

Balls in Constrained Urns and Cantor-Like Sets

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Abstract. Let $A_n(k)$ denote the number of different ways to distribute k indistinguishable balls into n constrained urns, with capacities c_1, \dots, c_n . We consider the *normalized counting functions* $\varphi_n(x) = \gamma_n A_n([\varrho_n x])$, where $\varphi_n, \varrho_n > 0$ are appropriate constants such that $\text{supp}(\varphi_n) = [0, 1]$ and $\int_0^1 \varphi_n(x) dx = 1$. It is shown here that, if $(c_n)_{n \in \mathbb{N}}$ is *asymptotically geometric with weight* $q > \frac{3}{2}$, i.e. if $q^{-n} c_n$ converges to some positive real number, then the functions φ_n converge to some C^∞ -function φ on \mathbb{R} . This function φ is the unique solution of the integral equation $\varphi(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \varphi(t) dt$ satisfying $\text{supp} \varphi \subset [0, 1]$ and $\int_0^1 \varphi(t) dt = 1$. Moreover, if $q > 2$, it is shown that φ is a polynomial on each interval outside a Cantor-like set in the interval $[0, 1]$.

Keywords: *Balls in constrained urns, special partitions of integers, asymptotically geometric sequences, sequences of integral operators, integral-functional equations, Cantor-like sets*

AMS subject classification: 47 G 10, 05 A 17, 45 D 05

0. Introduction

Suppose we are given k indistinguishable balls and n urns U_1, \dots, U_n . Suppose in addition that the capacity of each urn U_j is given by a non-negative integer c_j , for $j = 1, \dots, n$. Then the different ways to distribute the k balls into the n urns, subject to the constraint that each urn U_j cannot contain more than c_j balls, are represented by the set of partitions

$$\mathcal{P}_{(c_1, \dots, c_n)}(k) = \left\{ (k_1, \dots, k_n) \in \mathbb{Z} \left| \begin{array}{l} 0 \leq k_j \leq c_j \text{ for each } j \\ \text{and } k_1 + \dots + k_n = k \end{array} \right. \right\} \quad (0.1)$$

Let

$$A_n(k) = A_{(c_1, \dots, c_n)}(k) := \#\mathcal{P}_{(c_1, \dots, c_n)}(k). \quad (0.2)$$

Computer evaluations (see Figures 1 - 4 overleaf) of $A_n(k)$ for some special capacity sequences (c_1, \dots, c_n) make it plausible that the functions A_n converge 'in shape' (that is, after an appropriate normalization), as $n \rightarrow \infty$, provided the sequence of capacities (c_1, \dots, c_n) is continued appropriately. In order to normalize A_n to support $[0, 1]$ and total integral one, choose coefficients $\gamma_n, \varrho_n > 0$ such that

$$\varphi_n(x) := \gamma_n A_n([\varrho_n x]) \quad \text{fulfill } \text{supp}(\varphi_n) = [0, 1] \text{ and } \int_0^1 \varphi_n(x) dx = 1. \quad (0.3)$$

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(For a real number r , $[r]$ denotes its Gauß bracket, i.e. the greatest integer lesser or equal r .) The condition which we impose on the sequence of capacities (c_n) is that it should be *asymptotically geometric with weight q* , that means, the sequence $(q^{-n}c_n)_{n \in \mathbb{N}}$ should converge to some positive real number. The motivation for these investigations comes from my work on the $3n + 1$ problem (see [6, 7]). Averages of certain $3n + 1$ predecessor counting functions are given by $A_n(k)$, with the sequence of capacities defined by $c_n = 2 \cdot 3^{n-1} - 1$ for $n \geq 1$. This sequence (c_n) is asymptotically geometric with weight $q = 3$.

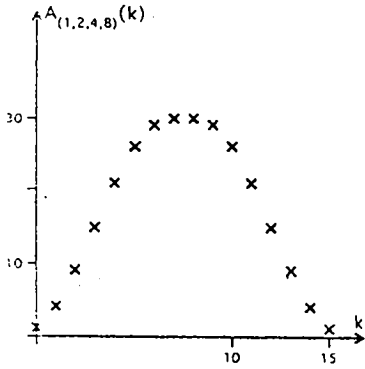


Figure 1

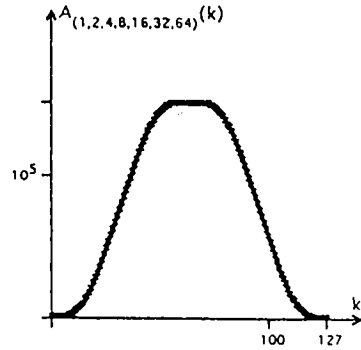


Figure 2

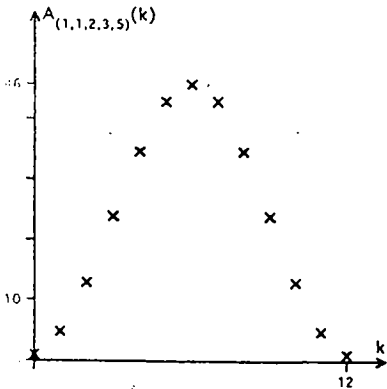


Figure 3

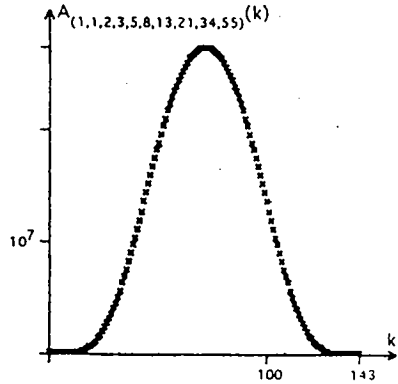


Figure 4

Here we show that, if (c_n) is asymptotically geometric with weight $q > \frac{3}{2}$, then the functions φ_n converge uniformly on $[0, 1]$ to the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ uniquely determined by

$$\varphi(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \varphi(\xi) d\xi, \quad \text{supp } \varphi \subset [0, 1], \quad \int_0^1 \varphi(t) dt = 1. \quad (0.4)$$

The basic idea is the following. Let S denote the operator on $L^1([0, 1])$ defined by

$$(Sf)(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \varphi(\xi) d\xi.$$

Berg and Krüppel [2] showed that, for any fixed $q > 1$, there is a unique function φ satisfying (0.4). Moreover, they proved that, for any starting function $f_0 \in L^1([0, 1])$ with $\int_0^1 f_0(x) dx = 1$, the sequence of iterates $f_n = Sf_{n-1}$ converges uniformly on $[0, 1]$ to φ .

