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Fourier Multipliers for Besicovitch Spaces

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Abstract. In this paper a generalization of some results from Fourier analysis on periodic function spaces to the almost periodic case is given. We consider almost periodic distributions which constitute a subclass of tempered distributions. Under suitable conditions on the spectrum $\Lambda \subset \mathbb{R}^s$, a distribution $T \in \mathcal{S}'(\mathbb{R}^s)$ is almost periodic if it can be represented as $\sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda\cdot x}$, where the sequence $(a_\lambda)_{\lambda\in\Lambda}$ is tempered. The main result states that any Fourier multipliers for $L^q(\mathbb{R}^s)$ of the Michlin-Hörmander type is also a Fourier multiplier for the Besicovich spaces $B^q_{ap}(\mathbb{R}^s, \Lambda)$, if it is restricted to the spectrum Λ. Finally, we prove that the Sobolev-Besicovich spaces $H^{N,q}_{ap}(\mathbb{R}^s, \Lambda)$ and $W^{N,q}_{ap}(\mathbb{R}^s, \Lambda)$ coincide if $N \in \mathbb{N}$.

Keywords: *Almost periodic functions, distributions, multipliers*

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0. Introduction

In recent years some spaces of generalized almost periodic functions attracted much interest. A reason for this is that the study of partial differential equations with almost periodic coefficients has been considerably developed, and almost periodic function spaces of Sobolev type are useful tools (see, for example, [3, 7, 15] and the references therein).

Unfortunately, the problem of regularity is delicate because of the lack of an appropriate version of the Sobolev embedding theorem in the whole space [2: Esempio 1]. However, in subspaces cut out by suitable spectral restrictions a similar result can be obtained (see [13]). The purpose of this paper is to apply some results from Fourier Analysis on periodic function spaces to the almost periodic case.

Our work is based essentially on the paper [2] and on the book [16]. Although the methods of Fourier Analysis are largely used in the theory of function spaces, it seems that this point of view is unusual in the context of almost periodic functions. Our principal goal is to handle almost periodic functions, also defined in the sense of Besicovitch, as elements of $\mathcal{S}'(\mathbb{R}^s)$, the space of tempered distributions on $\mathbb{R}^s$. To obtain this, we require a structural condition on the spectrum. This hypothesis leads us to a definition of Sobolev-Besicovitch spaces analogous to that of periodic Sobolev spaces [16, 19]. Moreover, Fourier multipliers for subspaces with given spectrum can be considered.

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Now, we sketch briefly the contents of the paper. In Section 1 we start with general definitions and notations. In particular, we define the spaces $B^q_{ap}(\mathbb{R}^s)$ of Besicovitch almost periodic functions. Section 2 is devoted to the relations between almost periodic distributions and tempered distributions. As we have mentioned before, the structure of the spectrum plays a crucial role in our proofs. These results may be considered as an analogue to the standard theory of distributions on the torus and periodic distributions (see, e.g., [11, 16]). In Section 3 we consider Fourier multipliers for trigonometric polynomials and derive a classical inequality. Our main result is stated and proved in Section 4. It states that every Fourier multiplier of Michlin-Hörmander type for $L^q(\mathbb{R}^s)$ is also a Fourier multiplier for $B^q_{ap}(\mathbb{R}^s,\Lambda)$ if it is restricted to the spectrum Λ. Finally, in Section 5 we conclude with a possible definition of Sobolev-Besicovitch spaces $H^{m,q}_{ap}(\mathbb{R}^s,\Lambda)$ and $W^{m,q}_{ap}(\mathbb{R}^s,\Lambda)$ and prove that $H^{N,q}_{ap}(\mathbb{R}^s,\Lambda) = W^{N,q}_{ap}(\mathbb{R}^s,\Lambda)$ if $N \in \mathbb{N}$.

1. Preliminaries

We refer the reader to the monographs [1, 7, 8, 10, 14] for the classical theory of uniformly almost periodic functions. A detailed account about the main properties of B^q_{ap} spaces may be found in the papers [2, 5, 9] and in the monograph [7] (where also other spaces of generalized almost periodic functions are considered). For a different point of view, see the papers [12, 17] and the monograph [15].

We recall that, for any $s \in \mathbb{N}$, $\mathcal{P}(\mathbb{R}^s)$ denotes the complex vector space of all (generalized) trigonometric polynomials of s variables, that is $P \in \mathcal{P}(\mathbb{R}^s)$ if and only if there exist $c_1, c_2, \ldots, c_w \in \mathbb{C}$ and $\lambda^1, \lambda^2, \ldots, \lambda^w \in \mathbb{R}^s$ such that

$$P(x) = \sum_{j=1}^{w} c_j e^{i\lambda^j \cdot x} \qquad (x \in \mathbb{R}^s) \tag{1.1}$$

where dot $\cdot$ stands for the usual inner product in $\mathbb{R}^s$, the $\lambda^1, \lambda^2, \ldots, \lambda^w$ are distinct and w is finite. If every c_j $(j = 1, 2, \ldots, w)$ is different from zero, the set $\sigma(P) = \{\lambda^1, \lambda^2, \ldots, \lambda^w\} \subset \mathbb{R}^s$ is called *spectrum* of P and the map

$$\lambda \longrightarrow a(\lambda; P) = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} P(x)\, e^{-i\lambda\cdot x} dx = \begin{cases} c_j & \text{if } \lambda = \lambda^j \in \sigma(P) \\ 0 & \text{if } \lambda \notin \sigma(P), \end{cases} \tag{1.2}$$

where $Q_T = [-T, T]^s$ and $|Q_T| = (2T)^s$, is called the *Bohr transform* of P.

A complex-valued function f, defined on $\mathbb{R}^s$, is called *uniformly almost periodic* in the sense of Bohr if for any $\varepsilon > 0$ there exists a trigonometric polynomial P_ε such that

$$|f(x) - P_\varepsilon(x)| < \varepsilon \qquad (x \in \mathbb{R}^s).$$

Thus the space $C^0_{ap}(\mathbb{R}^s)$ of all uniformly almost periodic functions is the completion of $\mathcal{P}(\mathbb{R}^s)$ with respect to the norm

$$\|P\|_\infty = \sup_{x\in\mathbb{R}^s} |P(x)| \qquad (P \in \mathcal{P}(\mathbb{R}^s)).$$

The spaces $C^m_{ap}(\mathbb{R}^s)$ $(m \in \mathbb{N})$ and $C^\infty_{ap}(\mathbb{R}^s)$ are defined in an obvious way.

If $f \in C^0_{ap}(\mathbb{R}^s)$, it is well-known that the *mean value*

$$\mathcal{M}(f) = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} f(x)\, dx$$

of F exists. Moreover, the limit

$$\mathcal{M}(f) = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T+y} f(x)\, dx \tag{1.3}$$

exists uniformly with respect to $y \in \mathbb{R}^s$.

For any fixed $q \in [1,\infty)$ we shall denote by $B^q_{ap}(\mathbb{R}^s)$ the completion of $\mathcal{P}(\mathbb{R}^s)$ with respect to the norm defined by

$$\|P\|_q^q = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} |P(x)|^q dx \qquad (P \in \mathcal{P}(\mathbb{R}^s)). \tag{1.4}$$

The space $B^q_{ap}(\mathbb{R}^s)$ is called *Besicovitch space of almost periodic functions*. An element $f \in B^q_{ap}(\mathbb{R}^s)$ is defined by a sequence of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$ such that

$$f = \lim_{n\to\infty} P_n$$

in the sense of $B^q_{ap}(\mathbb{R}^s)$ and

$$\|f\|_q = \lim_{T\to\infty} \left(\frac{1}{|Q_T|} \int_{Q_T} |f(x)|^q dx \right)^{1/q} = \lim_{n\to\infty} \|P_n\|_q .$$

By the Hölder inequality it follows that

$$C^0_{ap}(\mathbb{R}^s) \hookrightarrow B^{q_1}_{ap}(\mathbb{R}^s) \hookrightarrow B^{q_2}_{ap}(\mathbb{R}^s) \hookrightarrow B^1_{ap}(\mathbb{R}^s)$$

with $1 \le q_2 \le q_1 < \infty$.

