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Cantor Sets and Integral-Functional Equations

L. Berg and M. Krüppel

Abstract. In this paper, we continue our considerations in [1] on a homogeneous integral-functional equation with a parameter $a > 1$. In the case of $a > 2$ the solution ϕ satisfies relations containing polynomials. By means of these polynomial relations the solution can explicitly be computed on a Cantor set with Lebesgue measure 1. Thus the representation of the solution ϕ is immediately connected with the exploration of some Cantor sets, the corresponding singular functions of which can be characterized by a system of functional equations depending on a. In the limit case $a = 2$ we get a formula for the explicit computation of ϕ in all dyadic points. We also calculate the iterated kernels and approximate ϕ by splines in the general case $a > 1$.

Keywords: *Integral-functional equations, generating functions, Cantor sets, singular functions, relations containing polynomials, iterated kernels, approximation by splines*

AMS subject classification: 45 D 05, 39 B 22, 34 K 15, 26 A 30, 41 A 15

1. Introduction

In [1] we have shown that the homogeneous integral-functional equation

$$\phi(t) = L\phi(t), \qquad L\phi(t) = b \int_{at-a+1}^{at} \phi(\tau)\, d\tau \qquad \left(b = \tfrac{a}{a-1}\right), \tag{1.1}$$

where $a > 1$ is a fixed parameter and $t \in \mathbb{R}$, has a unique compactly supported solution up to a constant factor. Since the support is contained in [0,1], the constant factor can be fixed by the value of its integral:

$$\int_0^1 \phi(t)\, dt = 1\ . \tag{1.2}$$

G. J. Wirsching has considered in [12] the case $a = 3$ and in the paper [13] also the case $a > \frac{3}{2}$, where ϕ is the limiting density of a certain transition probability of a non-homogeneous Markov process arising in a combinatorial problem. The case $a = 2$ was considered by W. Volk in [11] in order to construct some subspaces of $C^\infty[a, b]$, which are spanned by translates of ϕ.

Both authors: FB Mathematik der Universität, Universitätspl. 1, D-18051 Rostock

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In this paper we continue our considerations in [1], primary for $a \geq 2$. For this reason, we list such results of [1] which we will need afterwards and, moreover, we make some supplements to them. The solution is infinitely often differentiable, symmetric with respect to the point $\frac{1}{2}$, and monotone at both sides of $\frac{1}{2}$. The solution has the support [0,1] and it is strictly positive for $t \in (0,1)$. For $a > 2$ it is a polynomial on each component of an open Cantor set with Lebesgue measure 1. The solution ϕ of (1.1) - (1.2) can be obtained by means of successive approximation. For every L-integrable function f_0 on the interval [0,1] with $f_0(t) = 0$ for $t \notin [0,1]$ and the property $\int_0^1 f_0(t)\,dt = 1$, the iterates $f_n = Lf_{n-1}$ $(n \geq 1)$ converge uniformly on [0,1] to the solution ϕ of (1.1) - (1.2). Hence, on account of a result of W. M. Gerstein and B. N. Sadowski, the operator L is contractive on a certain subspace of $C^1[0,1]$ equipped with a metric ρ which is equivalent to the maximum norm (cf. [8]).

The Laplace transform Φ of the compactly supported solution ϕ of (1.1) - (1.2) has the product representation

$$\Phi(p) = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} \tag{1.3}$$

and the power series representation

$$\Phi(p) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!}\, p^n \tag{1.4}$$

which are both convergent for all $p \in \mathbb{C}$. The coefficients of the series are rational functions with respect to a and, starting with $\rho_0(a) = 1$ for $n \geq 1$, they can be determined by means of the recursion formula

$$\rho_n(a) = \frac{1}{(n+1)(a^n - 1)} \sum_{\nu=0}^{n-1} \binom{n+1}{\nu} \rho_\nu(a)(1-a)^{n-\nu}\,. \tag{1.5}$$

Moreover, we have

$$\frac{1}{\Phi(p)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rho_n\Big(\frac{1}{a}\Big) p^n \qquad (|p| < 2b\pi) \tag{1.6}$$

and

$$\ln \Phi(p) = \sum_{n=1}^{\infty} \frac{B_n}{n!n} \frac{(a-1)^n}{a^n - 1}\, p^n \qquad (|p| < 2b\pi)\,, \tag{1.7}$$

where B_n are the Bernoulli numbers

$$B_0 = 1\,, \quad B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad B_3 = 0\,, \quad B_4 = -\frac{1}{30}\,, \ldots\,.$$

The polynomials

$$\psi_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \rho_{n-\nu}(a)\, t^\nu \tag{1.8}$$

will play an essential role later on. Note that in [1] we have used the abbreviation ψ_n for the polynomials (1.8) with $\frac{1}{a}$ instead of a. The polynomials ψ_n have the generating function

$$e^{tp}\Phi(p) = \sum_{n=0}^{\infty} \frac{\psi_n(t)}{n!} p^n \tag{1.9}$$

and the properties

$$\psi_n'(t) = n\,\psi_{n-1}(t) \tag{1.10}$$

$$\psi_n(1-t) = (-1)^n \psi_n(t)\ . \tag{1.11}$$

In the case of $a \geq 2$ the solution ϕ of (1.1) - (1.2) can be expressed by the polynomials ψ_n in the intervals $\frac{1}{a^{n+1}} \leq t \leq \frac{a-1}{a^{n+1}}$ $(n \geq 0)$, namely

$$\phi(t) = \frac{\psi_n(a^{n+1}t)}{n!\ a^{\frac{1}{2}(n+1)(n-2)}(a-1)^{n+1}}\ . \tag{1.12}$$

Also, the functions ϕ_n $(n \in \mathbb{N}_0)$ defined by $\phi_0 = \phi$ from (1.1) - (1.2) and

$$\phi_{n+1}(t) = \int_0^t \phi_n(\tau)\,d\tau \tag{1.13}$$

for $n \geq 0$ are needed in this note. We recall for arbitrary $a > 1$ the following relations between the functions ϕ, ψ_n and ϕ_n, namely

$$\phi_n(t) = a^{\frac{n(n-3)}{2}}(a-1)^n \sum_{\nu_1,\ldots,\nu_n \geq 0} \phi\Big(\frac{t}{a^n} - \frac{\nu_1}{a^{n-1}b} - \ldots - \frac{\nu_n}{b}\Big) \tag{1.14}$$

for all t and $n \in \mathbb{N}$, in particular

$$\phi_n(t) = a^{\frac{n(n-3)}{2}}(a-1)^n \phi(a^{-n}t) \qquad \text{for } \ t \leq a-1 \tag{1.15}$$

as well as

$$\phi_n(t) = \frac{1}{(n-1)!}\psi_{n-1}(t) \qquad \text{for } \ t \geq 1 \tag{1.16}$$

and

$$\sum_{\nu_1,\ldots,\nu_n \geq 0} \phi\Big(t - \frac{\nu_1}{a^{n-1}b} - \ldots - \frac{\nu_n}{b}\Big) = \frac{\psi_{n-1}(a^n t)}{(n-1)!\ a^{\frac{n(n-3)}{2}}\ (a-1)^n} \tag{1.17}$$

for $t \geq \frac{1}{a^n}$. The solution ϕ of (1.1) - (1.2) satisfies the equation

$$\sum_{\nu=-\infty}^{+\infty} \phi\left(t - \frac{\nu}{a^n b}\right) = a^n b$$

for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}_0$. In [1] this was proved only for $n = 0$, but the general form easily follows by means of (1.1) and induction.

The eigenvalue problem

$$\lambda f(t) = \int_{at-a+1}^{at} f(\tau)\,d\tau \tag{1.18}$$

with $a > 1$ has the solution $f = \phi = \phi_0$ from (1.1) - (1.2) for $\lambda_0 = \frac{1}{b}$, and for the eigenvalues $\lambda_n = \frac{a^n}{b}$ ($n \in \mathbb{N}$) the eigenfunctions $f = \psi_{n-1}$ and $f = \phi_n$ (cf. (1.8) and (1.13)), which have non-compact support.

The aim of this paper is to investigate in detail the Cantor intervals for $a > 2$, in which the solution ϕ of (1.1) - (1.2) is equal to certain polynomials, and to find these polynomials explicitly, i.e. to generalize (1.12) to the other Cantor intervals. The results are also valid in the limit case $a = 2$, where the Cantor intervals degenerate. In this connection we characterize the mapping between corresponding Cantor intervals for different a by Sierpiński-like functional equations. Moreover, for arbitrary $a > 1$ we find the iterated kernels of the integral equation (1.1), as well as new spline approximations for the solution.