Provided the sequence (c_n) is asymptotically geometric with weight $q > \frac{3}{2}$, we prove that the functions φ_n defined in (0.3) satisfy $\varphi_n = S_n \varphi_{n-1}$ where (S_n) is a sequence of operators on $L^1([0, 1])$ which converges in some sense to S . Then the proof of convergence of (φ_n) is completed by exploiting the fact that S is a contraction on an appropriate subspace of $L^1([0, 1])$ when $q > \frac{3}{2}$.

It remains open whether or not (φ_n) converges if (c_n) is assumed to be asymptotically geometric with weight satisfying $1 < q \leq \frac{3}{2}$. In view of the proofs given here, it is likely that, in order to ensure convergence of (φ_n) , one has to impose additional conditions on (c_n) concerning the speed of convergence of $(c_n q^{-n})$. In this context, it is worth mentioning that in the case $q = 1$, i.e. if the sequence (c_n) is eventually constant, one should expect convergence of the φ_n in the distributional sense to Dirac's distribution $\delta_{\frac{1}{2}}$ which is defined by $\delta_{\frac{1}{2}}(u) = u(\frac{1}{2})$ for any test function $u : [0, 1] \rightarrow \mathbb{R}$. This conjecture is based on the observation that it is true for the constant sequence $c_n = 1$: in this case we have $A_n(k) = \binom{n}{k}$, and convergence to $\delta_{\frac{1}{2}}$ in the distributional sense of the sequence of normalized counting functions

$$\varphi_n(x) = \frac{n+1}{2^n} \binom{n}{[(n+1)x]} \quad \text{for } 0 \leq x \leq 1$$

follows from Bernstein's proof of the Weierstraß approximation theorem [3]. In fact, Bernstein showed that, for any continuous function f on $[0, 1]$, the Bernstein polynomials

$$B_n^f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

converge uniformly on $[0, 1]$ to f . Inserting $x = \frac{1}{2}$ and interpreting $B_n^f(\frac{1}{2})$ as a Riemann sum for an integral, one infers

$$\int_0^1 f(\xi) \varphi_n(\xi) d\xi \approx B_n^f\left(\frac{1}{2}\right) \rightarrow f\left(\frac{1}{2}\right) = \delta_{\frac{1}{2}}(f)$$

as $n \rightarrow \infty$, for any continuous function f . As test-functions are continuous, this gives convergence in the distributional sense.

The plan of the present paper is as follows. The essential formula on the functions A_n is given in Section 1, it computes A_n using A_{n-1} and c_n . This formula is interpreted as an integral equation connecting φ_n to φ_{n-1} in Section 2, where we also explore the limiting behaviour of the coefficients occurring in that integral equation, provided the sequence (c_n) is asymptotically geometric. Section 3 gives a proof of convergence based on that integral equation. Finally, in Section 4 we describe a remarkable property of the limiting function φ which connects it to Cantor-like sets.

1. Combinatorics

For each $n \in \mathbb{N}$, denote by $s_n = c_1 + \dots + c_n$ the total capacity of the first n urns U_1, \dots, U_n . To be able to handle the empty case $n = 0$, we set

$$s_0 = 0 \quad \text{and} \quad A_0(k) = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}$$

In general, we see that

$$A_n(k) > 0 \iff 0 \leq k \leq s_n. \tag{1.1}$$

There is also the symmetry property

$$A_n(k) = A_n(s_n - k) \quad \text{for } k \in \mathbb{Z} \tag{1.2}$$

which follows from the bijection

$$\mathcal{P}_n(k) \rightarrow \mathcal{P}_n(s_n - k), \quad (k_1, \dots, k_n) \mapsto (c_1 - k_1, \dots, c_n - k_n).$$

Since we have $A_n(k) = \binom{n}{k}$ if $c_1 = \dots = c_n = 1$, the following lemma generalizes the production rule of Pascal's triangle.

Lemma 1.1. *There is the following reproduction formula for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$:*

$$A_n(k) = \sum_{j=k-c_n}^k A_{n-1}(j) = \sum_{j=\max\{0, k-c_n\}}^{\min\{k, s_{n-1}\}} A_{n-1}(j).$$

Proof. It is possible to exhibit the set $\mathcal{P}_n(k)$ in the following way as a disjoint union:

$$\mathcal{P}_n(k) = \bigcup_{j=k-c_n}^k \left\{ (k_1, \dots, k_n) \mid \begin{array}{l} 0 \leq k_i \leq c_i \text{ for } i = 1, \dots, n \\ k_1 + \dots + k_{n-1} = j, \quad k_n = k - j \end{array} \right\}$$

where the sets on the right-hand side admit bijective maps to $\mathcal{P}_{n-1}(j)$ ■

We compute some values of $A_n(k)$ using this lemma. First,

$$A_n(0) = 1 \quad \text{for each } n \in \mathbb{N}_0. \tag{1.3}$$

Second, using the set calculation

$$\bigcup_{k=0}^{s_n} \mathcal{P}_n(k) = \left\{ (k_1, \dots, k_n) : 0 \leq k_i \leq c_i \text{ for } i = 1, \dots, n \right\}$$

one arrives at once at the formula

$$\sum_{k=0}^{s_n} A_n(k) = \prod_{i=1}^n (c_i + 1). \tag{1.4}$$

Lemma 1.1 combined with (1.4) gives a plateau value of A_n , provided the sequence of capacities increases so fast that $c_n \geq s_{n-1}$:

$$\max_{\ell \in \mathbb{Z}} A_n(\ell) = \sum_{j=0}^{s_{n-1}} A_{n-1}(j) = \prod_{i=1}^{n-1} (c_i + 1) = A_n(k) \quad \text{if } s_{n-1} \leq k \leq c_n. \tag{1.5}$$

Another consequence of Lemma 1.1 is the following monotonicity property of the functions A_n .

Lemma 1.2. *For any $n \in \mathbb{N}$, the function A_n fulfills*

$$A_n(k) < A_n(\ell) \quad \text{whenever} \quad -1 \leq k < \ell \leq \min \{s_{n-1}, \frac{1}{2}s_n\}. \tag{1.6}$$

Proof. We proceed by induction on n .

If $n = 1$, we have $\min \{s_{n-1}, \frac{1}{2}s_n\} = 0$, and (1.6) is only to be checked for $k = -1$ and $\ell = 0$, whence it is true by (1.1) and (1.3).

