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Solutions and Periodic Solutions for Nonlinear Evolution Equations with Non-Monotone Perturbations

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Abstract. In this paper we solve periodic and Cauchy problems for nonlinear evolution equations driven by time-dependent, pseudomonotone operators and a non-monotone perturbation term. Our proof produces as a by-product a useful property of the solution map for maximal monotone problems. Two examples of nonlinear parabolic problems illustrate the applicability of our work.

Keywords: *Evolution triples, compact embeddings, coercive operators, pseudomonotone operators, L−pseudomonotone operators, surjectivity results, demicontinuity, parabolic problems*

AMS subject classification: 34 G 20

1. Introduction

In this paper we prove two existence theorems for evolution equations defined in the framework of an evolution triple $X \subseteq H \subseteq X^*$. The first theorem is about a periodic problem, while the second concerns a Cauchy problem. Our work here extends that of Hirano [8] who treats autonomous equations, with the operator $A : X \to X^*$ being monotone and the conditions on the perturbation term f being more restrictive. Recently Ahmed and Xiang [2] extended the result of Hirano [8]. Although some of their hypotheses are more general than ours (they do not assume that the embedding of X into H is compact and f takes values in X^*), nevertheless they still require A to be monotone (analogous results can also be found in the works of Ahmed [1] and Ahmed and Xiang [3]). Moreover, our method of proof is different from that of Hirano [8] and Ahmed and Xiang [2] (which move along similar paths) and uses a general surjectivity theorem for the sum of two operators of monotone type. The use of this surjectivity result is made possible by an intermediate result of independent interest, which roughly speaking says that the property of pseudomonotonicity of $A(t,\cdot)$ can be 'lifted' in some sense to the Nemitsky operator $\widehat{A}(\cdot)$ corresponding to $A(t,x)$.

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To complete our survey of the relevant literature, we should also mention the work of Vrabie [17] who considers autonomous systems assuming that the operator A generates a compact semigroup on H and the work of Gutman [7] in which A is an m-accretive operator on a Banach space X and the perturbation term is completely continuous from $C(T, X)$ into $L^p(T, X)$. Related to the work of Gutman [7] is the recent one by Kartsatos and Shin [9] who study functional evolution equations driven by a time-dependent m-accretive operator on a Banach space which generates a compact evolution operator (the evolution operator can be only equicontinuous, but then X and X^* are uniformly convex and the perturbation term is compact). In our hypotheses here, nothing implies that the operator $A(t, x)$ generates an evolution operator, let alone a compact one. Similarly, our hypotheses on the perturbation term $f(t, x)$ do not imply that its Nemytski operator is compact.

2. Preliminaries

By an *evolution triple* we mean three real spaces $X \subseteq H \subseteq X^*$ such that:

(a) X is a separable, reflexive Banach space.

(b) H is a separable Hilbert space identified with its dual (pivot space).

(c) The embedding of X into H is continuous and dense.

Hence $H^* = H$ is embedded into X^* continuously and densely, too. By $|\cdot|, \|\cdot\|$ and $\|\cdot\|_*$ we will denote the norm of H, X and X^*, respectively. Also, by $(\cdot,\cdot)$ we will denote the inner product of H and by $\langle\cdot,\cdot\rangle$ the duality brackets of the pair (X, X^*). The two are compatible in the sense that $\langle\cdot,\cdot\rangle|_{X\times H} = (\cdot,\cdot)$.

Given an intercal $T = [0, b]$, an evolution triple (X, H, X^*) and numbers $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we define

$$W_{pq}(T) = \left\{x \in L^p(T, X) : \dot{x} \in L^q(T, X^*)\right\}.$$

The time derivative of x involved in the above definition of $W_{pq}(T)$ is understood in the sense of vector-valued distributions. We furnish $W_{pq}(T)$ with the natural norm

$$\|x\|_{pq} = \left(\|x\|_p^2 + \|\dot{x}\|_q^2\right)^{1/2}.$$

Equipped with this norm $W_{pq}(T)$ becomes a separable, reflexive Banach space. It is embedded continuously into $C(T, H)$, i.e. every element in $W_{pq}(T)$ has a unique representative in $C(T, H)$. If in addition we assume that X is embedded compactly into H, then $W_{pq}(T)$ is embedded compactly into $L^p(T, H)$. For further details on these issues, we refer to Zeidler [18].

An operator $A : X \to X^*$ is said to be *pseudomonotone*, if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$ imply that $\langle A(x), x - y\rangle \leq \underline{\lim}\langle A(x_n), x_n - y\rangle$ for all $y \in X$. If A is bounded (i.e it maps bounded sets in X into bounded sets in X^*), then pseudomonotonicity is equivalent to saying that if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$, then $A(x_n) \xrightarrow{w} A(x)$ in X^* and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ (this property is usually known as *generalized pseudomonotonicity*, see Browder and Hess [6]). A monotone hemicontinuous operator or a completely continuous operator $A : X \to X^*$ is pseudomonotone, and pseudomonotonicity is preserved by addition.

Another closely related concept, suitable for the study of evolution equations, is that of L-pseudomonotonicity. So, let Y be a reflexive Banach space, $L : D \subseteq Y \to Y^*$ a closed, densely defined, linear operator and $K : Y \to Y^*$ a bounded nonlinear operator. We say that K is *L-pseudomonotone* if, for $\{y_n\}_{n\geq 1} \subseteq D$ such that $y_n \xrightarrow{w} y \in D$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^* as $n \to \infty$ and $\overline{\lim}(K(y_n), y_n - y)_{Y^*,Y} \leq 0$, then $K(y_n) \xrightarrow{w} K(y)$ in Y^* and $(K(y_n), y_n)_{Y^*Y} \to (K(y), y)_{Y^*Y}$ as $n \to \infty$. It is well-known (see, for example, Zeidler [18: p. 897]) that a linear operator $L : D \subseteq Y \to Y^*$ is maximal monotone if and only if it is densely defined, closed and both L and L^* are monotone.

Our existence theorems will be based on the following surjectivity result which can be found in Lions [10] or B.-A. Ton [16].