For any $f \in B^q_{ap}(\mathbb{R}^s)$ we call the map

$$\lambda \longrightarrow a(\lambda; f) = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} f(x)\, e^{-i\lambda\cdot x} dx = \lim_{n\to\infty} a(\lambda; P_n)$$

(where the sequence of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$ converges to f in $B^q_{ap}(\mathbb{R}^s)$) the *Bohr transform* of f.

We will call *spectrum* of an element $f \in B^q_{ap}(\mathbb{R}^s)$ the subset of $\mathbb{R}^s$ defined by

$$\sigma(f) = \{\lambda \in \mathbb{R}^s : a(\lambda; f) \neq 0\}.$$

Hence, in particular, when f is the polynomial P given by (1.1), we have

$$\sigma(P) = \{\lambda^1, \lambda^2, \ldots, \lambda^w\}.$$

For any $f \in B^q_{ap}(\mathbb{R}^s)$ one has:

(i) $\lim_{|\lambda|\to\infty} a(\lambda; f) = 0$.

(ii) $\sigma(f)$ is at most a countable set.

(iii) $\sigma(f) = \emptyset \iff a(\lambda; f) = 0 \iff f = 0 \in B^1_{ap}(\mathbb{R}^s)$.

We call the elements of $\sigma(f)$ *Fourier exponents* of f. Hence with each element $f \in B^q_{ap}(\mathbb{R}^s)$ we associate formally the *Bohr-Fourier series*

$$f \sim \sum_{\lambda\in\sigma(f)} a(\lambda; f)\, e^{i\lambda\cdot x}.$$

By virtue of (iii), each element in $B^q_{ap}(\mathbb{R}^s)$ can be identified with its Bohr-Fourier series.

Remark 1.1. If $q \in (0,1)$, (1.4) is a quasi-norm on the spaces $\mathcal{P}(\mathbb{R}^s)$. However, it defines a metric d_q on $\mathcal{P}(\mathbb{R}^s)$ by

$$d_q(P,Q) = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} |P(x) - Q(x)|^q dx = (\|P - Q\|_q)^q$$

We denote also by $B^q_{ap}(\mathbb{R}^s)$ $(q \in (0,1))$ the completion of the metric space $(\mathcal{P}(\mathbb{R}^s), d_q)$.

2. Almost periodic distributions and tempered distributions

Let Λ be a non-empty subset of $\mathbb{R}^s$. According to [2], we say that Λ satisfies the (α)-*condition* if the following holds:

(α) Λ is a countable semigroup in $\mathbb{R}^s(+)$ with a finite number of generators, which is contained in a convex cone in $\mathbb{R}^s(+)$, that is

$$\Lambda = \left\{ n_1\lambda_1^* + \ldots + n_d\lambda_d^* : (n_1, \ldots, n_d) \in \mathbb{N}_0^d \setminus \{0\} \right\}$$

with the hypotheses

(i) $\lambda_1^*, \ldots, \lambda_d^*$ are $\mathbb{Z}$-linearly independent

(ii) $\overline{\mathrm{conv}}\{\lambda_1^*, \ldots, \lambda_d^*\} \cap \{0\} = \emptyset$.

Moreover, we say that Λ satisfies the (β)-*condition* if there exists a positive number β such that

$$\sum_{\lambda\in\Lambda} \frac{1}{|\lambda|^\gamma} < +\infty, \qquad \text{for all } \gamma > \beta. \tag{2.1}$$

There exists a remarkable connection between the (α)- and (β)-conditions which is illustrated by the following result. For a proof see [13: Lemma 6.1].

Theorem 2.1. *If Λ satisfies the (α)-condition, then*

$$\sum_{\lambda\in\Lambda}\frac{1}{|\lambda|^{\gamma}}\begin{cases}<+\infty & \text{if } \gamma>d\\ =+\infty & \text{if } \gamma\le d,\end{cases}$$

that is, Λ satisfies the (β)-condition and the number β equals d.

Remark 2.2. In [13] some continuous embedding results in certain spaces of almost periodic functions with fixed spectrum Λ, quite similar to the classical Sobolev embedding theorem, are obtained. Here, the (β)-condition, that is the summability property of the spectrum, plays a crucial role (see, e.g., [13: Theorem 5.1]). These classes of almost periodic functions include the periodic functions and some classes of quasi periodic functions. Observe that in the periodic case we have $\beta = s$, since the series (2.1) has the same behaviour as the series

$$\sum_{k\in\mathbb{Z}^s_*}\frac{1}{|k|^{\gamma}}<+\infty,$$

where $\mathbb{Z}^s_* = \mathbb{Z}^s\setminus\{0\}$.

On the other hand, the (β)-condition is not really useful to approach problems in Nonlinear Analysis. For example, if $f,g\in B^q_{ap}(\mathbb{R}^s)$ with $\sigma(f),\sigma(g)\subset\Lambda$, it is not true in general that $\sigma(fg)\subset\Lambda$, without further assumptions on Λ. For this reason, it seems natural to consider at least a semigroup structure for Λ.

In this section, we shall consider almost periodic functions such that $\sigma(f)\subset\Lambda$ with Λ satisfying the (α)-condition. Therefore, we set

$$C^0_{ap}(\mathbb{R}^s,\Lambda)=\left\{f\in C^0_{ap}(\mathbb{R}^s):\sigma(f)\subseteq\Lambda\right\}$$

and, analogously, $\mathcal{P}(\mathbb{R}^s,\Lambda)$, $C^m_{ap}(\mathbb{R}^s,\Lambda)$, $C^\infty_{ap}(\mathbb{R}^s,\Lambda)$ and $B^q_{ap}(\mathbb{R}^s,\Lambda)$. Observe that these spaces are separable.

In [2] the space $\mathcal{S}_{ap}(\Lambda)$ of almost periodic test functions and the space $\mathcal{S}'_{ap}(\Lambda)$ of almost periodic distributions are defined and some of their properties are presented. Here, we will recall only basic facts. Since it is without meaning to consider almost periodic functions with compact support or rapidly decreasing, we consider almost periodic functions which possess uniformly convergent Bohr-Fourier series with all the series obtained from them by differentiation.

Remark 2.3. Let us consider $\varphi\in C^\infty_{ap}(\mathbb{R}^s)$. For every multi-index $\alpha\in\mathbb{N}^0$ we have

$$D^\alpha\varphi\sim\sum_{\lambda\in\sigma(\varphi)}a(\lambda;\varphi)(\lambda)^\alpha e^{i\lambda\cdot x}$$

and, by the Parseval equality,

$$\|D^\alpha\varphi\|_2^2=\sum_{\lambda\in\sigma(\varphi)}|a(\lambda;\varphi)|^2|(\lambda)^\alpha|^2.$$

A consequence is that the spectrum $\sigma(\varphi)$ must be unbounded, to guarantee (by the techniques used in [13]) that $D^\alpha\varphi$ may be represented in $C^0_{ap}(\mathbb{R}^s)$ by an uniformly convergent Bohr-Fourier series.

The set

$$\mathcal{S}_{ap}(\Lambda) = \bigcap_{m=1}^{\infty} C^m_{ap}(\mathbb{R}^s, \Lambda)$$

equipped with its natural topology

$$\varphi_n \xrightarrow{\mathcal{S}_{ap}} \varphi \iff D^\alpha\varphi_n \rightrightarrows D^\alpha\varphi \text{ for all } \alpha \in \mathbb{N}^s_0$$

is the *space of almost periodic test functions*. The embedding $\mathcal{S}_{ap}(\Lambda) \subset C^m_{ap}(\mathbb{R}^s, \Lambda)$ is continuous with respect to the natural topology for any $m \in \mathbb{N}$.

