)^T\end{aligned} \qquad (x \in (0,1))$$

and

$$m_j = \int_{\mathbb{R}} x^j\phi(x)\,dx \qquad (0 \le j \le N-1).$$

Let

$$A(x) = \big((x+k)^j\big)_{0\le j,k\le N-1} \qquad \text{and} \qquad \widetilde{A}(x) = \big((x+k)^j\big)_{\substack{0\le j\le N-1\\ 1\le k\le N}}.$$

Denote the transpose of a matrix (or a vector) A by A^T. Then we have the following

Lemma 2.1. *Let $M-1 > r$ and ϕ be as in Theorem 2.2. Then*

$$\left.\begin{aligned}A(x)\Phi(x) &= \big(m_0,\ldots,m_{N-1}\big)^T - \big(1, x+N,\ldots,(x+N)^{N-1}\big)^T\phi(x+N) \\ \widetilde{A}(x)\widetilde{\Phi}(x) &= \big(m_0,\ldots,m_{N-1}\big)^T - \big(1, x,\ldots,x^{N-1}\big)^T\phi(x)\end{aligned}\right\} \tag{2.1}$$

on $(0,1)$ and ϕ is polynomial on $\bigcup_{j=0}^{N-1}\left(j+\left(\frac{r}{M-1},1\right)\right)$.

Proof. We first note that ϕ is supported on $\left[0, N+\frac{r}{M-1}\right]$ and

$$\det A(x) = \prod_{0\le i<j\le N-1}(j-i) \neq 0.$$

Therefore, from the first formula in (2.1) we get

$$\Phi(x) = \left(\det A(x)\right)^{-1} A^*(x)(m_0, \ldots, m_{N-1})^T$$

on $\left(\frac{r}{M-1}, 1\right)$, where $A^*(x)$ denotes the adjoint matrix of $A(x)$. Then the second assertion follows from (2.1).

Now we prove (2.1). By taking the Fourier transform of both sides of the refinement equation (1.1), we obtain

$$\hat{\phi}(\xi) = H\left(\frac{\xi}{M}\right)\hat{\phi}\left(\frac{\xi}{M}\right). \tag{2.2}$$

Therefore, $D^j\hat{\phi}(2k\pi) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$ and $0 \le j \le N-1$, where $D = \frac{\partial}{\partial\xi}$ is a differential operator, and furthermore

$$\sum_{k\in\mathbb{Z}}(x+k)^j\phi(x+k) = \int_{\mathbb{R}} x^j\phi(x)\,dx = m_j \qquad (0 \le j \le N-1) \tag{2.3}$$

by the Poisson summation formula. Then the first assertion in (2.1) follows from (2.3) ∎

Lemma 2.2. *Let ϕ be the same as in Theorem 2.2. Then there exist real numbers $a(0), \ldots, a(r)$ and polynomials $P_1, \ldots, P_r$ with degree at most $N-1$ such that*

$$\phi\left(\frac{x+j}{M}\right) = a(j)\phi(x) + P_j(x) \qquad (0 \le j \le r,\ x \in (0,1)) \tag{2.4}$$

and

$$\begin{aligned}\phi\left(\sum_{j=1}^{k}\frac{\varepsilon_j}{M^j} + \frac{x}{M^k}\right) &= \prod_{j=1}^{k} a(\varepsilon_j)\phi(x) + P_{\varepsilon_k}\left(\sum_{j=2}^{k}\frac{\varepsilon_j}{M^{j-1}} + \frac{x}{M^{k-1}}\right)\\ &\quad + \sum_{i=0}^{k-2}\prod_{l=k-i}^{k} a(\varepsilon_l)P_{\varepsilon_{k-1-i}}\left(\sum_{j=i+3}^{k}\frac{\varepsilon_j}{M^{j-i-2}} + \frac{x}{M^{k-i-2}}\right)\end{aligned} \tag{2.5}$$

where $\varepsilon_j \in \{0, 1\ldots, r\}$ and $x \in (0,1)$.

Proof. By the refinement equation (1.1), we obtain

$$\phi\left(\frac{x+j}{M}\right) = \sum_{l=0}^{(M-1)N+r} c_l\phi(x+j-l) = \sum_{l=0}^{j} c_{j-l}\phi(x+l) \tag{2.6}$$

on (0,1). From Lemma 2.1, there exist polynomials $Q_j \in \Pi_{N-1}$ and numbers d_j $(1 \le j \le N)$ such that

$$\phi(x+j) = d_j\phi(x) + Q_j(x) \tag{2.7}$$

where Π_{N-1} denotes the class of polynomials with degrees at most $N-1$. Then (2.4) follows from (2.6) and (2.7), and (2.5) follows by using formula (2.4) k times ∎

