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Free Boundary Value Problem for the Axisymmetric Fluid's Flow with Surface Tension and Wedging Forces

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Abstract. A free boundary value problem with surface tension and wedging forces is considered. From the mathematical point of view it leads to a boundary value problem in which the mean and Gauss curvatures appear in the boundary condition on the unknown surface. The variational problem is formulated and it is proved that its solution is the solution of the free boundary value problem. Infinite smoothness of the free surface is proved and it is also proved that the curve generating the free surface is analytic for some values of the problem parameters.

Keywords: *Calderon-Zygmund inequality, conformal mapping, free boundaries, Gauss curvature, mean curvature, Orlicz spaces, Shauder estimates, variational problems, wedging forces*

AMS subject classification: 35

1. The boundary value problem

Let B_0 be the body obtained by rotation of a closed domain B from the upper half plane $E^+ = \{(x,y) \in \mathbb{R}^2 | y > 0\}$. We will suppose that the boundary ∂B of the domain B consists of the segment $C_1 = \{|x| \le k_0,\ y = h\}$, of the monotone arcs S^+ and S^-, which are the graphs of the monotone functions

$$\begin{aligned} y^+ &: [k_0, 1] \to \mathbb{R}, \qquad y^+(k_0) = h \quad \text{and } y^+(1) = 0 \\ y^- &: [-1, -k_0] \to \mathbb{R}, \qquad y^-(x) = y^+(-x), \end{aligned}$$

respectively, and of the segment $C_2 = \{(x,y) \in \mathbb{R}^2 |\, |x| \le 1 \text{ and } y = 0\}$.

We will suppose that the flow in question consists of the following three phases:

The phase of evaporized fluid, constituting the cavity W_0.

The mixed phase, determined by the mixture of the vapour and of the fluid filling in the set W_0'.

The phase of fluid past the "body" $B_0 \cup \bar{W}_0 \cup \bar{W}_0' = B'$.

As usual we denote by $\bar{A}$ the closure of the set A. We will study now the case of a flow for which the complement B'^* to the set B' is an axisymmetrical domain. Let us

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denote by D that part of the meridional sections of the set B'^* which is situated in the half plane E^+. We will consider a domain D such that ∂D consists of the part of the axis $\{y = 0\}$ which is complementar to the segment $C_2' = \{|x| < 1\}$, of the monotone arcs S^+ and S^-, of the "free" arc Σ with the endpoints $(-k, h)$ and (k, h) lying in the strip $S(k_0) = \{(x, y) \in \mathbb{E}^+ : |x| \leq k_0 \text{ and } y \geq h\}$, and of the two horizontal segments $\{-k_0 < x < -k, y = h\}$ and $\{k < x < k_0, y = h\}$.

In our study we are looking for a constant $k \leq k_0$, a continuous curve $\Sigma : [-k, k] \to S(k_0)$, twice differentiable over $(-k, k)$, and for a function Ψ in a domain D of type as just finished to be described, satisfying the conditions

$$\iint_D \left|\nabla\left(\Psi - \frac{y^2}{2}\right)\right|^2 \frac{dxdy}{y} < \infty \tag{1.1}$$

$$\left(\left(\frac{\Psi_x}{y}\right)_x + \left(\frac{\Psi_y}{y}\right)_y\right)(x, y) = 0 \qquad \text{for } (x, y) \in D \tag{1.2}$$

$$\Psi(x, y) = 0 \qquad \text{for } (x, y) \in \partial D \tag{1.3}$$

$$\frac{|\Delta\Psi(x, y)|^2}{2y^2} + \kappa H(x, y) + \theta K(x, y) = \lambda \qquad \text{for } (x, y) \in \Sigma, y > h \tag{1.4}$$

$$\Psi(x, y) \sim \frac{y^2}{2} \qquad \text{for } (x, y) \to \infty. \tag{1.5}$$

Here κ, θ and λ are real, non-negative numbers, $H(x, y)$ and $K(x, y)$ are the mean and Gaussian curvature of the surface S obtained by rotation of the free boundary curve Σ at the point (x, y) of its meridional section. We calculate principal curvatures in accordance with Frenet formulas for left handed orientation of coordinate system [5: p. 21].

The conditions (1.1) - (1.3) and (1.5) are well-known. The condition (1.4) is also famous when $\theta = 0$, it seems that Zhukovski was the first who studied it in the plain case (see [25: p. 489], but also [1, 20]). The term K in the present form of this condition is introduced to take into account the intermediate phase W_0'. It is supposed that the free curve is determined besides the usual forces also by the wedging force which is responsible for the appearence of this term in the boundary condition. We have borried the term "wedging force" from the work [16: p. 31] where the static condition

$$\kappa H + \theta K = \lambda$$

was introduced for the first time in the attempt to take into account the intermediate layers in the multiphased systems.

The whole Free boundary value problem (1.1) - (1.5) in the case of $\kappa = \theta = 0$ was studied in the well-known work [10]. The fast development of the theory of the Variational inequalities have overshadowed in some sense the importance of this work (see [14, 18]). Yet it seems to us that problem (1.1) - (1.5) never was studied even in the case of $\kappa \neq 0$ and $\theta = 0$. It was solved for the last case in the works [2 - 4]. The plane case for sufficiently large class of the boundaries was studied in the works of Kazhihov (see [13] and [19: pp. 184 – 189]).

2. Variational problem

In this section we will introduce the functional, whose minimum as it will be shown later is achieved over the solution of the free boundary value problem. From the works [7, 10] it is clear that this functional must be of the form

$$M + (1 - 2\lambda)V + \kappa S + 2\theta\mathcal{F}$$

where M is the virtual mass of the perturbed flow past the body, V is its volume, S is the area of the free surface and $\mathcal{F}$ is the functional, responsible for the appearence of the Gauss curvature on the free streamline. We now deduce the general form of this functional in some specific case and then we will formulate the variational problem. Later considerations will justify our approach.

Let A_Σ be the set of rectifiable Jordan curves connecting the points $P(-k, h)$ and $P(k, h)$ for some k with $0 < k < k_0$, whose bodies lie in the strip $S(k_0)$. For each $\Sigma \in A_\Sigma$ we will construct a domain D of the boundary described in Section 1, that is consisting of the same part of the axis $\{y = 0\}$, of the monotone arcs S^+ and S^-, of the free arc Σ, connecting the points $P_1(-k, h)$ and $P_2(k, h)$ and of the two horizontal segments $\sigma_1 = \{-k_0 < x < -k, y = h\}$ and $\sigma_2 = \{k < x < k_0, y = h\}$.

Let $B' = E^+\setminus\bar{D}$ and S be the surface obtained by rotation of Σ around the x-axis. When needed we will write $S = S(\Sigma)$, $D = D(\Sigma)$ and $B' = B'(\Sigma)$ to underline the dependence of S, D and B' on Σ. Let A_Ψ be the set of the functions $\Psi : D \to \mathbb{R}, D = D(\Sigma)$, which satisfies conditions (1.3) and (1.5) and which has generalized derivatives Ψ_x and Ψ_y in D satisfying (1.1). We will denote by D_L the set $\{(\Psi, \Sigma) | \Sigma \in A_\Sigma \text{ and } \Psi \in A_\Psi\}$.

Let now

$$\left.\begin{aligned} x &= x(s) \\ y &= y(s) \end{aligned}\right\} \qquad (s \in [0, |\Sigma|])$$

be the natural parametrization of the curve $\Sigma \in A_\Sigma$, $\Sigma_1 = \Sigma \cap \{x \le 0\}$ and $\Sigma_2 = \Sigma^+ \cap \{x \ge 0\}$ where Σ^+ is the parametrized curve defined as

$$\left.\begin{aligned} x &= x(|\Sigma| - t) \\ y &= y(|\Sigma| - t) \end{aligned}\right\} \qquad (t \in [0, |\Sigma|]).$$

We will consider the functional L defined by

$$L(\Psi, \Sigma) = M(\Psi, \Sigma) + (1 - 2\lambda)V(\Sigma) + \kappa|S|(\Sigma) + 2\theta\mathcal{F}(\Sigma) \tag{2.1}$$

over the set D_L. Here κ, θ and λ are the real non-negative numbers from Section 1 and

$$M = M(\Psi, \Sigma) = \iint_D \left|\nabla\left(\Psi - \frac{y^2}{2}\right)\right|^2 \frac{dxdy}{y} \tag{2.2}$$

$$V = V(\Sigma) = \pi \iint_{B'} y\, dxdy \tag{2.3}$$

$$|S| = |S|(\Sigma) = \int_{\Sigma} y\, ds \tag{2.4}$$

$$\mathcal{F}(\Sigma) = \int_{\Sigma_1 \cup \Sigma_2} f(\dot y)\, ds \tag{2.5}$$

$$f(\dot y) = \frac{1}{2}\left\{ -|\dot x| \int\limits_0^{|\dot y|} \left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\right)^{-3/2} d\sigma + E_0|\dot x| \right\}$$

where $|\dot x| = \sqrt{1-\dot y^2}$ and $E_0 > 0$ is an arbitrary number. It is easy to verify that $f(\dot y)$ is well defined even for $\dot y = 1$. To understand better the nature of the functional just introduced we will consider now a twice differentiable curve Γ with natural parametrization

$$\left.\begin{aligned} x &= x(s) \\ y &= y(s) \end{aligned}\right\} \qquad (s \in [0, |\Gamma|])$$

and suppose that there exists a point $z_0 \in \Gamma$ such that

$$\ddot y(s_0(z_0)) \neq 0, \qquad \dot y(s_0(z_0)) > 0, \qquad \dot x(s_0(z_0)) \neq 0.$$

Let us consider the functional

$$K(\Gamma) = \int_{\Gamma} f(\dot y)\, ds \qquad \text{for } f \in C^2([0,1]). \tag{2.5$'$}$$

In the variational approach to free boundary value problems the method of interior variations and its generalizations proved to be useful. In our consideration we also resort to this method. We consider local topological transformations of the complex plane $\mathbb{C}$ of the form

$$z^* = z + \varepsilon F(z, \bar z) \qquad \text{where } z = x + iy, \bar z = x - iy, F \in C_0^1(B(z_0, r)). \tag{2.6}$$

It is easy to see that to obtain the Gauss curvature in the boundary condition on the streamline the first variation of the functional we consider under the variations introduced should be of the form

$$\delta\, K(\Gamma) = -\int_{\Gamma} \operatorname{Re}\left(i\varepsilon \ddot y \bar{\dot z} F(s)\right) ds. \tag{2.7}$$

The following lemma gives us the class of functions which satisfy this condition.