About the representation of almost periodic test functions, the following proposition holds (for a proof see [2: Teorema 4.1]).

Proposition 2.4. *Every almost periodic test function $\varphi \in \mathcal{S}_{ap}(\Lambda)$ can be represented as*

$$\varphi(x) = \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda\cdot x}, \tag{2.2}$$

where the series converges in $\mathcal{S}_{ap}(\Lambda)$ and $(a_\lambda)_{\lambda\in\Lambda}$ is a sequence of complex numbers such that

$$|a_\lambda| \le M(1 + |\lambda|)^{-m} \qquad (\lambda \in \Lambda), \tag{2.3}$$

for all $m \in \mathbb{N} \cup \{0\}$. Here $M = M(m)$ is an appropriate positive constant. It holds

$$a_\lambda = a(\lambda; \varphi) \qquad (\lambda \in \Lambda).$$

Conversely, if $(a_\lambda)_{\lambda\in\Lambda}$ is a sequence of complex numbers which satisfies (2.3), *then the series $\sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda x}$ converges in $\mathcal{S}_{ap}(\Lambda)$. If φ is its sum, it holds $a(\lambda;\varphi) = a_\lambda$ $(\lambda \in \Lambda)$.*

Let $\mathcal{S}'_{ap}(\Lambda)$ denote the topological dual space of $\mathcal{S}_{ap}(\Lambda)$. The elements in this space are the continuous and sesquilinear functionals on $\mathcal{S}_{ap}(\Lambda)$. We call $T \in \mathcal{S}'_{ap}(\Lambda)$ an *almost periodic distribution*. For every $T \in \mathcal{S}'_{ap}(\Lambda)$ and $\varphi \in \mathcal{S}_{ap}(\Lambda)$ we set

$$T(\varphi) = \langle T \,|\, \varphi \rangle.$$

The *Bohr transform* of the distribution T and its *Bohr-Fourier series* are defined in the following way:

$$a(\lambda, T) = \begin{cases} \langle T \,|\, e^{i\lambda\cdot x} \rangle & \text{if } \lambda \in \Lambda \\ 0 & \text{if } \lambda \in \mathbb{R}^s \setminus \Lambda \end{cases}$$

and

$$T \sim \sum_{\lambda\in\Lambda} a(\lambda; T)\, e^{i\lambda\cdot x}.$$

Detailed information about the space $\mathcal{S}'_{ap}(\Lambda)$ and its natural topology may be found in [2, 3]. However, we point out that the next result holds (for a proof, see [2: Teorema 4.5]).

Proposition 2.5. *Every almost periodic distribution $T \in S'_{ap}(\Lambda)$ can be represented as*

$$T = \sum_{\lambda \in \Lambda} a_\lambda \, e^{i\lambda \cdot x}, \tag{2.4}$$

where the series converges in $S'_{ap}(\Lambda)$ and the sequence of complex numbers $(a_\lambda)_{\lambda \in \Lambda}$ is tempered, that is

$$\text{there exist } M, m \geq 0 \text{ such that} \qquad |a_\lambda| \leq M(1+|\lambda|)^m \quad (\lambda \in \Lambda). \tag{2.5}$$

It holds

$$a_\lambda = a(\lambda; T) \qquad (\lambda \in \Lambda).$$

Conversely, if $(a_\lambda)_{\lambda \in \Lambda}$ is a tempered sequence of complex numbers, then $\sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda x}$ converges in $S'_{ap}(\Lambda)$. If T is its sum, it holds $a(\lambda; T) = a_\lambda \ \ (\lambda \in \Lambda)$.

Let $\mathcal{D}(\mathbb{R}^s)$ denote the space of test functions and let $\mathcal{D}'(\mathbb{R}^s)$ be the space of distributions. By $\mathcal{S}(\mathbb{R}^s)$ we denote the *Schwartz space* of rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^s$ and by $\mathcal{S}'(\mathbb{R}^s)$ its topological dual, the space of *tempered distributions*. Then the Fourier transform F is given by

$$(Ff)(x) = (2\pi)^{-s/2} \int_{\mathbb{R}^s} f(y)\, e^{-ix\cdot y} dy \tag{2.6}$$

on $\mathcal{S}(\mathbb{R}^s)$ and the inverse Fourier transform $F^{-1}f$ of f is given by (2.6) where one must replace $-i$ by i. One extends F and F^{-1} from $\mathcal{S}(\mathbb{R}^s)$ to $\mathcal{S}'(\mathbb{R}^s)$ in the usual way.

Let $\delta_\lambda \in \mathcal{S}'(\mathbb{R})$ be the Dirac distribution with respect to $\lambda \in \Lambda$, that is

$$\langle \delta_\lambda \,|\, \varphi \rangle = \varphi(\lambda) \qquad \text{for all } \ \varphi \in \mathcal{S}(\mathbb{R}^s), \ \lambda \in \Lambda$$

and let $e^{i\lambda\cdot x} \in \mathcal{S}'(\mathbb{R}^s)$ be defined by

$$\langle e^{i\lambda\cdot x} \,|\, \varphi \rangle = \int_{\mathbb{R}^s} e^{i\lambda\cdot x} \varphi(x)\, dx \qquad \text{for all } \ \varphi \in \mathcal{S}(\mathbb{R}^s), \ \lambda \in \Lambda.$$

Then the equality $F^{-1}\delta_\lambda = (2\pi)^{-s/2} e^{i\lambda\cdot x} \ \ (\lambda \in \Lambda)$ holds.

It may happen that a sequence of tempered distributions converges to a tempered distribution in the sense of $\mathcal{D}'(\mathbb{R}^s)$, but this fact does not generally imply the convergence in the sense of $\mathcal{S}'(\mathbb{R}^s)$. However, the following result is true.

Proposition 2.6. *Let $(a_\lambda)_{\lambda \in \Lambda}$ be a sequence of complex numbers. Then the sum T of the series*

$$\sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \tag{2.7}$$

belongs to $\mathcal{S}'(\mathbb{R}^s)$ if and only if $(a_\lambda)_{\lambda \in \Lambda}$ is a tempered sequence, that is (2.5) holds. Moreover, if $(a_\lambda)_{\lambda \in \Lambda}$ is a tempered sequence, the series (2.7) converges in the sense of $\mathcal{S}'(\mathbb{R}^s)$.

Proof. Assume that $(a_\lambda)_{\lambda\in\Lambda}$ is a tempered sequence, i.e. (2.5) holds. If $\varphi \in \mathcal{S}(\mathbb{R}^s)$, we get

$$\begin{aligned}|a_\lambda \varphi(\lambda)| &\le M(1+|\lambda|)^m |\varphi(\lambda)|\\ &= \frac{M}{|\lambda|^\gamma}|\lambda|^\gamma (1+|\lambda|)^m|\varphi(\lambda)|\\ &\le \frac{M}{|\lambda|^\gamma}\sum_{|\alpha|=\gamma}\sup_{x\in\mathbb{R}^s}|x^\alpha|(1+|x|)^m|\varphi(x)| \qquad (\lambda\in\Lambda).\end{aligned}$$

Now, it follows easily that the series $\sum_{\lambda\in\Lambda} a_\lambda\varphi(\lambda)$ converges, provided $\gamma \in \mathbb{N}$ and $\gamma > \beta$. Therefore, the series (2.7) converges in the sense of $\mathcal{S}'(\mathbb{R}^s)$ and its sum T is a tempered distribution.