Theorem 1. *If Y is a reflexive Banach space, $L : D \subseteq Y \to Y^*$ is a linear, maximal monotone operator and $K : Y \to Y^*$ is an L-pseudomonotone operator which is coercive (i.e. $\lim_{\|y\|_Y \to \infty} \frac{(K(y),y)_{Y^*Y}}{\|y\|_Y} = +\infty$), then $R(L + K) = Y^*$ (i.e. $L + K$ is surjective).*

Let

$$L_1 : D_1 \subseteq L^p(T, X) \to L^q(T, X^*) \qquad \left(\tfrac{1}{p} + \tfrac{1}{q} = 1\right)$$

be defined by $L_1(x) = \dot{x}$ for all $x \in D_1$,

$$D_1 = \left\{y \in L^p(T, X) : \dot{y} \in L^q(T, X^*) \text{ and } y(0) = y(b)\right\}.$$

Here as before the time derivative of x is understood in the sense of vector-valued distributions. Also, since $W_{pq}(T) \subseteq C(T, H)$, we see that the pointwise evaluations at $t = 0$ and $t = b$ in the definition of D_1 make sense. Since the space $C_0^1(T, X)$ is dense in $L^p(T, X)$, we see at once that D_1 is dense in $L^p(T, X)$. Also,

$$L_1^* : D_1^* \subseteq L^p(T, X) \to L^q(T, X^*)$$

(recall that, for a reflexive Banach space Y and $1 \leq r < \infty$, $L^r(T, Y)^* = L^s(T, Y^*)$ with $\frac{1}{r} + \frac{1}{s} = 1$) is defined by $L_1(v) = -\dot{v}$ for all $v \in D_1^* = D_1$. Hence both L_1 and L_1^* are linear monotone and clearly L_1 is closed. Hence by what was said earlier, we have that L_1 is a maximal monotone linear operator. In a similar way we can show that

$$L_2 : D_2 \subseteq L^p(T, X) \to L^q(T, X^*)$$

defined by $L_2(x) = \dot{x}$ for all $x \in D_2$,

$$D_2 = \left\{y \in L^p(T, X) : \dot{y} \in L^q(T, X^*) \text{ and } y(0) = 0\right\}$$

is maximal monotone. Note that in this case

$$L_2^* : D_2^* \subseteq L^p(T, X) \to L^q(T, X^*)$$

is defined by $L_2(v) = -\dot{v}$ for all $v \in D_2^*$,

$$D_2^* = \Big\{w \in L^p(T,X) : \dot{w} \in L^q(T,X^*) \text{ and } w(b) = 0\Big\}.$$

Finally, recall that on operator $K : X \to X^*$ is said to be *demicontinuous* if $x_n \to x$ in X implies $K(x_n) \xrightarrow{w} K(x)$ in X^* as $n \to \infty$.

For the rest of this paper (X, H, X^*) is an evolution triple with X embedded compactly into H (hence H is embedded compactly into X^*). The next proposition will make possible the use of Theorem 1. The hypothesis on the operator $A(t,x)$ is the following:

H(A) $A : T \times X \to X^*$ is an operator such that:

(i) $t \to A(t,x)$ is measurable.

(ii) $x \to A(t,x)$ is pseudomonotone.

(iii) $\|A(t,x)\|_* \le a_1(t) + c_1\|x\|^{p-1}$ a.e. on T $(a_1 \in L^q(T), c_1 \ge 0, p \ge 2, \frac{1}{p} + \frac{1}{q} = 1)$.

(iv) $\langle A(t,x), x\rangle \ge c\|x\|^p - \vartheta(t)$ for a.a. $t \in T$ and all $x \in X$ $(c \le 0, \vartheta \in L^1(T))$.

Remark. The pseudomonotonicity of $A(t,\cdot)$ (hypothesis H(A)/(ii)) and the boundedness growth condition on $A(t,\cdot)$ (hypothesis H(A)/(iii)) imply that $A(t,\cdot)$ is demicontinuous.

Let $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ be defined by $\widehat{A}(x)(\cdot) = A(\cdot, x(\cdot))$ (the Nemitsky (superposition) operator corresponding to $A(t,x)$). Also, by $((\cdot,\cdot))$ we will denote the duality brackets of the pair $(L^q(T,X^*), L^p(T,X))$, i.e. $((u,y)) = \int_0^b \langle u(t), y(t)\rangle\, dt$ for all $y \in L^p(T,X)$ and all $u \in L^q(T,X^*)$.

Proposition 2. *If $A : T \times X \to X^*$ is an operator satisfying hypothesis* H(A) *and $L : D = W_{pq}(T) \subseteq L^p(T,X) \to L^q(T,X^*)$ is the closed, densely defined, linear operator given by $L(x) = \dot{x}$, then $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ is demicontinuous and L-pseudomonotone.*

Proof. First we will prove the demicontinuity of $\widehat{A}$. So let $x_n \to x$ in $L^p(T,X)$ as $n \to \infty$. By passing to a subsequence if necessary we may assume that $x_n(t) \to x(t)$ a.e. on T in X as $n \to \infty$. Because of hypothesis H(A)/(ii), for every $y \in L^p(T,X)$ we have $\langle A(t,x_n(t)), y(t)\rangle \to \langle A(t,x(t)), y(t)\rangle$ a.e. on T. Then using the extended dominated convergence theorem (see, for example, Ash [4: Theorem 7.5.2, p.295]), we have $((\widehat{A}(x_n), y)) \to ((\widehat{A}(x), y))$ as $n \to \infty$. Since $y \in L^p(T,X)$ was arbitrary we have that $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*)$ as $n \to \infty$, hence $\widehat{A}(\cdot)$ is demicontinuous.

Next we will show the L-pseudomonotonicity of $\widehat{A}$ for

$$L : D = W_{pq}(T) \subseteq L^p(T,X) \to L^q(T,X^*)$$

defined by $L(x) = \dot{x}$. So let $\{x_n, x\}_{n\ge 1} \subseteq W_{pq}(T)$ and suppose $x_n \xrightarrow{w} x$ in $L^p(T,X)$ and $L(x_n) \xrightarrow{w} L(x)$ in $L^q(T,X^*)$ as $n \to \infty$. Hence $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$. Assume that $\overline{\lim}((\widehat{A}(x_n), x_n - x)) \le 0$. Set $\xi_n(t) = \langle A(t, x_n(t)), x_n(t) - x(t)\rangle$. Since $W_{pq}(T)$ is embedded continuously into $C(T,H)$, we have $x_n \xrightarrow{w} x$ in $C(T,H)$ as $n \to \infty$. So for every $t \in T$, we have $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$. Also, let $N \subseteq T$ be the exceptional Lebesgue-null set outside of which hypothesis H(A)/(iiii) and (iv) holds. We have

$$\xi_n(t) \geq c\|x_n(t)\|^p - \vartheta(t) - \big(a_1(t) + c_1\|x_n(t)\|^{p-1}\big)\|x(t)\| \qquad (t \in T \backslash N). \tag{1}$$

Set $C = \{t \in T : \underline{\lim}\,\xi_n(t) < 0\}$. This is a Lebesgue measurable subset of T. Suppose that $\lambda(C) > 0$, with λ being the Lebesgue measure on T. From (1) it follows that for every $t \in C \cap (T\backslash N) \neq \emptyset$, the sequence $\{x_n(t)\}_{n\geq 1}$ is bounded in X. Since X is reflexive and $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$, we deduce that $x_n(t) \xrightarrow{w} x(t)$ in X as $n \to \infty$. We fix $t \in C \cap (T\backslash N)$ and consider a subsequence $\{\xi_{n_m}\}_{m\geq 1}$ of $\{\xi_n\}_{n\geq 1}$ such that $\lim \xi_{n_m}(t) = \underline{\lim}\,\xi_n(t) < 0$ (of course, the subsequence in general depends on t). Exploiting the fact that $A(t,\cdot)$ is pseudomonotone, we have that $\langle A(t, x_{n_m}(t)), x_{n_m}(t) - x(t)\rangle \to 0$ as $m \to \infty$, which is a contradiction to the hypothesis that $t \in C$.