For any $\varepsilon_i \in \{0,1,\dots,r\}$ and $1 \le i \le k$, define

$$A(\varepsilon_1,\dots,\varepsilon_k) = \left(\sum_{j=1}^{k} \frac{\varepsilon_j}{M^j} + \frac{r}{(M-1)M^k}, \sum_{j=1}^{k} \frac{\varepsilon_j}{M^j} + \frac{1}{M^k}\right).$$

Then $A(\varepsilon_1,\dots,\varepsilon_k) \subset (0, \frac{r}{M-1})$ when $\varepsilon_k \ne r$. Furthermore, we have the following

Lemma 2.3. *Let $A(\varepsilon_1,\dots,\varepsilon_k)$ be defined as above. Then*

$$A(\varepsilon_1,\dots,\varepsilon_k) \cap A(\varepsilon'_1,\dots,\varepsilon'_{k'}) = \emptyset$$

when $\varepsilon_k, \varepsilon'_{k'} \ne r$ except $k = k'$ and $(\varepsilon_1,\dots,\varepsilon_k) = (\varepsilon'_1,\dots,\varepsilon'_{k'})$.

Proof. Define

$$a(\varepsilon_1,\dots,\varepsilon_k) = \sum_{j=1}^{k} \frac{\varepsilon_j}{M^j} + \frac{r}{(M-1)M^k} \quad \text{and} \quad b(\varepsilon_1,\dots,\varepsilon_k) = \sum_{j=1}^{k} \frac{\varepsilon_j}{M^j} + \frac{1}{M^k}.$$

Then it suffices to prove that

$$a(\varepsilon'_1,\dots,\varepsilon'_{k'}) > a(\varepsilon_1,\dots,\varepsilon_k) \implies a(\varepsilon'_1,\dots,\varepsilon'_{k'}) \ge b(\varepsilon_1,\dots,\varepsilon_k)$$

We note that

$$M^j a(\varepsilon_1,\dots,\varepsilon_k) = M^j \sum_{j=1}^{j} \frac{\varepsilon_i}{M^i} + M^j a(\varepsilon_{j+1},\dots,\varepsilon_k) \subset M^j \sum_{j=1}^{j} \frac{\varepsilon_i}{M^i} + (0,1)$$

and

$$M^j b(\varepsilon_1,\dots,\varepsilon_k) = M^j \sum_{j=1}^{j} \frac{\varepsilon_i}{M^i} + M^j b(\varepsilon_{j+1},\dots,\varepsilon_k) \subset M^j \sum_{j=1}^{j} \frac{\varepsilon_i}{M^i} + (0,1].$$

Therefore the problem reduces to prove

$$b(\varepsilon_1,\dots,\varepsilon_k) \le a(\varepsilon'_1,\dots,\varepsilon'_{k'})$$

for the following two cases: (i) $\varepsilon'_1 \ne \varepsilon_1$ and (ii) $\varepsilon_1 = \varepsilon'_1$ and $k = 1$ or $k' = 1$.

For the case (i), we get $\varepsilon'_1 > \varepsilon_1$, otherwise

$$a(\varepsilon'_1,\dots,\varepsilon'_{k'}) < \frac{\varepsilon'_1 + 1}{M} \le a(\varepsilon_1,\dots,\varepsilon_k)$$

which is a contradiction. Therefore, we have

$$b(\varepsilon_1,\dots,\varepsilon_k) \le \frac{\varepsilon_1 + 1}{M} \le a(\varepsilon'_1,\dots,\varepsilon'_{k'}).$$

For the case (ii), k' must be one, otherwise

$$a(\varepsilon'_1,\dots,\varepsilon'_k) < \frac{\varepsilon_1}{M} + \sum_{j=2}^{k} \frac{r}{M^j} + \frac{r}{(M-1)M^k} = \frac{\varepsilon_1}{M} + \frac{r}{(M-1)M} = a(\varepsilon_1)$$

which is a contradiction. Therefore, we have

$$b(\varepsilon_1 \dots,\varepsilon_k) \le \frac{\varepsilon_1}{M} + \sum_{j=2}^{k-1} \frac{r}{M^j} + \frac{r-1}{M^k} + \frac{1}{M^k} \le a(\varepsilon'_1)$$

and the lemma is proved ∎

Proof of Theorem 2.2. We define

$$O=\bigcup_{k=1}^{\infty}\bigcup_{\substack{(\varepsilon_1,\ldots,\varepsilon_{k-1})\in\{0,1,\ldots,r\}^{k-1}\\ \varepsilon_k\in\{0,1\ldots,r-1\}}} A(\varepsilon_1,\ldots,\varepsilon_k)$$

and

$$A=\left(\bigcup_{i=0}^{N}(O+i)\right)\bigcup\left(\bigcup_{i=0}^{N-1}\left(i+\left(\frac{r}{M-1},1\right)\right)\right).$$