Lemma 2.1. *Let K be a functional of the form (2.5)$'$ whose variation under the transformation (2.6) in the neighbourhood of the point z_0 we have already described is of the form (2.7). Then we have*

$$\begin{aligned} f(\dot y) = \frac{1}{2}\Big\{ &- (\operatorname{sgn} \dot x)\sqrt{1-\dot y^2} \\ &\times \int\limits_0^{\dot y} \left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} + E\right)(1-\sigma^2)^{-3/2} d\sigma + E_0\sqrt{1-\dot y^2}\operatorname{sgn}\dot x \Big\} \end{aligned} \tag{2.8}$$

where E and E_0 are arbitrary constants.

Proof. Let Γ^* be the variation of the curve Γ under the transformation (2.6). Then we easily get

$$\begin{aligned}\int_{\Gamma^*} f(\dot{y}^*)\,ds^* - \int_\Gamma f(\dot{y})\,ds \\ = \int_\Gamma \Big[- f_{\dot{y}}(\dot{y})\dot{x}\,\mathrm{Re}\Big\{ i\varepsilon\bar{z}\,\frac{dF}{ds}\Big\} + f(\dot{y})\,\mathrm{Re}\Big\{\varepsilon\bar{z}\,\frac{dF}{ds}\Big\}\Big] ds + o(\varepsilon)\end{aligned} \tag{2.9}$$

as $\varepsilon \to 0$. Using lengthy but simple calculations we obtain from (2.7) and (2.9) that the function $f = f(t)$ satisfies over $[0,1)$ the ordinary differential equation of second order

$$\frac{d^2 f}{dt^2}\sqrt{1-t^2} - \frac{df}{dt}\,\frac{t}{\sqrt{1-t^2}} + f\,\frac{1}{\sqrt{1-t^2}} = \mp 1. \tag{2.10}$$

The sign of the right part depends on the sign of the function $\dot{x}$, we have minus if $\dot{x} > 0$ and plus if $\dot{x} < 0$. Integrating (2.10) we get (2.8) ∎

The lemma we have proved justifies in some sense our selection of the functionals $\mathcal{F}$ and L. Now we can formulate the variational problem:

We are looking for a point $(\Psi_o, \Sigma_0) \in D_L$ such that

$$L(\Psi_0, \Sigma_0) = \inf\big\{L(\Psi,\Sigma) : (\Psi,\Sigma) \in D_L\big\}. \tag{2.11}$$

3. Minimal sequences of the variational problem

It is easy to prove that

$$1_L = \inf\big\{L(\Psi,\Sigma) : (\Psi,\Sigma) \in D_L\big\} > -\infty \tag{3.1}$$

(see, for example, [10: §5]). Let $\{m_n\}$, $m_n = (\Psi_n, \Sigma_n) \in D_L$, be a minimizing sequence. We wish to prove that the functions Ψ_n and the curves Σ_n can be symmetrized in some sense giving us some new minimizing sequence. To this end we prove now some auxiliary results.

Lemma 3.1. *Let $\gamma \in A_\Sigma$ and $x = x(s)$, $y = y(s)$ its natural parametrization. Then there exists a sequence $\{\gamma_n\}$ of analytical Jordan curves with parametrization $x = x_n(s)$, $y = y_n(s)$ ($s \in [0,\gamma']$, γ' being the length of the curve γ) such that*

a) *$y_n(0) = y(0) = h, y_n(\gamma') = y(\gamma') = h$ and $y_n(s) \geq h$ for $s \in [0,\gamma']$.*

b) *$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in the sense of convergence in the space $W_1^1([0,\gamma']) \cap C([0,\gamma'])$.*

Proof. We can add to the curve γ some other curve γ_0 in such a way that $\gamma_0 \cup \gamma$ will constitute a rectifiable Jordan curve limiting some domain D_0. Mapping the unit disk $\{|z| < 1\}$ over D_0 by the conformal mapping Φ and taking into account that Φ' belongs to the space H^1 (see [15: Chapter II, D]), we can construct the necessary approximation of γ ∎

Lemma 3.2. *Let L be the functional from* (2.1) *and* 1_L *the number from* (3.1). *Then there exists a sequence* $\{m_n\}$, *with* $m_n \in D_L$ *and* $m_n = (\Psi_n, \Sigma_n)$, *such that*

$$\lim_{n\to\infty} L(m_n) = 1_L$$

and Σ_n *is an analytical curve and* Ψ_n *is the streamline function of the domain* $D_n = D_n(\Sigma_n)$.

Proof. Let $\{(\Psi_n, \Sigma_n)\}$ be some minimizing sequence for problem (2.11). We will consider, omitting the index n, an arbitrary curve Σ of this sequence. We can assume that the endpoints of Σ are located inside of C_1. From Lemma 3.1 it follows immediately that there exists a sequence $\{\gamma_n\}$ of analytical curves from A_Σ converging to Σ. It is sufficient to prove now that

$$\lim_{n\to\infty} \mathcal{F}(\gamma_n) = \mathcal{F}(\Sigma).$$

From Lemma 3.1 it follows that the sequence of lengths $|\gamma_n|$ of γ_n converges to the length of Σ, that is $\lim_{n\to\infty} |\gamma_n| = |\Sigma|$. Let

$$\left.\begin{aligned} x &= u_n(\sigma) \\ y &= v_n(\sigma) \end{aligned}\right\} \qquad (\sigma \in [0, |\gamma_n|])$$

be the natural parametrization of the curve γ_n and $\theta_n : [0, |\Sigma|] \to [0, |\gamma_n|]$ a monotone mapping such that $x_n = u_n \circ \theta_n$ and $y_n = v_n \circ \theta_n$, where $x = x_n(s), y = y_n(s)$ $(s \in [0, |\Sigma|])$ is the parametrization of the curve γ_n from Lemma 3.1, converging in $W_1^1([0, |\Sigma|]) \cap C([0, |\Sigma|])$ to the natural parametrization $x = x(s), y = y(s)$ of the curve Σ. It is clear that $\theta_n' = \frac{d\theta_n}{ds} \to 1$ a.e. on $[0, |\Sigma|]$ and

$$\int_0^{|\gamma_n|} f(\dot{v}_n)\, d\sigma = \int_0^{|\Sigma|} f(\dot{y}_n(\theta_n^{-1})')\theta_n' ds. \tag{3.2}$$

Now applying to these integrals the reasoning of the Lebesgue theorem [6: §2.4] on the convergence of integrals, we get the necessary result ∎

As it was done in [9: p. 377] we can now introduce the symmetrizations of the functions Ψ_n (and consequently the corresponding symmetrizations of the domains $D_n = D_n(\Psi_n)$) of the extremal sequence $\{m_n\}$ to the x-axis and to the (y, θ)-plane (in cylindrical coordinate system (y, θ, x)) to prove that the curves Σ_n can be considered as monotone ones in each quadrant. The monotone behaviour of the functionals M, V and $|S|$ under symmetrization in question is well-known (see [9, 10, 22]). Now we will study the behaviour of the functional $\mathcal{F}$ under these symmetrizations.

Let us consider for the first the symmetrization of the function $\Psi : D \to \mathbb{R}$ in the (y, θ)-plane. Under this symmetrization we substitute each x-section B_x by the circle in the (y, θ)-plane of the same area. This symmetrization evidently leads to the symmetrization of the domain $E^+ \backslash D$ to the x-axis. Hence, to study the behaviour of the functional L under this symmetrization we are to study the behaviour of the functional $\mathcal{F}$ under the symmetrization of the domain $\tilde{B} = B' \cup B''$, where $B' = E^+ \backslash D$ and B'' is its reflection in the x-axis.