Conversely, assume that the sum T of the series (2.7) belongs to $\mathcal{S}'(\mathbb{R}^s)$ and by way of contradiction that $(a_\lambda)_{\lambda\in\Lambda}$ is not a tempered sequence. Then, also the statement

$$|a_\lambda| \le M(1+|\lambda|)^m \qquad \text{if } |\lambda| > p$$

for some $M, m, p \ge 0$, which is equivalent to (2.5), is false. Therefore, for any $M, m, p \ge 0$ there exists $\lambda \in \Lambda$ such that

$$|\lambda| > p \qquad \text{and} \qquad |a_\lambda| > M(1+|\lambda|)^m.$$

Choose $M = 1, m = 1$ and $p = 1$. Then there exists $\lambda_1 \in \Lambda$ such that

$$|\lambda_1| > 1 \qquad \text{and} \qquad |a_{\lambda_1}| > M(1+|\lambda_1|).$$

Now, choose $M = 1, m = 2$ and $p = |\lambda_1| + 1$. We find $\lambda_2 \in \Lambda$ such that

$$|\lambda_2| > |\lambda_1|, \qquad |\lambda_2 - \lambda_1| > 1, \qquad |a_{\lambda_2}| > M(1+|\lambda_2|)^2.$$

Continuing in this way, we obtain a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \Lambda$ such that, for all k,

$$|\lambda_{k+1}| > |\lambda_k|, \qquad |\lambda_{k+1} - \lambda_k| > 1, \qquad |a_{\lambda_k}| > M(1+|\lambda_k|)^k.$$

Let $\zeta \in \mathcal{S}(\mathbb{R}^s)$ such that $\operatorname{supp}\zeta \subset B(0,\frac{1}{2}) = \{x \in \mathbb{R}^s : |x| < \frac{1}{2}\}$ and $\zeta(0) = 1$. Set

$$v_k(x) = \sum_{j=1}^{k}\frac{1}{a_{\lambda_j}}\zeta(x-\lambda_j) \qquad \text{and} \qquad v(x) = \sum_{j=1}^{\infty}\frac{1}{a_{\lambda_j}}\zeta(x-\lambda_j),$$

with

$$\operatorname{supp}\zeta(x-\lambda_j) \cap \operatorname{supp}\zeta(x-\lambda_h) = \emptyset \tag{2.8}$$

if $j \ne h$, by virtue of what has been said about the function ζ. In particular, $v_k \in \mathcal{D}(\mathbb{R}^s)$ and the series $\sum_{j=1}^{\infty}\frac{1}{a_{\lambda_j}}\zeta(x-\lambda_j)$ converges. Moreover, $v \in C^\infty(\mathbb{R}^s)$ and we shall prove that $v \in \mathcal{S}(\mathbb{R}^s)$. To this end, if $\alpha, \beta \in \mathbb{N}_0^s$ are arbitrarily fixed, we must verify that $x^\alpha D^\beta v(x)$ is bounded. By virtue of (2.8) we get

$$\sup_{x\in\mathbb{R}}|x^\alpha D^\beta v(x)| = \sup_{j\ge 1}\ \sup_{x\in\overline{B}(\lambda_j,\frac{1}{2})}\left|\frac{x^\alpha}{a_{\lambda_j}}D^\beta\zeta(x-\lambda_j)\right|$$

and consequently, we need to prove that there exists a constant $C > 0$, independent from x and j, such that

$$\left|\frac{x^\alpha}{a_{\lambda_j}} D^\beta \zeta(x-\lambda_j)\right| < C$$

if x and j satisfy $x \in \overline{B}(\lambda_j, \frac{1}{2})$. Since $v \in C^\infty(\mathbb{R}^s)$, then it is bounded over any compactum, so that we can consider only $j \geq m = |\alpha|$. Now, if $x \in \overline{B}(\lambda_j, \frac{1}{2})$ and $j \geq m$, we obtain

$$\left|\frac{x^\alpha}{a_{\lambda_j}} D^\beta \zeta(x-\lambda_j)\right| \leq \frac{(\frac{1}{2}+|\lambda_j|)^m}{|a_{\lambda_j}|} \|D^\beta\zeta\|_\infty \leq \frac{(1+|\lambda_j|)^m}{(1+|\lambda_j|)^j} \|D^\beta\zeta\|_\infty.$$

Therefore, we find that

$$\left|\frac{x^\alpha}{a_{\lambda_j}} D^\beta \zeta(x-\lambda_j)\right| \leq \|D^\beta\zeta\|_\infty (1+|\lambda_j|)^{m-j} \tag{2.9}$$

if $x \in \overline{B}(\lambda_j, \frac{1}{2})$ and $j \geq m$. From the fact that $(1+|\lambda_j|)^{m-j} \leq 1$ if $j \geq m$ the required boundedness is proved, and since α, β are arbitrary we deduce that $v \in \mathcal{S}(\mathbb{R}^s)$.

Now we show that $v_k \to v$ in the sense of $\mathcal{S}(\mathbb{R}^s)$. Continuing as above with the same notations, we must prove that the quantity

$$\sup_{x\in\mathbb{R}}\left|x^\alpha D^\beta v(x) - x^\alpha D^\beta v_k(x)\right| = \sup_{j\geq k} \sup_{x\in\overline{B}(\lambda_j,\frac{1}{2})} \left|\frac{x^\alpha}{a_{\lambda_j}} D^\beta \zeta(x-\lambda_j)\right|$$

converges to zero. But if we suppose $k \geq m = |\alpha|$, from (2.5) it follows that

$$\left|\frac{x^m}{a_{\lambda_j}} D^\beta \zeta(x-\lambda_j)\right| \leq \|D^\beta\zeta\|_\infty (1+|\lambda_j|)^{m-j} \leq (1+|\lambda_j|)^{m-k} \|D^\beta\zeta\|_\infty$$

provided $x \in \overline{B}(\lambda_j, \frac{1}{2})$ and $j \geq k$. Since $(1+|\lambda_j|)^{m-k} \to 0$ if $k \to \infty$, we get the desired convergence. Finally, we verify that $\langle T \,|\, v_k \rangle \to \infty$ as $k \to \infty$. Indeed,

$$\langle T \,|\, v_k \rangle = \sum_{\lambda\in\Lambda} a_\lambda v_k(\lambda) = \sum_{j=1}^{k} a_{\lambda_j} v_k(\lambda_j) = \sum_{j=1}^{k} a_{\lambda_j} \frac{1}{a_{\lambda_j}} = k$$

and the proposition is proved ∎

We note that a trigonometric polynomial $P \in \mathcal{P}(\mathbb{R}^s, \Lambda)$ (which is an almost periodic function in the sense of Bohr) with spectrum $\sigma(P) \subset \Lambda$, that is

$$P(x) = \sum_{\lambda\in\sigma(P)} c_\lambda e^{i\lambda\cdot x}$$

defines an element $P \in \mathcal{S}'(\mathbb{R}^s)$ if we set

$$\langle P \,|\, \varphi \rangle = \int_{\mathbb{R}^s} P(x)\varphi(x)\,dx \qquad (\varphi \in \mathcal{S}(\mathbb{R}^s))$$

and, moreover, $\operatorname{supp} FP = \sigma(P) \subset \Lambda$. More generally, we observe that, by virtue of Proposition 2.6, if the series $\sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ converges in $\mathcal{S}'(\mathbb{R}^s)$, then the sequence $(a_\lambda)_{\lambda\in\Lambda}$ is tempered. Since

$$F^{-1}\Big(\sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda\Big) = \sum_{\lambda\in\Lambda} a_\lambda F^{-1}\delta_\lambda = (2\pi)^{-s/2} \sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x},$$

the series $\sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x}$ also converges in $\mathcal{S}'(\mathbb{R}^s)$. On the other hand, let us consider $T \in \mathcal{S}'(\mathbb{R}^s)$ such that

$$T = \sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x},$$

where the series converges in $\mathcal{S}'(\mathbb{R}^s)$ and the sequence of complex numbers $(a_\lambda)_{\lambda\in\Lambda}$ is *a priori* arbitrary. Since

$$FT = F\Big(\sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x}\Big) = \sum_{\lambda\in\Lambda} a_\lambda F e^{i\lambda\cdot x} = (2\pi)^{s/2} \sum_{\lambda\in\Lambda} a_\lambda\, \delta_\lambda,$$

by Proposition 2.6 we obtain that the sequence $(a_\lambda)_{\lambda\in\Lambda}$ is tempered, and $\operatorname{supp} FT \subset \Lambda$. The above argument gives us

Corollary 2.7. *Let $(a_\lambda)_{\lambda\in\Lambda}$ be a sequence of complex numbers. Then the series*

$$\sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x}$$

converges in the sense of $\mathcal{S}'(\mathbb{R}^s)$ if and only if $(a_\lambda)_{\lambda\in\Lambda}$ is a tempered sequence, that is (2.4) *holds.*

Hence, the following definition seems to be natural.