So $\lambda(C) = 0$, which means that $0 \leq \underline{\lim}\,\xi_n(t)$ a.e. on T. Then from the extended Fatou lemma (see Ash [4: Theorem 7.5.2, p. 295]) we obtain

$$0 \leq \int_0^b \underline{\lim}\,\xi_n(t)\,dt \leq \underline{\lim}\int_0^b \xi_n(t)\,dt \leq \overline{\lim}\int_0^b \xi_n(t)\,dt \leq 0.$$

Hence $\int_0^b \xi_n(t)\,dt \to 0$ as $n \to \infty$. Since $0 \leq \underline{\lim}\,\xi_n(t)$ a.e. on T, we deduce that $\xi_n^-(t) \to 0$ a.e. on T. Moreover, from (1) it is evident that $\vartheta_n(t) \leq \xi_n(t)$ a.e. on T with $\{\vartheta_n\}_{n\geq 1}$ being a uniformly integrable sequence. Thus $0 \leq \xi_n^-(t) \leq \vartheta_n^-(t)$ a.e. on T and of course $\{\vartheta_n^-\}_{n\geq 1}$ is uniformly integrable. Thus a new application of the extended dominated convergence implies that $\int_0^b \xi_n^-(t)\,dt \to 0$ as $n \to \infty$. So finally we have

$$\int_0^b |\xi_n(t)|\,dt = \int_0^b \big(\xi_n(t) + 2\xi_n^-(t)\big)dt \to 0 \qquad \text{as } n \to \infty$$

and thus by passing to a subsequence if necessary, we may assume that $\xi_n(t) \to 0$ a.e. on T as $n \to \infty$. Because $A(t,\cdot)$ is pseudomonotone, we have

$$A(t, x_n(t)) \xrightarrow{w} A(t, x(t)) \qquad \text{a.e. on } T \text{ in } X^*$$

and

$$\langle A(t, x_n(t)), x_n(t)\rangle \to \langle A(t, x(t)), x(t)\rangle \qquad \text{as } n \to \infty.$$

Then a final application of the extended dominated convergence theorem implies that

$$\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x) \ \text{ in } L^q(T, X^*) \qquad \text{and} \qquad ((\widehat{A}(x_n), x_n)) \to ((\widehat{A}(x), x))$$

as $n \to \infty$. Therefore $\widehat{A}(\cdot)$ is L-pseudomonotone ∎

3. Existence theorems

In this section we prove existence theorems for the two problems

$$\left.\begin{aligned} \dot{x}(t) + A(t,x(t)) = 0 \quad \text{a.e. on } T \\ x(0) = x(b) \end{aligned}\right\} \tag{2}$$

and

$$\left.\begin{aligned} \dot{x}(t) + A(t,x(t)) = 0 \quad \text{a.e. on } T \\ x(0) = x_0. \end{aligned}\right\} \tag{3}$$

The hypothesis on $A(t,x)$ is that introduced in Section 2, namely H(A).

In Hirano [8] and Ahmed and Xiang [2], in both problems the nonlinear term has the form $A(t,x) + f(t,x)$ with $f(t,x)$ satisfying hypothesis $H(f)$ below.

H(f) $f : T \times X \to H$ is a function such that:

(i) $t \to f(t,x)$ is measurable.

(ii) $x \to f(t,x)$ is sequentially weakly continuous.

(iii) $|f(t,x)| \le a_2(t) + c_2\|x\|^{p-1}$ a.e. on T $(a_2 \in L^q(T), c_2 > 0)$.

(iv) $(f(t,x),x) \ge -c_3$ for a.a. $t \in T$ and all $x \in X$ $(c_3 > 0)$.

In our case no extra generality is achieved by such a decomposition since the term $A(t,x) + f(t,x)$ still satisfies hypothesis H(A).

We start with the periodic problem (2).

Theorem 3. *If hypothesis* H(A) *holds, then problem* (2) *has a solution* $x \in W_{pq}(T)$.

Proof. Let

$$L_1 : D_1 \subseteq L^p(T,X) \to L^q(T,X^*)$$

be the linear maximal monotone operator defined by $L(x) = \dot{x}$ for $x \in D$,

$$D = \left\{ x \in W_{pq}(T) : x(0) = x(b) \right\}.$$

Also, let $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ be the Nemytski operator corresponding to $A(t,x)$, i.e. $\widehat{A}(x)(\cdot) = A(\cdot, x(\cdot))$, and let $K = \widehat{A} : L^p(T,X) \to L^q(T,X^*)$.

Claim 1 : *K is L_1-pseudomonotone.* This is proved as Proposition 2.

Claim 2 : *$K(\cdot)$ is coercive.* We have

$$((K(x),x)) = ((\widehat{A}(x),x)) \ge c\|x\|^p_{L^p(T,X)} - \|\vartheta\|_1$$

(see hypothesis H(A)/(iv)). From this it follows that $K(\cdot)$ is coercive. Rewrite problem (2) as the equivalent abstract operator equation $L_1(x) + K(x) = 0$. By Theorem 1 this equation has a solution $x \in D_1$. So $x \in W_{pq}(T)$ is the desired solution of problem (2) ∎

Now we turn to the Cauchy problem (3). We have the following existence result.

Theorem 4. *If hypothesis* H(A) *holds and* $x_0 \in H$, *then problem* (3) *has a nonempty solution set which is compact in* $C(T, H)$.

Proof. In the first part of the proof we assume that $x_0 \in X$. Let $A_1 : T \times X \to X^*$ be defined by $A_1(t,x) = A(t, x + x_0)$. Evidently, $t \to A_1(t,x)$ is measurable and $x \to A_1(t,x)$ is demicontinuous.