For the step $n \rightarrow n + 1$, first note that by the definition of A_n in (0.2) and (0.1), A_{n+1} does not depend on the arrangement of the c_j . Therefore we may assume

$$c_1 \leq \dots \leq c_{n+1}. \tag{1.7}$$

Now observe that it suffices to prove

$$A_{n+1}(k) < A_{n+1}(k+1) \quad \text{if} \quad 0 \leq k+1 \leq \min \{s_n, \frac{1}{2}s_{n+1}\}. \tag{1.8}$$

Now Lemma 1.1 implies that

$$A_{n+1}(k) < A_{n+1}(k+1) \iff A_n(k - c_{n+1}) < A_n(k+1).$$

If $k < c_{n+1}$, we have $A_n(k - c_{n+1}) = 0$ by (1.1). On the other hand, the condition on k in (1.8) gives $0 \leq k+1 \leq s_n$, and we obtain $A_n(k+1) > 0$ again by (1.1). If $k \geq c_{n+1}$, we use the induction hypothesis

$$A_n(\ell_1) < A_n(\ell_2) \quad \text{if} \quad -1 \leq \ell_1 < \ell_2 \leq \min \{s_{n-1}, \frac{1}{2}s_n\}.$$

We know by (1.2) that $A_n(\ell_2) = A_n(s_n - \ell_2)$, hence

$$A_n(\ell_1) < A_n(\ell_2) \quad \text{if} \quad -1 \leq \ell_1 < \min \{\ell_2, s_n - \ell_2\} \leq \min \{s_{n-1}, \frac{1}{2}s_n\}. \tag{1.9}$$

Clearly, we have

$$\min \{\ell_2, s_n - \ell_2, s_{n-1}, \frac{1}{2}s_n\} = \min \{\ell_2, s_n - \ell_2, s_{n-1}\}.$$

Assume that both $\ell_2 > s_{n-1}$ and $s_n - \ell_2 > s_{n-1}$. Then the definition of s_n gives $s_{n-1} < \ell_2 < c_n$ and $s_{n-1} < \frac{1}{2}s_n$. Hence (1.5) implies $A_n(\ell_2) = A_n(s_{n-1})$ in this case, and we can rewrite (1.9) to

$$A_n(\ell_1) < A_n(\ell_2) \quad \text{whenever} \quad -1 \leq \ell_1 < \min \{\ell_2, s_n - \ell_2, s_{n-1}\}.$$

Now put $\ell_1 = k - c_{n+1} \geq 0$ and $\ell_2 = k + 1$. It remains to check $\ell_1 < \min \{\ell_2, s_n - \ell_2, s_{n-1}\}$. Clearly, $\ell_1 < \ell_2$ and

$$\ell_1 < s_n - \ell_2 \iff k - c_{n+1} < s_n - (k + 1) \iff 2k + 1 < s_{n+1}$$

which is true since the condition on k in (1.8) implies $k + 1 \leq \frac{1}{2}s_{n+1}$. Moreover, that condition combines with (1.7) to imply

$$k + 1 \leq s_n = s_{n-1} + c_n \leq s_{n-1} + c_{n+1}$$

which in turn gives $\ell_1 = k - c_{n+1} < s_{n-1}$, and the proof is completed ■

Corollary 1.3. For each $n \in \mathbb{N}$, $A_n(k)$ is non-decreasing for $-1 \leq k \leq \frac{1}{2}s_n$, and non-increasing for $\frac{1}{2}s_n \leq k \leq s_n + 1$.

Proof. Combine (1.5) with Lemma 1.2 for the first claim. The second claim now follows from (1.2) ■

Finally, there is the following more subtle symmetry property.

Lemma 1.4. If $c_n \geq s_{n-1}$, we have

$$A_n(k - 1) + A_n(s_{n-1} - k) = A_n(s_{n-1}) \quad \text{for } 0 \leq k \leq s_{n-1} + 1.$$

Proof. Using the sets

$$\begin{aligned} S_+(k) &= \{(k_1, \dots, k_n) \in \mathcal{P}_n(s_{n-1}) : k_n \geq k\} \\ S_-(k) &= \{(k_1, \dots, k_n) \in \mathcal{P}_n(s_{n-1}) : k_n \leq k - 1\} \end{aligned}$$

we have a splitting of the set defining $A_n(k)$: $\mathcal{P}_n(k) = S_+(k) \cup S_-(k)$. Now there is a bijection

$$S_-(k) \rightarrow \mathcal{P}_n(k - 1), \quad (k_1, \dots, k_n) \mapsto (c_1 - k_1, \dots, c_{n-1} - k_{n-1}, k - 1 - k_n),$$

which is well-defined since

$$(c_1 - k_1) + \dots + (c_{n-1} - k_{n-1}) + (k - 1 - k_n) = k - 1.$$

In addition, there is another bijection

$$S_+(k) \rightarrow \mathcal{P}_n(s_{n-1} - k), \quad (k_1, \dots, k_n) \mapsto (k_1, \dots, k_{n-1}, k_n - k)$$

which is surjective because $s_{n-1} \leq c_n$ ■

2. Normalization and limiting

Having collected some information on the counting functions A_n in the last section, we now describe the crucial normalization procedure. We do this by first seeking constants γ_n and ϱ_n such that each of the “normalized counting functions” $\varphi_n(x) = \gamma_n A_n([\varrho_n x])$ is supported by the unit interval $[0, 1]$ and has total integral 1. Then the next task is to transfer the basic Lemma 1.1 to the new setting and to investigate convergence properties of the quantities emerging during the normalization process.

By (1.1) we have

$$\varrho_n = s_n + 1 = c_1 + \dots + c_n + 1. \tag{2.1}$$

To compute γ_n , we use (1.4) to obtain

$$\frac{1}{\gamma_n} = \int_0^1 A_n([\varrho_n x]) dx = \sum_{k=0}^{s_n} \frac{A_n(k)}{\varrho_n} = \frac{(c_1 + 1) \cdots (c_n + 1)}{s_n + 1}.$$

Hence the functions defined by

$$\varphi_n(x) := \gamma_n A_n([\varrho_n x]) = \frac{s_n + 1}{(c_1 + 1) \cdots (c_n + 1)} A_n([(s_n + 1)x]) \tag{2.2}$$

have the required properties $\text{supp } \varphi_n = [0, 1]$ and $\int_0^1 \varphi_n(x) dx = 1$. We call φ_n the *n-th normalized counting function* associated to the sequence $(c_n)_{n \in \mathbb{N}}$.