Then ϕ is local polynomial on A by Lemmas 2.1 and 2.2. By Lemma 2.3, we obtain

$$\begin{aligned}|A|&=(N-1)\left(1-\frac{r}{M-1}\right)+N\sum_{k=1}^{\infty}\sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_{k-1})\in\{0,1,\ldots,r\}^{k-1}\\ \varepsilon_k\in\{0,1\ldots,r-1\}}}|A(\varepsilon_1,\ldots,\varepsilon_k)|\\ &=(N-1)\left(1-\frac{r}{M-1}\right)+N\sum_{k=1}^{\infty}\left(1-\frac{r}{M-1}\right)\frac{r}{M}\left(\frac{r+1}{M}\right)^{k-1}\\ &=N+\frac{r}{M-1}.\end{aligned}$$

This proves the theorem ∎

3. Asymptotic behavior of M-band symbol

We write

$${}^M_N H(\xi)=\left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)^N {}^M_N\widetilde{H}(\xi) \tag{3.1}$$

and

$${}^M_N\widetilde{H}(\xi)=\sum_{s=0}^{N-1}a_M(s)(e^{i\xi}-1)^s. \tag{3.2}$$

Then we have the following

Theorem 3.1. *Let* ${}^M_N\widetilde{H}(\xi)$ *be defined by* (3.2). *Then* $a_M(0)=1$ *and the limit of* $a_M(s)M^{-s}$ *exists for* $1\le s\le N-1$ *and*

$$\lim_{M\to\infty}{}^M_N H\left(\frac{\xi}{M}\right)=\left(\frac{1-e^{i\xi}}{-i\xi}\right)^N\sum_{s=0}^{N-1}\alpha(s)(i\xi)^s$$

where $\alpha(s)=\lim_{M\to\infty}a_M(s)M^{-s}$ $(0\le s\le N-1)$. *Furthermore,*

$$\left|M^{-(N-1)}a_M(N-1)\right|\le 2^{-N+1}\left(1-\left|\frac{2}{\pi}\right|^{2N}\right)^{1/2}.$$

To prove Theorem 3.1, we need some lemmas.

Define

$$A(k,s) = \sum_{l=0}^{s} \binom{2N-1+l}{l} (2k\pi)^{-2l} A(k-1, s-l) \qquad (k \geq 2)$$

and

$$A(1,s) = \binom{2N-1+s}{s} (2\pi)^{-2s}.$$

Then $A(k,s) \geq A(k-1,s)$ and $|A(k,s) - A(k-1,s)| \leq C_s k^{-2}$ holds for all $k \geq 2$, where C_s is a constant depending on s only. Therefore, $\lim_{k\to\infty} A(k,s)$ exists for all $0 \leq s \leq N-1$. Denote its limit by A_s $(0 \leq s \leq N-1)$ (the explicit computation of A_s will be given in Section 5). Then we have the following:

Lemma 3.1. *Let* ${}_N^M a(s)$ *be defined by* (1.6). *Then*

$$\lim_{M\to\infty} {}_N^M a(s) M^{-2s} = A_s \qquad (0 \leq s \leq N-1). \tag{3.3}$$

Proof. First we prove the assertion when M is odd. Denote $M' = \frac{M-1}{2}$. Then we may write

$$\begin{aligned}
{}_N^M a(s) &= M^{2s} \sum_{s_1+\ldots+s_{M'}=s} \left(\prod_{j=1}^{M'} \binom{2N-1+s_j}{s_j} (2j\pi)^{-2s_j} \right) \\
&\quad \times \left(\prod_{j=1}^{M'} \left(1 + O\left(\frac{s_j j}{M}\right)^2 \right) \right) \\
&= M^{2s} \sum_{l=1}^{M'} \sum_{s_1+\ldots+s_{M'}=s} \left(\prod_{j=1}^{M'} \binom{2N-1+s_j}{s_j} (2j\pi)^{-2s_j} \right) \\
&\quad \times \left(O\left(\frac{s_l l}{M}\right)^2 \prod_{j=1}^{l-1} \left(1 + O\left(\frac{s_j j}{M}\right)^2 \right) \right) \\
&\quad + M^{2s} \sum_{s_1+\ldots+s_{M'}=s} \prod_{j=1}^{M'} \binom{2N-1+s_j}{s_j} (2j\pi)^{-2s_j} \\
&= M^{2s} \left(\sum_{l=1}^{M'} I_{l,M}(s) + A(M', s) \right)
\end{aligned}$$

where $O(\frac{s_j j}{M})^2$ denotes a term bounded by $C(\frac{s_j j}{M})^2$ for some constant C independent of

M. Obviously, we have

$$\begin{aligned}
0 &\le I_{l,M}(s) \\
&\le C\sum_{s_l=1}^{s}\binom{2N-1+s_l}{s_l}(2l\pi)^{-2(s_l-1)}M^{-2} \\
&\times \sum_{s_1+\ldots+s_{l-1}+s_{l+1}+\ldots+s_{M'}=s-s_l}\left(\prod_{j\neq l,1\le j\le M'}\binom{2N-1+s_j}{s_j}(2j\pi)^{-2s_j}\right) \\
&\le CA(M',s)\sum_{s_l=1}^{s}(2l\pi)^{-2(s_l-1)}M^{-2} \\
&\le CM^{-2}A(M',s).
\end{aligned}$$

Hereafter the letter C would denote a constant independent of M which may be different at different instances. Therefore, we get $M^{-2s}\,{}_N^Ma(s) \to A_s$ as $M \to \infty$ when M is odd.