We claim that $x \to A_1(t,x)$ is also pseudomonotone. By what was said in Section 2 (see also Browder and Hess [6: Proposition 4]) it suffices to show that if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\langle A_1(t,x_n), x_n - x\rangle \le 0$, then

$$A_1(t,x_n) \xrightarrow{w} A_1(t,x) \text{ in } X^* \qquad \text{and} \qquad \langle A_1(t,x_n), x_n\rangle \to \langle A_1(t,x), x\rangle$$

as $n \to \infty$. Note that

$$\overline{\lim}\big\langle A(t, x_n + x_0), x_n + x_0 - (x_n + x_0)\big\rangle = \overline{\lim}\big\langle A_1(t,x_n), x_n - x\big\rangle \le 0$$

and since $A(t,\cdot)$ is pseudomonotone, we have

$$A_1(t,x_n) = A(t, x_n + x_0) \xrightarrow{w} A(t, x + x_0) = A_1(t,x)$$

in X^* and

$$\langle A(t, x_n + x_0), x_n + x_0\rangle \to \langle A(t, x + x_0), x_n + x_0\rangle$$

as $n \to \infty$. Hence we have

$$\underline{\lim}\langle A(t, x_n + x_0), x_n\rangle + \langle A(t, x + x_0), x_0\rangle \ge \langle A(t, x + x_0), x\rangle + \langle A(t, x + x_0), x_0\rangle$$

from which it follows that $\langle A_1(t,x_n), x_n\rangle \to \langle A_1(t,x), x\rangle$. So indeed $x \to A_1(t,x)$ is pseudomonotone. Also, it is easy to check using hypothesis H(A)/(iii) and (iv) that

$$\|A_1(t,x)\|_* \le \widehat{a}_1(t) + \widehat{c}_1\|x\|^{p-1} \qquad \text{a.e. on } T \ \ (\widehat{a}_1 \in L^q(T), \widehat{c}_1 > 0)$$

and

$$\langle A_1(t,x), x\rangle \ge \widehat{c}\|x\|^p - \vartheta(t) \qquad \text{a.e. on } T \ \ (\widehat{c} > 0, \vartheta \in L^1(T)).$$

Thus $A_1(t,x)$ satisfies the same kind of hypothesis as $A(t,x)$.

So if $\widehat{A}_1 : L^p(T,X) \to L^q(T,X^*)$ is the Nemytski operator corresponding to $A_1(t,x)$ (i.e. $\widehat{A}_1(x)(\cdot) = A_1(\cdot, x(\cdot))$), by Proposition 2, $\widehat{A}_1(\cdot)$ is L_2-pseudomonotone, where recall that $L_2 : D_2 \subseteq L^p(T,X) \to L^q(T,X^*)$ is the linear maximal monotone operator defined by $L_2(x) = \dot{x}$ for all $x \in D_2$,

$$D_2 = \Big\{ y \in L^p(T,X) : \dot{y} \in L^q(T,X^*) \text{ and } y(0) = 0 \Big\}.$$

Now let $K_1 : L^p(T,X) \to L^q(T,X)$ be defined by $K_1(x) = \widehat{A}_1(x)$.

Claim 1 : *K_1 is L_2-pseudomonotone.* This is proved using the same arguments as Proposition 2.

Claim 2 : K_1 *is coercive.* Note that

$$((K_1(x), x)) = ((\widehat{A}_1(x), x)) \geq \widehat{c}_4 \|x\|^p_{L^p(T,X)} - \hat{\gamma}$$

where $\widehat{c}_4, \hat{\gamma} > 0$. Therefore K_1 is coercive. Then consider the equivalent operator equation $L_2(x) + K_1(x) = 0$. Invoking Theorem 1, we infer that the operator equation has a solution $\widehat{x} \in D_2$. Set $x(\cdot) = \widehat{x}(\cdot) + x_0$. Then $x \in W_{pq}(T)$ and x is a solution of the Cauchy problem (3) when the initial condition x_0 belongs to X.

Now we remove the regularity condition on x_0 and assume that in general $x_0 \in H$. Let $\{x_0^n\}_{n\geq 1} \subseteq X$ and assume that $x_0^n \to x_0$ in H as $n \to \infty$. Consider the Cauchy problem

$$\left.\begin{aligned} \dot{x}(t) + A(t, x(t)) = 0 \quad \text{a.e. on } T \\ x(0) = x_0^n \quad (n \geq 1). \end{aligned}\right\} \tag{4}$$

From the first part of the proof, we know that for every $n \geq 1$, the evolution equation (4) has a solution $x_n \in W_{pq}(T)$. Multiply by $x_n(t)$ and integrate over T to obtain

$$((\dot{x}_n, x_n)) + ((\widehat{A}(x_n), x_n)) = 0 \qquad (n \geq 1). \tag{5}$$

From integration by parts for functions in $W_{pq}(T)$ (see Zeidler [18: Proposition 23.23, p.422 – 423]) we have

$$((\dot{x}_n, x_n)) = \frac{1}{2}\int_0^b \frac{d}{dt}|x_n(t)|^2 dt = \frac{1}{2}|x_n(b)|^2 - \frac{1}{2}|x_0^n|^2 \geq \frac{1}{2}|x_n(b)|^2 - \beta \tag{6}$$

for some $\beta > 0$. Also, from hypothesis H(A)/(iv) it follows that

$$((\widehat{A}(x_n), x_n)) \geq c\,\|x_n\|^p_{L^p(T,X)} - \|\vartheta\|_1. \tag{7}$$

Using (6) and (7) in (5), we obtain

$$c\,\|x_n\|^p_{L^p(T,X)} \leq \|h\|_{L^q(T,X^*)}\|x_n\|_{L^p(T,X)} + \beta_1 \qquad \text{with } \beta_1 = \beta + \|\vartheta\|_1.$$

From this inequality it is clear that the sequence $\{x_n\}_{n\geq 1}$ is bounded in $L^p(T, X)$. Then using hypothesis H(A)/(iii) we show easily that the sequence $\{\dot{x}_n\}_{n\geq 1}$ is bounded in $L^q(T, X^*)$. Therefore we conclude that $\{x_n\}_{n\geq 1}$ is bounded in $W_{pq}(T)$ and so, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$. Then we have

$$\overline{\lim}((\widehat{A}(x_n) + \widehat{f}(x_n), x_n - x)) \leq \overline{\lim}((\dot{x}_n, x - x_n)). \tag{8}$$

Employing once again integration by parts for functions in $W_{pq}(T)$, we have

$$((\dot{x}_n, x - x_n)) = -\frac{1}{2}|x(b) - x_n(b)|^2 + \frac{1}{2}|x(0) - x_0^n|^2 + ((\dot{x}, x - x_n)). \tag{9}$$