2. The sequences γ_n and ε_n

Besides of the foregoing results from [1], for the piecewise representation of the solution ϕ of (1.1) - (1.2) by polynomials and for the approximation of ϕ by splines we need an auxiliary sequence $\gamma_n = \gamma_n(a)$ defined as follows: If n has the dyadic representation $n = d_s \cdots d_1 d_0$ with $d_s = 1$ and $d_\nu \in \{0, 1\}$, then

$$\gamma_n = (a-1)\sum_{\nu=0}^{s} d_\nu\, a^\nu. \tag{2.1}$$

The first elements of this sequence are

$$\gamma_0 = 0\,, \quad \gamma_1 = a-1\,, \quad \gamma_2 = (a-1)a\,, \quad \gamma_3 = (a-1)(a+1)$$
$$\gamma_4 = (a-1)a^2\,, \quad \gamma_5 = (a-1)(a^2+1)\,, \quad \gamma_6 = (a-1)(a^2+a)$$
$$\gamma_7 = (a-1)(a^2+a+1)\,, \quad \gamma_8 = (a-1)a^3\,, \quad \gamma_9 = (a-1)(a^3+1)\,, \;\ldots\,.$$

For integers $a \geq 2$ also the numbers γ_n are integers. In particular, for $a = 2$ we have $\gamma_n = n$. It is easy to see that the sequence γ_n has the property

$$\left.\begin{aligned} \gamma_{2n} &= a\,\gamma_n \\ \gamma_{2n+1} &= a\,\gamma_n + a - 1 \end{aligned}\right\} \qquad (n \in \mathbb{N}_0). \tag{2.2}$$

In view of $a \neq 1$ the sequence γ_n can also be defined by (2.2), because the first equation implies $\gamma_0 = 0$, and the next terms of the sequence are determined recursively by (2.2). According to (2.2), the generating function

$$g(z) = \sum_{n=0}^{\infty} \gamma_n z^n$$

satisfies the equation

$$g(z) = a(1+z)g(z^2) + \frac{z(a-1)}{1-z^2} .$$

Defining $(Tg)(z) = g(z^2)$, we find for the solution the series

$$g(z) = (a-1)\sum_{n=0}^{\infty} a^n((1+z)T)^n \frac{z}{1-z^2} = \frac{a-1}{1-z}\sum_{n=0}^{\infty} \frac{a^n z^{2^n}}{1+z^{2^n}} \tag{2.3}$$

which is convergent for $|z| < 1$. For $a = 2$ we have, of course, $g(z) = \frac{z}{(1-z)^2}$ (cf. also [7: p. 451]). For later purpose we list some further properties of γ_n.

Lemma 2.1. *The sequence γ_n has the following properties:*

(i) $\gamma_{2k+1} = \gamma_{2k} + \gamma_1 \quad (k \geq 0)$.

(ii) $a^l\gamma_k = \gamma_{2^lk}$ *and* $a^l(\gamma_k + 1) = \gamma_{2^l(k+1)-1} + 1 \quad (k, l \geq 0)$.

(iii) $\gamma_k + \gamma_l + 1 = a^m$ *if* $k + l + 1 = 2^m \quad (k, l \geq 0)$.

Proof. Statement (i) and the first equality in (ii) follow immediately from (2.2). The second equality in (ii) can easily be proved by induction with respect to l, since it is an identity for $l = 0$ and the induction step reads in view of (2.2)

$$a^{l+1}(\gamma_k + 1) = a\,\gamma_{2^l(k+1)-1} + a = \gamma_{2^{l+1}(k+1)-1} + 1 .$$

In order to show statement (iii) we assume without loss of generality that $k > l$ and that k has the representation $k = d_0 + 2d_1 + \ldots + 2^{m-1}d_{m-1}$ with $d_{m-1} = 1$ and $d_\nu \in \{0,1\}$, i.e. the dyadic representation $k = d_{m-1}d_{m-2}\cdots d_0$. This implies that l has the representation $l = \overline{d}_0 + 2\overline{d}_1 + \ldots + 2^{m-2}\overline{d}_{m-2}$ with $\overline{d}_\nu = 1 - d_\nu$ since

$$k + l = \sum_{\nu=0}^{m-1} 2^\nu = 2^m - 1 .$$

In view of (2.1) with $s = m-1$ and $n = k$, resp. $n = l$ and $\overline{d}_\nu$ instead of d_ν, we get

$$\gamma_k + \gamma_l = (a-1)\sum_{\nu=0}^{m-1} a^\nu = a^m - 1 .$$

This completes the proof ∎

Lemma 2.2. *In the case of $a \geq 2$ we have $\gamma_{n+1} \geq \gamma_n + a - 1 \quad (n \in \mathbb{N}_0)$.*

Proof. For $n = 2k$ the inequality is even an equality in view of Lemma 2.1/(i). Moreover, it is true also for $n = 1$. Assume that $\gamma_{m+1} \geq \gamma_m + a - 1$ is true for $m < n = 2k+1$. Then in view of (2.2) and $a \geq 2$ we get $\gamma_{2k+2} = a\gamma_{k+1} \geq a(\gamma_k + a - 1) = \gamma_{2k+1} + a - 1$ and the assertion is proved by induction ∎

Moreover, we need the sign sequence $\varepsilon_n = (-1)^{\nu(n)}$, where $\nu(n)$ denotes the number of "1" in the dyadic representation of n, i.e. $\nu(n)$ is the binary sum-of-digits function (cf. [4]).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\nu(n)$	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4
ε_n	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1

Table 1: The first numbers $\nu(n)$ and ε_n

Considering (2.1) and (2.3) we get in view of $\frac{\gamma_n}{a-1} \to \nu(n)$ for $a \to 1$ the generating function

$$\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{z^{2^n}}{1+z^{2^n}} = \sum_{n=0}^{\infty}\nu(n)\,z^n \qquad (|z|<1)\,. \tag{2.4}$$

The sequence $\nu(n)$ mod 2 with values from $\{0,1\}$ is the Morse sequence (cf. [5]) which is equivalent to ε_n by the mapping $1 \mapsto -1$ and $0 \mapsto 1$. It is easy to see that the sequence ε_n can be also defined recursively by

$$\left.\begin{aligned} &\varepsilon_0 = 1 \\ &\varepsilon_{2n} = \varepsilon_n \text{ and } \varepsilon_{2n+1} = -\varepsilon_n \quad (n \geq 0). \end{aligned}\right\} \tag{2.5}$$

According to (2.5) the generating function

$$f(z) = \sum_{n=0}^{\infty}\varepsilon_n z^n$$

satisfies the equation $f(z) = (1-z)f(z^2)$. Hence, we get in view of $f(0) = \varepsilon_0 = 1$ the representation

$$f(z) = \prod_{n=0}^{\infty}(1-z^{2^n}) \qquad (|z|<1)\,. \tag{2.6}$$

The sequence ε_n was already used in [1] for the determination of the signs of the Fourier coefficients of the solution ϕ of (1.1) - (1.2) in the case of $a = 2$.

In view of (2.5) it is easy to show by means of induction that the sequence ε_n has the properties

$$\sum_{\nu=0}^{2n}\varepsilon_\nu = \varepsilon_n \qquad \text{and} \qquad \sum_{\nu=0}^{2n+1}\varepsilon_\nu = 0 \tag{2.7}$$

as well as

$$\varepsilon_\nu - \varepsilon_{\nu-1} = \begin{cases} 0 & \text{for } \nu = 2^{2\mu-1} \bmod 2^{2\mu} \\ 2\varepsilon_\nu & \text{else} \end{cases} \qquad (\nu,\mu \in \mathbb{N}) \tag{2.8}$$

where the signs of the non-vanishing differences alternate. Furthermore, we have

$$\sum_{\nu=1}^{k}\varepsilon_\nu\gamma_\nu = \begin{cases} \varepsilon_n a^2\gamma_n & \text{for } k = 4n \\ -\varepsilon_n\gamma_1 & \text{for } k = 4n+1 \\ -\varepsilon_n\gamma_{4n+3} & \text{for } k = 4n+2 \\ 0 & \text{for } k = 4n+3 \end{cases}$$

which follows from (2.2) and (2.5) by induction.

Both sequences γ_n and ε_n appear in the following connection.

Lemma 2.3. *We have the identity*

$$\prod_{k=0}^{n-1}\left(1-e^{-p/(ba^k)}\right)=\sum_{\nu=0}^{2^n-1}\varepsilon_\nu e^{-\frac{\gamma_\nu}{a^n}p}\,. \tag{2.9}$$

Proof. This formula is true for $n=1$ in view of $\frac{\gamma_1}{a}=\frac{a-1}{a}=\frac{1}{b}$. If (2.9) is true for a certain n, then it follows

$$\begin{aligned}\prod_{k=0}^{n}\left(1-e^{-p/(ba^k)}\right)&=\left(1-e^{-p/(ba^n)}\right)\sum_{\nu=0}^{2^n-1}\varepsilon_\nu e^{-\frac{\gamma_\nu}{a^n}p}\\&=\sum_{\nu=0}^{2^n-1}\varepsilon_\nu e^{-\frac{a\gamma_\nu}{a^{n+1}}p}-\sum_{\nu=0}^{2^n-1}\varepsilon_\nu e^{-\frac{a\gamma_\nu+a-1}{a^{n+1}}p}\\&=\sum_{\nu=0}^{2^{n+1}-1}\varepsilon_\nu e^{-\frac{\gamma_\nu}{a^{n+1}}p}\end{aligned}$$

where we have used (2.2) and (2.5). Thus, assertion (2.9) is proved by induction ∎

We remark that

$$\prod_{k=0}^{n-1}\left(1-e^{-p/(ba^k)}\right)=\sum_{\nu=0}^{2^n-1}\varepsilon_\nu e^{-\frac{\gamma_\nu}{a^n}p}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!a^{nm}}\sum_{\nu=0}^{2^n-1}\varepsilon_\nu\gamma_\nu^m p^m \tag{2.10}$$

implies that the sum

$$s_n(m)=\sum_{\nu=0}^{2^n-1}\varepsilon_\nu\gamma_\nu^m \tag{2.11}$$

equals to 0 for $m=0,1,\dots,n-1$.