Lemma 3.3. *Let Σ be an analytical curve from A_Ψ, $B' = B'(\Psi)$,*

$$B'^* = \bigcup_{x_0 \in \mathrm{pr}_x B'} C_{x_0} \quad \text{with } C_{x_0} = \Big\{(x,y) \in \mathbb{R}^2 : x = x_0 \text{ and } 0 < y < \operatorname{mes} B'_{x_0}\Big\}$$

and

$$B'_{x_0} = B' \cap \{(x,y) \in \mathbb{R}^2 : x = x_0\}$$

the upper half of the body $\tilde{B}^*$ representing the symmetrization of $\tilde{B}$ to the x-axis. Let $\Sigma^* = \partial\tilde{B}^* \setminus \partial B$. Then

$$\mathcal{F}(\Sigma^*) \le \mathcal{F}(\Sigma). \tag{3.3}$$

Proof. Let

$$\Lambda(t) = -\int_0^t \Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}d\sigma \qquad (t \in [0,1]).$$

Then

$$\left.\begin{aligned} \Lambda'(t) > 0 \\ \Lambda''(t) < 0 \end{aligned}\right\} \quad \text{for all } t \in (0,1).$$

We will denote by $2m(x)+1$ the cardinal number of $\bar{B}'_x \cap \partial B'$. This number is finite because of the analyticity of the curve Σ. The function $m = m(x)$ is the step function, whose domain $D_{m(x)}$ is the reunion of the segments D_i, i.e. $D_{m(x)} = \cup_i D_i$. We have

$$\mathcal{F}(\Sigma) = \int_0^{k_0} \sum_{\nu=1}^{2m(x)+1} \big(\Lambda_1(|y'_\nu(x)|) + E_0\big)dx$$

where $y_\nu = y_\nu(x)$ represents the connected part of Σ over D_i and

$$\Lambda_1(t) = \Lambda(l_1(t)) \qquad \text{with } l_1(t) = \frac{t}{\sqrt{1+t^2}}$$

and $y'_\nu(x) = \frac{dy_\nu}{dx}$. We get now due to (3.4)

$$\left.\begin{aligned} \Lambda_1(t) > 0 \\ \Lambda''(t) < 0 \end{aligned}\right\} \quad \text{for all } t > 0. \tag{3.4$'$}$$

This means that

$$\begin{aligned} \mathcal{F}(\Sigma) &\ge \frac{1}{2}\int_{-k_0}^{k_0} \left[\Lambda_1\bigg(\sum_{\nu=1}^{2m(x)+1} |y'_\nu(x)|\bigg) + E_0\right]dx \\ &\ge \frac{1}{2}\int_{-k_0}^{k_0} \left[\Lambda_1\bigg(\bigg|\sum_{\nu=1}^{2m(x)+1} (-1)^{\nu+1}y'_\nu(x)\bigg|\bigg) + E_0\right]dx \\ &= \mathcal{F}(\Sigma^*). \end{aligned}$$

Thus the lemma is proved ∎

Lemma 3.3 means that we can consider curves $\Sigma_n \in A_\sigma$ as analytical ones parametrized in the form

$$\left.\begin{aligned} x &= x \\ y &= y_n(x) \end{aligned}\right\} \qquad (x \in [-k,k], n \in \mathbb{N}).$$

Without loss of generality we can assume that

$$y_n(-x) = y_n(x) \qquad (x \in [-k,0]). \tag{3.5}$$

Lemma 3.4. *Let γ be any analytical curve from A_Σ, parametrized in the form (3.5), $\bar{\gamma}$ its reflection in the x-axis, parametrized in the form*

$$\left.\begin{aligned} x &= x \\ y &= \bar{y}(x),\ \ \bar{y}(x) = y(-x) \end{aligned}\right\} \qquad (x \in [-k,k])$$

and

$$\sigma_1 = y \vee \bar{y} = \sigma_1(x) = \max\{y(x), \bar{y}(x)\}$$
$$\sigma_2 = y \wedge \bar{y} = \sigma_2(x) = \min\{y(x), \bar{y}(x)\}.$$

Let $D_1 = D_1(\gamma), D_2 = D_2(\bar{\gamma})$ and Ψ_1, Ψ_2 the corresponding stream functions which we consider extended by zero to the domains $E^+ \setminus \bar{D}_1$ and $E^+ \setminus \bar{D}_2$, respectively. Then for $v_1 = \Psi_1 \wedge \Psi_2$ and $v_2 = \Psi_1 \vee \Psi_2$ we get

$$L(\Psi_1, \gamma) + L(\Psi_2, \bar{\gamma}) = L(v_1, \sigma_1) + L(v_2, \sigma_2),$$

that is

$$L(v_1, \sigma_1) \le L(\Psi_1, \Gamma_1) \qquad or \qquad L(v_2, \sigma_2) \le L(\Psi_2, \Gamma_2).$$

Proof. It does not differ very much from that of [8] and it is not necessary to bring details ∎

Lemma 3.5. *Let Σ be an analytical curve whose parametrization $x = x, y = y(x)$ $(x \in [-k,k])$ satisfies the condition $y(x) = y(-x)$ $(x \in [0,k])$ and $B' = B'(\Sigma)$. Let $B'' = B' \cap \{x \ge 0\}$ and*

$$B''^* = \bigcup_{y_0 \in \mathrm{pr}_y B''} C_{y_0}$$

with

$$C_{y_0} = \Big\{(x,y) \in \mathbb{R}^2 \Big| 0 \le x \le \operatorname{mes} B''_{y_0} \ \ and\ \ B''_{y_0} = B'' \cap \{y = y_0\}\Big\}.$$

Let B'^* be the Domain B''^* completed by its reflection in the y-axis and Σ^* the part of its boundary different from S^+, S^- and C_2. Then

$$\mathcal{F}(\Sigma^*) \le \mathcal{F}(\Sigma).$$

Proof. Let $2m(\bar{y}) + 1$ be the cardinal number of the set $\bar{B}''_{\bar{y}} \cap \partial B'', \bar{y} \in \mathrm{pr}_y B''$ and $B = \operatorname{mes} \mathrm{pr}_y B''$. Then $m = m(\bar{y})$ is a step function and $D_{m(\bar{y})} = \cup_i \Delta_i$. Now

$$\mathcal{F}(\Sigma) = \int_0^B \sum_{\nu=1}^{2m(\bar{y})+1} M(|x'_\nu(\bar{y})|)\, d\bar{y} \tag{3.6}$$

where $x_\nu = x_\nu(\bar{y})$ represents the connected part of Σ over Δ_i,

$$M = M(t) = t \wedge (\mu(t)) + E_0 t, \qquad \mu(t) = (1+t^2)^{-1/2}, \qquad x'_\nu = \frac{dx_\nu}{d\bar{y}}.$$

Using simple calculations it is easy to prove that

$$M'(t) > 0 \qquad \text{for all } \; E_0 > 0 \text{ and } t > 0 \tag{3.7}$$
$$M''(t) < 0 \qquad \text{for all } \; E_0 \in \mathbb{R} \text{ and } t > 0. \tag{3.8}$$

From (3.6) - (3.8) we now have

$$\mathcal{F}(\Sigma) \geq \int_0^B M\left(\sum_{\nu=1}^{2m(\bar{y})+1} |x'_\nu(\bar{y})|\right) d\bar{y} \geq \int_0^B M\left(\left|\sum_{\nu=1}^{2m(\bar{y})+1} (-1)^{\nu+1} x'_\nu(\bar{y})\right|\right) d\bar{y} = \mathcal{F}(\Sigma^*).$$

Thus the lemma is proved ∎

Theorem 3.1. *Let* 1_L *be the number from* (3.1). *Then there exists a minimal sequence* $\{m_n\}$, $m_n \in D_n$ *and* $m_n = (\Psi_n, \Sigma_n)$, *such that*

$$1_L = \lim_{n\to\infty} L(m_n)$$

and Σ_n *are the graphs of monotone functions in each quadrant and* Ψ_n *are the stream functions of the domains* $D_n = D_n(\Sigma_n)$.

Proof. From Lemmas 3.2 - 3.5 and from known results on the behaviour of M, V and $|S|$ under symmetrization to the x-axis and to the (y,θ)-plane we get that we can consider the curves Σ_n as graphs of functions monotone in each quadrant. From the Dirichlet principle, valid for equation (1.2), it follows that we can consider Ψ_n as stream functions of the domains D_n, $D_n = D_n(\Sigma_n)$. Thus the theorem is proved ∎

4. Existence of the solution of the variational problem

The considerations of the previous part permit us to prove the following theorem.