Definition 2.8. An element $T \in \mathcal{S}'(\mathbb{R}^s)$ is said to be an *almost periodic distribution* on $\mathbb{R}^s$ with spectrum $\sigma(T) \subset \Lambda$ if there exists a tempered sequence of complex numbers $(a_\lambda)_{\lambda\in\Lambda}$ such that

$$T = \sum_{\lambda\in\Lambda} a_\lambda\, e^{i\lambda\cdot x}$$

where the series converges in $\mathcal{S}'(\mathbb{R}^s)$. The collection of all these tempered distributions is denoted by $\mathcal{S}'_\Lambda(\mathbb{R}^s)$.

Remark 2.9. Observe that $\mathcal{S}'_\Lambda(\mathbb{R}^s)$ is non-empty. Moreover, there exists a correspondence between $\mathcal{S}'_{ap}(\Lambda)$ and $\mathcal{S}'_\Lambda(\mathbb{R}^s)$ which is one-to-one. This allows to identify $T \in \mathcal{S}'_{ap}(\Lambda)$, with $T \in \mathcal{S}'_\Lambda(\mathbb{R}^s)$, via (2.4).

3. Fourier multipliers for trigonometric polynomials

Let $\Lambda \subset \mathbb{R}^s$ be a finite subset and let $(M_\lambda)_{\lambda\in\Lambda}$ be a set of complex numbers.

Definition 3.1. Let $0 < q \le \infty$. Then $(M_\lambda)_{\lambda\in\Lambda}$ is a *Fourier multiplier* for $\mathcal{P}(\mathbb{R}^s, \Lambda)$ if there exists a positive constant c such that

$$\Big\| \sum_{\lambda\in\Lambda} M_\lambda a(\lambda; P)\, e^{i\lambda\cdot x} \Big\|_q \le c\,\|P\|_q \tag{3.1}$$

for any trigonometric polynomial $P \in \mathcal{P}(\mathbb{R}^s, \Lambda)$.

If $M \in S'(\mathbb{R}^s)$ with $F^{-1}M \in L^1(\mathbb{R}^s)$, then, by properties of the Fourier transform, $M = M(x)$ is a continuous function on $\mathbb{R}^s$. Therefore $M_\lambda = M(\lambda)$ $(\lambda \in \Lambda)$ makes sense. We have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} M_\lambda a(\lambda; P)\, e^{i\lambda\cdot x} \\
&\quad= \sum_{\lambda\in\Lambda} [F(F^{-1}M)](\lambda)\, a(\lambda; P)\, e^{i\lambda\cdot x} \\
&\quad= \sum_{\lambda\in\Lambda} \Big((2\pi)^{-s/2} \int_{\mathbb{R}^s} (F^{-1}M)(y)\, e^{-i\lambda\cdot y} dy\Big) a(\lambda; P)\, e^{i\lambda\cdot x} \\
&\quad= (2\pi)^{-s/2} \int_{\mathbb{R}^s} (F^{-1}M)(y) \Big(\sum_{\lambda\in\Lambda} a(\lambda; P)\, e^{i\lambda\cdot(x-y)} \Big) dy \\
&\quad= (2\pi)^{-s/2} \int_{\mathbb{R}^s} (F^{-1}M)(y) P(x-y)\, dy.
\end{aligned} \tag{3.2}$$

Let $m \ge 0$. We recall that $H^{m,2}(\mathbb{R}^s)$ are the usual Bessel potential spaces. These spaces can be defined via

$$H^{m,2}(\mathbb{R}^s) = \Big\{ f \in S'(\mathbb{R}^s) : \|f\|_{m,2} = \|(1+|x|^2)^{m/2} Ff\|_{L^2} < \infty \Big\}.$$

Theorem 3.2. *Let $\Lambda \subset \mathbb{R}^s$ be a finite subset such that*

$$d_\Lambda = \max_{\lambda,\mu\in\Lambda} |\lambda - \mu| > 0.$$

Let $0 < q \le \infty$ and $0 < \chi < \infty$ with

$$\chi > \sigma_q = s\Big(\frac{1}{\min(1,q)} - \frac{1}{2}\Big). \tag{3.3}$$

Then there exists a positive constant c depending on q and s such that

$$\Big\| \sum_{\lambda\in\Lambda} M_\lambda a(\lambda; P)\, e^{i\lambda\cdot x} \Big\|_q \le c\,\|M(d_\lambda \cdot)\|_{\chi,2} \|P\|_q \tag{3.4}$$

for all $M \in H^{\chi,2}(\mathbb{R})$ *and all* $P \in \mathcal{P}(\mathbb{R}^s, \Lambda)$.

Proof. We follow [16: Theorem 3.3.4]. Without loss of generality we may assume $0 \in \Lambda$ (otherwise consider $e^{-i\lambda_0 \cdot x}P(x)$). If $0 < q \le \infty$, then in any case $\chi > \frac{s}{2}$. Then, by [16: Proposition 1.7.5], $F^{-1}M \in L^1(\mathbb{R}^s)$. Then properties of the Fourier transform yield that M must be a continuous function on $\mathbb{R}^s$. Thus, the left-hand side of (3.2) makes sense. Choose a function $\psi \in \mathcal{S}(\mathbb{R}^s)$ with $\operatorname{supp}\psi \subset \{x : |x| \le 2\}$ and $\psi(x) = 1$ if $|x| \le 1$. Then $\psi M \in H^{\chi,2}(\mathbb{R}^s)$ and

$$\|\psi M\|_{\chi,2} \le c_\psi \|M\|_{\chi,2}. \tag{3.5}$$

Furthermore, [16: Proposition 1.7.5] yields

$$F^{-1}(\psi(d_\Lambda^{-1}\cdot)M) \in L^1(\mathbb{R}^s).$$

Therefore, formula (3.2) can be applied and we get

$$\Big|\sum_{\lambda\in\Lambda} M(\lambda)a(\lambda;P)\,e^{i\lambda\cdot x}\Big| \le c\int_{\mathbb{R}^s} \big|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)(y)P(x-y)\big|\,dy. \tag{3.6}$$

Let $\overline{q} = \min(1,q)$. For fixed $x \in \mathbb{R}^s$ we have by the Nikolskij inequality for entire analytic functions [9: Section 1.3.2/Remark 1]

$$\big\|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)(y)P(x-y)\big\|_{L^{\overline{q}}} \le \|P\|_\infty \|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M\|_{L^{\overline{q}}} < \infty.$$

Furthermore,

$$\operatorname{supp} F\big[F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)P(x-\cdot)\big] \subset \big\{y : |y| \le 3d_\Lambda\big\}.$$