Proposition 2.1. *For $m\ge n$ we have*

$$s_n(m)=\frac{(-1)^m m!}{(m-n)!}\frac{a^{\frac{n(2m-n+1)}{2}}}{b^n}\sum_{\mu=0}^{m-n}(-1)^\mu\binom{m-n}{\mu}\frac{\rho_\mu(\frac{1}{a})\rho_{m-n-\mu}(a)}{a^{n\mu}}\,.$$

Proof. From (1.3) we get

$$\frac{\Phi(p)}{\Phi(\frac{p}{a^n})}=\prod_{k=0}^{n-1}\frac{1-e^{-p/(ba^k)}}{p/(ba^k)}=\frac{a^{\frac{n(n-1)}{2}}b^n}{p^n}\prod_{k=0}^{n-1}\left(1-e^{-p/(ba^k)}\right),$$

and in view of (2.10) and (2.11) we find

$$\phi(p)\,\frac{1}{\phi(\frac{p}{a^n})}=a^{\frac{n(n-1)}{2}}b^n\sum_{m=0}^{\infty}\frac{(-1)^m s_n(m)}{m!\,a^{mn}}\,p^{m-n}\,.$$

Using the representations (1.4) and (1.6), the last with $\frac{p}{a^n}$ instead of p, we obtain the assertion by means of the Cauchy product and comparison of coefficients ∎

In particular, we have

$$\sum_{\nu=0}^{2^n-1} \varepsilon_\nu \gamma_\nu^n = (-1)^n\, n!\,(a-1)^n\, a^{\frac{n(n-1)}{2}} \tag{2.12}$$

and

$$\sum_{\nu=0}^{2^n-1} \varepsilon_\nu \gamma_\nu^{n+1} = \frac{1}{2}(-1)^n (n+1)!\,(a-1)^n (a^n-1) a^{\frac{n(n-1)}{2}}\ . \tag{2.13}$$

3. Cantor sets and singular functions

In this section, we explore Cantor sets which are immediately connected with the solution ϕ of (1.1) - (1.2) in the case of $a > 2$. First, we note that in the case of $a > 2$ Lemma 2.2 implies

$$\gamma_n + 1 < \gamma_{n+1}\ . \tag{3.1}$$

Hence, we can define the following open intervals G_{kn} $(k = 0, 1, \ldots, 2^n - 1;\ n \in \mathbb{N}_0)$ and the corresponding union G_m:

$$G_{kn} = \left(\frac{\gamma_{2k}+1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right), \qquad G_m = \bigcup_{n=0}^{m} \bigcup_{k=0}^{2^n-1} G_{kn}\ . \tag{3.2}$$

In order to show that all G_{kn} are disjoint, we consider the following closed intervals F_{kn} $(k = 0, 1, \ldots, 2^n - 1;\ n \in \mathbb{N}_0)$ and the corresponding union F_n:

$$F_{kn} = \left[\frac{\gamma_k}{a^n}, \frac{\gamma_k + 1}{a^n}\right], \qquad F_n = \bigcup_{k=0}^{2^n-1} F_{kn}\ . \tag{3.3}$$

Note that $F_0 = [0,1]$ and in view of (3.1), all F_{kn} with a fixed n are disjoint. From Lemma 2.1/(ii) we see that F_{kn} and $F_{2^l k, n+l}$ have the same left end-points and, analogously, F_{kn} and $F_{2^l(k+1)-1, n+l}$ the same right end-points for all $l \in \mathbb{N}_0$.

Lemma 3.1. *In the case of $a > 2$, for all $n \in \mathbb{N}$ and $k = 0, 1, \ldots, 2^n - 1$ we have $G_{kn} \subset F_{kn}$ and the disjoint decomposition*

$$F_{kn} = F_{2k,n+1} \cup G_{kn} \cup F_{2k+1,n+1}\ . \tag{3.4}$$

Proof. In view of (2.2), we have

$$F_{kn} = \left[\frac{a\gamma_k}{a^{n+1}}, \frac{a\gamma_k + a}{a^{n+1}}\right] = \left[\frac{\gamma_{2k}}{a^{n+1}}, \frac{\gamma_{2k+1}+1}{a^{n+1}}\right]\ .$$

According to (3.2), we see that from the intervals $G_{\nu n}$ $(\nu = 0, 1, \ldots, 2^n - 1)$ exactly the interval G_{kn} lies in F_{kn}, since $\gamma_{2k} < \gamma_{2k} + 1 < \gamma_{2k+1} < \gamma_{2k+1} + 1$. In view of (3.3) this implies the decomposition (3.4) (cf. Figure 1) ∎

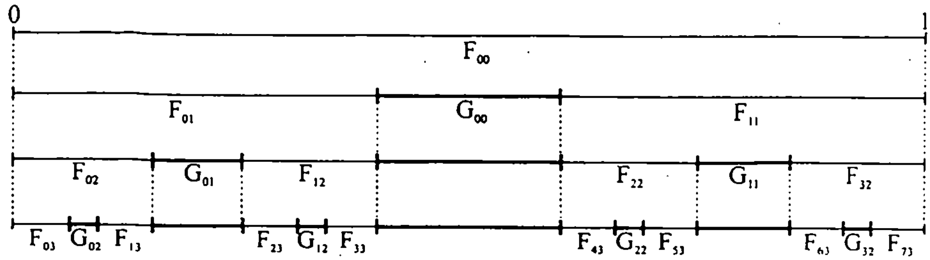

Figure 1: The first Cantor intervals

The disjoint composition (3.4) shows that also all G_{kn} are disjoint and, moreover, that $F_{m+1} = [0,1] \setminus G_m$. Since $\gamma_{2k+1} - \gamma_{2k} = a - 1$, we get for the measure of G_{kn} that $|G_{kn}| = \frac{a-2}{a^{n+1}}$, and for the measure of the open Cantor set

$$G = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} G_{kn}$$

we have

$$|G| = \sum_{n=0}^{\infty} 2^n \frac{a-2}{a^{n+1}} = 1 ,$$

and hence for the perfect Cantor set $F = [0,1] \setminus G$ the measure $|F| = 0$ as in the original construction of Cantor, i.e. in the case of $a = 3$. We remark that the Cantor set G can be generated from [0,1] by iteration of the functions $f_1(x) = \frac{x}{a}$ and $f_2(x) = \frac{x+a-1}{a}$ (cf. [3: p. 6], [1] or [13]). For $a = 2$ the intervals G_{kn} are empty and $F_n = F = [0,1]$.

Next, for arbitrary $a > 1$ we introduce numbers $x = x(a)$ of the form

$$x = (a-1) \sum_{\nu=1}^{\infty} \frac{\xi_\nu}{a^\nu} \qquad (\xi_\nu \in \{0,1\}) \tag{3.5}$$

which lie in [0,1] in view of

$$(a-1) \sum_{\nu=1}^{\infty} \frac{1}{a^\nu} = 1 . \tag{3.6}$$

In the case of $\xi_\nu = 0$ for $\nu \geq n+1$ we write

$$x_n = (a-1) \sum_{\nu=1}^{n} \frac{\xi_\nu}{a^\nu} \qquad (\xi_\nu \in \{0,1\}) \tag{3.7}$$

for $n \in \mathbb{N}_0$. Denoting $\xi_\nu = d_{n-\nu}$ for $\nu = 1, \ldots, n$ and $k = d_0 + 2d_1 + \ldots + 2^{n-1} d_{n-1}$, we see from (2.1) that $x_n = \frac{\gamma_k}{a^n}$ with a certain index $k \in \{0, 1, \ldots, 2^n - 1\}$, i.e. x_n is the left end-point of F_{kn} if we use the notation (3.3) also for $1 < a \leq 2$. Clearly, in the case of $a = 2$ these numbers are equal to $\frac{k}{2^n}$ $(n \in \mathbb{N}; k = 0, 1, \ldots, 2^n - 1)$ and they lie densely in [0,1]. The points (3.5) fill the whole interval [0,1] not only for $a = 2$, but also for $1 < a < 2$. In order to see this we remark that in the case of $1 < a < 2$ the intervals $F_{2k,n}$ and $F_{2k+1,n}$ are overlapping with $F_{2k,n} \cup F_{2k+1,n} = F_{k,n-1}$, so that $F_0 = [0,1]$ implies that $F_n = [0,1]$ for all $n \in \mathbb{N}$ (cf. (3.3)). Hence, the left end-points (3.7) of the intervals F_{kn} $(k = 0,1,\ldots,2^n-1)$ form an ε-net ($\varepsilon = \frac{1}{2a^n}$) for the interval [0,1] since for every fixed n each $x \in [0,1]$ is contained in at least one F_{kn}, i.e. $x_n \le x \le x_n + \frac{1}{a^n}$ with x_n from (3.7). Having already determined x_n for a given $x \in [0,1]$, the next number ξ_{n+1} in (3.5) reads

$$\xi_{n+1} = \begin{cases} 0 & \text{for } x_n \le x < x_n + \frac{a-1}{a^{n+1}}, & \text{i.e. } x \in F_{2k,n+1} \setminus F_{2k+1,n+1} \\ 1 & \text{for } x_n + \frac{1}{a^{n+1}} < x \le x_n + \frac{1}{a^n}, & \text{i.e. } x \in F_{2k+1,n+1} \setminus F_{2k,n+1} \end{cases}$$

whereas ξ_{n+1} can be chosen arbitrarily for $x_n + \frac{a-1}{a^{n+1}} \le x \le x_n + \frac{1}{a^{n+1}}$, i.e. for $x \in F_{2k,n+1} \cap F_{2k+1,n+1}$.