Theorem 4.1. *Let* 1_L *be the number from* (3.1). *Then there exists a constant* $c_0 > 0$ *such that for all numbers* κ *and* θ *satisfying the condition*

$$\kappa - c_0\theta > 0 \tag{4.1}$$

there exists an element $m_0 \in D_L$ *such that*

$$L(m_0) = 1_L. \tag{4.2}$$

Proof. The semicontinuity of M is well known (see [10: §4] and [22: A.5]). Thus it remains to prove that $\kappa|S| + \theta\mathcal{F}$ is also "semicontinuous". It is easy to see that in the coordinate system (σ, τ) rotated through -45^o to the (x, y)-system we will have the parametric representations

$$\left.\begin{aligned} x = x_n(\sigma) &= \frac{\sqrt{2}}{2}(\sigma + h_n(\sigma)) \\ y = y_n(\sigma) &= \frac{\sqrt{2}}{2}(\sigma - h_n(\sigma)) \end{aligned}\right\} \qquad (\sigma \in \Delta) \tag{4.3}$$

for the curves Σ_n, where Δ is some segment over the σ-axis. We can assume that this segment is the same for all curves Σ_n and also for the limit (if needed we can extend the domains of the functions from (4.3)). Using the parametric representations for the curves Σ_n and an analogous one for the curve Σ we will now have

$$\mathcal{F}(\Sigma_n) = \frac{1}{2}\int_\Delta \frac{\sqrt{2}}{2}(h_n'(\sigma) + 1)\, d\sigma \int_0^{\frac{\sqrt{2}}{2}(1-h_n')/\sqrt{1+h_n'^2}} f_0(u)\, du + \frac{1}{2}E_0 \int_\Delta (h_n' + 1)\frac{\sqrt{2}}{2}\, d\sigma$$

$$f_0(u) = -\Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2} \qquad (u \in (0,1))$$

with an analogous expression for the curve Σ. Let us consider the difference

$$\delta_n = \kappa|S|(\Sigma_n) + 2\theta\mathcal{F}(\Sigma_n) - \big(\kappa|S|(\Sigma) + 2\theta\mathcal{F}(\Sigma)\big)$$

which can be written in the form

$$\delta_n = \int_\Delta \big(\Lambda_3(h_n') - \Lambda_3(h')\big)\, d\sigma + E_0\theta\frac{\sqrt{2}}{2}\int_\Delta (h_n' - h')\, d\sigma \tag{4.4}$$

where

$$\Lambda_3(t) = \kappa\sqrt{1+t^2} + \frac{\sqrt{2}}{2}\theta(1+t)\int_0^{N(t)} f_0(u)\, du \quad \text{with } N(t) = \frac{\sqrt{2}}{2}\frac{1-t}{\sqrt{1+t^2}}.$$

It is clear that

$$\Lambda_3'(t) = \kappa\frac{t}{\sqrt{1+t^2}} + \frac{\sqrt{2}}{2}\theta\int_0^{N(t)} f_0(u)\, du + \frac{\sqrt{2}}{2}\theta(1+t)f_0(N(t))N'(t)$$

$$\begin{aligned} \Lambda_3''(t) &= \kappa\frac{1}{(1+t^2)^{3/2}} + \sqrt{2}\theta f_0(N(t))(N(t))' + \theta(N'(t))^2 f_0'(N(t))\frac{\sqrt{2}}{2}(1+t) \\ &\quad + \frac{\sqrt{2}}{2}\theta f_0(N(t))\Big(\frac{1-t}{\sqrt{1+t^2}}\Big)''\frac{\sqrt{2}}{2}(1+t) \\ &= \kappa\frac{1}{(1+t^2)^{3/2}} + \theta\left\{-\frac{3(1-t^2)}{2(1+t^2)^{5/2}}f_0(N(t)) + \frac{\sqrt{2}}{4}\frac{(1+t)^3}{(1+t^2)^3}f_0(N'(t))\right\} \end{aligned} \tag{4.5}$$

where

$$
\begin{aligned}
f_0'(N(t)) = & -\frac{4(1+t^2)}{(1+t)^2} - \frac{3\sqrt{2}}{2}\frac{1-t}{\sqrt{1+t^2}} \\
& \times \frac{\arcsin N(t) + N(t)\sqrt{1-N^2} - \frac{\pi}{2}}{(1-N^2(t))^{3/2}} \frac{1+t^2}{(1+t)^2}.
\end{aligned}
\tag{4.6}
$$

It is easy to verify that the function $h = h_n(\sigma)$ satisfies the Lipschitz condition

$$|h_n(\sigma_1) - h_n(\sigma_2)| \le |\sigma_2 - \sigma_1|.$$

Hence $|h_n'(\sigma)| \le 1$ a.e. on Δ, which signifies that the domain of the function $f'(N(t))$ is the segment $[-1,1]$. We see from (4.6) that the function $f'(N(t))$ has a pole of second order at $t=-1$. The representation for Λ_3'' thus shows that this function can be considered as a continuous one over $[-1,1]$ for any fixed numbers κ and θ, and it can be represented as

$$\Lambda_3''(t) = \kappa\lambda_3(t) - \theta\lambda_4(t)$$

where λ_3 and λ_4,

$$
\begin{aligned}
\lambda_3(t) &= \frac{1}{(1+t^2)^{3/2}} \\
\lambda_4(t) &= 3(1-t^2)\frac{f_0(N(t))}{2(1+t^2)^{5/2}} - \sqrt{2}(1+t)^3\frac{f_0'(N(t))}{4(1+t^2)^3}
\end{aligned}
$$

are functions positive and continuous over $[-1,1]$. Let $c_0 = \sup\{\lambda_4(t) : t \in [-1,1]\}$ and let κ and θ such that $\kappa - 2c_0\theta > 0$. Then

$$\Lambda_3''(t) \ge \frac{1}{2}\kappa - \theta c_0 = \frac{1}{2}(\kappa - 2c_0\theta) > 0. \tag{4.7}$$

Returning now to the study of the difference δ_n we can now write

$$
\begin{aligned}
\delta_n = & \int_\Delta \Lambda_3'(h')(h_n' - h')\,d\sigma \\
& + \int_\Delta \Lambda_3''(\tilde{\theta}h')\frac{(h_n'-h')^2}{2}\,d\sigma + \frac{E_0\theta\sqrt{2}}{2}\int_\Delta (h_n' - h')\,d\sigma
\end{aligned}
\tag{4.8}
$$

for $-1 \le \tilde{\theta} \le 1$. The functions h_n are uniformly limited in the space $W_2^1(\Delta)$. We can select a sequence $\{h_n\}$ uniformly converging over Δ. Then using the Banach-Alaoglu theorem [12: §3] we can assume that the sequence $\{h_n'\}$ converges weakly to the function h'. Now taking into account (4.7) we get from (4.8) that $\lim_{n\to\infty}\delta_n \ge 0$, that is

$$\kappa|S|(\Sigma) + 2\theta\mathcal{F}(\Sigma) \le \lim_{n\to\infty}\Big(\kappa|S|(\Sigma_n) + 2\theta\mathcal{F}(\Sigma_n)\Big).$$

Combining this with the above mentioned semicontinuity of M we get (4.2). Thus the theorem is proved ∎

5. A necessary condition for existence of a solution of the variational problem

Here we will consider a necessary condition for the solution of the problem from Section 2. Let $z_0 \in \Sigma \setminus \{P_1, P_2\}$ and $K = B(z_0, r)$ where $r > 0$ is such that there exists no intersection of $B(z_0, r)$ with $\partial S(k_0)$. For any function $F = F(z, \bar{z}) : \mathbb{C} \to \mathbb{C}$ of class $C^{0,1}(K)$ with support in K we consider the variation $z^*(z, \bar{z})$ of the curve Σ in the form

$$z^* = z + \varepsilon F(z, \bar{z}) \qquad (\varepsilon > 0, z = x + iy). \tag{5.1}$$

It is clear that for $\varepsilon > 0$ sufficiently small the transformation (5.1) is a topological one. It can be easily shown that for the transformed curve Σ^* we have

$$\begin{aligned} &|S|(\Sigma^*) - |S|(\Sigma) \\ &\quad = \int_{\Sigma^* \cap K} y^* ds - \int_{\Sigma \cap K} y\, ds \\ &\quad = \operatorname{Re} \varepsilon \left\{ \iint_{\Sigma \cap K} y \left(\frac{\partial F}{\partial z} \frac{dz}{ds} + \frac{\partial F}{\partial \bar{z}} \frac{d\bar{z}}{ds} \right) ds + \iint_{\Sigma \cap K} \frac{F}{i}\, ds \right\} + o(\varepsilon) \end{aligned} \tag{5.2}$$

as $\varepsilon \to 0$ (see [3: p. 110]).

Lemma 5.1. *Let $m = (\Psi, \Sigma)$ be any solution of the variational problem from Section 2. Then*

$$\begin{aligned} \mathcal{F}(\Sigma^*) - \mathcal{F}(\Sigma) &= \int_{\Sigma^* \cap K} f(\dot{y}^*)\, ds^* - \int_{\Sigma \cap K} f(\dot{y})\, ds \\ &= \int_{\Sigma \cap K} \left\{ -f_{\dot{y}} \dot{x} \operatorname{Re} \left(i\bar{\dot{z}} \frac{dF}{ds} \varepsilon \right) + f(\dot{y}) \operatorname{Re} \left(\bar{\dot{z}} \frac{dF}{ds} \varepsilon \right) \right\} ds + o(\varepsilon) \end{aligned} \tag{5.3}$$

as $\varepsilon \to 0$, where $2f_{\dot{y}}(\dot{y})\dot{x}(s)$ is the function defined a.e. by

$$\begin{aligned} 2f_{\dot{y}}(\dot{y})\dot{x} &= \dot{y} \int_0^{\dot{y}} \left(\arcsin \sigma + \sigma\sqrt{1 - \sigma^2} - \frac{\pi}{2} \right) (1 - \sigma^2)^{-3/2} d\sigma \\ &\quad - \frac{1}{\sqrt{1 - \dot{y}^2}} \left(\arcsin \dot{y} + \dot{y}\sqrt{1 - \dot{y}^2} - \frac{\pi}{2} \right) - E_0 \dot{y} \\ &\quad (\dot{x} \neq 0 \ \ (\dot{y} \neq 1)) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} 2f_{\dot{y}}(\dot{y})\dot{x} &= \lim_{t \to 1} 2f_t(t)\sqrt{1 - t^2} \\ &= \int_0^1 \left(\arcsin \sigma + \sigma\sqrt{1 - \sigma^2} - \frac{\pi}{2} \right) (1 - \sigma^2)^{-3/2} d\sigma - E_0 \end{aligned}$$

for $\dot{y}(s) = 1$.