Consequently, the right-hand side of (3.6) can be estimated again with the help of the Nikolskij inequality [19: Section 1.3.2/Remark 1]. We get

$$\begin{aligned} &\Big|\sum_{\lambda\in\Lambda} M(\lambda)a(\lambda;P)\,e^{i\lambda\cdot x}\Big| \\ &\quad \le c' d_\Lambda^{s(1/\overline{q}-1)}\Big(\int_{\mathbb{R}^s} \big|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)(y)P(x-y)\big|^{\overline{q}}\,dy\Big)^{1/\overline{q}}. \end{aligned} \tag{3.7}$$

Taking the quasi-norms in $B^q_{ap}(\mathbb{R}^s)$ $(0 < q \le \infty)$ on both sides we obtain

$$\begin{aligned} &\lim_{T\to\infty}\Big(\frac{1}{|Q_T|}\int_{Q_T}\Big|\sum_{\lambda\in\Lambda} M(\lambda)a(\lambda;P)\,e^{i\lambda\cdot x}\Big|^q dx\Big)^{1/q} \\ &\le c' d_\Lambda^{s(1/\overline{q}-1)}\big\|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)\big\|_{L^{\overline{q}}}\lim_{T\to\infty}\Big(\frac{1}{|Q_T|}\int_{Q_T}|P(x)|^q\,dx\Big)^{1/q} \\ &= c''\big\|F^{-1}(\psi(d_\Lambda^{-1}\cdot)M)\big\|_{L^{\overline{q}}}\lim_{T\to\infty}\Big(\frac{1}{|Q_T|}\int_{Q_T}|P(x)|^q dx\Big)^{1/q} \\ &\le c''\big\|M(d_\lambda\cdot)\big\|_{\chi,2}\lim_{T\to\infty}\Big(\frac{1}{|Q_T|}\int_{Q_T}|P(x)|^q dx\Big)^{1/q}. \end{aligned} \tag{3.8}$$

In the last estimate we used [16: Proposition 1.7.5], (1.3) and (3.5). Thus the theorem is proved ∎

As a application of (3.4) we can deduce an inequality of Bernstein type for trigonometric polynomials.

Corollary 3.3. *Let $0 < q \le \infty$ and let $\Lambda \subset \mathbb{R}^s$ be a finite subset with*

$$\Lambda \subset \{x \in \mathbb{R}^s : |x| \le N\}$$

where N is a given natural number. Then there exists a constant $c > 0$ such that

$$\|D^\alpha P\|_q \le cN^\alpha \|P\|_q \tag{3.9}$$

for all $P \in \mathcal{P}(\mathbb{R}^s, \Lambda)$ and all multi-indices $\alpha \in \mathbb{N}_0^s$.

Proof. We choose a function $\psi \in \mathcal{S}(\mathbb{R}^s)$ with compact support and $\psi(x) = 1$ if $|x| \le 1$. Then (2.4) yields

$$\|D^\alpha P\|_q = \Big\| \sum_{\lambda\in\Lambda} \psi\Big(\frac{\lambda}{N}\Big) \lambda^\alpha a(\lambda; P) e^{i\lambda\cdot x} \Big\|_q \le cN^\alpha \|x^\alpha \psi\|_{\chi,2} \|P\|_q \tag{3.10}$$

where

$$\chi > \sigma_q = s\Big(\frac{1}{\min(1,q)} - \frac{1}{2}\Big).$$

This proves (3.9) ∎

Remark 3.4. The use of the Nikolskij inequality for entire analytic functions in the proof of Proposition 3.2 allows us to consider $0 < q \le \infty$.

4. Fourier multipliers for Besicovitch spaces

First of all, we recall the definition of Fourier multipliers for $L^q(\mathbb{R}^s)$ (see [16: p. 155]).

Definition 4.1. Let $1 < q < \infty$ and $M \in L^\infty(\mathbb{R}^s)$. Then M is a *Fourier multiplier* for $L^q(\mathbb{R}^s)$ if there exists a positive constant c_M such that

$$\|F^{-1}MFf\|_{L^q} \le c_M \|f\|_{L^q} \tag{4.1}$$

for any $f \in \mathcal{S}(\mathbb{R}^s)$.

Our aim is to extend the above definition to Besicovitch spaces with fixed spectrum Λ satisfying the (α)-condition. Therefore, it seems to be natural to give the following definition (see [16: p. 155/Definition 3.4.2.1 and Remark 3.4.2.2]).

Definition 4.2. Let $1 < q < \infty$ and $\widetilde{M} = (\widetilde{M}_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$. Then $\widetilde{M}$ is a *Fourier multiplier* for $B^q_{ap}(\mathbb{R}^s, \Lambda)$ if there exists a positive constant $c_{\widetilde{M}}$ such that

$$\Big\| \sum_{\lambda\in\Lambda} \widetilde{M}_\lambda a(\lambda; f) e^{i\lambda\cdot x} \Big\|_q \le c_{\widetilde{M}} \|f\|_q \tag{4.2}$$

for any $f \in B^q_{ap}(\mathbb{R}^s, \Lambda)$.

The following is the main result of this section.

Theorem 4.3. *Let $1 < q < \infty$ and let $M \in L^\infty(\mathbb{R}^s)$ be a Fourier multiplier for $L^q(\mathbb{R}^s)$. Suppose additionally that M is continuous at all points $\lambda \in \Lambda$. Then $\widetilde{M} = (M(\lambda))_{\lambda\in\Lambda}$ is a Fourier multiplier for $B^q_{ap}(\mathbb{R}^s, \Lambda)$ with $c_{\widetilde{M}} \le c_M$ for the constants from* (4.1) *and* (4.2).

Here, we assume that c_M and $c_{\widetilde{M}}$ are the minimal constants in the corresponding inequalities.

We begin the proof by considering some lemmas.

Lemma 4.4. *Let $f \in C^0_{ap}(\mathbb{R}^s)$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{s/2} \int_{\mathbb{R}^s} f(x)\, e^{-\pi\varepsilon|x|^2} dx = \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} f(x)\, dx. \tag{4.3}$$

Proof. Equality (4.3) is true if $f(x) = e^{i\lambda\cdot x}$ $(\lambda \in \mathbb{R}^s)$ because, by virtue of Lemma 4.6, we have

$$\varepsilon^{s/2} \int_{\mathbb{R}^s} e^{i\lambda\cdot x} e^{-\pi\varepsilon|x|^2}\, dx = e^{-|\lambda|^2/4\pi\varepsilon}$$

if $\varepsilon > 0$. On the other hand,

$$\lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} e^{i\lambda\cdot x}\, dx = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0. \end{cases}$$

Therefore, (4.3) is true for any trigonometric polynomial. The claim now follows by approximating an arbitrary $f \in C^0_{ap}(\mathbb{R}^s)$ uniformly on $\mathbb{R}^s$ by using such polynomials ∎

We recall the following lemma (see [18: Theorem 1.13]).

Lemma 4.5. *The Fourier transform of $f(x) = e^{-a|x|^2}$ $(a > 0, x \in \mathbb{R}^s)$ is*

$$(Ff)(\zeta) = (2\pi)^{-s/2} \int_{\mathbb{R}^s} e^{-a|x|^2} e^{-i\zeta\cdot x}\, dx = (2a)^{-s/2} e^{-|\zeta|^2/4a}. \tag{4.4}$$

Then, the following result also holds.