Lemma 3.2. *In the case of $a > 2$ the numbers (3.5) and $y = (a-1)\sum_{\nu=1}^{\infty} \frac{\eta_\nu}{a^\nu}$ with $\eta_\nu \in \{0,1\}$ have the following properties:*

(i) *The usual order of x and y is equivalent to the lexicographic order of $(\xi_1,\xi_2,\ldots)$ and $(\eta_1,\eta_2,\ldots)$.*

(ii) *The 2^n intervals G_{kn} with fixed n are exactly the intervals (x,y) with $\xi_\nu = \eta_\nu$ $(\nu = 1,2,\ldots,n)$, $\xi_{n+1} = 0$, $\xi_{n+2} = \xi_{n+3} = \ldots = 1$ and $\eta_{n+1} = 1$, $\eta_{n+2} = \eta_{n+3} = \ldots = 0$.*

Proof. Let be $(\xi_1,\xi_2,\ldots) < (\eta_1,\eta_2,\ldots)$ lexicographically, i.e. $\xi_\nu = \eta_\nu$ for $1 \le \nu \le m-1$ and $\xi_m < \eta_m$ for a certain $m \in \mathbb{N}$, which is only possible for $\xi_m = 0$ and $\eta_m = 1$. Then we have in view of $a > 2$ the inequality

$$y - x \ge \frac{a-1}{a^m} - (a-1)\sum_{\nu=m+1}^{\infty} \frac{1}{a^\nu} = \frac{a-2}{a^m} > 0.$$

Vice versa, $(\xi_1,\xi_2,\ldots) > (\eta_1,\eta_2,\ldots)$ implies analogously $x > y$, so that property (i) is valid.

In order to show property (ii), we first remark that for $k \le 2^n - 1$ the dyadic representation of k has at most n digits, i.e. $k = d_0 + 2d_1 + \ldots + 2^{n-1}d_{n-1}$ with $d_\mu \in \{0,1\}$. Hence, for the left end-point of G_{kn} we have as in the foregoing case of F_{kn} and in view of (3.6) the representation

$$\frac{\gamma_{2k}+1}{a^{n+1}} = (a-1)\sum_{\nu=1}^{n} \frac{\xi_\nu}{a^\nu} + \frac{1}{a^{n+1}} = (a-1)\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{a^\nu}$$

with $\xi_\nu = d_{n-\nu}$ for $\nu = 1,2,\ldots,n$, $\xi_{n+1} = 0$ and $\xi_\nu = 1$ for $\nu \ge n+2$. For the right end-point of G_{kn} we have analogously

$$\frac{\gamma_{2k+1}}{a^{n+1}} = (a-1)\left(\sum_{\nu=1}^{n} \frac{\xi_\nu}{a^\nu} + \frac{1}{a^{n+1}}\right) = (a-1)\sum_{\nu=1}^{n+1} \frac{\eta_\nu}{a^\nu}$$

with $\eta_\nu = d_{n-\nu}$ for $\nu = 1,2,\ldots,n$ and $\eta_{n+1} = 1$, so that property (ii) is proved ∎

Remark. In the case of $a = 2$, $(\xi_1, \xi_2, \ldots) < (\eta_1, \eta_2, \ldots)$ implies only $x \leq y$.

Lemma 3.2 shows once more that all G_{kn} are disjoint. Property (ii) from Lemma 3.2 means that the left end-points x^- of G_{kn} and the corresponding right end-points x^+ can be written in the form

$$x^- = (a-1)\sum_{\nu=1}^{n} \frac{\xi_\nu}{a^\nu} + \frac{1}{a^{n+1}} \qquad \text{and} \qquad x^+ = (a-1)\left(\sum_{\nu=1}^{n} \frac{\xi_\nu}{a^\nu} + \frac{1}{a^{n+1}}\right)$$

with $\xi_\nu \in \{0,1\}$. Since these end-points belong to the closed set F, also all points of the form (3.5) belong to F.

Now, for a fixed $a > 2$ and a fixed $c \geq 2$ we define a function $g_0 : F \mapsto [0,1]$ by

$$g_0(x) = (c-1)\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{c^\nu} \tag{3.8}$$

with $x = x(a)$ from (3.5), i.e. $g_0 : x(a) \mapsto x(c)$. According to property (i) from Lemma 2.3, this function is strictly increasing and, obviously, it is also continuous. We extend g_0 to the whole interval [0,1] by the definition

$$g_0\Big(x^- + \frac{a-2}{a^{n+1}}\,t\Big) = g_0(x^-) + \frac{c-2}{c^{n+1}}\,t \qquad (0 < t < 1)\,, \tag{3.9}$$

i.e. in view of $x^- + \frac{a-2}{a^{n+1}} = x^+$ we extend the function g_0 linearly on the intervals G_{kn}, so that it remains continuous and increasing (but only for $c > 2$ strictly increasing). Moreover, replacing t by $1-t$ in (3.9) we get

$$g_0\Big(x^+ - \frac{a-2}{a^{n+1}}\,t\Big) = g_0(x^+) - \frac{c-2}{c^{n+1}}\,t \qquad (0 < t < 1)\,. \tag{3.10}$$

Next, we show that the function $g = g_0$ satisfies for $0 \leq t \leq 1$ the following system of functional equations:

(i) $g(\frac{1}{a} + \frac{a-2}{a}\,t) = \frac{1}{c} + \frac{c-2}{c}\,t$.

(ii) $g(\frac{t}{a}) = \frac{1}{c} g(t)$.

(iii) $g(t) + g(1-t) = 1$.

The general solution of (ii) alone reads $g(t) = t^\alpha Q(\frac{\ln t}{\ln a})$, where $Q(x+1) = Q(x)$ is an arbitrary 1-periodic function and $\alpha = \frac{\ln c}{\ln a}$.

Proposition 3.1. *The function $g = g_0$ is the unique bounded solution of the functional equations* (i) - (iii) *in* $[0,1]$.

Proof. **1.** First, we show that the function g_0 satisfies equations (i) - (iii). Clearly, g_0 satisfies (i) in view of (3.9) with $n = 0$, and (ii) follows immediately from (3.5), (3.8) and (3.9). In order to show that g_0 satisfies also equation (iii), we assume first that $x \in F$, i.e. x is of the form (3.5). Then in view of (3.6) we have

$$1 - x = (a-1)\sum_{\nu=1}^{\infty} \frac{\bar{\xi}_\nu}{a^\nu}$$

with $\bar{\xi}_\nu = 1 - \xi_\nu$, and in view of (3.8) we get

$$g_0(x) + g_0(1-x) = (c-1)\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{c^\nu} + (c-1)\sum_{\nu=1}^{\infty} \frac{\bar{\xi}_\nu}{c^\nu} = (c-1)\sum_{\nu=1}^{\infty} \frac{1}{c^\nu} = 1 .$$

In the case of $x \notin F$, i.e. $x \in G_{kn}$, x has the representation $x = x^- + \frac{a-2}{a^{n+1}} t$ with $0 < t < 1$, so that $g_0(x)$ is given by (3.9). In view of $1 - x = 1 - x^- - \frac{a-2}{a^{n+1}} t$, where $1 - x^- = y^+$ for a certain right end-point y^+, we get according to (3.10) that

$$g_0(1-x) = g_0(1-x^-) - \frac{c-2}{c^{n+1}} t .$$

This together with (3.9) implies that

$$g_0(x) + g_0(1-x) = g_0(x^-) + g_0(1-x^-) = 1$$

since $x^- \in F$.