Since $W_{pq}(T)$ is embedded continuously into $C(T,H)$, we have $x_n \xrightarrow{w} x$ in $C(T,H)$ and so $x_n(0) = x_0^n \xrightarrow{w} x(0)$ in H as $n \to \infty$. Hence $x(0) = x_0$. Also, from (9) we have

$$\overline{\lim}((\dot{x}_n, x - x_n)) \leq 0. \tag{10}$$

Moreover, since $W_{pq}(T)$ is embedded compactly into $L^p(T,H)$, we have $x_n \to x$ in $L^p(T,H)$ as $n \to \infty$. So using (10) in (8), we have $\overline{\lim}((\widehat{A}(x_n), x_n - x)) \leq 0$. But by Proposition 2, $\widehat{A}(\cdot)$ is L-pseudomonotone with $L : D = W_{pq}(T) \subseteq L^p(T,X) \to L^q(T,X^*)$ defined by $L(x) = \dot{x}$. So we have $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*)$ as $n \to \infty$. Hence in the limit as $n \to \infty$, we have

$$\left.\begin{aligned} \dot{x} + \widehat{A}(x) &= 0 \\ x(0) &= x_0 \end{aligned}\right\}$$

which shows that $x \in W_{pq}(T)$ is a solution of problem (3) when $x_0 \in H$.

Finally we will show that the solution set $S \subseteq W_{pq}(T) \subseteq C(T,H)$ of problem (3) is compact in $C(T,H)$. To this end let $\{x_n\}_{n\geq 1} \subseteq S$. Then we have

$$\left.\begin{aligned} \dot{x}_n + \widehat{A}(x_n) &= h \\ x_n(0) &= x_0. \end{aligned}\right\}$$

From the previous estimation we know that the sequence $\{x_n\}_{n\geq 1}$ is bounded in $W_{pq}(T)$. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$, $x_n \to x$ in $L^p(T,H)$ and $x_n(t) \to x(t)$ in H as $n \to \infty$. As above we can show that

$$\left.\begin{aligned} \dot{x} + \widehat{A}(x) &= 0 \\ x(0) &= x_0, \end{aligned}\right\}$$

i.e. $x \in S$. Now we will show that $x_n \to x$ in $C(T,H)$.

In what follows, for any $t \in T$ by $((\cdot,\cdot))_t$ we will denote the duality brackets of the pair $(L^q([0,t],X^*), L^p([0,t],X))$. Also, by $(\cdot,\cdot)_{L^q,L^p,t}$ we will denote the duality brackets of the pair $(L^q([0,t],H), L^p([0,t],H))$. From integration by parts for functions in $W_{pq}(T)$ we have that

$$\frac{1}{2}|x_n(t) - x(t)|^2 + \left((\widehat{A}(x_n) - \widehat{A}(x), x_n - x)\right)_t = 0$$

implies

$$\frac{1}{2}|x_n(t) - x(t)|^2 \leq \int_0^b \left|\langle A(t, x_n(t)), x_n(t) - x(t)\rangle\right| dt + \left((\widehat{A}(x), x_n - x)\right)_t.$$

From the proof of Proposition 2 we know that

$$\int_0^b \left|\langle A(t, x_n(t)), x_n(t) - x(t)\rangle\right| dt = \int_0^b |\xi_n(t)|\, dt \to 0 \qquad \text{as } n \to \infty.$$

Next we examine the limit behaviour of the sequence

$$\left\{\sup_{t\in T}\left((\widehat{A}(x), x_n - x)\right)_t\right\}_{n\geq 1}.$$

We have

$$\sup_{t\in T}\left((\widehat{A}(x), x_n - x)\right)_t = \sup_{t\in T}\int_0^t \langle A(s, x(s)), x_n(s) - x(s)\rangle ds.$$

Let

$$\varphi_n(t) = \int_0^t \langle A(s, x(s)), x_n(s) - x(s)\rangle ds \qquad (n \geq 1).$$

Then $\varphi_n \in AC^1(T)$. Let $t_n \in T$ be such that $\varphi_n(t_n) = \sup_{t\in T}\varphi_n(t)$. We may assume that $t_n \to t \in T$. Then

$$\begin{aligned}\varphi_n(t_n) &= \int_0^{t_n} \langle A(s, x(s)), x_n(s) - x(s)\rangle ds \\ &= \int_0^b \langle \chi_{[0,t_n]}(s)A(s, x(s)), x_n(s) - x(s)\rangle ds \\ &= \left((\chi_{[0,t_n]}\widehat{A}(x), x_n - x)\right).\end{aligned}$$

Note that

$$\int_0^b \left\|\chi_{[0,t_n]}(s)A(s, x(s)) - \chi_{[0,t]}(s)A(s, x(s))\right\|_*^q ds = \int_{t\wedge t_n}^{t\vee t_n} \|A(s, x(s))\|_*^q ds \to 0$$

as $n \to \infty$. Hence $\chi_{[0,t_n]}\widehat{A}(x) \to \chi_{[0,t]}\widehat{A}(x)$ in $L^q(T, X)$ as $n \to \infty$. Therefore

$$\left((\widehat{A}(x), x_n - x)\right)_{t_n} = \sup_{t\in T}\left((\widehat{A}(x), x_n - x)\right)_t \to 0$$

as $n \to \infty$. Thus finally we have that $\sup_{t\in T}|x_n(t) - x(t)| \to 0$ as $n \to \infty$, i.e. $x_n \to x$ in $C(T, H)$ as $n \to \infty$ and $x \in S$. This proves that the solution set S of problem (3) is compact ∎

The last part of the previous proof has an interesting consequence. More specifically, consider the evolution equation

$$\left.\begin{aligned}\dot{x}(t) + A(t, x(t)) = g(t) \quad \text{a.e. on } T \\ x(0) = x_0 \in H.\end{aligned}\right\} \tag{11}$$

Under a monotonicity condition on $A(t,\cdot)$, for every $g \in L^q(T,H)$ problem (11) has a unique solution $x \in W_{pq}(T) \subseteq C(T,H)$. So we can define the map $w : L^q(T,H) \to C(T,H)$, which to each $g \in L^q(T,H)$ assigns the unique solution $x \in W_{pq}(T) \subseteq C(T,H)$ of problem (11). The next proposition establishes a useful property of that map w. But first we formulate the precise hypothesis $\mathrm{H(A)}_1$ on $A(t,x)$.

$\mathbf{H(A)_1}$ $A : T \times X \to X^*$ is an operator such that:

(i) $t \to A(t,x)$ is measurable.

(ii) $x \to A(t,x)$ is demicontinuous and monotone.

(iii) $\|A(t,x)\|_* \leq a_1(t) + c_1\|x\|^{p-1}$ a.e. on T $(a_1 \in L^q(T), c_1 \geq 0, p \geq 2, \frac{1}{p} + \frac{1}{q} = 1)$.