Proof. It is clear that at points with $\dot{y}(s) \neq 1$ $(\dot{x}(s) \neq 0)$ the function $f_{\dot{y}}(\dot{y}(s))$ is well defined, so that at such points we have

$$f(\dot{y}^*)\,ds^* - f(\dot{y})\,ds = -f_{\dot{y}}(\dot{y})\dot{x}\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) + f(\dot{y})\,\mathrm{Re}\left(\bar{z}\frac{dF}{ds}\varepsilon\right)ds + o(\varepsilon) \qquad (5.5)$$

as $\varepsilon \to 0$, the principal part of (5.5) being the integrand from (5.3). The second part of this integrand is the same even at points where $\dot{y}(s) = 1$. The first part of it corresponds to the difference Δf, and at points with $\dot{y}(s) = 1$ we have

$$\begin{aligned}
&2(f(\dot{y}^*) - f(\dot{y})\\
&= \dot{x}^* \int_0^{\dot{y}^*} \Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}\,d\sigma + E_0\dot{x}^*\\
&= \left\{-\dot{y}\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\right) + o(\varepsilon)\right\} \int_0^{1+o(\varepsilon)} \Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}\,d\sigma\\
&\quad + E_0\dot{y}\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) + o(\varepsilon)\\
&= -\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) \int_0^1 \Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}\,d\sigma\\
&\quad + E_0\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) + o(\varepsilon)\\
&= -\lim_{t\to 1}\sqrt{1-t^2}\,f_t(t)\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) + o(\varepsilon)
\end{aligned}$$

as $\varepsilon \to 0$ whence (5.4) follows and the lemma is proved ∎

Using the calculations made we arrive at the following result.

Theorem 5.1. *Let (Ψ, Σ) be any solution of the variational problem from Section 2 and F the function from* (5.1). *Then*

$$\begin{aligned}
4\iint_{D\cap K} \frac{F}{i}\Psi_z\Psi_{\bar{z}}\frac{dxdy}{y^2} &+ 8\iint_{D\cap K}\frac{\partial F}{\partial \bar{z}}\Psi_z^2\frac{dxdy}{y^2}\\
&+2\lambda\iint_{D\cap K}\left(\frac{F}{i} + 2y\frac{\partial F}{\partial z}\right)dxdy\\
-\kappa\int_{D\cap K} \dot{y}\left(\frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial \bar{z}}\dot{\bar{z}}\right)d\bar{z} &- \kappa\int_{D\cap K}\frac{F}{i}ds\\
-2\theta\int_{\Sigma\cap K}\left\{-f_{\dot{y}}(\dot{y})\dot{x}\,\mathrm{Re}\left(i\bar{z}\frac{dF}{ds}\varepsilon\right) + f(\dot{y})\,\mathrm{Re}\left(\bar{z}\frac{dF}{ds}\varepsilon\right)\right\}&ds = 0
\end{aligned} \qquad (5.6)$$

where $W = B' \setminus \bar{B}$.

Proof. It follows from Lemma 5.1, representation (5.2) and from known results on the variation of virtual mass under the transformation (5.1) (see [3: p. 110]) ∎

6. Conformal mappings, boundary estimates for $|\nabla\Psi|$

Using in (5.6) the Privalov theorem for integrals of Cauchy type we get that for any interior arc Σ' of Σ there exists a function $\mathcal{P}$ and a succession $\{l_n\}$ of arcs in D tending to Σ' such that

$$\lim_{n\to\infty}\int_{l_n}|\nabla\Psi|^2 ds = \int_{\Sigma'}\mathcal{P}\, ds.$$

To prove this we also need the inversion of the Federer's theorem (see [6: Theorem 3.2.6]) on the substitution of the variables under the Lebesgue's integral, which can be proved in our case using the known Lewy's arguments on the convergence of the integrals. This result and also condition (5.6) give us the generalized boundary value condition on the arcs of the free boundary (see [2]). But we will proceed here in other direction. We will get now a stronger result on the boundary behaviour of $|\nabla\Psi|$. This result was suggested to us by P. I. Plotnikov (see [3: pp. 136 - 141]).

We start now with the study of the conformal mappings of the domains we have in the variational study.

Lemma 6.1 (see [3: p. 136]). *Let $D\subset E^+$ be an infinite domain whose boundary consists of an arc α symmetrical to the y-axis and of two semi-infinite segments of the real axis constituting the complement to the projection of α on $\{y=0\}$. Let $w: D\to E_w^+$ be a conformal mapping with $w(\infty)=\infty$ and $\mathrm{Re}\, w(iy)=0$, and let $z=z(w)$ be its inverse. Then for a, sufficiently small, and any $q\in(0,2)$ there exists a number $c_1(a,q,v')$ depending only on a, q and $v'>0$ sufficiently small such that for the derivative $z_w(w)$ of $z(w)$ we get*

$$\int_{-a}^{a}|z_w(u+iv)|^{\pm q}du \le c_1(a,q,v') \tag{6.1}$$

for all $v\in[0,v']$.

Proof. As α is a rectifiable Jordan curve, it is easy to see that the function $\ln\frac{dz}{dw}$ has angular boundary values a.e. on $\{u=0\}$, defined by some function θ_0 and

$$i\ln\frac{dz}{dw} = -S\theta_0$$

where S is the integral operator defined by

$$Sf = \int_{-\infty}^{+\infty}\frac{f(u)}{w-u}\, du.$$

Let $\eta=\eta(u)$ be an infinite differentiable function, with values in $[0,1]$, equal one in some neighbourhood of $[u_0-2a, u_0+2a]$ and equal zero in the exterior of (u_0-3a, u_0+3a) for some $u_0>0$, $a>0$ fixed. Then

$$i\ln z_w = \Phi_0-\Phi_1+\Phi_2 \tag{6.2}$$

where

$$\Phi_0 = \ln \frac{dz}{dw} - S(\eta\theta_0), \qquad \Phi_1 = S\Big(\frac{\pi}{4}\eta\Big), \qquad \Phi_2 = S\Big(\eta\Big(\theta_0 + \frac{\pi}{4}\Big)\Big).$$

As the arc α is monotone in each quadrant, then for $a > 0$ sufficiently small we get

$$|\theta_0| \leq \frac{\pi}{2} \qquad \text{a.e. on } \{u = 0\} \tag{6.3}$$

and $\theta_2 = |\eta(\theta_4 + \frac{\pi}{4})| \leq \frac{\pi}{4}$ there. The function Φ_0 has imaginary part equal to zero on $\{u = 0\}$ and is infinitely differentiable, hence

$$|exp(\pm i\Phi_j(w))| \leq c \qquad \big(j = 0, 1;\ w \in Q_{2a} = (u_0 - 2a, u_0 + 2a) \times (0, a)\big). \tag{6.4}$$

Now let us consider $\Phi_2(w) = \phi_2(w) + i\Psi_2(w)$. It is clear that $\Phi_2(\infty) = 0$. Let us define $F_2(w) = \exp(\pm qi\Phi_2(w))$ $(0 \leq q < 2)$. For any $w_0 \in E_w^+$, $w_0 = u_0 + iv_0$, there exists some constant c_0 such that for any $h \in (0, |w_0|)$ we have

$$F_2(w_0 + ih) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{v_0 F_2(u + ih)}{(u_0 - u)^2 + v_0^2}\, du + c_h \qquad (c_h \leq c_0) \tag{6.5}$$

(see, for example, [15: Chapter VI, D]). Now using the Zygmund idea on studying singular integrals [19: p. 71], we get from (6.3) and (6.5) that, for any $v' > 0$ sufficiently small,

$$\int_{-3a}^{+3a} \exp\big(\pm q\Psi_2(u + ih)\big)du \leq c''(a, q, v') \qquad (h < v'). \tag{6.6}$$

Combining (6.2), (6.4) and (6.6) we get (6.1). Thus the lemma is proved ∎

Lemma 6.2. *Let D be the domain from Lemma 6.1, $\Psi = \Psi(z)$ its stream function, $z : E^+ \to D$ the conformal mapping from Lemma 6.1 and $\hat\Psi(w) = \Psi(z(w))$. Then for $a > 0$ sufficiently small and any $r > 1$ there exists a constant $c = c(r, a)$ such that*

$$\int_{-a}^{+a} |\nabla \hat\Psi(u + iv)|^r du \leq c.$$

Proof. Let $\omega = \hat\Psi_w$. Then

$$4\frac{\partial \omega}{\partial \bar z} = A\omega \qquad \text{and} \qquad A = A_1 + A_2 \frac{\bar\omega}{\omega} \tag{6.7}$$

where $A, A_1, A_2 \in L_{2,loc}(E_w^+)$. Let $T_1 f$ be the Vekua operator (see [3: p. 139] and [19: p. 210]) defined on finite in E_w^+ functions, summable with degree $p > 1$ and $\zeta = \zeta(w)$ an infinitely differentiable function equal to 1 in some neighbourhood of Q_{2a} and with compact support in Q_{3a}. Then

$$T_1(\zeta A) \in W_2^1(Q_{3a}) \qquad \text{and} \qquad \operatorname{Im} T_1(\zeta A)(w) = 0 \text{ when } \operatorname{Im} w = 0.$$