2. Let g be a further solution of equations (i) - (iii). For $0 \le t \le 1$ we put $d(t) = |g_0(t) - g(t)|$. In view of (i) we have $d(t) = 0$ for $\frac{1}{a} \le t \le 1 - \frac{1}{a}$. Hence, if there exists a point $t_0 \in [0,1]$ with $d(t_0) > 0$, then for $t_1 = a \min\{t_0, 1-t_0\}$ we have $t_1 \in [0,1]$. We show that $d(t_1) = c\, d(t_0)$. In the case of $t_0 < \frac{1}{a}$ this follows immediately from (ii). In the case of $t_0 > 1 - \frac{1}{a}$ we first get from (iii) that $d(1-t_0) = d(t_0)$ and afterwards from (ii) that $d(t_1) = c\, d(t_0)$. Thus for the sequence $t_n = a \min\{t_{n-1}, 1 - t_{n-1}\}$ we obtain $d(t_n) = c^n d(t_0)$ and in view of $c \ge 2$ a contradiction to the boundedness of g ∎

Proposition 3.2. *Suppose that g satisfies properties* (ii) *and* (iii). *Then we have*

$$g\left(x_n + \frac{t}{a^n}\right) = g_0(x_n) + \frac{1}{c^n} g(t) \tag{3.11}$$

for $0 \le t \le 1$, with x_n from (3.7). *Moreover, $g(x_n) = g_0(x_n)$.*

Proof. Equation (ii) for $t = 0$ implies $g(0) = 0$, hence in view of $g_0(0) = 0$ we have an identity for $n = 0$. Assume that the assertion is true for a certain $n - 1$. Since $x_n \in F$, we have either $x_n \le \frac{1}{a}$ or $x_n \ge 1 - \frac{1}{a}$. In the first case $x_n = \frac{x_{n-1}}{a}$ and we get from (ii) and (3.8) that

$$g\left(x_n + \frac{t}{a^n}\right) = \frac{1}{c} g_0(x_{n-1}) + \frac{1}{c^n} g(t) = g_0(x_n) + \frac{1}{c^n} g(t)$$

for $0 \le t \le 1$. In the case of $x_n \ge 1 - \frac{1}{a}$ we have in view of $\xi_n = 1$ the representation

$$1 - x_n = (a-1)\sum_{\nu=1}^{n-1} \frac{1-\xi_\nu}{a^\nu} + (a-1)\sum_{\nu=n+1}^{\infty} \frac{1}{a^\nu} = y_{n-1} + \frac{1}{a^n}$$

with $y_{n-1} \le \frac{1}{a}$ and in view of (iii) and $g(1) = 1 - g(0) = 1$ the relation

$$1 - g(x_n) = g(1 - x_n) = g\left(y_{n-1} + \frac{1}{a^n}\right) = g_0(y_{n-1}) + \frac{1}{c^n}$$

which implies that

$$g(x_n) = 1 - g_0(y_{n-1}) - \frac{1}{c^n} = g_0(x_n)\,.$$

Now, we get by application of (iii) the relation

$$\begin{aligned} g\Big(x_n + \frac{t}{a^n}\Big) &= g\Big(1 - y_{n-1} - \frac{1}{a^n} + \frac{t}{a^n}\Big) = 1 - g\Big(y_{n-1} + \frac{1-t}{a^n}\Big) \\ &= 1 - g_0(y_{n-1}) - \frac{g(1-t)}{c^n} = g_0(x_n) + \frac{1}{c^n}\, g(t) \end{aligned}$$

for $0 \le t \le 1$, which proves (3.11) by induction. The second assertion of the proposition follows from (3.11) for $t = 0$ ∎

Remarks. 1. For $g = g_0$ and $t = (a-1)\sum_{\nu=1}^{\infty} \frac{\varepsilon_{n+\nu}}{a^\nu}$ equation (3.11) easily follows from $x = x_n + \frac{t}{a^n}$ and (3.8).

2. Equations (iii) and (3.11) imply

$$g\Big(z_n - \frac{t}{a^n}\Big) = g_0(z_n) - \frac{1}{c^n}\, g(t)$$

for $0 \le t \le 1$ with $z_n = 1 - x_n$ and $g(z_n) = g_0(z_n)$.

3. The statement of Proposition 3.1 is also valid if we replace (iii) by

$$g\Big(\frac{a-1}{a} + \frac{t}{a}\Big) = \frac{c-1}{c} + \frac{1}{c}\, g(t) \qquad (0 \le t \le 1)\,,$$

i.e. by (3.11) with $n = 1$. Proposition 3.2 implies that $g = g_0$ satisfies this equation. The proof of the uniqueness can be carried out analogously as in the second part of the proof of Proposition 3.1, however, with the sequence

$$t_n = \begin{cases} a\,t_{n-1} & \text{if } t_{n-1} < \frac{1}{a} \\ a\,t_{n-1} - a + 1 & \text{if } t_{n-1} > 1 - \frac{1}{a}. \end{cases}$$

Thus we have a generalization of a result of W. Sierpiński [10] concerning the case of $a = 3$ and $c = 2$, where g_0 is Cantor's singular function (cf. also [9: p. 241]). A non-constant $g : [0,1] \mapsto [0,1]$ is called (*strictly*) *singular*, if it is continuous and (strictly) increasing with $g'(t) = 0$ a.e. (cf. [6], where also some examples of strictly singular functions are given). In the case of $c = 2$, g_0 is a singular function which is constant on the closed intervals $\overline{G}_{kn}$, more precisely, (3.10) implies in view of $\gamma_k(2) = k$ that

$$g_0(t) = \frac{2k+1}{2^{n+1}} \qquad \text{for } t \in \overline{G}_{kn}\,. \tag{3.12}$$

Proposition 3.3. *In the case of $c = 2$, $g = g_0$ is the unique function of bounded variation on $[0,1]$ satisfying only*

$$g\Big(\frac{t}{a}\Big) = \frac{1}{2}\, g(t) \qquad \text{and} \qquad g(t) + g(1-t) = 1. \tag{3.13}$$

Proof. We show that every function g of bounded variation on [0,1] satisfying (3.13) has the property

$$g(t) = \frac{1}{2} \qquad \text{for } \frac{1}{a} \le t \le \frac{a-1}{a}. \tag{3.14}$$

Let D denote the total variation of g in the interval $G_0 = G_{00}$. In view of (3.11) with $c = 2$ and Lemma 3.1 we have

$$\bigvee_{G_{kn}}(g) = \frac{1}{2^n} D$$

for $k = 0, 1, \ldots, 2^n - 1$ and all $n \in \mathbb{N}$. Since the intervals G_{kn} are disjoint, for the total variation of g on the set G_m defined by (3.2) we get

$$\bigvee_{G_m}(g) = \sum_{n=0}^{m} \sum_{k=0}^{2^n-1} \frac{1}{2^n} D = (m+1) D .$$

For $m \to \infty$ this implies $D = 0$ since g has bounded variation, i.e. (3.14) is valid. Now the statement follows from Proposition 3.1 ∎

4. Properties of the eigenfunctions

In order to obtain relations between eigenfunctions of the integral equation (1.18), we first remember that a solution f of (1.18) with $a > 1$ is infinitely often differentiable and that we get by differentiation

$$\lambda f^{(n)}(t) = a^n \int_{at-a+1}^{at} f^{(n)}(\tau)\, d\tau .$$

Hence, the n-th derivative $f^{(n)}$ is also an eigenfunction of (1.18) to the eigenvalue λa^{-n}, so far as $f^{(n)}$ does not vanish identically (cf. [1: Formula (6.6) for $\lambda = \frac{1}{b}$]). Next, we shall see that each derivative of f can be expressed as a linear combination of f with different arguments. For the first derivative f' we have

$$f'(t) = \frac{a}{\lambda}[f(at) - f(at-a+1)] . \tag{4.1}$$

In order to obtain a representation for the higher derivatives, we need the former sequences γ_n and ε_n.

Lemma 4.1. *Suppose that f is an eigenfunction of (1.18) with the eigenvalue λ and $n \in \mathbb{N}_0$. Then we have*

$$f^{(n)}(t) = \lambda^{-n} a^{\frac{n(n+1)}{2}} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu f(a^n t - \gamma_\nu) . \tag{4.2}$$

Proof. For $n = 0$ this equation is an identity. If (4.2) is true for an integer n, then we have in view of (4.1), (2.2) and (2.5) that

$$\begin{aligned} f^{(n+1)}(t) &= \lambda^{-n} a^{\frac{n(n+1)}{2}+n} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu f'(a^n t - \gamma_\nu) \\ &= \lambda^{-n-1} a^{\frac{n(n+1)}{2}+n+1} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu \Big(f(a^{n+1}t - a\gamma_\nu) - f(a^{n+1}t - a\gamma_\nu - \gamma_1) \Big) \\ &= \lambda^{-n-1} a^{\frac{(n+1)(n+2)}{2}} \sum_{\nu=0}^{2^{n+1}-1} \varepsilon_\nu f(a^{n+1}t - \gamma_\nu) , \end{aligned}$$

such that (4.2) is proved by induction ∎

Taking into account that $f = \phi_n$ is an eigenfunction of (1.18) to the eigenvalue $\lambda = \frac{a^n}{b}$, and considering

$$\lambda^{-n} a^{\frac{n(n+1)}{2}} = \frac{a^{\frac{n(n+1)}{2}} b^n}{a^{n^2}} = \frac{b^n}{a^{\frac{n(n-1)}{2}}}$$

as well as $\phi(t) = \phi_n^{(n)}(t)$ for all t, we get the following inversion of (1.14).

Corollary 4.1. *For all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}_0$, the solution ϕ of (1.1) – (1.2) has the representation*

$$\phi(t) = \frac{b^n}{a^{\frac{n(n-1)}{2}}} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu \phi_n(a^n t - \gamma_\nu) . \tag{4.3}$$

Proposition 4.1. *The polynomials ψ_n have the property*

$$\sum_{\nu=0}^{2^n-1} \varepsilon_\nu \psi_m(t - \gamma_\nu) = \frac{m!}{(m-n)!} \frac{a^{\frac{n(2m-n+1)}{2}}}{b^n} \psi_{m-n}\Big(\frac{t}{a^n}\Big) \tag{4.4}$$

for arbitrary $m \geq n \geq 0$.