When M is even, we may write

$$\begin{aligned}
&M^{-2s}\,{}_N^Ma(s) \\
&= \sum_{s_{M/2}=0}^{s}\binom{N-1+s_{M/2}}{s_{M/2}}(2M)^{-2s_{M/2}} \\
&\quad\times\left(M^{-2(s-s_{M/2})}\sum_{s_1+\ldots+s_{M/2-1}=s-s_{M/2}}\prod_{j=1}^{M/2-1}\binom{2N-1+s_j}{s_j}\left(2\sin\frac{j\pi}{M}\right)^{-2s_j}\right) \\
&= \sum_{s_{M/2}=0}^{s}\binom{N-1+s_{M/2}}{s_{M/2}}4^{-s_{M/2}}M^{-2s_{M/2}}I(s-s_{M/2}).
\end{aligned}$$

Using the same procedure to prove the assertion when M is odd, we may prove that $I(t) \to A_t$ as $M \to \infty$ for all $0 \le t \le N-1$. Thus also $M^{-2s}\,{}_N^Ma(s) \to A_s$ as $M \to \infty$ when M is even ∎

Lemma 3.2. *Let ${}_N^Ma(N-1)$ be defined by* (1.11). *Then*

$$M^{-2N+2}\,{}_N^Ma(N-1) \le \left(1-\left(\frac{2}{\pi}\right)^{2N}\right)2^{-2(N-1)}.$$

Proof. It is proved by Bi et al. in [1] that

$$\sum_{s=0}^{M-1}\left|{}_N^MH\left(\xi+\frac{2\pi s}{M}\right)\right|^2 = 1. \tag{3.4}$$

We note that

$$\sum_{s=0}^{M-1}\left|\frac{1-e^{iM\xi}}{M(1-e^{i(\xi+2s\pi/M)})}\right|^{2N} \ge \left|\frac{\sin\frac{M\xi}{2}}{M\sin\frac{\xi}{2}}\right|^{2N} \ge \left(\frac{2}{\pi}\right)^{2N}$$

when $|\xi| \le \frac{\pi}{M}$. Therefore, we get

$$\sum_{s=0}^{M-1} \left| \frac{1 - e^{iM\xi}}{M(1 - e^{i(\xi + 2s\pi/M)})} \right|^{2N} \ge \left(\frac{2}{\pi}\right)^{2N} \qquad (\xi \in \mathbb{R}). \tag{3.5}$$

We recall that

$$|{}_N^M H(\xi)|^2 = \left(\frac{\sin\frac{M\xi}{2}}{M\sin\frac{\xi}{2}}\right)^{2N} \sum_{s=0}^{N-1} 2^{2s}\,{}_N^M a(s) \left(\sin\frac{\xi}{2}\right)^{2s}$$

and ${}_N^M a(0) = 1$. Then it follows from (3.4) and (3.5) that

$$\begin{aligned} 1 - \left(\frac{2}{\pi}\right)^{2N} &\ge M^{-2N+2}\,{}_N^M a(N-1)\left(\sin\frac{M\xi}{2}\right)^{2(N-1)} \\ &\quad \times 2^{2(N-1)} \sum_{s=0}^{M-1} \frac{\sin^2\frac{M\xi}{2}}{M^2\sin^2(\frac{\xi}{2} + \frac{s\pi}{M})} \\ &= M^{-2N+2}\,{}_N^M a(N-1) 2^{2(N-1)} \left(\sin\frac{M\xi}{2}\right)^{2(N-1)} \end{aligned} \tag{3.6}$$

Substituting $\xi = \frac{\pi}{M}$ in (3.6) gives the lemma ∎

Proof of Theorem 3.1. Let

$$Q(t) = \sum_{s=0}^{N-1} A_s t^s. \tag{3.7}$$

By Lemma 3.1, we can write

$$\widetilde{P}(t) = \sum_{s=0}^{N-1} {}_N^M a(s) t^s = \sum_{s=0}^{N-1} \beta_M(s)(M^2 t)^s$$

where $\beta_M(s) \to A_s$ as $M \to \infty$. We then set

$$Q(t) = \prod_{j=1}^{N-1} \left(\frac{t - t_j}{-t_j}\right).$$

Then there exists a sequence $\{t_{j,M}\}_{j=1}^{N-1}$ such that

$$\widetilde{P}(t) = \prod_{j=1}^{N-1} \left(\frac{M^2 t - t_{j,M}}{-t_{j,M}}\right)$$

and $t_{j,M} \to t_j$ as $M \to \infty$.