Clearly, φ_n inherits the properties of $A_n(k)$. For example, (1.2) implies for $n \geq 0$

$$\varphi_n(x) = \varphi_n(1 - x) \quad \text{if } \varrho_n x \notin \mathbb{Z}. \tag{2.3}$$

The counterpart of Lemma 1.1 is the following reproduction formula.

Lemma 2.1. *For each $n \in \mathbb{N}$, we have*

$$\varphi_n(x) = \tau_n \int_{\alpha_n(x)}^{\omega_n(x)} \varphi_{n-1}(\xi) d\xi = \tau_n \int_{I_n(x)} \varphi_{n-1}(\xi) d\xi$$

with the notation

$$\tau_n = \frac{s_n + 1}{c_n + 1}, \quad \alpha_n(x) = \frac{[\varrho_n x] - c_n}{\varrho_{n-1}}, \quad \omega_n(x) = \frac{[\varrho_n x] + 1}{\varrho_{n-1}}$$

and $I_n(x) = [\alpha_n(x), \omega_n(x)] \cap [0, 1]$.

Proof. Putting $k = [\varrho_n x]$ and using Lemma 1.1, we get

$$\begin{aligned} \varphi_n(x) &= \gamma_n A_n(k) \\ &= \gamma_n \sum_{j=k-c_n}^k A_{n-1}(j) \\ &= \frac{\gamma_n}{\gamma_{n-1}} \sum_{j=-c_n}^0 \gamma_{n-1} A_{n-1}(k+j) \\ &= \frac{\varrho_n}{\varrho_{n-1}(c_n + 1)} \sum_{j=-c_n}^0 \varphi_{n-1}\left(\frac{k+j}{\varrho_{n-1}}\right). \end{aligned}$$

Since φ_{n-1} is a constant on each of the intervals $[\frac{k+j}{\varrho_{n-1}}, \frac{k+j+1}{\varrho_{n-1}})$, we have

$$\frac{1}{\varrho_{n-1}} \varphi_{n-1}\left(\frac{k+j}{\varrho_{n-1}}\right) = \int_{\frac{k+j}{\varrho_{n-1}}}^{\frac{k+j+1}{\varrho_{n-1}}} \varphi_{n-1}(\xi) d\xi$$

which leads to the first equality. The second one is due to the fact that φ_{n-1} is supported by $[0, 1]$ ■

The following reformulation of Corollary 1.3 is necessary to conclude uniform convergence of the sequence (φ_n) from its L^1 -convergence (see the remark at the end of Section 3).

Lemma 2.2. *Each φ_n is non-decreasing on $[0, \frac{1}{2}]$ and non-increasing on $[\frac{1}{2}, 1]$.*

The second symmetry property, Lemma 1.4, is transformed into the following

Lemma 2.3. *Suppose $c_n \geq s_{n-1}$. With the notation $\sigma_n = \frac{s_{n-1}}{\varrho_n}$, we have*

$$\varphi_n(x) + \varphi_n(\sigma_n - x) = \varphi_n(\sigma_n) \quad \text{if } 0 < x < \sigma_n, \varrho_n x \notin \mathbb{Z}.$$

Proof. Put $k = [\varrho_n x] + 1$. Then we have if $\varrho_n x \notin \mathbb{Z}$

$$[\varrho_n(\sigma_n - x)] = [s_{n-1} - \varrho_n x] = s_{n-1} - k.$$

In addition, $1 \leq k \leq s_{n-1}$ is a consequence of $0 < x < \sigma_n$. Now Lemma 1.4 gives

$$\begin{aligned} \varphi_n(x) + \varphi_n(\sigma_n - x) &= \gamma_n A_n([\varrho_n x]) + \gamma_n A_n([\varrho_n(\sigma_n - x)]) \\ &= \gamma_n (A_n(k - 1) + A_n(s_{n-1} - k)) \\ &= \gamma_n A_n(s_{n-1}) \\ &= \varphi_n(\sigma_n). \end{aligned}$$

and the lemma is proved ■

We want to know whether or not the normalized counting functions φ_n defined in (2.2) converge to a limiting function as $n \rightarrow \infty$. This limiting function would have to fulfill a formula which is a “limit” of the formula given in Lemma 2.1. In order to get a feeling for such a “limiting formula”, we now start to explore the limiting behaviour of the coefficients $\alpha_n(x)$, $\omega_n(x)$ and τ_n occurring in Lemma 2.1. Our next aim is to give a sufficient condition on the sequence of capacities (c_n) to insure convergence of the sequences defined by those coefficients.

A sequence $(a_n)_{n \in \mathbb{N}}$ will be called *asymptotically geometric with weight $q > 0$* if the weighted sequence $(q^{-n} a_n)_{n \in \mathbb{N}}$ converges to a positive real number.

Lemma 2.4. *Suppose that the sequence $(c_n)_{n \in \mathbb{N}}$ is asymptotically geometric with weight $q > 1$ and satisfies $c_n > 0$ for each n , and use the notation $s_n = c_1 + \dots + c_n$. Then*

$$\lim_{n \rightarrow \infty} \frac{s_n}{c_n} = \frac{q}{q - 1}.$$

This lemma is a special case of [1: S. 47, Satz 6.5]. For convenience, we include a proof which is based on the following simple observation.

Lemma 2.5. *Let $(\alpha_{nk})_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ be a double sequence of real numbers, and let c and β be reals with $c > 0$ and $0 < \beta < 1$. If*

- (i) $|\alpha_{nk}| \leq c \cdot \beta^k$ for each $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and
- (ii) $\lim_{n \rightarrow \infty} \alpha_{nk} = \beta^k$ for each fixed k ,

then we have

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \alpha_{nk} = \frac{1}{1 - \beta}.$$

Proof. The right-hand side of

$$\left| \sum_{k=0}^{\infty} \beta^k - \sum_{k=0}^{n-1} \alpha_{nk} \right| \leq \sum_{k=N}^{\infty} \beta^k + \sum_{k=N}^{n-1} |\alpha_{nk}| + \sum_{k=0}^{N-1} |\beta^k - \alpha_{nk}|$$

can be made arbitrarily small if first N and then n are chosen sufficiently large ■

Proof of Lemma 2.4. Put $\alpha_{nk} = \frac{c_{n-k}}{c_n}$ and $\beta = q^{-1}$. Because $(c_n)_{n \in \mathbb{N}}$ is supposed to be asymptotically geometric, and because $c_n > 0$, there are constants A and B such that

$$Aq^n \leq c_n \leq Bq^n \quad \text{for each } n \in \mathbb{N}. \tag{2.4}$$

From this we infer condition (i) of the preceding lemma:

$$|\alpha_{nk}| = \frac{c_{n-k}}{c_n} \leq \frac{Bq^{n-k}}{Aq^n} = \frac{B}{A} \cdot \beta^k.$$

A consequence of the convergence of $(q^{-n}c_n)_{n \in \mathbb{N}}$ is the convergence of consecutive quotients:

$$\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = q^{-1}. \tag{2.5}$$

This implies condition (ii):

$$\alpha_{nk} = \frac{c_{n-k}}{c_n} = \prod_{j=1}^k \frac{c_{n-j}}{c_{n-j+1}} \rightarrow q^{-k} = \beta^k \quad \text{as } n \rightarrow \infty.$$

Applying Lemma 2.5 gives the claim

$$\lim_{n \rightarrow \infty} \frac{s_n}{c_n} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \alpha_{nk} = \frac{1}{1 - \beta} = \frac{q}{q - 1}$$

and the lemma is proved ■

Now it is easy to compute the limits of the coefficients of Lemma 2.1.