Proof. We apply Lemma 4.1 with $f = \psi_m$ and $\lambda = \frac{a^{m+1}}{b}$ and use that

$$\psi_m^{(n)}(t) = \frac{m!}{(m-n)!} \psi_{m-n}(t) \quad \text{and} \quad \lambda^{-n} a^{\frac{n(n+1)}{2}} = \frac{a^{\frac{n(n+1)}{2}} b^n}{a^{n(m+1)}} = \frac{b^n}{a^{\frac{n(2m-n+1)}{2}}} .$$

Relation (4.4) is proved after replacing t by $\frac{t}{a^n}$ ∎

We remark that for $n > m \geq 0$ the left-hand side of (4.4) vanishes, since the sums (2.11) vanish for these m and n. This is also the reason why for $m > n$ the degree of the polynomials (4.4) reduces from m to $m - n$. In particular, for $m \geq n = 1$ we have

$$\psi_m(t) - \psi_m(t - a + 1) = m a^{m-1}(a-1)\psi_{m-1}\Big(\frac{t}{a}\Big). \tag{4.5}$$

By analytic continuation this equation is even valid for all a different from the poles of ψ_m as a function of a (these poles lie on the circle $|a| = 1$). For $t = \frac{a}{2}$ (4.5) simplifies in view of (1.1) to

$$\psi_m\Big(\frac{a}{2}\Big) = \frac{m}{2}a^{m-1}(a-1)\psi_{m-1}\Big(\frac{1}{2}\Big) \tag{4.6}$$

for m odd. For all $m \in \mathbb{N}_0$, considering (1.3) and the generating function $\frac{p}{e^p-1}$ for the Bernoulli numbers B_μ, we can derive from (1.9) the representation

$$\psi_m(t) = \sum_{\mu=0}^{m} \binom{m}{\mu} B_\mu (1-a)^\mu a^{m-\mu} \psi_{m-\mu}\Big(\frac{t}{a}\Big) ,$$

which for $t = \frac{a}{2}$ contains (4.6) as a special case. Moreover, for all $m \in \mathbb{N}$, (1.7), (1.9) and

$$\frac{\partial}{\partial p} e^{tp}\Phi(p) = e^{tp}\Phi(p)\Big(t + \frac{d}{dp}\ln\Phi(p)\Big)$$

imply by comparison of coefficients the recursion formula

$$\psi_m(t) = \Big(t - \frac{1}{2}\Big)\psi_{m-1}(t) + \frac{1}{m}\sum_{\mu=2}^{m}\binom{m}{\mu} B_\mu \frac{(a-1)^\mu}{a^\mu - 1}\psi_{m-\mu}(t) ,$$

which for $t = 0$ is already known from [1].

In the following, we once more restrict ourselves to $a \geq 2$ and apply Lemma 4.1 to the solution ϕ of (1.1) - (1.2), i.e. we consider $f = \phi$ and $\lambda = \frac{1}{b}$. For $t \in F_{kn}$, i.e. according to (3.3) for

$$\frac{\gamma_k}{a^n} \leq t \leq \frac{\gamma_k + 1}{a^n} ,$$

we have $0 \leq a^n t - \gamma_k \leq 1$, but in view of (3.1) $a^n t - \gamma_\nu \notin (0,1)$ for $\nu \neq k$. Hence, for the solution ϕ of (1.1) - (1.2) with $a \geq 2$, which vanishes outside of (0,1), we get from Lemma 4.1

$$\phi^{(n)}(t) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^n \phi(a^n t - \gamma_k) \qquad \text{for } \ t \in F_{kn} , \tag{4.7}$$

and otherwise we have $\phi^{(n)}(t) = 0$, namely for $t \in G_{n-1}$ with $n \geq 1$. In view of $\phi(t) > 0$ for $t \in (0,1)$ this result implies [13: Proposition 4.1] that F_n is the support of $\phi^{(n)}$ and, moreover, for $n = 2$ that $\phi(t)$ is strictly convex for t in F_{02} or F_{32}, and strictly concave for t in F_{12} or F_{22}. In the case of $a = 2$ where $\gamma_k = k$ formula (4.7) reduces to

$$\phi^{(n)}(t) = \varepsilon_k 2^{\frac{n(n+3)}{2}} \phi(2^n t - k) \qquad (k = [2^n t]) , \tag{4.8}$$

in particular to $\phi^{(n)}(\frac{k}{2^n}) = 0$ (cf. [11]). Formula (4.7) is very useful for the calculation of the L_2-norms of $\phi^{(n)}$, namely

$$\|\phi^{(n)}\|^2 = a^{n(n+1)} b^{2n} \sum_{k=0}^{2^n-1} \int_{\gamma_k/a^n}^{(\gamma_k+1)/a^n} \phi^2(a^n t - \gamma_k)\, dt = 2^n a^{n^2} b^{2n} \|\phi\|^2 .$$

Moreover, we find for the corresponding scalar product by m partial integrations

$$\big(\phi^{(n)}, \phi^{(n+2m)}\big) = (-1)^m \big(\phi^{(n+m)}, \phi^{(n+m)}\big) = (-1)^m 2^{n+m} a^{(n+m)^2} b^{2(n+m)} \|\phi\|^2 ,$$

whereas $\big(\phi^{(n)}, \phi^{(n+2m+1)}\big) = 0$ in view of the symmetry $\phi(t) = \phi(1-t)$.

5. Relations with polynomials

For $a > 2$ and $t \in \overline{G}_{kn}$ given by (3.2) we have the inequality $\frac{1}{a} \le a^n t - \gamma_k \le 1 - \frac{1}{a}$. Hence, we get in view of $\overline{G}_{kn} \subset F_{kn}$, (4.7) and $\phi(\tau) = b$ for $\frac{1}{a} \le \tau \le 1 - \frac{1}{a}$ that

$$\phi^{(n)}(t) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^n \phi(a^n t - \gamma_k) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^{n+1} \qquad (t \in \overline{G}_{kn}) .$$

Thus for $t \in \overline{G}_{kn}$, $\phi(t)$ is a polynomial of degree n, a fact which is already known from [1], but now we also know the main term of this polynomial:

$$\phi(t) = \varepsilon_k \frac{a^{\frac{n(n+1)}{2}} b^{n+1}}{n!} t^n + \dots \qquad (t \in \overline{G}_{kn}) . \tag{5.1}$$

Moreover, we can even determine the complete polynomials and include the limit case $a = 2$, where the intervals $\overline{G}_{kn}$ degenerate to single points $\frac{2k+1}{2^n+1}$. Since G lies densely in $[0,1]$, the function ϕ is uniquely determined by means of these polynomials and continuity.

Theorem 5.1. *In the case of $a \ge 2$ and t in one of the closed intervals $\overline{G}_{kn}$ for $k = 0, 1, \dots, 2^n - 1$ $(n \in \mathbb{N})$, the solution ϕ of (1.1) – (1.2) has the representation*

$$\phi(t) = c_n \sum_{\nu=0}^{2k} \varepsilon_\nu \psi_n(a^{n+1} t - \gamma_\nu) \qquad (t \in \overline{G}_{kn}) \tag{5.2}$$

where c_n is given by

$$c_n = \frac{b^{n+1}}{a^{\frac{n(n+1)}{2}} n!} = \frac{1}{a^{\frac{(n+1)(n-2)}{2}} (a-1)^{n+1} n!} . \tag{5.3}$$

Proof. We use the representation (4.3) with $n+1$ instead of n. For $t \in \overline{G}_{kn}$, i.e.

$$\frac{\gamma_{2k} + 1}{a^{n+1}} \le t \le \frac{\gamma_{2k+1}}{a^{n+1}} ,$$

we have the inequalities $a^{n+1} t - \gamma_{2k+1} \le 0$ and $a^{n+1} t - \gamma_{2k} \ge 1$. According to (3.1) and $\phi_{n+1}(\tau) = 0$ for $\tau \le 0$, the terms $\phi_{n+1}(a^{n+1} t - \gamma_\nu)$ vanish for $\nu \ge 2k+1$, but for $\nu \le 2k$, in view of (1.16) with $n+1$ instead of n, we have the representations

$$\phi_{n+1}(a^{n+1} t - \gamma_\nu) = \frac{1}{n!} \psi_n(a^{n+1} t - \gamma_\nu)$$

for $\nu = 0, 1, \dots, 2k$. This altogether implies the assertion ∎

We remark that, for $k = 0$, formula (5.2) reduces to (1.12).

Next, we are going to extend (5.2) to the larger interval $F_{kn} \supset G_{kn}$.

Proposition 5.1. *For $a \geq 2$ and $t \in F_{kn}$, i.e. $t = \frac{\gamma_k+\tau}{a^n}$ with $0 \leq \tau \leq 1$, the solution ϕ of* (1.1) – (1.2) *has the property*

$$\phi\Big(\frac{\gamma_k+\tau}{a^n}\Big) - \varepsilon_k \phi\Big(\frac{\tau}{a^n}\Big) = c_{n-1}\sum_{\nu=0}^{k-1} \varepsilon_\nu \psi_{n-1}(\gamma_k+\tau-\gamma_\nu) \qquad (0 \leq \tau \leq 1) \tag{5.4}$$

where c_n is given by (5.3).