Using the Pohozhaev theorem on Orlich and Sobolev spaces (see [17: p. 290] and [23: p. 193 in the Russ. transl.]) we have

$$\| \exp |T_1(\zeta A)| \|_{L_r(Q_{3a})} \leq c(r,a) \tag{6.8}$$

for all $r > 1$. Let us set

$$\Phi(w) = \omega \exp(-T_1(\zeta A))(w).$$

It is easy to prove that Φ is an analytical in Q_{3a} function whose imaginary part is zero on $\{u = 0\}$, which means in accordance with (6.8) that

$$\|\omega\|_{L_r(Q_{3a})} \leq c^1(r,a) \tag{6.9}$$

for all $r > 1$. Returning to the function $\hat{\Psi}$ we see that it satisfies the Poisson equation

$$\Delta\hat{\Psi}(w) = A\omega$$

where

$$\|A\omega\|_{L_q(Q_{2a})} \leq c(a)\, \|A\|_{L_2(Q_{2a})} \|\omega\|_{L_{\frac{2q}{2-q}}(Q_{2a})}$$

for any $q \in (0,2)$ and for some constant $c(a)$ depending on a only [3: p. 140]. This means due to the Calderon-Zygmund inequality [19: p. 198] that $\hat{\Psi} \in W_q^2(Q_{2a})$. Using now imbedding theorems for Sobolev spaces we get the result we need ∎

Lemma 6.3. *Let D be the domain from Lemma 6.1, $z = z(w)$ the conformal mapping defined there, l_v the image of the segment $s_v = \{(u,v) : |u-u_0| < a$ and $l_v = z(s_v)\}$ and Ψ the streamline function of the domain D. Then for any $v' > 0$ sufficiently small there exist numbers $c > 0$ and $p > 1$ such that*

$$\int_{l_v} |\nabla\Psi|^{2p} ds < c \qquad (v < v')$$

where $c = c(p,v')$.

Proof. It follows immediately from Lemmas 6.1 and 6.2 ∎

7. Generalized boundary conditions

The estimates we have made in the previous section permit us to study boundary conditions on a free boundary.

Lemma 7.1. *Let $m = (\Psi, \Sigma)$ be the solution of the variational problem from Section 2 and let $x = x(s), y = y(s)$ be the natural parametrization of the curve Σ. Then the function*

$$\kappa y\dot{\bar{z}} - 2i\theta f_{\dot{y}}\dot{x}\dot{\bar{z}} + 2\theta f\dot{\bar{z}}$$

has a generalized derivative over $[0, |\Sigma|]$ satisfying the condition

$$\frac{d}{ds}\Big(\kappa y\dot{\bar{z}} - 2i\theta f_{\dot{y}}\dot{x}\dot{\bar{z}} + 2\theta f\dot{\bar{z}}\Big) = -2i\lambda y\dot{\bar{z}} - 4i\frac{\Psi_z^2}{y}\dot{z} + \frac{\kappa}{i}. \tag{7.1}$$

Proof. Let z_0 be any point of $\Sigma \setminus \{P_1, P_2\}$, $D_1^\varepsilon \subset D$ the domain whose boundary consists of the arc $\gamma_\varepsilon = \Gamma_\varepsilon \cap \bar{D}_1$ with $\Gamma_\varepsilon = \{|z - z_0| = \varepsilon\}$ and $\Sigma_\varepsilon = \Sigma \cap \{|z - z_0| < \varepsilon\}$. We transform the double integrals in (5.6) into integrals over lines using the Green formula over the domain D_1^ε. Then passing to the limit and using the results of Section 6 we easily get from (5.6) for any function $F \in C_0^\infty([0, |\Sigma_\varepsilon|)$ the condition

$$\int_0^{|\Sigma_\varepsilon|} \left(\left(2i\lambda y\bar{\dot{z}} + 4i\frac{\Psi_z^2}{y}\dot{z} - \frac{\kappa}{i} \right) F - \left(\kappa y\bar{\dot{z}} - 2i\theta f_{\dot{y}}\dot{x}\bar{\dot{z}} + 2\theta f\bar{\dot{z}} \right) \frac{dF}{ds} \right) ds = 0. \tag{7.2}$$

It is worthwhile to note here that it is the Björke theorem which permits us to extend functions of this type, defined over monotone arcs, to functions from $C^{0,1}(K)$ (see [17: p. 48]). Condition (7.2), evidently, implies the existence of the generalized derivative for the function

$$\kappa y\bar{\dot{z}} - 2i\theta f_{\dot{y}}\dot{x}\bar{\dot{z}} + 2\theta f\bar{\dot{z}} \in L_2([0, |\Sigma_\varepsilon|])$$

which leads to condition (7.1). Thus the lemma is proved ∎

We still cannot differentiate inside of the brackets in the left side of (7.1). The following lemma permits us to do it sometimes.

Lemma 7.2. *Let us suppose that the numbers κ and θ satisfy the condition*

$$\kappa h - 2\theta > 0. \tag{7.3}$$

Then for the extremal element $m = (\Psi, \Sigma)$ the functions $\dot{x} = \dot{x}(s)$ and $\dot{y} = \dot{y}(s)$ are differentiable a.e. over $(0, |\Sigma|)$.

Proof. From Lemma 7.1 we have that the functions

$$\Phi_1(s) = \kappa y\dot{x} - 2\theta f_{\dot{y}}\dot{x}\dot{y} + 2\theta f\dot{x} \tag{7.4}$$

$$\Phi_2(s) = -\kappa y\dot{y} - 2\theta f_{\dot{y}}\dot{x}^2 - 2\theta f\dot{y} \tag{7.5}$$

are absolutely continuous over $(0, |\Sigma|)$ (the functions Φ_1 and Φ_2 are defined over the whole interval $(0, |\Sigma|)$, yet the functions $\dot{x}$ and $\dot{y}$ are defined a.e. there). Let us consider an arbitrary point $z_0 = z(s_0) \in \Sigma_1 \setminus \{(0, y(0))\}$ such that $\dot{y}(s_0)$ and $\dot{\Phi}_2(s_0)$ exist. Then we can rewrite (7.5) in some neighbourhood of z_0 in the form

$$\Phi(s, \dot{y}) = 0 \tag{7.5$'$}$$

for a.a. points of some neighbourhood of s_0. Here the function Φ is defined as

$$\Phi(s,t) = -\kappa y_0 t - \kappa\dot{y}(s_0)t(s - s_0) + \theta\Big(\arcsin t + t\sqrt{1 - t^2} - \frac{\pi}{2}\Big) - \Phi_3(s)$$

where

$$\Phi_3(s) = \Phi_2(s) + 0(s - s_0) \quad \text{and} \quad 0(s - s_0) = y(s) - \big(y(s_0) + \dot{y}(s_0)(s - s_0)\big).$$

Let us consider now the equation

$$\Phi(s,t) = 0. \tag{7.5''}$$

As inequality (7.3) holds, we have

$$\frac{\partial \Phi}{\partial t}(s_0, t_0) = -\kappa y_0 + 2\theta\sqrt{1-t_0^2} \neq 0$$

for all $t_0 \in [0,1]$ (if necessary we extend the function $\arcsin t + t\sqrt{1-t^2} - \frac{\pi}{2}$ by zero outside of the interval $[0,1]$). This means that in some neighbourhood of $(s_0, \dot{y}(s_0))$ the set of solutions of equation (7.5)″ can be considered as the graph of a unique (continuous) function $t = t(s)$. Let

$$\begin{aligned} G_s &= g_s(t) \\ &= \begin{cases} -\kappa y_0 t - \kappa \dot{y}(s_0)t(s-s_0) & \text{for } t > 0 \\ -\kappa y_0 t - \kappa \dot{y}(s_0)t(s-s_0) + \theta(\arcsin t + t\sqrt{1-t^2} - \frac{\pi}{2}) & \text{for } 0 \le t \le 1. \end{cases} \end{aligned}$$

Then from (7.5)′ and (7.5)″ we have $G_s(\dot{y}(s)) = G_s(t(s))$ a.e. in some neighbourhood U_1 of s_0. As

$$\frac{dG_s}{dt} = -\kappa y_0 - \kappa \dot{y}(s_0)(s-s_0) + 2\theta\sqrt{1-t^2}[(1-t)]^+,$$

then there exists a neighbourhood U_2 of s_0, $U_2 \subset \bar{U}_2 \subset U_1$, such that $\frac{dG_s}{dt}(s) \neq 0$ for all $s \in U_2$ and all $t \in [0, +\infty)$. This means that $\dot{y}$ can be extended continuously on U_2. As the function $y = y(s)$ is absolutely continuous on $[0, |\Sigma|]$, we get in correspondence with the Newton-Leibnitz formula for functions of such type that the function $\dot{y}(s)$ is defined over U and is continuous here. This means that the function

$$\Phi_4(s) = \Phi_3(s) + \kappa \dot{y}(s_0)\dot{y}(s)(s-s_0)$$

is differentiable in the point $s_0 \in U_2$. Now we evidently have that

$$\bar{G}(\dot{y}(s)) = \Phi_4(s) \qquad (s \in U_2) \tag{7.6}$$

where the function

$$\bar{G}(t) = -\kappa y_0 t + \theta\Big[\Big(\arcsin t + t\sqrt{1-t^2} - \frac{\pi}{2}\Big)\Big]^+$$

is monotone on $[0, +\infty)$. From (7.6) we have that the function $\dot{y}(s) = \bar{G}^{-1} \circ \Phi_4(s)$ is differentiable in the point s_0. In the same way we can prove that the function $\dot{x}(s)$ is differentiable a.e. in Σ. Thus the lemma is proved ∎

Now we can prove the theorem on the existence of a generalized solution for problem (1.1) - (1.5).