(iv) $\langle A(t,x), x\rangle \geq c\|x\|^p - \vartheta(t)$ for a.a. $t \in T$ and all $x \in X$ $(c > 0, \vartheta \in L^1(T))$.

Proposition 5. *If hypothesis* H(A) *holds and* $x_0 \in H$, *then the map*

$$w : L^q(T,H) \to C(T,H)$$

is completely continuous, i.e. continuous and maps bounded sets into relatively compact sets.

Remark. In the light of the recent counterexample to the embedding theorem of Nagy [12], due to Migorski [11], it is this proposition that should be used in [13 - 15] instead of Nagy's embedding theorem. In fact, the restriction that X is a Hilbert space too is no longer necessary. So we can improve the results of [13 - 15].

4. Examples

In this last section, we present two examples from parabolic partial differential equations, which illustrate the applicability of our results.

Example 1. Let $T = [0,b]$ and $Z \subseteq \mathbb{R}^N$ $(N \leq 3)$ a bounded domain with Lipschitz boundary Γ. Let $D_k = \frac{\partial}{\partial z_k}$ $(k = 1,\dots,N)$ and $D = (D_k)_{k=1}^N$. We consider the nonlinear parabolic problem

$$\left.\begin{aligned} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} \big(D_k a_k(t,z,x,Dx) + a_0(t,z,x)D_k x\big) + f(t,z,x(t,z)) &= h(t,z) \text{ in } T \times Z \\ x(0,z) &= x_0(z) \text{ a.e. on } Z \\ x &= 0 \text{ on } T \times \Gamma. \end{aligned}\right\} \quad (12)$$

The presence of the first-order derivatives makes it more difficult to establish the pseudomonotonicity of the operator $A(t,\cdot)$, which uses critically the compact embedding of $W_0^{1,p}(Z)$ into $L^{2q}(Z)$. A typical example of the first order term is the term $\gamma \sum_{k=1}^N (\sin x) D_k x$ with $\gamma \in \mathbb{R}$.

The hypotheses on the data of problem (12) are the followings.

H(a) $a_k : T \times Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ $(k = 1,\dots,N)$ are functions such that:

(i) $(t,z) \to a_k(t,z,x,\eta)$ is measurable.

(ii) $(x,\eta) \to a_k(t,z,x,\eta)$ is continuous.

(iii) $|a_k(t,z,x,\eta)| \le \beta_1(t,z)+c_1\left(|x|^{p-1}+\|\eta\|^{p-1}\right)$ a.e. on $T\times Z$ $(\beta_1 \in L^q(T\times Z), c_1 \ge 0, 2 \le p < \infty, \frac{1}{p}+\frac{1}{q}=1)$.

(iv) $\sum_{k=1}^N \left(a_k(t,z,x,\eta)-a_k(t,z,x,\eta')\right)(\eta_k-\eta'_k) > 0$ for a.a. $(t,z) \in T\times Z$, all $x \in \mathbb{R}$ and all $\eta,\eta' \in \mathbb{R}^N$ with $\eta \ne \eta'$.

(v) $\sum_{k=1}^N a_k(t,z,x,\eta)\eta_k \ge c_2\|\eta\|^p - \vartheta(t)$ for a.a. $(t,z) \in T\times Z$, all $x \in \mathbb{R}$ and all $\eta \in \mathbb{R}^N$ $(\vartheta \in L^1(T))$.

H(a$_0$) $a_0 : T\times Z\times\mathbb{R} \to \mathbb{R}$ is a function such that:

(i) $(t,z) \to a_0(t,z,x)$ is measurable.

(ii) $|a_0(t,z,x)-a_0(t,z,y)| \le k(t,z)|x-y|$ a.e. on $T\times Z$ $(k \in L^\infty(T\times Z))$.

(iii) $|a_0(t,z,x)| \le \beta_2$ a.e. on $T\times Z$ $(\beta_2 > 0)$.

H(f)$_1$ $f : T\times Z\times\mathbb{R} \to \mathbb{R}$ is a function such that:

(i) $(t,z) \to f(t,z,x)$ is measurable.

(ii) $x \to f(t,z,x)$ is continuous.

(iii) $|f(t,z,x)| \le \beta_3(t,z)+c_3|x|^{p/2}$ a.e. on $T\times Z$ $(\beta_3 \in L^q(T,L^2(Z)), c_3 > 0)$.

(iv) $f(t,z,x)x \ge -c_4$ a.e. on $T\times Z$ $(c_4 > 0)$.

Proposition 6. *If hypotheses* H(a), H(a$_0$) *and* H(f)$_1$ *hold,* $h \in L^q(T\times Z)$ *and* $x_0 \in L^2(Z)$, *then problem* (12) *has at least one solution*

$$x \in L^p(T,W_0^{1,p}(Z)) \times C(T,L^2(Z)) \qquad \text{such that } \frac{\partial x}{\partial t} \in L^q(T,W^{-1,q}(Z))$$

and the set of such solutions is compact in $L^p(T,L^2(Z))$.

Proof. For this problem the evolution triple consists of $X = W_0^{1,p}(Z), H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. Since $2 \le p < \infty$, by the Sobolev embedding theorem, X is embedded compactly into H.

Let $u_1 : T\times W_0^{1,p}(Z)\times W_0^{1,p}(Z) \to \mathbb{R}$ be the time-dependent semilinear form defined by

$$u(t,x,y) = \int_Z \sum_{k=1}^N a_k(t,z,z,Dx)D_k y(z)\,dz.$$

Because of hypothesis H(a)/(iii), $u(t,x,\cdot)$ is bounded and linear, and so we can define a nonlinear operator $A : T\times X \to X^*$ by setting

$$u(t,x,y) = \langle A(t,x),y\rangle.$$

Note that by Fubini's theorem, $t \to \langle A(t,x),y\rangle$ is measurable, and since $y \in X$ is arbitrary, we have that $t \to A(t,x)$ is weakly measurable, and because X^* is separable, by the Pettis measurability theorem $t \to A(t,x)$ is measurable. Also, because of hypothesis

H(A)/(ii) and (iii), it is straightforward to check that $x \to A(t,x)$ is demicontinuous. Moreover, from Browder [5] we know that $x \to A(t,x)$ is pseudomonotone.