Recall that $\tilde{P}(t) > 0$ when $t > 0$. Therefore, $t_j \notin (0, \infty)$, and $t_{j,M} \notin (0, \infty)$ as M is sufficiently large. When $t = 2 - e^{i\xi} - e^{-i\xi}$, we may write

$$M^2 t - t_{j,M} = [M(e^{i\xi} - 1) - \theta_{j,M}][M(e^{-i\xi} - 1) - \theta_{j,M}] \times \beta_{j,M}$$

where

$$\left.\begin{aligned} \theta_{j,M} &= \frac{-t_{j,M} - \sqrt{t_{j,M}^2 - 4t_{j,M}M^2}}{2M} \to -\sqrt{-t_j} \\ \beta_{j,M} &= \frac{M}{M + \theta_{j,M}} \to 1 \end{aligned}\right\} \quad \text{as } M \to \infty.$$

Furthermore, the real part of $\theta_{j,M}$ is always less than zero when M is large enough. Therefore, the root of $M(z-1) - \theta_{j,M}$ is contained in the open unit disk and

$$\prod_{j=1}^{N} \frac{M(e^{i\xi} - 1) - \theta_{j,M}}{-\theta_{j,M}}$$

is a trigonometrical polynomial with real coefficients. By the Riesz lemma [2: p. 172/Lemma 6.1.3] we obtain

$${}_N^M\widetilde{H}(\xi) = \prod_{j=1}^{N} \left[\frac{M(e^{i\xi} - 1) - \theta_{j,M}}{-\theta_{j,M}} \right].$$

Hence

$$\lim_{M\to\infty} \sum_{s=0}^{N-1} M^{-s} a_M(s) t^s = \prod_{j=1}^{N-1} \left(\frac{t + \sqrt{-t_j}}{\sqrt{-t_j}} \right)$$

and the limit $\lim_{M\to\infty} M^{-s} a_M(s)$ exists for all $0 \le s \le N-1$.

We observe that

$${}_N^M H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{M(1 - e^{i\xi/M})} \right)^N \sum_{s=0}^{N-1} (a_M(s) M^{-s}) \times (M(e^{i\xi/M} - 1))^s.$$

Then we find that

$$\lim_{M\to\infty} {}_N^M H\left(\frac{\xi}{M}\right) = \left(\frac{1 - e^{i\xi}}{-i\xi} \right)^N \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s.$$

We recall from Lemma 3.2 that

$$\prod_{j=1}^{N-1} \frac{1}{-t_{j,M}} = M^{-2(N-1)} {}_N^M a(N-1) \le 2^{-2(N-1)} \left(1 - \left(\frac{2}{\pi}\right)^{2N} \right).$$

Then we obtain

$$|M^{-N+1} a_M(N-1)| = \left| \prod_{j=1}^{N} \frac{1}{\sqrt{-t_{j,M}}} \right| \le 2^{-N+1} \left(1 - \left(\frac{2}{\pi}\right)^{2N} \right)^{1/2}. \tag{3.8}$$

Theorem 3.1 is thus proved ∎

4. Pointwise convergence and L^p-convergence

The B-spline B_N of degree $N-1$ is defined with the help of the Fourier transform by

$$\widehat{B_N}(\xi) = \left(\frac{1-e^{i\xi}}{-i\xi}\right)^N.$$

Theorem 4.1. *Let ${}^M_N\phi$ be the solution of the refinement equation* (1.1) *with symbol ${}^M_N H$ and $1 \le p < \infty$. Then ${}^M_N\phi$ converges pointwisely and in L^p-norm to*

$$g(x) = B_N(x) + \sum_{s=1}^{N-1} \alpha(s) B_N^{(s)}(x) \tag{4.1}$$

where $\alpha(s) = \lim_{M\to\infty} M^{-s} a_M(s)$ and $B_N^{(s)}$ is the s-th derivative of B_N. Furthermore, g is orthonormal.

We need the following lemmas to prove this theorem whose proof will be given later. For a compactly supported integrable function f, the k-moment of f is defined by

$$m_k(f) = \int_{\mathbb{R}} x^k f(x)\,dx \qquad (0 \le k \le N-1).$$

Then we have the following

Lemma 4.1. *Let ${}^M_N\phi$ be the solution of the refinement equation* (1.1) *with symbol ${}^M_N H$. Then*

$$\lim_{M\to\infty} m_k({}^M_N\phi) = m_k(B_N) + \sum_{j=1}^{N-1} \alpha(j) m_k(B_N^{(j)}) \qquad (0 \le k \le N-1) \tag{4.2}$$

holds.