Proof. According to $0 \leq a^n t - \gamma_k = \tau \leq 1$ for $t \in F_{kn}$ and $\gamma_\nu + 1 \leq \gamma_{\nu+1}$ for $\nu \geq 0$ (cf. Lemma 2.2) we have the relations

$$a^n t - \gamma_\nu = \gamma_k + \tau - \gamma_\nu \begin{cases} \geq 1 & \text{for } \nu < k \\ \in [0,1] & \text{for } \nu = k \\ \leq 0 & \text{for } \nu > k. \end{cases}$$

Hence, (5.4) follows from (4.3) in view of $\phi_n(t) = 0$ for $t \leq 0$, as well as (1.15) and (1.16) ∎

By m differentiations of (5.4) we get in view of $\phi^{(m)}(0) = 0$ and $\psi'_m = m\psi_{m-1}$ the

Corollary 5.1. *In the case of $a \geq 2$ and $n > m \geq 0$ the derivatives $\phi^{(m)}$ of the solution of* (1.1) – (1.2) *have the values*

$$\phi^{(m)}\Big(\frac{\gamma_k}{a^n}\Big) = \frac{a^{mn}\, c_{n-1}\,(n-1)!}{(n-m-1)!}\sum_{\nu=0}^{k-1} \varepsilon_\nu \psi_{n-m-1}(\gamma_k - \gamma_\nu) \tag{5.5}$$

with $0 \leq k \leq 2^n - 1$.

In particular, in the case of $a = 2$ where $\gamma_\nu = \nu$ the values (5.5) with $m = 0$ simplify to

$$\phi\Big(\frac{k}{2^n}\Big) = \frac{1}{2^{\frac{n(n-3)}{2}}(n-1)!}\sum_{\nu=0}^{k-1} \varepsilon_\nu \psi_{n-1}(k-\nu)\,. \tag{5.6}$$

Thus in the case of $a = 2$ we obtain, for example,

$$\phi\Big(\frac{1}{2}\Big) = 2\,, \quad \phi\Big(\frac{1}{4}\Big) = 1\,, \quad \phi\Big(\frac{1}{8}\Big) = \frac{1}{9}\,, \quad \phi\Big(\frac{3}{8}\Big) = \frac{17}{9}\,,$$

$$\phi\Big(\frac{1}{16}\Big) = \frac{1}{24}\,, \quad \phi\Big(\frac{3}{16}\Big) = \frac{145}{288}\,, \quad \phi\Big(\frac{5}{16}\Big) = \frac{431}{288}\,, \quad \phi\Big(\frac{7}{16}\Big) = \frac{575}{288}\,, \ldots\,.$$

We remark that the particular formula (5.6) can also be derived from (1.17). Namely, in the case of $a = 2$ the left-hand side from (1.17) can be written in the form

$$\begin{aligned} &\sum_{\nu_i \geq 0} \phi\Big(t - \frac{\nu_1}{2^n} - \ldots - \frac{\nu_n}{2}\Big) \\ &\quad = \phi(t) + \phi\Big(t - \frac{1}{2^n}\Big) \\ &\qquad + 2\sum_{\nu=0}^{\infty}(\nu^2+\nu+1)\Big(\phi\Big(t - \frac{4\nu+2}{2^n}\Big) + \phi\Big(t - \frac{4\nu+3}{2^n}\Big)\Big) \\ &\qquad + 2\sum_{\nu=1}^{\infty}(\nu^2+1)\Big(\phi\Big(t - \frac{4\nu}{2^n}\Big) + \phi\Big(t - \frac{4\nu+1}{2^n}\Big)\Big). \end{aligned}$$

Putting in (1.17) with $a = 2$ successively

$$t = \frac{1}{2^n}, \qquad t = \frac{2}{2^n}, \qquad t = \frac{3}{2^n}, \quad \ldots$$

we obtain for the values $\phi(\frac{k}{2^n})$ $(k = 0, 1, 2, \ldots)$ a linear system of equations with a Toeplitz matrix T, which is the inverse of the Toeplitz matrix (ε_{i-j}) $(\varepsilon_i = 0$ for $i < 0)$

$$T = \begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 & 1 \\ . & . & . & . & . & . & . \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 \\ -1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ . & . & . & . & . & . \end{pmatrix}.$$

Since the right-hand side of (1.17) with $a = 2$ is $c_{n-1}\psi_{n-1}(2^n t)$ (cf. (5.3)), we obtain (5.6) after simple calculations.

6. Reduced representations

The polynomial relation (5.4) reads for $k = 1$

$$\phi\Big(\frac{\gamma_1 + \tau}{a^n}\Big) + \phi\Big(\frac{\tau}{a^n}\Big) = c_{n-1}\psi_{n-1}(\gamma_1 + \tau) \qquad (0 \le \tau \le 1)\,, \tag{6.1}$$

where c_n is given by (5.3). For large k, (5.4) is rather redundant so that we want to derive a reduced representation. For convenience, the first parameters l_ν, ε_{l_ν} and k_ν appearing in the later formula (6.2) are shown in Table 2 for the interesting indices ν with $d_\nu \neq 0$.

Proposition 6.1. *Assume that $a \ge 2$ and that the number $k \in \mathbb{N}$ has the dyadic representation $k = d_0 + d_1 2 + d_2 2^2 + \ldots + d_s 2^s$, $d_s = 1$ and $d_\sigma \in \{0, 1\}$. Then with the notations $k_\nu = d_0 + d_1 2 + \ldots + d_\nu 2^\nu$ and $l_\nu = d_{\nu+1} + d_{\nu+2} 2 + \ldots + d_s 2^{s-\nu-1}$ for $0 \le \nu \le s$ we have the relation*

$$\phi\Big(\frac{\gamma_k + \tau}{a^n}\Big) - \varepsilon_k \phi\Big(\frac{\tau}{a^n}\Big) = \sum_{\nu=0}^{s} \varepsilon_{l_\nu} d_\nu c_{n-\nu-1}\psi_{n-\nu-1}\Big(\frac{\gamma_{k_\nu} + \tau}{a^\nu}\Big) \tag{6.2}$$

for $0 \le \tau \le a^\sigma$, whenever $d_0 = d_1 = \ldots = d_{\sigma-1} = 0$ and $d_\sigma \neq 0$.

Proof. Equation (6.2) can be derived by successive application of (4.5) to (5.4). But an inductive proof is more lucid. For $k = 1$, the representation is true in view of (6.1). In order to prove the assertion by induction, we assume that (6.2) is valid for a fixed k and take into consideration that the parameters d_ν, k_ν, l_ν, s and σ depend on k. Moreover, we recognize that $k = k_\nu + l_\nu 2^{\nu+1}$, i.e. $k \equiv k_\nu \bmod 2^{\nu+1}$, $0 \le k_\nu < 2^{\nu+1}$.

k	dyadic	ε_k	l_0	ε_{l_0}	k_0	l_1	ε_{l_1}	k_1	l_2	ε_{l_2}	k_2	l_3	ε_{l_3}	k_3
0	0	1												
1	1	-1	0	1	1									
2	10	-1				0	1	2						
3	11	1	1	-1	1	0	1	3						
4	100	-1							0	1	4			
5	101	1	2	-1	1				0	1	5			
6	110	1				1	-1	2	0	1	6			
7	111	-1	3	1	1	1	-1	3	0	1	7			
8	1000	-1										0	1	8
9	1001	1	4	-1	1							0	1	9
10	1010	1				2	-1	2				0	1	10
11	1011	-1	5	1	1	2	-1	3				0	1	11
12	1100	1							1	-1	4	0	1	12
13	1101	-1	6	1	1				1	-1	5	0	1	13
14	1110	-1				3	1	2	1	-1	6	0	1	14
15	1111	1	7	-1	1	3	1	3	1	-1	7	0	1	15

Table 2: The first parameters l_ν, ε_{l_ν} and k_ν

1. *Induction from k to $2k$*: In view of $2k = d_0 2 + d_1 2^2 + \ldots + d_s 2^{s+1} = 2k_\nu + 2^{\nu+2} l_\nu$, the parameters of $2k$ depend on the parameters of k in the following way:

k	d_ν	k_ν	l_ν	s	σ
$2k$	$d_{\nu-1}$	$2k_{\nu-1}$	$l_{\nu-1}$	$s+1$	$\sigma+1$

Table 3: The parameters of $2k$ expressed by those of k

where $d_{-1} = k_{-1} = l_s = 0$ and $l_{-1} = k_s = k$. Making in (6.2) the substitution $n \mapsto n-1$, $\nu \mapsto \nu - 1$, $\tau \mapsto \frac{\tau}{a}$, so that $0 \le \tau \le a^{\sigma+1}$ for the new τ, we obtain

$$\phi\Big(\frac{a\gamma_k + \tau}{a^n}\Big) - \varepsilon_k \phi\Big(\frac{\tau}{a^n}\Big) = \sum_{\nu=0}^{s+1} \varepsilon_{l_{\nu-1}} d_{\nu-1} c_{n-\nu-1} \psi_{n-\nu-1}\Big(\frac{a\gamma_{k_{\nu-1}} + \tau}{a^\nu}\Big),$$

and in view of $a\gamma_k = \gamma_{2k}$, $\varepsilon_k = \varepsilon_{2k}$ and Table 3 this is nothing else than (6.2) with $2k$ instead of k.