Lemma 2.6. Suppose $(c_n)_{n \in \mathbb{N}}$ is asymptotically geometric with weight $q > 1$ and $c_n > 0$ for each n . Then, with the notation of Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \tau_n = \frac{q}{q - 1}, \quad \lim_{n \rightarrow \infty} \alpha_n(x) = qx - q + 1, \quad \lim_{n \rightarrow \infty} \omega_n(x) = qx.$$

Proof. These formulae are computational consequences of Lemma 2.4 ■

In addition, the limit of the sequence $(\sigma_n)_{n \in \mathbb{N}}$ defined in Lemma 2.3 is easily calculated:

$$\lim_{n \rightarrow \infty} \sigma_n = \left(1 + \lim_{n \rightarrow \infty} \frac{c_n + 1}{s_{n-1}} \right)^{-1} = \frac{1}{q}. \tag{2.6}$$

Remark. An inspection of the proof of Lemma 2.4 shows that the condition on the sequence (c_n) to be asymptotically geometric is only needed through its consequences (2.4) and (2.5). Hence, in order to get the conclusion of Lemma 2.6, it suffices to postulate as conditions on (c_n) only (2.4) and (2.5). The combination of these two conditions is strictly weaker than the assumption that (c_n) should be asymptotically geometric. The following example is due to an anonymous referee: Put $c_n = 2 + \sin(\ln n)$. Then (c_n) satisfies (2.4) and (2.5), but is not asymptotically geometric.

3. L^1 -convergence

Recall the normalized counting functions φ_n associated to a given capacity sequence $(c_n)_{n \in \mathbb{N}}$ which are defined in (2.2). In order to investigate their limiting behaviour, we take the formula given in Lemma 2.1 as definition of a sequence of linear operators $(S_n)_{n \in \mathbb{N}}$ on $L^1([0, 1])$ satisfying

$$\varphi_n = S_n \varphi_{n-1}. \tag{3.1}$$

It is possible to show for the operators S_n defined in (3.2) below that they are integral preserving. This means that we can view the φ_n as densities of probability measures and the S_n as transition probabilities of a non-homogeneous Markov process with discrete time parameter n and state space $[0, 1]$.

Under the condition that the previously fixed capacity sequence (c_n) is asymptotically geometric with some weight q , we shall show in this section that the sequence (S_n) converges to some limiting transition probability S , interpreted again as an integral preserving linear operator on $L^1([0, 1])$. In addition, we prove that S is a contraction on an appropriate subspace if $q > \frac{3}{2}$. From this we conclude that the sequence (φ_n) converges to some uniquely determined fixed point of S .

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of capacities $c_n \in \mathbb{N}$, asymptotically geometric with weight $q > 1$, which we assume to be fixed throughout this section. Having in mind Lemma 2.1, we define the n -th successor operator by

$$S_n f(x) = \tau_n \int_{I_n(x)} f(\xi) d\xi \quad \text{for } f \in L^1([0, 1]). \tag{3.2}$$

According to Lemma 2.6, the candidate for the limiting operator of the sequence (S_n) is given by

$$S f(x) = \tau \int_{I(x)} f(\xi) d\xi \quad \text{for } f \in L^1([0, 1]) \tag{3.3}$$

with the limiting ingredients

$$\tau = \lim_{n \rightarrow \infty} \tau_n = \frac{q}{q - 1} \tag{3.4}$$

$$I(x) = \left[\lim_{n \rightarrow \infty} \alpha_n(x), \lim_{n \rightarrow \infty} \omega_n(x) \right] \cap [0, 1] = [qx - q + 1, qx] \cap [0, 1]. \tag{3.5}$$

This is indeed a good choice:

Lemma 3.1. *For each $f \in L^1([0, 1])$, the sequence $(S_n f)_{n \in \mathbb{N}}$ converges in L^1 to Sf . In addition, this L^1 -convergence is uniform on sets of uniformly essentially bounded L^1 -functions.*

Proof. Denote by $\|f\|_\infty$ the essential supremum of $|f|$, and put

$$h_n(x) = \int_0^1 |\tau_n \chi_{I_n(x)}(\xi) - \tau \chi_{I(x)}(\xi)| d\xi \quad \text{for } x \in [0, 1].$$

Then we have

$$\|S_n f - S f\|_1 = \int_0^1 \left| \tau_n \int_{I_n(x)} f(\xi) d\xi - \tau \int_{I(x)} f(\xi) d\xi \right| dx \leq \|f\|_\infty \int_0^1 h_n(x) dx.$$

It remains to estimate $h_n(x)$:

$$\begin{aligned} 0 \leq h_n(x) &\leq \int_0^1 \left(\tau_n |\chi_{I_n(x)}(\xi) - \chi_{I(x)}(\xi)| + |\tau_n - \tau| \chi_{I(x)}(\xi) \right) d\xi \\ &\leq \tau_n \lambda(I_n(x) \Delta I(x)) + |\tau_n - \tau| \lambda(I(x)) \end{aligned}$$

where $\lambda(A)$ denotes the Lebesgue measure of a set A and

$$I_n(x) \Delta I(x) = (I_n(x) \setminus I(x)) \cup (I(x) \setminus I_n(x))$$

is the symmetric difference of the two intervals. We have that

$$\lambda(I_n(x) \Delta I(x)) \leq |\alpha_n(x) - qx + q - 1| + |\omega_n(x) - qx|$$

and Lemma 2.6 implies that the right-hand side tends to zero as $n \rightarrow \infty$. Employing also the convergence $\tau_n \rightarrow \tau$ proved in that lemma, we conclude that $h_n \rightarrow 0$ pointwise on the interval $[0, 1]$. Using Lebesgue's dominated convergence theorem, we infer that

$$\lim_{n \rightarrow \infty} \int_0^1 h_n(x) dx = 0.$$

Hence $\lim_{n \rightarrow \infty} \|S_n f - S f\|_1 = 0$, the limit being uniform on sets where $\|f\|_\infty \leq B$, for any previously fixed bound $B > 0$ ■

It is possible to visualize the situation by drawing the regions $\{\xi \in I_n(x)\}$ and $\{\xi \in I(x)\}$ in (ξ, x) -diagrams. We have done this in Figures 5 and 6 for the capacity

sequence $(c_1, \dots, c_4) = (1, 2, 4, 8)$.