Proof. Let $D = i\frac{d}{d\xi}$ be a differential operator. Then for any compactly supported integrable function f, we have $m_k(f) = (D^k \widehat{f})(0)$. Define

$$h_M(\xi) = \prod_{j=1}^{\infty} {}^M_N\widetilde{H}\left(\frac{\xi}{M^j}\right).$$

Then we have

$$h_M(\xi) = {}^M_N\widetilde{H}\left(\frac{\xi}{M}\right) h_M\left(\frac{\xi}{M}\right)$$

and

$$D^k h_M(\xi) = M^{-k} \sum_{j=0}^{k} \binom{k}{j} (D^j\, {}^M_N\widetilde{H})\left(\frac{\xi}{M}\right) (D^{k-j} h_M)\left(\frac{\xi}{M}\right).$$

Hence, we find

$$(1-M^{-k})(D^k h_M)(0) = M^{-k}\sum_{j=1}^{k}\binom{k}{j}(D^j\, {}_N^M\widetilde{H})(0)(D^{k-j}h_M)(0).$$

From Theorem 3.1, $M^{-j}(D^j\, {}_N^M\widetilde{H})(0) \to (-1)^j j!\alpha(j)$ as $M \to \infty$ $(0 \le j \le N-1)$. Hence $D^k h_M(0) \to (-1)^k k!\alpha(k)$ as $M \to \infty$ $(0 \le k \le N-1)$.

We recall that

$${}_N^M H(\xi) = \left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)^N {}_N^M\widetilde{H}(\xi).$$

Therefore, $\widehat{{}_N^M\phi}(\xi) = \widehat{B_N}(\xi)h_M(\xi)$, and

$$(D^k\widehat{{}_N^M\phi})(0) = \sum_{j=0}^{k}\binom{k}{j}(D^j\widehat{B_N})(0)(D^{k-j}h_M)(0).$$

Hence

$$(D^k\widehat{{}_N^M\phi})(0) \to \sum_{j=0}^{k}\frac{k!}{(k-j)!}(-1)^j\alpha(j)D^{k-j}\widehat{B_N}(0) = \sum_{j=0}^{N-1}\alpha(j)\int_{\mathbb{R}} x^k B_N^{(j)}(x)\,dx$$

and the lemma is proved ∎

Lemma 4.2. *Let $d_{j,M}$ $(0 \le j \le N-1)$ be numbers such that*

$${}_N^M\phi\left(\frac{x+j}{M}\right) = d_{j,M}\, {}_N^M\phi(x) + Q_{j,M}(x) \qquad (x \in (0,1),\ 0 \le j \le N-1) \tag{4.3}$$

holds for a polynomial $Q_{j,M} \in \Pi_{N-1}$. Then

$$\beta_j = \lim_{M\to\infty} d_{j,M} = (-1)^{N-1}\alpha(N-1)\sum_{s=0}^{j}\frac{(-1)^s N!}{s!(N-s)!}$$

and

$$|\beta_j| \le 2^{N-1}|\alpha(N-1)| \le \left(1-\left(\frac{2}{\pi}\right)^{2N}\right)^{1/2} \qquad (0 \le j \le N-1).$$

Proof. We recall that

$$\sum_{k\in\mathbf{Z}}(x+k)^j\, {}_N^M\phi(x+k) = m_j({}_N^M\phi) \qquad (0 \le j \le N-1).$$

Then we obtain

$$\sum_{k\in\mathbf{Z}} k^j\, {}_N^M\phi(\cdot+k) = Q_j \in \Pi_{N-1} \qquad (0 \le j \le N-1)$$

or, in matrix form,

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{N-1} & \dots & N^{N-1} \end{pmatrix} \begin{pmatrix} {}_N^M\phi(x+1) \\ {}_N^M\phi(x+2) \\ \vdots \\ {}_N^M\phi(x+N) \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} {}_N^M\phi(x) + \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_{N-1}(x) \end{pmatrix}$$

on $(0,1)$. Therefore, we get

$$ {}_N^M\phi(x+j) = \frac{(-1)^j N!}{j!(N-j)!} {}_N^M\phi(x) + \tilde{Q}_j(x) \tag{4.4}$$

where $\widetilde{Q}_j \in \Pi_{N-1}$. It follows from Theorem 3.1 that

$$\begin{aligned} {}_N^M H(\xi) &= \left(\frac{e^{iM\xi}-1}{M(e^{i\xi}-1)}\right)^N \sum_{s=0}^{N-1} a_M(s)(e^{i\xi}-1)^s \\ &= \frac{1}{M}\alpha(N-1)\left(\frac{1-e^{iM\xi}}{1-e^{i\xi}}\right)(e^{iM\xi}-1)^{N-1} + \frac{1}{M}\sum_{k=0}^{MN-1} o(1)e^{ik\xi} \end{aligned}$$

