# **Cantor Sets and Integral-Functional Equations**

### L. **Berg and** M. **Krüppel**

**Abstract.** In this paper, we continue our considerations in [1] on a homogeneous integralfunctional equation with a parameter  $a > 1$ . In the case of  $a > 2$  the solution  $\phi$  satisfies relations containing polynomials. By means of these polynomial relations the solution can explicitly be computed on a Cantor set with Lebesgue measure 1. Thus the representation of the solution  $\phi$ is immediately connected with the exploration of some Cantor sets, the corresponding singular functions of which can be characterized by a system of functional equations depending on a. In the limit case  $a = 2$  we get a formula for the explicit computation of  $\phi$  in all dyadic points. We also calculate the iterated kernels and approximate  $\phi$  by splines in the general case  $a > 1$ .

Keywords: *Integral-functional equations, generating functions, Cantor sets, singular functions, relations containing polynomials, iterated kernels, approximation by splines* 

AMS subject classification: 45 D 05, 39 B 22, 34 K 15, 26 A 30, 41 A 15

### 1. Introduction

In [1] we have shown that the homogeneous integral-functional equation

**ction**

\nshown that the homogeneous integral-functional equation

\n
$$
\phi(t) = L\phi(t), \qquad L\phi(t) = b \int_{a^t - a + 1}^{a^t} \phi(\tau) d\tau \qquad (b = \frac{a}{a - 1}), \tag{1.1}
$$

where  $a > 1$  is a fixed parameter and  $t \in \mathbb{R}$ , has a unique compactly supported solution up to a constant factor. Since the support is contained in  $[0,1]$ , the constant factor can be fixed by the value of its integral:

geneous integral-functional equation  
\n
$$
t = b \int_{at-a+1}^{at} \phi(\tau) d\tau \qquad (b = \frac{a}{a-1}), \qquad (1.1)
$$
\n
$$
t \in \mathbb{R}, \text{ has a unique compactly supported solution\nimport is contained in [0,1], the constant factor can\n
$$
\int_{0}^{1} \phi(t) dt = 1.
$$
\n
$$
\int_{0}^{1} \phi(t) dt = 1.
$$
\n
$$
(1.2)
$$
\n
$$
12
$$
\n
$$
t = 2, \qquad (1.3)
$$
$$

G. J. Wirsching has considered in  $[12]$  the case  $a = 3$  and in the paper  $[13]$  also the case  $a > \frac{3}{2}$ , where  $\phi$  is the limiting density of a certain transition probability of a non-homogeneous Markov process arising in a combinatorial problem. The case  $a = 2$ was considered by W. Volk in [11] in order to construct some subspaces of  $C^{\infty}[a, b]$ , which are spanned by translates of  $\phi$ .

Both authors: FB Mathematik der Universität, Universitatspl. 1, D-18051 Rostock

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In this paper we continue our considerations in [1], primary for  $a \geq 2$ . For this reason, we list such results of [1] which we will need afterwards and, moreover, we make some supplements to them. The solution is infinitely often differentiable, symmetric with respect to the point  $\frac{1}{2}$ , and monotone at both sides of  $\frac{1}{2}$ . The solution has the support  $[0,1]$  and it is strictly positive for  $t \in (0,1)$ . For  $a > 2$  it is a polynomial on each component of an open Cantor set with Lebesgue measure 1. The solution  $\phi$ of (1.1) - ( *1.2)* can be obtained by means of successive approximation. For every *L*integrable function  $f_0$  on the interval  $[0,1]$  with  $f_0(t) = 0$  for  $t \notin [0,1]$  and the property  $\int_0^1 f_0(t) dt = 1$ , the iterates  $f_n = Lf_{n-1}$   $(n \ge 1)$  converge uniformly on [0,1] to the solution  $\phi$  of (1.1) - (1.2). Hence, on account of a result of W. M. Gerstein and B. N. Sadowski, the operator *L* is contractive on a certain subspace of  $C^1[0,1]$  equipped with a metric  $\rho$  which is equivalent to the maximum norm (cf. [8]).  $e^{(0,1)}$ . For a<br> *p* ith Lebesgue meantle is the set of  $b$  of  $t \neq 2$ <br> *p* (*b*) = 0 for *t*  $\neq 2$ <br> *p* (*c*) = 0 for *t*  $\neq 2$ <br> *p* (*ba*<sup>k</sup>)<br> *p* (*ba*<sup>k</sup>)<br> *p* (*ba*<sup>k</sup>)

The Laplace transform  $\Phi$  of the compactly supported solution  $\phi$  of (1.1) - (1.2) has the product representation

to the maximum norm (cf. [8]).