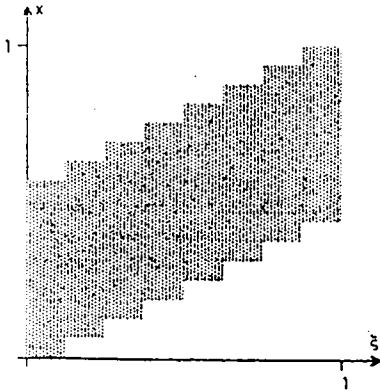


Figure 5

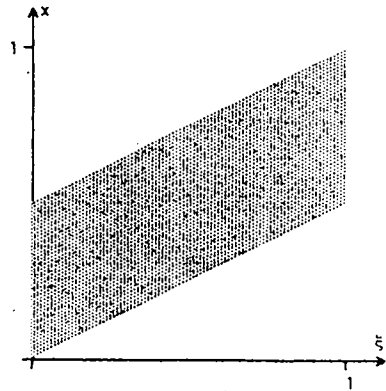


Figure 6

Now we can start to prove L^1 -convergence of the sequence $(\varphi_n)_{n \in \mathbb{N}}$, provided $q > \frac{3}{2}$. Besides the preceding Lemma 3.1, there is another fact crucial to this proof: namely, that the limiting operator S is a contraction on certain subspaces of $L^1([0, 1])$.

We use the notation

$$\mathcal{M}^0 = \left\{ f \in L^1([0, 1]) : \int_0^1 f(x) dx = 0 \right\} \tag{3.6}$$

for the L^1 -functions of vanishing total integral, and

$$\mathcal{M}_{\text{sym}}^0 = \left\{ f \in \mathcal{M}^0 : f(x) = f(1-x) \text{ a.e.} \right\} \tag{3.7}$$

for the symmetric L^1 -functions of vanishing total integral.

Let us first show that these two subspaces are left invariant by S .

Lemma 3.2. *Suppose $q > 1$. Then the operator S is integral-preserving:*

$$\int_0^1 (Sf)(x) dx = \int_0^1 f(x) dx \quad \text{for each } f \in L^1([0, 1]).$$

Proof. We have

$$\int_0^1 (Sf)(x) dx = \int_0^1 \tau \int_{I(x)} f(\xi) d\xi dx = \tau \int_0^1 \sum_{k=0}^{m(x, \lambda)} \int_{I(x+k\lambda)} f(\xi) d\xi dx \tag{3.8}$$

whenever $0 < \lambda < 1$, the sum being extended over all non-negative integers k such that $x + k\lambda \leq 1$, i.e.

$$0 \leq k \leq m(x, \lambda) := \left\lfloor \frac{1-x}{\lambda} \right\rfloor.$$

The device is to choose λ such that the intervals $I(x + k\lambda)$ ($0 \leq k \leq m(x, \lambda)$) form a partition of the unit interval. We want to have a formula

$$[0, 1] = \bigcup_{k=0}^{m(x, \lambda)} I(x + k\lambda) \tag{3.9}$$

such that, for each k , the endpoint of $I(x + k\lambda)$ should either be equal to 1 or match the starting point of $I(x + (k + 1)\lambda)$. By (3.5), this amounts to the equations

$$q(x + k\lambda) = q(x + (k + 1)\lambda) - q + 1 \quad \text{for } 0 \leq k < m(x, \lambda)$$

which give $\lambda = \frac{q-1}{q} = \frac{1}{r}$ by (3.4). Inserting this into (3.8), we obtain the desired equation

$$\int_0^1 (Sf)(x) dx = \tau \int_0^{\frac{q-1}{q}} \int_0^1 f(\xi) d\xi dx = \int_0^1 f(x) dx$$

and the lemma is proved ■

Lemma 3.3. *Let $f \in L^1([0, 1])$. If $f(x) = f(1-x)$ a.e., then $(Sf)(x) = (Sf)(1-x)$ a.e.*

Proof. Use (3.5) to verify the relation $\xi \in I(1-x) \iff 1-\xi \in I(x)$ ■

The following lemma improves upon the remark in Section 1 of Berg and Krüppel [2].

Lemma 3.4. *Let \mathcal{M}^0 and $\mathcal{M}_{\text{sym}}^0$ be given by (3.6) and (3.7), respectively.*

(1) *Suppose $q \geq 2$. Then $\|Sf\|_1 \leq \frac{1}{q-1} \|f\|_1$ for $f \in \mathcal{M}^0$.*

(2) *Suppose $q \geq \frac{3}{2}$. Then $\|Sf\|_1 \leq \frac{1}{2(q-1)} \|f\|_1$ for $f \in \mathcal{M}_{\text{sym}}^0$.*

Proof. Item (1): By the definition of $I(x)$ in (3.5), we see that $I(x) = [0, 1]$ if $\frac{1}{q} \leq x \leq \frac{q-1}{q}$. Using in addition the partition (3.9) with $\lambda = \frac{q-1}{q}$ and the assumption that the total integral of f vanishes, we calculate

$$\begin{aligned} \|Sf\|_1 &= \int_0^1 \left| \tau \int_{I(x)} f(\xi) d\xi \right| dx \\ &= \frac{q}{q-1} \int_0^{\frac{q-1}{q}} \sum_{k=0}^{1/q} \left| \int_{I(x+k\lambda)} f(\xi) d\xi \right| dx \\ &\leq \frac{q}{q-1} \int_0^{\frac{q-1}{q}} \int_0^1 |f(\xi)| d\xi dx \\ &= \frac{1}{q-1} \|f\|_1. \end{aligned}$$