Theorem 7.1. *There exists a number $c_0 > 0$ such that for all κ and θ satisfying the condition*

$$\kappa - c_0\theta > 0$$

there exists an element $m = (\Psi, \Sigma) \in D_L$ giving a solution of the variational problem of Section 2. The function $y = y(x)$ $(x \in [-k, +k])$ representing the curve Σ is monotone in each quadrant of E^+ and the functions $x = x(s)$ and $y = y(s)$ from the natural parametrization of Σ have derivatives $\dot{x} = \dot{x}(s)$ and $\dot{y} = \dot{y}(s)$ absolutely continous on $(0, |\Sigma|)$. The function $\Psi = \Psi(z)$ is the streamline function satisfying the boundary condition (1.5) and its gradient $\nabla\Psi$ has angular values a.e. on $(0, |\Sigma|)$ which satisfy there the boundary condition (1.4).

Proof. It is clear that there exist numbers κ, θ, c_0 which satisfy each of the conditions (4.1) and (7.3). From Theorem 4.1 it follows that an extremal element $m = (\Psi, \Sigma)$ exists. It is easy to prove that the function Ψ satisfies conditions (1.1) - (1.3) and (1.5) (see [10: §5]). From Theorem 3.1 we have now that Σ is a curve monotone in each quadrant. It remains to prove that boundary condition (1.4) holds. Lemmas 7.1 and 7.2 show that the function Ψ_z satisfies a.e. on the curve Σ the condition

$$\begin{aligned}\kappa y\ddot{\bar{z}} + \kappa\dot{y}\dot{\bar{z}} - 2i\theta f_{\dot{y}\dot{y}}(\dot{y})\dot{x} - 2i\theta f_{\dot{y}}\dot{x}\ddot{\bar{z}} - 2i\theta f_{\dot{y}}\ddot{x}\dot{\bar{z}} + 2i\theta f\ddot{\bar{z}} + 2i\theta f_{\dot{y}}(\dot{y})\ddot{y}\dot{\bar{z}} \\ = -2i\lambda y\dot{\bar{z}} - 4i\frac{\Psi_z^2}{y}\dot{z} + \frac{\kappa}{i}.\end{aligned} \tag{7.7}$$

At points $s \in (0, |\Sigma|)$ such that $\dot{x}(s) \neq 0$ and $\dot{y}(s) \neq 0$ the right-hand side Z of (7.7) can be represented in the form

$$\begin{aligned}Z &= \frac{d}{ds}(\Phi_1 + i\Phi_2) \\ &= -i\kappa\dot{\bar{z}}yk(z) + \kappa\dot{y}\dot{\bar{z}} - 2i\theta\ddot{y}\dot{\bar{z}}\Big[f_{\dot{y}\dot{y}}(\dot{y})\dot{x} - f_{\dot{y}}(\dot{y})\frac{\dot{y}}{\dot{x}} + f(\dot{y})\frac{1}{\dot{x}}\Big]\end{aligned} \tag{7.8}$$

where $k = k(z)$ is the curvature of the plane curve Σ. The function $f = f(t)$ satisfies the equation

$$\frac{d^2f}{dt^2}\sqrt{1-t^2} - \frac{df}{dt}\frac{t}{\sqrt{1-t^2}} + f\frac{1}{\sqrt{1-t^2}} = -1.$$

This permits us to receive from (7.8)

$$-i\kappa\dot{z}yk(z) + \kappa\dot{y}\dot{\bar{z}} + 2i\theta\ddot{y}\dot{\bar{z}} = -2i\lambda y\dot{\bar{z}} - 4i\frac{\Psi_z^2}{y}\dot{z} + \frac{\kappa}{i}. \tag{7.9}$$

Now it is well known that the mean and the Gaussian curvatures can be represented as

$$2H(z) = k(z) + \frac{1}{y\sqrt{1+y'^2}} = k(z) + \frac{\Psi_y}{y\sqrt{\Psi_x^2 + \Psi_y^2}}$$

and

$$K(z(s)) = \frac{-\ddot{y}(s)}{y(s)}$$

(see, for example, [5: p. 162] and [21: §7.4]). Substituting these expressions into (7.9) we get

$$\frac{1}{2y^2}|\nabla\Psi|^2 + \kappa H + \theta K = \lambda$$

for a.a. points of Σ in consideration. Now let us consider those points for which $z = z(s) \in \Sigma_1 \setminus \{(0, y(0))\}$ and such that $\dot{x} = \dot{x}(s) = 0$ and $\ddot{z}(s)$ exists. It is clear that as the function y attains its maximum value at such points, then we will have $\ddot{y}(s) = \ddot{x}(s) = 0$. It is easy to see that a.e. on $(0, |\Sigma|)$ independently on the values $\dot{x}(s)$ assumed we will have

$$-f_{\dot{y}}(\dot{y}(s))\dot{x}^2 + f_{\dot{y}}(\dot{y})\dot{y} = \frac{1}{2}\Big(\arcsin\dot{y}(s) + \dot{y}\sqrt{1-\dot{y}^2} - \frac{\pi}{2}\Big). \tag{7.10}$$

Besides

$$\begin{aligned}\kappa y\dot{x} - \theta\int_0^{\dot{y}}\Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}d\sigma \\ -\theta\dot{y}(s)h(\dot{y}) - E_0\theta = \Phi_1(s)\end{aligned} \tag{7.11}$$

where

$$h(t) = \begin{cases} 0 & \text{for } t = 1 \\ -\big(\arcsin t + t\sqrt{1-t^2}(1-t^2)^{-1/2} - \frac{\pi}{2}\big) & \text{for } t \in [0,1) \end{cases}$$

is a function differentiable over $[0,1]$. Now we see that from (7.10) and (7.11) it follows

$$\begin{aligned}
&\frac{d}{ds}(\Phi_1(s) + i\Phi_2(s) - i\kappa y\bar{\dot{z}}) \\
&= \frac{d}{ds}\bigg\{\frac{1}{2}\bigg[-\theta\int_0^{\dot{y}}\Big(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\Big)(1-\sigma^2)^{-3/2}d\sigma - \theta\dot{y}h(\dot{y}) + E_0\bigg] \\
&\quad + \frac{i}{2}\theta\Big(\arcsin\dot{y} + \dot{y}\sqrt{1-\dot{y}^2} - \frac{\pi}{2}\Big) \\
&= -\theta\lim_{\dot{y}\to 1}\bigg\{\Big(\arcsin\dot{y} + \dot{y}\sqrt{1-\dot{y}^2} - \frac{\pi}{2}\Big)(1-\dot{y}^2)^{-3/2}\bigg\}\ddot{y}(s) \\
&\quad - \theta\ddot{y}h(\dot{y}) - \theta\dot{y}h'(\dot{y})\ddot{y}(s) + i\theta\sqrt{1-\dot{y}^2(s)}\ddot{y}(s) \\
&= 0 \\
&= 2i\theta\ddot{y}\bar{\dot{z}}
\end{aligned}$$

whence it follows that

$$Z = 2i\theta\ddot{y}\bar{\dot{z}} + \frac{d}{ds}(i\kappa y\bar{\dot{z}}) = 2i\theta yK + 2i\kappa Hy. \tag{7.12}$$

As before we have from (7.12) that condition (1.4) is also satisfied at points s where $\dot{x}(s) = 0$ and $\ddot{z}(s)$ exists. This and the preceeding result mean that condition (1.4) is satisfied a.e. over Σ. Thus the theorem is proved ∎

8. Analyticity of the free boundary

Here we will prove the following result.

Theorem 8.1. *Let c_0, κ, θ and $m = (\Psi, \Sigma)$ be as in Theorem 7.1. Then any open arc $\Sigma' \subset \Sigma$ is an analytical curve and the function Ψ is the classical solution of problem* (1.1) – (1.5).

Proof. To start with we prove the infinitely differentiability of the curve Σ'. From Lemma 6.4 it follows that Σ' is a curve of Liapunov type, i.e. it belongs to some class $C^{1,\alpha}$ $(0 < \alpha < 1)$. Really, from (7.7) and (7.10) we get for arbitrary points $s_1, s_2 \in (0, |\Sigma|)$ that

$$\begin{aligned} &|\kappa y(s_1) - 2\theta|\, |\dot{y}(s_1) - \dot{y}(s_2)| \\ &\quad \le \kappa\, |y(s_1) - y(s_2)| + \left[\int_{s_1}^{s_2} \left| -2i\lambda y \bar{z} - 4i \frac{\Psi_z^2}{y} \dot{z} + \frac{\kappa}{i} \right|^{\gamma} (\sigma)\, d\sigma \right]^{\frac{1}{\gamma}} |s_1 - s_2|^{\frac{1}{\gamma'}} \end{aligned} \tag{8.1}$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Now applying the Shauder estimates (see [3: p. 142] and [19: p. 198]) we get the infinitely differentiability of the curve Σ'.