Lemma 4.6. *For any $\alpha > 0$,*

$$\int_{\mathbb{R}^s} e^{-\pi\alpha|y|^2} e^{-it\cdot y}\, dy = \alpha^{-s/2} e^{-|t|^2/4\pi\alpha}. \tag{4.5}$$

Proof. The claim follows easily from (4.4) with $a = \pi\alpha$ ∎

Lemma 4.7. *Let $P, Q \in \mathcal{P}(\mathbb{R}^s, \Lambda)$ be trigonometric polynomials, let $M \in L^\infty(\mathbb{R}^s)$ be continuous at every point $\lambda \in \Lambda$ and set*

$$\omega_\delta(x) = e^{-\pi\delta|x|^2} \qquad (\delta > 0, x \in \mathbb{R}^s).$$

Then

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} \Big(\sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; P)\, e^{i\lambda\cdot x} \Big) \overline{Q(x)}\, dx \\ &= \lim_{\varepsilon\to 0} \varepsilon^{s/2} \int_{\mathbb{R}^s} [F^{-1} M F(P\omega_{\varepsilon\alpha})](x) \overline{Q(x)}\, \omega_{\varepsilon\beta}(x)\, dx \end{aligned} \tag{4.6}$$

whenever $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. The expressions in (4.6) are linear in P and Q, therefore, it suffices to prove (4.6) when $P(x) = e^{i\lambda\cdot x}$ and $Q(x) = e^{i\mu\cdot x}$ for $\lambda, \mu \in \Lambda$. By the Plancherel theorem we have

$$\begin{aligned} \varepsilon^{s/2} &\int_{\mathbb{R}^s} [F^{-1}MF(P\omega_{\varepsilon\alpha})](x)\overline{Q(x)}\omega_{\varepsilon\beta}(x)\,dx \\ &= \varepsilon^{s/2}\int_{\mathbb{R}^s} M(x)[F(P\omega_{\varepsilon\alpha})](x)\overline{F(Q\omega_{\varepsilon\beta})}(x)\,dx. \end{aligned}$$

Since

$$(P\omega_{\varepsilon\alpha})(x) = e^{i\lambda\cdot x}e^{-\pi\varepsilon\alpha|x|^2} \qquad \text{and} \qquad (Q\omega_{\varepsilon\beta})(x) = e^{i\mu\cdot x}e^{-\pi\varepsilon\beta|x|^2},$$

by virtue of Lemma 4.5 we obtain

$$\begin{aligned} [F(P\omega_{\varepsilon\alpha})](x) &= e^{-(|x-\lambda|^2/4\pi\alpha\varepsilon)}(2\pi\alpha\varepsilon)^{-s/2} \\ [F(Q\omega_{\varepsilon\beta})](x) &= e^{-(|x-\mu|^2/4\pi\beta\varepsilon)}(2\pi\beta\varepsilon)^{-s/2}. \end{aligned}$$

Now we assume that $\lambda \neq \mu$, and consequently $|\lambda - \mu| \geq l > 0$. Since $|M(x)| \leq A$ for a suitable constant A, we obtain

$$\begin{aligned} \varepsilon^{s/2} &\int_{\mathbb{R}^s} M(x)[F(P\omega_{\varepsilon\alpha})](x)\overline{F(Q\omega_{\varepsilon\beta})}(x)\,dx \\ &= \varepsilon^{s/2}\int_{\mathbb{R}^s} M(x)e^{-(|x-\lambda|^2/4\pi\alpha\varepsilon)}(2\pi\alpha\varepsilon)^{-s/2}e^{-(|x-\mu|^2/4\pi\beta\varepsilon)}(2\pi\beta\varepsilon)^{-s/2}\,dx \\ &\leq A\varepsilon^{s/2}\int_{\mathbb{R}^s} e^{-(|x-\lambda|^2/4\pi\alpha\varepsilon)}(2\pi\alpha\varepsilon)^{-s/2}e^{-(|x-\mu|^2/4\pi\beta\varepsilon)}(2\pi\beta\varepsilon)^{-s/2}\,dx \\ &\leq A\varepsilon^{s/2}\left[\int_{|x-\lambda|\geq l/2} + \int_{|x-\mu|\geq l/2}\right]. \end{aligned}$$

In the integral extended over $\{x \in \mathbb{R}^s : |x - \lambda| \geq \frac{l}{2}\}$ the factor

$$\varepsilon^{s/2}e^{-(|x-\lambda|^2/4\pi\alpha\varepsilon)}(2\pi\alpha\varepsilon)^{-s/2}$$

tends uniformly to 0 as $\varepsilon \to 0$, while the factor $e^{-(|x-\mu|^2/4\pi\beta\varepsilon)}(2\pi\beta\varepsilon)^{-s/2}$ has total integral $(2\pi)^{s/2}$ when extended over $\mathbb{R}^s$. It follows that $\varepsilon^{s/2}\int_{|x-\lambda|\geq l/2} \to 0$ as $\varepsilon \to 0$. The same argument, with the roles of λ and μ interchanged, shows that $\varepsilon^{s/2}\int_{|x-\mu|\geq l/2} \to 0$ as $\varepsilon \to 0$. Since, for $\lambda \neq \mu$,

$$\lim_{T\to\infty}\frac{1}{|Q_T|}\int_{Q_T}[M(\lambda)e^{i\lambda\cdot x}]e^{-i\mu\cdot x}\,dx = 0,$$

equality (4.6) is established. If $\lambda = \mu$, the right-hand side of (4.6) equals

$$\lim_{\varepsilon\to 0}(4\pi^2\varepsilon\alpha\beta)^{-s/2}\int_{\mathbb{R}^s} M(x)e^{-(|x-\lambda|^2/4\pi\varepsilon)(1/\alpha+1/\beta)}\,dx.$$

Since $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\alpha\beta}$, the limit above is the limit, as $\varepsilon \to 0$, of the Gauss-Weierstrass integral of M. It is well known (see [18]) that this limit is $M(\lambda)$ provided λ belongs to the Lebesgue set of M. But this is the case since M is assumed to be continuous at λ. This proves equality (4.6) when $P(x) = e^{i\lambda\cdot x} = Q(x)$, since

$$\lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} M(\lambda) e^{i\lambda\cdot x} e^{-i\lambda\cdot x}\, dx = M(\lambda).$$

Thus the statement is proved ∎

Proof of Theorem 4.3. Assume that $1 < q < \infty$. Let q' be the conjugate exponent to q. Then $\frac{1}{q} + \frac{1}{q'} = 1$ and $1 < q < \infty$. We first prove that there exists $c_{\widetilde{M}} \le c_M$ such that

$$\begin{aligned} &\lim_{T\to\infty} \left(\frac{1}{|Q_T|} \int_{Q_T} \left| \sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; P)\, e^{i\lambda\cdot x} \right|^q dx \right)^{1/q} \\ &\qquad \le c_{\widetilde{M}} \lim_{T\to\infty} \left(\frac{1}{|Q_T|} \int_{Q_T} |P(x)|^q dx \right)^{1/q} \end{aligned} \tag{4.7}$$

for all trigonometric polynomials $P \in \mathcal{P}(\mathbb{R}^s, \Lambda)$. If $Q \in \mathcal{P}(\mathbb{R}^s, \Lambda)$, then

$$\left| \int_{\mathbb{R}^s} [F^{-1} M F(P\omega_{\varepsilon\alpha})](x) \overline{Q(x)} \omega_{\varepsilon\beta}(x)\, dx \right| \le c_M \|P\omega_{\varepsilon\alpha}\|_{L^q} \|Q\omega_{\varepsilon\beta}\|_{L^{q'}}, \tag{4.8}$$

where ω_δ, for $\delta > 0$, is the function introduced in Lemma 3.7. Set $\alpha = \frac{1}{q}$ and $\beta = \frac{1}{q'}$, multiply both sides by $\varepsilon^{s/2}$ and let $\varepsilon \to 0$. By Lemma 4.7, the left-hand side converges to

$$\lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} \left(\sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; P)\, e^{i\lambda\cdot x} \right) \overline{Q(x)}\, dx.$$