2. *Induction from $2k$ to $2k+1$*: Formula (6.2) reads for $2k$ instead of k and $\gamma_1 + \tau$ instead of τ

$$\phi\Big(\frac{\gamma_{2k} + \gamma_1 + \tau}{a^n}\Big) - \varepsilon_{2k} \phi\Big(\frac{\gamma_1 + \tau}{a^n}\Big) = \sum_{\nu=0}^{s} \varepsilon_{l_\nu} d_\nu c_{n-\nu-1} \psi_{n-\nu-1}\Big(\frac{\gamma_{k_\nu} + \gamma_1 + \tau}{a^\nu}\Big)$$

where the parameters are those belonging to $2k$. According to $d_0 = 0$ we have $\sigma \ge 1$, so that the last equation is valid at least for $0 \le \gamma_1 + \tau \le a = \gamma_1 + 1$, i.e. at least for

$0 \le \tau \le 1$. Multiplying (6.1) by ε_{2k} and adding the result to the foregoing equation, we obtain

$$\phi\Big(\frac{\gamma_{2k}+\gamma_1+\tau}{a^n}\Big)+\varepsilon_{2k}\phi\Big(\frac{\tau}{a^n}\Big)$$
$$=\varepsilon_{2k}c_{n-1}\psi_n(\gamma_1+\tau)+\sum_{\nu=1}^{s}\varepsilon_{l_\nu}d_\nu c_{n-\nu-1}\psi_{n-\nu-1}\Big(\frac{\gamma_{k_\nu}+\gamma_1+\tau}{a^\nu}\Big).$$

But this is nothing else than (6.2) with $2k+1$ instead of $2k$, since $\gamma_{2k}+\gamma_1=\gamma_{2k+1}$, $\varepsilon_{2k}=-\varepsilon_{2k+1}$, and k_ν of $2k$ is even so that $\gamma_{k_\nu}+\gamma_1=\gamma_{k_\nu+1}$ for $\nu\ge 1$, and the parameters of $2k+1$ depend on the parameters of $2k$ in the following way:

$2k$	d_ν	k_ν	l_ν	s	σ
$2k+1$	d_ν	$k_\nu+1$	l_ν	s	0

Table 4: The parameters of $2k+1$ expressed by those of $2k$

for $\nu\ge 1$, whereas $d_0=1$ for $2k+1$ and $\varepsilon_{l_0}=\varepsilon_{2k}$ for the parameter l_0 of $2k+1$ ∎

Remark. For large k formula (6.2) has the advantage that the sum on the right-hand side consists of $O(\ln k)$ terms only compared to the k terms in the sum of (5.4). Moreover, many d_ν in (6.2) can vanish. If the terms with $d_\nu=0$ are cancelled, then the remaining terms have alternating signs ending with $\varepsilon_{l_s}=1$ in view of $l_s=0$. Hence, (6.2) implies

$$\phi\Big(\frac{\gamma_k+\tau}{a^n}\Big)+\phi\Big(\frac{\gamma_m+\tau}{a^n}\Big)=c_{n-s-1}\psi_{n-s-1}\Big(\frac{\gamma_k+\tau}{a^s}\Big)\qquad(0\le\tau\le a^\sigma)$$

with $m=k_{s-1}$, i.e. $k=m+2^s$ and $\gamma_k=\gamma_m+a^s\gamma_1$.

For $t\in\overline{G}_{kn}$, from $\overline{G}_{kn}\subset F_{kn}$, (6.2), (5.3) and (1.12) we obtain instead of (5.2) the reduced polynomial representation

$$\phi\Big(\frac{\gamma_{2k}+\tau}{a^{n+1}}\Big)=\varepsilon_k c_n\psi_n(\tau)+\sum_{\nu=1}^{s}\varepsilon_{l_\nu}d_\nu c_{n-\nu}\psi_{n-\nu}\Big(\frac{\gamma_{k_\nu}+\tau}{a^\nu}\Big)\tag{6.3}$$

where $1\le\tau\le a-1$, and the parameters d_ν, k_ν, l_ν and s are those of $2k$. The first term of (6.3) cannot be included into the sum with $\nu=0$ in view of $d_0=0$.

7. Approximation by splines

Finally, we return to the general case $a>1$. From (1.3) we observe that the Laplace transform Φ of the solution ϕ of (1.1) - (1.2) is the limit of

$$G_n(p)=\prod_{k=0}^{n-1}\frac{1-e^{-p/(ba^k)}}{p/(ba^k)}\tag{7.1}$$

for $n \to \infty$. On account of Lemma 2.3 we have for $n \geq 1$

$$G_n(p) = \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu e^{-\frac{\gamma_\nu}{a^n} p} . \tag{7.2}$$

According to $\mathcal{L}^{-1}\{p^{-n}\} = \frac{t^{n-1}}{(n-1)!}$ and the shift property of the Laplace transform, we obtain for the original function g_n of G_n the representation

$$g_n(t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu (a^n t - \gamma_\nu)_+^{n-1} \tag{7.3}$$

where c_n is given by (5.3) and $t_+ = t$ for $t \geq 0$ and $t_+ = 0$ elsewhere. We see that the functions g_n are splines consisting of piecewise polynomials of degree at most $n-1$. Moreover, $g_n(t) = 0$ for $t \notin (0,1)$ since the sums (2.11) vanish for $m < n$, and according to $G_n(0) = 1$ we have $\int_0^1 g_n(t)\,dt = 1$. In view of $G_n(p) \to \Phi(p)$ we get

$$\lim_{n\to\infty} \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu e^{-\frac{\gamma_\nu}{a^n} p} = \int_0^1 e^{-pt}\phi(t)\,dt$$

and, moreover, from the proof of [1: Theorem 3.1] we know that g_n is uniformly convergent to the solution ϕ of (1.1) - (1.2), i.e.

$$\phi(t) = \lim_{n\to\infty} c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu (a^n t - \gamma_\nu)_+^{n-1} . \tag{7.4}$$

If we introduce the kernel

$$k_1(s,t) = \begin{cases} b & \text{for } \frac{s}{a} \leq t \leq \frac{s+a-1}{a} \\ 0 & \text{elsewhere,} \end{cases} \tag{7.5}$$

then equation (1.1) can be written as Fredholm integral equation

$$\phi(t) = \int_0^1 k_1(s,t)\phi(s)\,ds .$$

It is possible to calculate also the iterated kernels k_n defined by

$$k_{n+1}(s,t) = \int_0^1 k_1(s,\tau)k_n(\tau,t)\,d\tau .$$

Proposition 7.1. *For the iterated kernels k_n $(n \geq 1)$ we have the representation*

$$k_n(s,t) = g_n\Big(t - \frac{s}{a^n}\Big)$$

where the splines g_n are given by (7.3), i.e.

$$k_n(s,t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu (a^n t - s - \gamma_\nu)_+^{n-1} . \tag{7.6}$$

Proof. Formula (7.6) is true for $n = 1$. Assume that (7.6) is valid for a fixed $n \geq 1$. In view of (7.5) we have

$$\begin{aligned} &\varepsilon_\nu \int_0^1 k_1(s,\tau)(a^n t - \tau - \gamma_\nu)_+^{n-1} d\tau \\ &= b\varepsilon_\nu \int_{s/a}^{(s+a-1)/a} (a^n t - \tau - \gamma_\nu)_+^{n-1} d\tau \\ &= \frac{b\varepsilon_{2\nu+1}}{a^n n}\left(a^{n+1}t - s - \gamma_{2\nu+1}\right)_+^n + \frac{b\varepsilon_{2\nu}}{a^n n}\left(a^{n+1}t - s - \gamma_{2\nu}\right)_+^n, \end{aligned}$$

where we have used (2.2) and (2.5). Hence (7.6) follows by $c_n = \frac{b}{a^n n} c_{n-1}$ and induction ∎

Starting with $f_0(t) = k_1(0,t)$ and calculating the iterates $f_n = Lf_{n-1}$, we find $f_n(t) = g_{n+1}(t)$, and (7.4) follows once more from [1: Theorem 3.1].

The iterates f_n of the function f_0, $f_0(t) = 1$ for $t \in [0,1]$ and $f_0(t) = 0$ elsewhere, have the similar representations

$$f_n(t) = \int_0^1 k_n(s,t)\, ds = \frac{c_{n-1}}{n} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu \left((a^n t - \gamma_\nu)_+^n - (a^n t - \gamma_\nu - 1)_+^n\right)$$

with t_+ defined as before, and they also converge to the solution ϕ of (1.1) - (1.2). In the case of $a = 2$ where $\gamma_\nu = \nu$ the last representations reduce to

$$f_n(t) = \frac{1}{2^{\frac{n(n-3)}{2}} n!} \sum_{\nu=0}^{2^n-1} (\varepsilon_\nu - \varepsilon_{\nu-1})(2^n t - \nu)_+^n \tag{7.7}$$

with $\varepsilon_{-1} = 0$, where the coefficients $\varepsilon_\nu - \varepsilon_{\nu-1}$ for $\nu \geq 1$ were calculated by (2.8). Let us mention that the function $f = f_n$ of (7.7) is the (unique up to a constant factor) non-vanishing L-integrable solution of a particular two-scale difference equation, which arises from (1.1) with $a = 2$ by means of the trapezoidal rule (cf. [2]).

Corrections. Unfortunately, [1] contains some misprints. On p. 164^1 replace $\Phi(0,p)$ by $\Phi(0,a)$. On p. 164^9 cancel: *quad*. On p. 165_3 replace n at the top of the product by $n-1$. On p. 176^7 replace (6.8) by (6.7). Moreover, the proof of the corollary on p. 176 becomes more lucid, if one recognizes that the first relation in (8.1) is also valid for $t < 0$.

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