Next, following Garabedian's idea we will consider the function Ψ from (1.2) as solution of a hyperbolic equation in two independent complex variables and (1.2) - (1.4) as problem with initial datas. This point of view proved to be usefull in many cases (see, for example, [3: p. 143] and [10: §8]). Let now $S(z, \bar{z}; \zeta, \bar{\zeta})$ be the fundamental solution of equation (1.2):

$$S(z, \bar{z}; \zeta, \bar{\zeta}) = A(z, \bar{z}; \zeta, \bar{\zeta}) \log(z - \zeta)(\bar{z} - \bar{\zeta}) + B(z, \bar{z}; \zeta, \bar{\zeta})$$

where $A(z, \bar{z}; \zeta, \bar{\zeta})$ is the Riemann function of the same equation written formally as a hyperbolic one in two complex variables z and $\bar{z}$. Let $z_0 \in \Sigma$ be an arbitrary point and σ_0 an arc containing this point, whose size will be discussed later. In usual way we introduce the function

$$F(\zeta) = \int_{\sigma_0} S(z, \zeta; \bar{z}, \bar{\zeta}) \big(-2\kappa H - 2\theta K + 2\lambda \big)^{1/2} \bar{z}\, dz.$$

(see [3: p. 143] and [10: §8]). The functions F and F' have boundary values F^+ and $(F')^+$ over σ_0 and

$$\frac{dF^+}{dt} = (F')^+(t). \tag{8.2}$$

Using condition (8.2) we obtain an integral equation over σ_0 for which $\bar{z}$ is the unique solution. We "extend" this equation as system of nonlinear integral equations to some neighbourhood $B'(z_0, \varepsilon) = B(z_0, \varepsilon) \cap D$ of the point z_0 to get

$$2iV = z - z_0 - \int_{z_0}^{z} U^2(t)\, dt$$

where

$$U = P^{-1}\Bigg\{ \int\limits_{z_0}^{z} W^2 U(t)\,dt + \frac{2\theta}{3i\kappa}\int\limits_{z_0}^{z} U^4 V^{-2}(t)\,dt + \frac{2\theta}{i\kappa}\int\limits_{z_0}^{z} U^2 V^{-2}(t)\,dt + t\int\limits_{z_0}^{z} U^2 V^{-1}(t)\,dt + i\int\limits_{z_0}^{z} U^4 V^{-4}(t)\,dt \Bigg\} \tag{8.3}$$

and

$$W = \frac{(F^+)'}{2\pi i}\big(A(z,z-2iV,z,\bar z_0)\big)^{-1} - \int\limits_{z_0}^{z} \frac{A_t(t,t-2iV,z,\bar z_0)}{(A(z,z-2iV,z,\bar z_0)}W(t)\,dt.$$

Here

$$P = P(\zeta) = \frac{2\theta}{i\kappa y_0}(\zeta - \bar{\dot z}) + \frac{2\theta}{3i\kappa y_0}(\zeta^3 - \bar{\dot z}^3) + (\zeta^2 - \bar{\dot z}^2) \qquad (\zeta \in \mathbb{C})$$

is an analytical function and P^{-1} is its inverse whose existence is guaranteed in the case of $\kappa h - 2\theta > 0$. Using slightly modified Picard method we prove the existence of a fixed point for the operator defined by (8.3) in the space of analytical functions considered in $B'(z_0,\varepsilon)$, for σ_0 sufficiently small (i.e. for $\varepsilon > 0$ sufficiently small). The iterations we need can be defined as follows:

$$\begin{aligned} U_{n+1}U_n &= -\frac{2\theta}{i\kappa}V_n^{-1}(U_n - \bar z_0) - \frac{2\theta}{3i\kappa}V_n^{-1}(U_n^3 - z_0^3) - \frac{2}{i\kappa}\int\limits_{z_0}^{z} W_n^2 U_n\,dt \\ &\quad + \frac{2\theta}{3i\kappa}\int\limits_{z_0}^{z} U_n^4 V_n^{-2}dt + \frac{2\theta}{i\kappa}\int\limits_{z_0}^{z} U_n^2 V_n^{-2}dt + i\int\limits_{z_0}^{z} U_n^2 V_n\,dt + i\int\limits_{z_0}^{z} U_n^4 V_n\,dt + \bar{\dot z}_0^2 \\ W_{n+1}(z) &= \frac{(F^+)'}{2\pi i}\big(A(z,z-2iV_n,z,\bar z_0)\big)^{-1} - \int\limits_{z_0}^{z} \frac{A_t(t,t-2iV_n,z,\bar z_0)}{A(z,z-2iV_n,z,\bar z_0)}W_{n+1}dt \\ 2iV_{n+1} &= z - \bar z_0 - \int\limits_{z_0}^{z} U_n^2(t)\,dt \end{aligned}$$

where $V_0(z) = y_0$ $(z \in \sigma_0)$ and W_0 is the solution of the integral equation

$$W_0 = \frac{(F^+)'}{2\pi i}\big(A(z,z-y_0,z,\bar z_0)\big)^{-1} - \int\limits_{z_0}^{z} \frac{A_t(t,t-y_0,z,\bar z_0)}{A(z,z-y_0,z,\bar z_0)}W_0(t)\,dt.$$

The "trace" of system (8.3) is the integral equation (8.2) on σ_0 which we extended to the neighbourhood of z_0 to get (8.3). This means that the function $\bar{\dot z}$ is the trace of an analytical function defined in $B'(z_0,\varepsilon)$. In this usual way we now receive the analyticity of Σ (see [3: p. 143] and [10: §8]). Thus the theorem is proved ∎

9. Appendix

Here we will give, following the reasoning of V. P. Korovkin, G. V. Secrieru and F. M. Sazhin [16] a justification to condition (1.4)

Let us consider an isolated system consisting of two sybsystems in two different coexistent faces L and V, separated by a surface layer M. Let us denote by P_L and P_V the pressures inside of the L-face and V-face and by σ the coefficient of surface tension characterizing the surface forces acting in the layer M. We will consider the process of face transition which takes place along the surface layer which can be characterized approximately by a surface S, given by the function $\vec{\rho} = \vec{f}(u,v)$. This process is characterized by absorbtion velocities $\vec{\omega}_1$ and $\vec{\omega}_2$, which can be considered as dislocations of the L- and V-boundaries, respectively. The process leads to a distortion of the layer which can be described by a function $\vec{r} = \vec{r}(u,v)$,

$$\vec{r}(u,v) = \vec{f}(u,v) + \vec{n}\int_0^T |\vec{\omega}_a|\, dt.$$

Here $\vec{n}$ is the unit normal vector to the surface S, $\vec{\omega}_a = \vec{\omega}_1 + \vec{\omega}_r$ and we suppose that $\vec{\omega}_1$ and $\vec{\omega}_r$ are oriented along the normals to S. Let us denote by dA_1 the area of a surface element of S and by dA_2 its distortion during the absorbtion process and by $dV_{V,L}$ the variation of one of the faces. Then from the second law of thermodynamics we can easily get the equilibrium condition

$$P_L - P_V = \pm\sigma\,\frac{d(A_2 - A_1)}{dV_{V,L}}. \tag{A.1}$$

Now

$$dA_2 = |\vec{r}_u \times \vec{r}_v|\, du dv$$

and using the Olinde-Rodrigues theorem we obtain

$$d\left(\vec{n}\int_0^T |\vec{\omega}_a|\, dt\right) = \vec{n} d\left(\int_0^T |\vec{\omega}_a|\, dt\right) - k\left(\int_0^T |\vec{\omega}_a|\, dt\right) d\vec{r},$$

where k is the principal curvature of the surface in the principal $d\vec{r}$-direction (see [5: p. 145] and [21: §7.3]). This gives us for the variation $d(A_1 - A_2)$ the expression

$$d(A_1 - A_2) = dA_2\left(2H + K\int_0^T |\vec{\omega}_a|\, dt\right)\left(\int_0^T |\vec{\omega}_a|\, dt\right) \tag{A.2}$$

where H is the mean curvarure and K the Gaussian curvature of the surface. Now the variation of the face volume $dV_{V,L}$ can be written in the form

$$dV_{V,L} = dA_2 \int_0^T |\vec{\omega}_a|\, dt. \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1) we will have

$$P_L - P_V = 2\sigma H + l_p \sigma K \tag{A.4}$$

where

$$l_p = \lim_{\tau \to \infty} \int_0^\tau |\vec{\omega}_a|\, dt$$

is the equilibrium width of the surface layer. When l_p can be neglected, we get the classical equilibrium condition of Laplace. In the general case this term can not be dropped, for example, when l_p is comparable with one of the radii of the surface curvature. The term $l_p\sigma K$ can be considered as an attempt to take into account wedging forces which arise due to non-homogeneous distributions of surface forces across the surface layer.

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