By Lemma 4.4,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{s/2} \|P\omega_{\varepsilon\alpha}\|_{L^q} \|Q\omega_{\varepsilon\beta}\|_{L^{q'}} \\ &= \lim_{\varepsilon\to 0} \left(\varepsilon^{s/2} \int_{\mathbb{R}^s} |P(x)|^q e^{-\pi\varepsilon|x|^2}\, dx \right)^{1/q} \left(\varepsilon^{s/2} \int_{\mathbb{R}^s} |Q(x)|^{q'} e^{-\pi\varepsilon|x|^2}\, dx \right)^{1/q'} \\ &= \lim_{T\to\infty} \left(\frac{1}{|Q_T|} \int_{Q_T} |P(x)|^q\, dx \right)^{1/q} \left(\frac{1}{|Q_T|} \int_{Q_T} |Q(x)|^{q'}\, dx \right)^{1/q'}. \end{aligned}$$

Together with (4.8) this implies that

$$\left| \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} \left(\sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; P)\, e^{i\lambda\cdot x} \right) \overline{Q(x)}\, dx \right| \le c_M \|P\|_q \|Q\|_{q'}.$$

Finally, taking the supremum over all polynomials Q satisfying $\|Q\|_{q'} \le 1$, we obtain (4.7) (see, for example, [4]). Hence, the linear operator T_M defined on the class of trigonometric polynomials $\mathcal{P}(\mathbb{R}^s, \Lambda)$ by

$$T_M(P) = \sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; P)\, e^{i\lambda\cdot x}$$

is bounded, with bound not exceeding c_M. Then it has a unique bounded extension to the whole $B^q_{ap}(\mathbb{R}^s, \Lambda)$ and it is this extension that satisfies the required assertion ∎

Remark 4.8. Similar results are well known in the periodic case (see, e.g., [18]).

Theorem 4.9. *Let* $1 < q < \infty$. *Let* $\psi \in \mathcal{S}(\mathbb{R}^s)$ *with*

$$0 \le \psi(x) \le 1, \quad \operatorname{supp}\psi \subset \{y : \tfrac{1}{4} \le |y| \le 4\}, \quad \psi(x) = 1 \ \text{ if } \tfrac{1}{2} \le |x| \le 2.$$

If $\chi > \frac{s}{2}$, *then there exists a positive constant c such that*

$$\left\| \sum_{\lambda\in\Lambda} M(\lambda) a(\lambda; f)\, e^{i\lambda\cdot x} \right\|_q \le c \sup_{j\in\mathbf{Z}} \|\psi(\cdot) M(2^j \cdot)\|_{\chi,2} \|f\|_q \tag{4.9}$$

for all $M \in H^{\chi,2}(\mathbb{R}^s)$ *and for all* $f \in B^q_{ap}(\mathbb{R}^s, \Lambda)$.

Proof. We recall a version of the Fourier multiplier theorem of Michlin-Hörmander type for $L^q(\mathbb{R}^s)$ as it is stated in [19] (for a detailed proof see [20]):

$$\|F^{-1} M F f\|_{L^q} \le c \sup_{j\in\mathbf{Z}} \|\psi(\cdot) M(2^j \cdot)\|_{\chi,2} \|f\|_{L^q}$$

if $\chi > \frac{s}{2}$ and $1 < q < \infty$. Then M satisfies the assumptions of the theorem. As an immediate consequence we obtain (4.9) ∎

Remark 4.10. Theorem 4.9 says that every Fourier multiplier of Michlin-Hörmander type for $L^q(\mathbb{R}^s)$ is also a Fourier multiplier for $B^q_{ap}(\mathbb{R}^s, \Lambda)$ if it is restricted to Λ.

5. Sobolev-Besicovitch spaces

In this section we consider a set $\Lambda \subset \mathbb{R}^s$ satisfying the (α)-condition, and some classes of almost periodic distributions.

Definition 5.1. Let $1 < q < \infty$ and $m \in \mathbb{R}$. We set

$$H^{m,q}_{ap}(\mathbb{R}^s, \Lambda) = \left\{ f \in \mathcal{S}'_\Lambda(\mathbb{R}^s) : \ \|f\|_{(m),q} = \left\| \sum_{\lambda\in\Lambda} (1 + |\lambda|^2)^{m/2} a(\lambda; f)\, e^{i\lambda\cdot x} \right\|_q < \infty \right\}.$$

These spaces are called *Sobolev-Besicovitch spaces of order m and type H.*

Definition 5.2. Let $1 < q < \infty$ and $m \in \mathbb{N}$. We set

$$W^{m,q}_{ap}(\mathbb{R}^s, \Lambda) = \left\{ f \in B^q_{ap}(\mathbb{R}^s, \Lambda) : \ \|f\|_{m,q} = \sum_{|\alpha|\le m} \|D^\alpha f\|_q < \infty \right\}$$

where the derivatives are intended in the distributional sense. These spaces are called *Sobolev-Besicovitch spaces of order m and type W.*

Theorem 5.3. *Let* $1 < q < \infty$ *and let* $N \ge 1$ *be an integer. Then*

$$H^{N,q}_{ap}(\mathbb{R}^s, \Lambda) = W^{N,q}_{ap}(\mathbb{R}^s, \Lambda).$$

Proof. It is quite similar to that used in [6: p. 142] in the case of ordinary Sobolev spaces. Here we give only the principal steps for reader's convenience.

We invoke the Michlin-Hörmander multiplier theorem to obtain that the function $x_j^N(1+|x|^2)^{-N/2}$ is a Fourier multiplier for $L^q(\mathbb{R}^s)$ $(1<q<\infty)$. Therefore, it is also a Fourier multiplier for $B^q_{ap}(\mathbb{R}^s,\Lambda)$ if it is restricted to Λ. We get

$$\begin{aligned}
\left\|\frac{\partial^N f}{\partial x_j^N}\right\|_q &= \left\|\sum_{\lambda\in\Lambda}\lambda_j^N a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q \\
&= \left\|\sum_{\lambda\in\Lambda}\lambda_j^N(1+|\lambda|^2)^{-N/2}(1+|\lambda|^2)^{N/2}a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q \\
&\le c\left\|\sum_{\lambda\in\Lambda}(1+|\lambda|^2)^{N/2}a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q \\
&= c\|f\|_{(N),q} \qquad (1\le j\le s).
\end{aligned}$$

Using the Michlin-Hörmander multiplier theorem once more and a suitable function χ on $\mathbb{R}$, we obtain that the functions

$$(1+|x|^2)^{N/2}\Big(1+\sum_{j=1}^{s}\chi(x_j)|x_j|^N\Big)^{-1} \qquad \text{and} \qquad \chi(x_j)|x_j|^N x_j^{-N}$$

are Fourier multipliers for $L^q(\mathbb{R}^s)$ $(1<q<\infty)$. Then they are also Fourier multipliers for $B^q_{ap}(\mathbb{R}^s,\Lambda)$ if they are restricted to Λ. Thus

$$\begin{aligned}
\|f\|_{(N),q} &= \left\|\sum_{\lambda\in\Lambda}(1+|\lambda|^2)^{N/2}a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q \\
&\le c\left\|\sum_{\lambda\in\Lambda}\Big(1+\sum_{j=1}^{s}\chi(\lambda_j)|\lambda_j|^N\Big)a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q \\
&\le c\left(\left\|\sum_{\lambda\in\Lambda}a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q + \left\|\sum_{\lambda\in\Lambda}\sum_{j=1}^{s}\chi(\lambda_j)|\lambda_j|^N\lambda_j^{-N}\lambda_j^N a(\lambda;f)\,e^{i\lambda\cdot x}\right\|_q\right) \\
&\le c\left(\|f\|_q + \sum_{j=1}^{s}\left\|\frac{\partial^N f}{\partial x_j^N}\right\|_q\right).
\end{aligned}$$

The proof is now complete ∎

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