where $o(1)$ means a number tending to zero as $M \to \infty$. From (2.4) and (4.4), there exist polynomials $Q_j \in \Pi_{N-1}$ such that

$$\begin{aligned} {}_N^M\phi\left(\frac{x+j}{M}\right) &= \sum_{l=0}^{j} c_{j-l}\, {}_N^M\phi(x+l) \\ &= (-1)^{N-1}\alpha(N-1)\sum_{l=0}^{j}(1+o(1))\, {}_N^M\phi(x+l) \\ &= \left((-1)^{N-1}\alpha(N-1)\sum_{l=0}^{j}(-1)^l\binom{N}{l}\right) {}_N^M\phi(x) + o(1)\, {}_N^M\phi(x) + Q_j(x). \end{aligned}$$

This proves the first assertion.

Observe that $\sum_{s=0}^{N}(-1)^s\binom{N}{s} = 0$. Then we have

$$\left|\sum_{s=0}^{j}(-1)^s\binom{N}{s}\right| = \left|\sum_{s=0}^{N-j-1}(-1)^s\binom{N}{s}\right|.$$

It is easy to see that

$$\left|\sum_{s=0}^{j}(-1)^s\binom{N}{s}\right| \le \binom{N}{j}$$

when $j < \frac{N}{2}$. Then from the identity $2^N = (1+1)^N = \sum_{s=0}^{N}\binom{N}{s}$ we get $\binom{N}{(N-1)/2} \le 2^{N-1}$ when N is odd and $\binom{N}{N/2-1} \le 2^{N-1}$. Observe that

$$\binom{N}{j} \le \begin{cases} \binom{N}{(N-1)/2} & \text{when } N \text{ is odd} \\ \binom{N}{N/2-1} & \text{when } N \text{ is even.} \end{cases} \qquad (0 \le j < \tfrac{N}{2}).$$

Thus, the second assertion follows from Theorem 3.1 ∎

Proof of Theorem 4.1. It follows from (2.1) that

$$A(x){}_N^M\Phi(x) = \left(m_0({}_N^M\phi), \dots, m_{N-1}({}_N^M\phi)\right)^T \qquad \text{on } \left(\frac{N-1}{M-1}, 1\right)$$

where ${}_N^M\Phi(x) = \left({}_N^M\phi(x), \dots, {}_N^M\phi(x+N-1)\right)$. Recall that

$$m_k({}_N^M\phi) \to \sum_{j=0}^{N-1} \alpha(j) m_k(B_N^{(j)})$$

by Lemma 4.1. Therefore,

$${}_N^M\Phi(x) \to \sum_{j=0}^{N-1} \alpha(j) A^{-1}(x) \left(m_0(B_N^{(j)}), \dots, m_{N-1}(B_N^{(j)})\right)^T$$

pointwisely on $(0,1)$. This proves that

$${}_N^M\phi(x) \to g(x) = \sum_{j=0}^{N-1} \alpha(j) B_N^{(j)}(x).$$

Obviously, by the dominated convergence theorem, the L^p-convergence $(1 \le p < \infty)$ of ${}_N^M\phi(x)$ reduces to prove that ${}_N^M\phi(x)$ is uniformly bounded. Recall that

$${}_N^M\phi\left(\frac{x+j}{M}\right) = d_{j,M}\, {}_N^M\phi(x) + Q_{j,M}(x) \quad \text{and} \quad |d_{j,M}| \le \left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{1/2}$$

when M is sufficiently large and $Q_{j,M}$ is uniformly bounded by C. Thus we get

$$\begin{aligned} \sup_{x \in A(\epsilon_1, \dots, \epsilon_k)} |{}_N^M\phi(x)| &\le C\left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{\frac{1}{2}k} + C\sum_{j=0}^{k-1}\left(1 - 2\left(\frac{2}{\pi}\right)^{2N}\right)^{\frac{1}{2}j} \\ &\le C\left(\frac{\pi}{2}\right)^{2N}. \end{aligned}$$

It follows from the proof of Theorem 2.1 that

$$\bigcup_{k=1}^{\infty} \bigcup_{\substack{\epsilon_i \in \{0,1,\dots,N-1\},\ 1 \le i \le k-1 \\ \epsilon_k \in \{0,1,\dots,N-2\}}} A(\epsilon_1, \dots, \epsilon_k) \cup \left(\frac{N-1}{M-1}, 1\right)$$

has Lebesgue measure 1. This proves that ${}_N^M\phi$ is uniformly bounded. Recall that ${}_N^M\phi$ is orthonormal. Then the limit g of ${}_N^M\phi$ in the L^2-norm is also orthonormal ∎

5. The limit function

In this section we will give a method to construct the limit function g in Theorem 4.1. Let

$$G(z) = \sum_{s=0}^{N-1} \alpha(s) z^s \qquad \text{and} \qquad Q(z) = \sum_{s=0}^{N-1} A_s z^{2s}.$$

Then, by the proof of Theorem 3.1, we get

$$G(iz)G(-iz) = Q(z). \tag{5.1}$$

Let g be the limit of ${}_N^M\phi$ in Theorem 4.1. Then g is unique determined by G.