Next, let $v : T \times X \to H \subseteq X^*$ be defined by

$$v(t,x)(\cdot) = a_0(t,\cdot,x(\cdot)) \sum_{k=1}^{N} D_k x(\cdot).$$

Again we can check that $t \to v(t,x)$ is measurable. We claim that $x \to v(t,x)$ is completely continuous from $W_0^{1,p}(Z)$ into $W^{-1,q}(Z)$. Because of the reflexivity of the spaces, we need to show that if $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, then $v(t,x_n) \to v(t,x)$ in $W^{-1,q}(Z)$ as $n \to \infty$. Suppose not. Then we can find $\varepsilon > 0$, a subsequence $\{x_m\}_{m\geq 1}$ of $\{x_n\}_{n\geq 1}$ and a sequence $\{y_m\}_{m\geq 1} \subseteq X = W_0^{1,p}(Z)$ with $\|y_m\| = 1$ such that for all $m \geq 1$ we have

$$\langle v(t,x_m) - v(t,x), y_m \rangle \geq \varepsilon. \tag{13}$$

Since $N \leq 3$, from the Sobolev embedding theorem we have that $W_0^{1,p}(Z)$ is embedded compactly into $L^{2p}(Z) \subseteq L^{2q}(Z)$, and so we may assume that $x_m \to x$ and $y_m \to y$ in $L^{2q}(Z)$ as $m \to \infty$. For every $k = 1, \ldots, N$ we have

$$\begin{aligned}
\varepsilon &\leq \int_Z a_0(t,z,x_m(z)) D_k x_m(z) y_m(z)\,dz - \int_Z a_0(t,z,x(z)) D_k x(z) y_m(z)\,dz \\
&= \int_Z \big(a_0(t,z,x_m(z)) - a_0(t,z,x(z))\big) D_k x_m(z) y_m(z)\,dz \\
&\quad + \int_Z a_0(t,z,x(z)) D_k x_m(z)(y_m(z) - y(z))\,dz \\
&\quad + \int_Z a_0(t,z,x(z))(D_k x_m(z) - D_k x(z)) y(z)\,dz \\
&\quad + \int_Z a_0(t,z,x(z)) D_k x(z)(y(z) - y_m(z))\,dz.
\end{aligned}$$

From Hölder's inequality with three factors we have

$$\begin{aligned}
&\left| \int_Z \big(a_0(t,z,x_m(z)) - a_0(t,z,x(z))\big) D_k x_m(z) y_m(z)\,dz \right| \\
&\qquad \leq \int_Z k(t,z) |x_m(z) - x(z)|\,|D_k x_m(z)|\,|y_m(z)|\,dz \\
&\qquad \leq \|k\|_\infty \|x_m - x\|_{2q} \|D_k x_m\|_p \|y_m\|_{2q} \\
&\qquad \to 0 \qquad \text{as } m \to \infty.
\end{aligned}$$

Also, we have

$$\left| \int_Z a_0(t,z,x(z)) D_k x_m(z)(y_m(z) - y(z))dz \right| \leq \beta_2 |Z|^{1/2q} \|D_k x_m\|_p \|y_m - y\|_{2q} \to 0$$

$$\left| \int_Z a_0(t,z,x(z))(D_k x_m(z) - D_k x(z)) y(z)\,dz \right| \to 0$$

$$\left| \int_Z a_0(t,z,x(z)) D_k x(z)(y(z) - y_m(z))dz \right| \leq \beta_2 |Z|^{1/2q} \|D_k x\|_p \|y - y_m\|_{2q} \to 0$$

as $m \to \infty$, where $|Z|$ is the Lebesgue measure of the domain Z. So finally we have

$$\langle v(t, x_m) - v(t, x), y_m \rangle \to 0 \qquad \text{as } m \to \infty$$

which contradicts (13). Therefore $x \to v(t,x)$ is completely continuous as claimed.

Now, if we define $A_2(t,x) = A_1(t,x) + v(t,x)$, we have that $t \to A_2(t,x)$ is measurable and $x \to A_2(t,x)$ is demicontinuous and pseudomonotone (see Zeidler [18: Proposition 27.6/(f), p. 586]). In addition, we have

$$\|A_2(t,x)\|_* \le \|A_1(t,x)\|_* + \|v(t,x)\|_* \le a_1(t) + c_1\|x\|^{p-1} \qquad \text{a.e. on } T$$

with $a_1 \in L^q(T)$ and $c_1 \ge 0$. Note that because of hypothesis H(A)/(iv)

$$\langle A_1(t,x), x \rangle \ge \widehat{c}\|x\|^p - \vartheta(t) \qquad \text{a.e. on } T$$

for some $\widehat{c} > 0$ and $\vartheta \in L^1(T)$. Also, for all $y \in W_0^{1,p}(Z)$, we have

$$\begin{aligned} |\langle v(t,x), y \rangle| &= \left| \int_Z a_0(t,z,x(z)) \left(\sum_{k=1}^N D_k x(z) \right) y(z)\, dz \right| \\ &\le \beta_2 \int_Z \sum_{k=1}^N |D_k x(z)|\, |y(z)|\, dz \\ &\le \beta_2 \|Dx\|_p \|y\|_q \\ &\le \beta_2' \|x\|\, \|y\| \end{aligned}$$

for some $\beta_2' > 0$. Since $y \in W_0^{1,p}(Z)$ was arbitrary, it follows that $\|v(t,x)\|_* \le \beta_2'\|x\|$. Applying Young's inequality with $\varepsilon > 0$, we obtain $v(t,x)\|_* \le \beta_2'(\varepsilon) + \widehat{c}(\varepsilon)\|x\|^{p-1}$ with $\widehat{\beta}_2'(\varepsilon), \widehat{c}(\varepsilon) > 0$. So we have

$$\langle v(t,x), x \rangle \ge -\|v(t,x)\|_* \|x\| \ge -\widehat{c}(\varepsilon)\|x\|^p - \widehat{\beta}_2'(\varepsilon)\|x\|.$$

Finally we have

$$\begin{aligned} \langle A(t,x), x \rangle &= \langle A_1(t,x) + v(t,x), x \rangle \\ &\ge \widehat{c}\|x\|^p - \vartheta(t) - \widehat{c}(\varepsilon)\|x\|^p - \widehat{\beta}_2(\varepsilon)\|x\| \\ &= (\widehat{c} - \widehat{c}(\varepsilon))\|x\|^p - \widehat{\beta}_2(\varepsilon)\|x\| - \vartheta(t) \qquad \text{((a.e. on } T) \\ &= (\widehat{c} - \widehat{c}(\varepsilon))\|x\|^p - \vartheta_\varepsilon(t) \qquad \text{(by Young's inequality).} \end{aligned}$$

Choose $\varepsilon > 0$ so that $\widehat{c}(\varepsilon) < \widehat{c}$. Thus we have checked that $A_2(t,x)$ satisfies hypothesis H(A).