Item (2): Because $f \in \mathcal{M}_{\text{sym}}^0$, we know that

$$\int_0^{1/2} f(\xi) d\xi = \frac{1}{2} \int_0^1 f(\xi) d\xi = 0.$$

Analogously to part (1), observe that

$$[0, \frac{1}{2}] \subset I(x) \text{ if } \frac{1}{2q} \leq x \leq \frac{q-1}{q} \quad \text{and} \quad [\frac{1}{2}, 1] \subset I(x) \text{ if } \frac{1}{q} \leq x \leq \frac{2q-1}{2q}.$$

Combining these two facts, we obtain for $\lambda = \frac{q-1}{q}$

$$\sum_{k=0}^{m(x,\lambda)} \left| \int_{I(x+k\lambda)} f(\xi) d\xi \right| \leq \begin{cases} \int_0^1 |f(\xi)| d\xi & \text{if } 0 \leq x \leq \frac{2-q}{q} \\ \int_0^{1/2} |f(\xi)| d\xi & \text{if } \frac{2-q}{q} \leq x \leq \frac{1}{2q} \\ \int_{1/2}^1 |f(\xi)| d\xi & \text{if } \frac{1}{2q} \leq x \leq \frac{q-1}{q}. \end{cases} \tag{3.10}$$

Since

$$\int_0^{1/2} |f(\xi)| d\xi = \int_{1/2}^1 |f(\xi)| d\xi = \frac{1}{2} \|f\|_1,$$

we conclude for $\frac{3}{2} \leq q \leq 2$ that

$$\begin{aligned} \|Sf\|_1 &= \frac{q}{q-1} \int_0^1 \sum_{k=0}^{(q-1)/q m(x,\lambda)} \left| \int_{I(x+k\lambda)} f(\xi) d\xi \right| dx \\ &\leq \frac{q}{q-1} \left(\frac{2-q}{q} + \left(\frac{q-1}{q} - \frac{2-q}{q} \right) \frac{1}{2} \right) \|f\|_1 \\ &= \frac{1}{2(q-1)} \|f\|_1. \end{aligned}$$

If $q > 2$, our calculation in part 1 and the last two branches of (3.10) combine to

$$\|Sf\|_1 = \frac{q}{q-1} \int_0^1 \sum_{k=0}^{1/q} \left| \int_{I(x+k\lambda)} f(\xi) d\xi \right| dx \leq \frac{1}{2(q-1)} \|f\|_1$$

and the lemma is proved ■

Theorem 3.5. *Suppose $q > \frac{3}{2}$, and let S_n and S be given by (3.2) and (3.3), respectively. Then the sequence $(\varphi_n)_{n \in \mathbb{N}_0} \subset L^1([0,1])$ defined by $\varphi_0 = 1$ and $\varphi_n = S_n \varphi_{n-1}$ for $n \geq 1$ converges in L^1 to a fixed point of S .*

Proof. The preceding Lemma 3.4 gives $\|Sf\|_1 \leq \theta\|f\|_1$ with $\theta = \frac{1}{2(q-1)}$, for any $f \in \mathcal{M}_{\text{sym}}^0$. Because $q > \frac{3}{2}$, we have $\theta < 1$, and S is a strict contraction on the metric space

$$\mathcal{M}_{\text{sym}}^1 = \left\{ f \in L^1([0, 1]) \mid f(x) = f(1-x) \text{ a.e. and } \int_0^1 f(x) dx = 1 \right\}$$

with metric induced by the L^1 -norm. Now Banach’s fixed point theorem implies that there is a unique $\varphi \in \mathcal{M}_{\text{sym}}^1$ with $S\varphi = \varphi$.

Now put $\varepsilon_n = \varphi_n - \varphi$ ($n \in \mathbb{N}_0$) for the error terms. Observe that we constructed the operators S_n in such a way that the functions φ_n given by

$$\varphi_0 = 1 \quad \text{and} \quad \varphi_n = S_n\varphi_{n-1} \quad \text{for } n \geq 1 \tag{3.11}$$

coincide with those defined in (2.2). Hence, by (2.3) we conclude that $\varepsilon_n \in \mathcal{M}_{\text{sym}}^0$. According to (3.11), we have $\varepsilon_{n+1} = S\varepsilon_n + (S_{n+1} - S)\varphi_n$, from which we infer

$$\|\varepsilon_{n+1}\|_1 \leq \theta\|\varepsilon_n\|_1 + \|(S_{n+1} - S)\varphi_n\|_1$$

and, inductively,

$$\|\varepsilon_{n+k}\|_1 \leq \theta^k\|\varepsilon_n\|_1 + \sum_{\ell=1}^k \theta^{k-\ell}\|(S_{n+\ell} - S)\varphi_{n+\ell-1}\|_1. \tag{3.12}$$

We know by construction and by convergence of τ_n that

$$B := \max_{n \in \mathbb{N}_0} \left(\max_{0 \leq x \leq 1} \varphi_n(x) \right) = \max_{n \in \mathbb{N}} \tau_n < \infty.$$

Hence, by the uniformity part of Lemma 3.1, we can make the sum in (3.12) arbitrarily small if n is sufficiently large. This implies $\lim_{n \rightarrow \infty} \|\varepsilon_n\|_1 = 0$ which completes the proof ■

To see what is the meaning of this theorem, let us briefly summarize what we have done up to now.

We started out with a sequence $(c_n)_{n \in \mathbb{N}}$ of positive integers, viewed as a sequence of capacities of constrained urns $(U_n)_{n \in \mathbb{N}}$, and we have been concerned with the “shapes” of the functions $A_n(k)$ defined in (0.2) which count the number of different ways to distribute k balls into the first n urns, subject to the constraint that each urn U_j only has place for at most c_j balls, $j = 1, \dots, n$. Then we normalized the $A_n(k)$ according to (2.2) to obtain a sequence of *normalized counting functions* $(\varphi_n)_{n \in \mathbb{N}}$, each with support $[0, 1]$ and total integral 1.

The essential point was to use the reproduction Lemma 1.1 and its “normalized” counterpart, Lemma 2.1, to define the successor operators S_n which allow to consider the sequence $(\varphi_n)_{n \in \mathbb{N}}$ as being defined by

$$\varphi_0 = 1 \quad \text{and} \quad \varphi_n = S_n\varphi_{n-1}.$$

Now, if we want to say something about the limiting behaviour of the φ_n , we can do this through two steps:

- Firstly, make sure that the S_n converge appropriately to some S .
- Secondly, investigate what happens if S is iterated.

In this setting, Theorem 3.5 says the following:

If we assume that the initial capacity sequence $(c_n)_{n \in \mathbb{N}}$ is asymptotically geometric with weight $q > \frac{3}{2}$, then the S_n converge to the limiting operator S given by (3.3), and the sequence (φ_n) of normalized counting functions converges in L^1 to a fixed point of that limiting operator.