Now we compute A_s $(0 \le s \le N-1)$ explicitly. Observe that g is orthonormal by Theorem 4.1. Then we have

$$\sum_{k\in\mathbf{Z}} |\hat{g}(\xi + 2k\pi)|^2 = 1 \qquad (\xi \in \mathbb{R}).$$

By Theorem 4.1 and by the orthonormality of ${}_N^M\phi$, we get $\hat{g}(\xi) = G(i\xi)\widehat{B_N}(\xi)$ and

$$\sum_{k\in\mathbf{Z}} Q(\xi + 2k\pi)|\widehat{B_N}(\xi + 2k\pi)|^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}. \tag{5.2}$$

Let $\widetilde{B}_{2N}(x) = B_{2N}(x+N)$. Then $\widehat{\widetilde{B}_{2N}}(\xi) = |\widehat{B}_N(\xi)|^2$ and the function $\tilde{g}$ defined by $\tilde{g}(x) = \sum_{s=0}^{N-1} A_s \widetilde{B}_{2N}^{(2s)}(x)$ satisfies the interpolation condition, that means $\tilde{g}$ takes the value zero at integer lattice except $\tilde{g}(0) = 1$. Hence A_s satisfies the equation

$$\begin{pmatrix} \tilde{B}_{2N}(0) & \tilde{B}''_{2N}(0) & \cdots & \tilde{B}_{2N}^{(2N-2)}(0) \\ \tilde{B}_{2N}(1) & \tilde{B}''_{2N}(1) & \cdots & \tilde{B}_{2N}^{(2N-2)}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{2N}(N-1) & \tilde{B}''_{2N}(N-1) & \cdots & \tilde{B}_{2N}^{(2N-2)}(N-1) \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{N-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.3}$$

The above equation can be solved by the following iterative algorithm.

ALGORITHM:

Step 1. Define $B(s,\xi) = \sum_{k\in\mathbf{Z}} \widetilde{B}_{2N-2s}(k)e^{ik\xi}$ and $G_0(\xi) = 1$.

Step 2. Define $A_s = G_s(0)$.

Step 3. Define $G_{s+1}(\xi) = (2 - e^{-i\xi} - e^{i\xi})^{-1}(G_s(\xi) - A_s B(s,\xi))$.

Step 4. Return to Step 2 if $s \le N-2$ and stop if $s = N-1$.

From the above equation, we see that the solution of equation (5.2) is unique, and it is just equal to A_s. This gives explicit description of A_s where $0 \le s \le N-1$.

Now we can show how to construct the coefficient $\alpha(s)$. First, we write

$$Q(z) = \sum_{s=0}^{N-1} A_s z^{2s} = \prod_{j=0}^{N-1} \left(\frac{z^2 - t_j}{-t_j} \right).$$

Then $\alpha(s)$ satisfies

$$\sum_{s=0}^{N-1} \alpha(s) z^s = \prod_{j=0}^{N-1} \left(\frac{z + \sqrt{-t_j}}{\sqrt{-t_j}} \right).$$

This give a explicit construction of g in Theorem 4.1.

Remark 1. From Theorem 3.1, we see that

$$\lim_{M\to\infty} {}_N^M H\left(\frac{\xi}{M}\right) = \left(\frac{1-e^{i\xi}}{-i\xi}\right)^N \sum_{s=0}^{N-1} \alpha(s)(i\xi)^s = \hat{g}(\xi).$$

Therefore, $\widehat{{}_N^M\phi}(\xi) \to \hat{g}(\xi)$ uniformly on any bounded set.

Remark 2. Observe that the solution of equation (5.1) is not unique. In particular, the polynomial

$$\tilde{Q}(z) = \prod_{j=0}^{N-1} \left(\frac{z \pm \sqrt{-t_j}}{\pm\sqrt{-t_j}}\right) \tag{5.4}$$

also satisfies equation (5.1). After careful choice of positive or negative sign in (5.4), we can make $\tilde{Q}$ to be a polynomial with real coefficients. Using the method of Theorem 4.1, we may find a class of scaling functions ${}_N^M\tilde{\phi}$ with the symbol

$${}_N^M\tilde{H}(\xi) = \left(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\right)^N \sum_{s=0}^{N-1} \tilde{a}_M(s)e^{i\xi}$$

satisfying equation (1.8) such that its limit function is $\sum_{s=0}^{N-1} \tilde{\alpha}(s)B_N^{(s)}(x)$ where $\tilde{Q}(z) = \sum_{s=0}^{N-1} \tilde{\alpha}(s)z^s$.

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