Next let $\widehat{f} : T \times X \to H$ be defined by

$$\widehat{f}(t,x)(\cdot) = f(t, \cdot, x(\cdot)).$$

Using the compact embedding of $W_0^{1,p}(Z)$ into $L^2(Z)$ and hypothesis $H(f)$, we can easily see that $\widehat{f}(t,\cdot)$ is continuous. Set

$$A(t,x(t)) = A_2(t,x(t)) + \widehat{f}(t,x).$$

Clearly, $A(t,x(t))$ satisfies hypothesis H(A). Rewrite (12) as equivalent abstract evolution equation

$$\left.\begin{aligned} \dot{x}(t) + A(t,x(t)) = h(t) \quad \text{a.e. on } T \\ x(0) = x_0. \end{aligned}\right\}$$

Invoking Theorem 4, we conclude that problem (12) has a solution

$$x \in L^p(T, W_0^{1,p}(Z)) \cap C(T, L^2(Z)) \qquad \text{with } \frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$$

and the solution set is compact in $C(T, L^2(Z))$.

Example 2. Again let $T = [0,b]$ and $Z \subseteq \mathbb{R}^N$ a bounded domain with Lipschitz boundary Γ. In what follows $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^N$ will be an N-multiindex and $|\alpha| = \sum_{k=1}^N \alpha_k$. Also, for $x \in W^{m,p}(Z)$ we set $\eta(x) = (D^\alpha x)_{|\alpha| \leq m-1}$, $\vartheta(x) = (D^\alpha x)_{|\alpha|=m}$ and $\xi(x) = (D^\alpha x)_{0<|\alpha|\leq m-1}$. As before we assume that $2 \leq p < \infty$.

The problem under consideration is the following:

$$\left.\begin{aligned} \frac{\partial x}{\partial t} - \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x),\vartheta(x)) + f(t,z,x,\xi(x)) = h(t,z) \text{ in } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z \\ \text{For all } |\beta| \leq m-1,\ D^\beta x = 0 \text{ on } T \times \Gamma. \end{aligned}\right\} \quad (14)$$

The hypotheses on the data of that problem are the followings:

$\mathbf{H(A)_2}$ $A_\alpha : T \times Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R}^{\widehat{N}_m} \to \mathbb{R}$ with $N_{m-1} = \frac{(N+m-1)!}{N!(m-1)!}$, $N_m = \frac{(N+m)!}{N!m!}$ and $\widehat{N}_m = N_m - N_{m-1}$ are functions such that:

(i) $(t,z) \to A_\alpha(t,z,\eta,\vartheta)$ is measurable.

(ii) $(\eta,\vartheta) \to A_\alpha(t,z,\eta,\vartheta)$ is continuous.

(iii) $|A_\alpha(t,z,\eta,\vartheta)| \leq a_1(t,z) + c_1(\|\eta\|^{p-1} + \|\vartheta\|^{p-1})$ a.e. on $T \times Z$ $(a_1 \in L^q(T \times Z), c_1 \geq 0)$.

(iv) $\sum_{|\alpha|=m} \big(A_\alpha(t,z,\eta,\vartheta) - A_\alpha(t,z,\eta,\vartheta')\big)(\vartheta_\alpha - \vartheta'_\alpha) > 0$ a.e. on $T \times Z$, for all $\eta \in \mathbb{R}^{N_{m-1}}$ and all $\vartheta, \vartheta' \in \mathbb{R}^{\widehat{N}_m}$ with $\vartheta \neq \vartheta'$.

(v) $\sum_{|\alpha| \leq m} A_\alpha(t,z,\eta,\vartheta) \geq c_2\|\vartheta\|^p - \varphi(t,z)$ for a.a. $(t,z) \in T \times Z$, all $\eta \in \mathbb{R}^{N_{m-1}}$ and $\vartheta \in \mathbb{R}^{\widehat{N}_m}$ $(c_2 > 0, \varphi \in L^1(T \times Z))$.

$\mathbf{H(f)_2}$ $f : T \times Z \times \mathbb{R}^{N_{m-1}} \to \mathbb{R}$ is a function such that:

(i) $(t,z) \to f(t,z,x,\xi)$ is measurable.

(ii) $(x,\xi) \to f(t,z,x,\xi)$ is continuous.

(iii) $|f(t,z,x,\xi)| \le a_2(t,z) + c_2(|x|^{p/2} + \|\xi\|^{p/2})$ a.e. on $T \times Z$ $(a_2 \in L^q(T \times Z), c_2 \ge 0)$.

(iv) $f(t,z,x,\xi)x \ge -c_3 > 0$ for a.a. $(t,z) \in T \times Z$, all $x \in \mathbb{R}$ and all $\xi \in \mathbb{R}^{N_{m-1}}$.

Proposition 7. *If hypotheses* $\mathrm{H(A)}_2$ *and* $\mathrm{H(f)}_2$ *hold and* $h \in L^q(T \times Z)$, *then problem* (14) *has a solution*

$$x \in L^p(T, W_0^{1,p}(Z)) \cap C(T, L^2(Z)) \qquad \text{with } \frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z)).$$

Proof. For this problem the evolution triple is $X = W_0^{m,p}(Z), H = L^2(Z)$ and $X^* = W^{-m,q}(Z)$. Note that X is embedded compactly into X.

Let $A_1 : T \times X \to X^*$ be defined by

$$\langle A_1(t,x), y\rangle = \sum_{|\alpha|\le m} \int_Z A_\alpha(t,z,\eta(x),\vartheta(x)) D^\alpha y(z)\, dz.$$

Again $t \to A_1(t,x)$ is measurable and $x \to A_1(t,x)$ is demicontinuous and pseudomonotone (see Browder [5]). Moreover, by virtue of hypothesis $\mathrm{H(A)}_2$/(iii) and (iv) we can check that $A_1(t,x)$ satisfies hypothesis H(A). Also, let $\widehat{f} : T \times X \to H$ be the Nemitsky operator corresponding to $f(t,z,x,\xi)$, i.e.

$$\widehat{f}(t,x)(z) = f(t,z,x(z),\xi(x(z))) \qquad (x \in W_0^{1,p}(Z)).$$

Exploiting the compact embedding of $W_0^{m,p}(Z)$ into $W_0^{m-1,p}(Z)$, we can easily see that $\widehat{f}(t,\cdot)$ is complete. Then

$$A(t,x(t)) = A_1(t,x(t)) + \widehat{f}(t,x)$$

satisfies hypotheses H(A). We rewrite problem (14) as equivalent abstract evolution equation

$$\left.\begin{aligned} \dot{x}(t) + A(t,x(t)) &= h(t) \quad \text{a.e. on } T \\ x(0) &= x(b). \end{aligned}\right\}$$

By Theorem 3, this problem has a solution $x \in L^p(T, W_0^{m,p}(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$.

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