Remark. It is possible to use L^1 -convergence of the functions φ_n to show that they in fact converge uniformly on the interval $[0, 1]$: Knowing that the L^1 -limit φ is a fixed point of S , it is easy to see that φ can be extended to a C^∞ -function on \mathbb{R} with support $[0, 1]$. Hence φ is uniformly Lipschitz, and, as we mentioned in Lemma 2.2, the functions φ_n are monotone on each of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Now it is not difficult to prove estimates showing that a sequence of monotone functions converging in L^1 to some function which is uniformly Lipschitz converges in fact uniformly.

4. The limiting density and Cantor-like sets

Let us assume that the initial sequence (c_n) of capacities is asymptotically geometric with weight $q > \frac{3}{2}$. Then, according to Theorem 3.5, the limiting function φ of our normalized counting functions φ_n fulfills

$$\varphi(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \varphi(\xi) d\xi \quad \text{for } x \in \mathbb{R} \tag{4.1}$$

and the normalization conditions

$$\text{supp}(\varphi) = [0, 1] \quad \text{and} \quad \int_0^1 \varphi(x) dx = 1. \tag{4.2}$$

We shall see in this section that in the case $q > 2$ this limiting function φ has some nice connection to the construction of Cantor-like sets, where the weight q is used to compute the construction parameter.

The construction of a Cantor-like set [5: p. 70] is a straightforward generalization of the construction of the well-known ‘‘Cantor middle thirds set’’. For our purpose, the most appropriate way is to consider the dilations (cf. [4: p. 6])

$$f_1(x) = \frac{x}{q} \quad \text{and} \quad f_2(x) = \frac{x+q-1}{q}.$$

Now put

$$E_0 = [0, 1] \quad \text{and} \quad E_n = f_1(E_{n-1}) \cup f_2(E_{n-1}) \quad \text{for } n \geq 1. \tag{4.3}$$

In other words: to obtain E_1 , we have to remove the set

$$M = \left\{ \frac{1}{q} < x < \frac{q-1}{q} \right\}$$

from the unit interval E_0 . Clearly, M is an interval of length $\delta = \frac{q-2}{q}$, centered about the center of $[0, 1]$. To construct E_n out of E_{n-1} according to (4.3), we have to squeeze each interval of E_{n-1} using f_1 and f_2 and put it once on the left and once on the right of M . The effect is the same as if we had removed an interval of proportional length δ from the middle of each interval of E_{n-1} .

Note that M is non-empty if and only if $q > 2$, and that the classical ‘‘Cantor middle thirds set’’ corresponds to $q = 3$.

Proposition 4.1. *Assume $q > 2$. Then E_n is the support of the n -th derivative of φ .*

Proof. First note that φ is a C^∞ -function because it is a solution of integral equation (4.1). Let Φ be a primitive of φ . Then (4.1) implies

$$\varphi(x) = \frac{q}{q-1} (\Phi(qx) - \Phi(qx - q + 1))$$

and differentiation gives

$$\varphi'(x) = \frac{q^2}{q-1} (\varphi(qx) - \varphi(qx - q + 1)). \tag{4.4}$$

Denoting the interior of E_n by $\text{int}E_n$, we now prove by induction on n that $\varphi^{(n)}(x) \neq 0$ if and only if $x \in \text{int}E_n$.

$n = 0$: By (4.2) we have $\text{supp}(\varphi) = [0, 1] = E_0$. Together with (4.1), this implies that $\varphi(x) \neq 0$ for any $x \in \text{int}E_0$.

$n - 1 \rightarrow n$: Differentiating (4.4) $(n - 1)$ times, we get

$$\varphi^{(n)}(x) = \frac{q^{n+1}}{q-1} (\varphi^{(n-1)}(qx) - \varphi^{(n-1)}(qx - q + 1)). \tag{4.5}$$

By induction hypothesis, we know that $\text{supp}(\varphi^{(n-1)}) = E_{n-1} \subset [0, 1]$. Because we assumed $q > 2$, we infer from this that at most one term in the difference in (4.5) is non-zero. Again by the induction hypothesis, we have

$$\begin{aligned} \varphi^{(n-1)}(qx) \neq 0 &\iff qx \in \text{int}E_{n-1} &\iff x \in f_1(\text{int}E_{n-1}) \\ \varphi^{(n-1)}(qx - q + 1) \neq 0 &\iff qx - q + 1 \in \text{int}E_{n-1} &\iff x \in f_2(\text{int}E_{n-1}). \end{aligned}$$

By (4.3), this completes the proof ■

The function φ defined by (4.1) and (4.2) (for $q > 2$) is really an astonishing one: it is clearly a C^∞ -function with compact support. But on the other hand, we derive from Proposition 4.1 that φ is "piecewise a polynomial". More precisely: if we define a sequence $(V_n)_{n \in \mathbb{N}_0}$ of open subsets of $[0, 1]$ inductively by

$$V_0 = \emptyset \quad \text{and} \quad V_n = [0, 1] \setminus (E_n \cup V_{n-1}) \quad \text{for } n \geq 1,$$

then we have the following facts:

- (1) The union of the V_n covers $[0, 1]$ up to a set of Lebesgue measure 0.
- (2) Each V_n consist of finitely many (exactly 2^{n-1} for $n \geq 1$) intervals.
- (3) φ is a polynomial of degree $n - 1$ on each interval of V_n .

Similar conclusions are also derived in [2: S. 175], using a different method.

References

- [1] Berg, L.: *Asymptotische Darstellungen und Entwicklungen*. Berlin: Dt. Verlag Wiss. 1968.
- [2] Berg, L. and M. Krüppel: *On the solutions of an integral-functional equation with a parameter*. J. Anal. Appl. 17 (1998), 159 - 181.
- [3] Bernstein, S. N.: *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*. Communic. Soc. Math. Charkow (Serie 2) 13 (1912), 1 - 2.
- [4] Edgar, G. A.: *Measure, Topology, and Fractal Geometry*. New York: Springer 1990.
- [5] Hewitt, E. and K. Stromberg: *Real and Abstract Analysis*. New York: Springer 1965.
- [6] Wirsching, G. J.: *On the combinatorial structure of $3n+1$ predecessor sets*. Discrete Math. 148 (1996), 265 - 286.
- [7] Wirsching, G. J.: *The Dynamical System Generated by the $3n + 1$ Function*. Lect. Notes Math. 1681 (1998), 1 - 158.

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