\nof the company supported solution 
$$
\phi
$$
 of (1.1) - (1.2) has

\n
$$
\Phi(p) = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}
$$

\nto find that  $p$  is the number of numbers.

and the power series representation

$$
\Phi(p) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} p^n \tag{1.4}
$$

which are both convergent for all  $p \in \mathbb{C}$ . The coefficients of the series are rational functions with respect to *a* and, starting with  $\rho_0(a) = 1$  for  $n \ge 1$ , they can be determined by means of the recursion formula

$$
\Phi(p) = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(ba^2)}}{p/(ba^k)}
$$
(1.3)  
series representation  

$$
\Phi(p) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} p^n
$$
(1.4)  
convergent for all  $p \in \mathbb{C}$ . The coefficients of the series are rational func-  
ect to *a* and, starting with  $\rho_0(a) = 1$  for  $n \ge 1$ , they can be determined  
recursion formula  

$$
\rho_n(a) = \frac{1}{(n+1)(a^n-1)} \sum_{\nu=0}^{n-1} {n+1 \choose \nu} \rho_{\nu}(a)(1-a)^{n-\nu}
$$
(1.5)  
ave  

$$
\frac{1}{\Phi(p)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rho_n(\frac{1}{a}) p^n
$$
  $(|p| < 2b\pi)$  (1.6)  

$$
\ln \Phi(p) = \sum_{n=1}^{\infty} \frac{B_n}{n!n} \frac{(a-1)^n}{a^n-1} p^n
$$
  $(|p| < 2b\pi)$ , (1.7)  
ne Bernoulli numbers

Moreover, we have

$$
\frac{1}{\Phi(p)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rho_n \left(\frac{1}{a}\right) p^n \qquad (|p| < 2b\pi) \tag{1.6}
$$

and

$$
\ln \Phi(p) = \sum_{n=1}^{\infty} \frac{B_n}{n!n} \frac{(a-1)^n}{a^n - 1} p^n \qquad (|p| < 2b\pi) \;, \tag{1.7}
$$

where  $B_n$  are the Bernoulli numbers

l,

$$
\ln \Psi(p) = \sum_{n=1}^{\infty} \frac{1}{n!n} \frac{1}{a^n - 1} p^n \qquad (|p| < 2b\pi) ,
$$
\nre the Bernoulli numbers

\n
$$
B_0 = 1 \ , \quad B_1 = -\frac{1}{2} \ , \quad B_2 = \frac{1}{6} \ , \quad B_3 = 0 \ , \quad B_4 = -\frac{1}{30} \ , \dots
$$

The polynomials

$$
= \sum_{n=0}^{\infty} \frac{B_n}{n!} \rho_n \left(\frac{1}{a}\right) p^n \qquad (|p| < 2b\pi) \tag{1.6}
$$
\n
$$
= \sum_{n=1}^{\infty} \frac{B_n}{n!n} \frac{(a-1)^n}{a^n - 1} p^n \qquad (|p| < 2b\pi) \,, \tag{1.7}
$$
\nnumbers\n
$$
-\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0 \,, \quad B_4 = -\frac{1}{30} \,, \dots \,.
$$
\n
$$
\psi_n(t) = \sum_{\nu=0}^n {n \choose \nu} \rho_{n-\nu}(a) t^{\nu} \tag{1.8}
$$

will play an essential role later on. Note that in [1] we have used the abbreviation  $\psi_n$ for the polynomials (1.8) with  $\frac{1}{a}$  instead of a. The polynomials  $\psi_n$  have the generating function tor Sets and Integral-Functional Equations 999<br>
e that in [1] we have used the abbreviation  $\psi_n$ <br>
of a. The polynomials  $\psi_n$  have the generating<br>  $=\sum_{n=0}^{\infty} \frac{\psi_n(t)}{n!} p^n$  (1.9) antor Sets and Integral-Functional Equations 999<br>
ote that in [1] we have used the abbreviation  $\psi_n$ <br>
ad of a. The polynomials  $\psi_n$  have the generating<br>  $\psi_p$ ) =  $\sum_{n=0}^{\infty} \frac{\psi_n(t)}{n!} p^n$  (1.9)<br>
(t) =  $n \psi_{n-1}(t)$  (1. Cantor Sets and Integral-Functional Equations 999<br>
Note that in [1] we have used the abbreviation  $\psi_n$ <br>
ead of a. The polynomials  $\psi_n$  have the generating<br>  $(p) = \sum_{n=0}^{\infty} \frac{\psi_n(t)}{n!} p^n$  (1.9)<br>  $\frac{1}{n}(t) = n \psi_{n-1}(t)$  (1

$$
\frac{1}{a} \text{ instead of } a. \text{ The polynomials } \psi_n \text{ have the generating}
$$
\n
$$
e^{tp}\Phi(p) = \sum_{n=0}^{\infty} \frac{\psi_n(t)}{n!} p^n
$$
\n
$$
\psi'_n(t) = n \psi_{n-1}(t) \qquad (1.10)
$$
\n
$$
\psi_n(1-t) = (-1)^n \psi_n(t) \qquad (1.11)
$$
\n
$$
\text{in } \phi \text{ of } (1.1) \cdot (1.2) \text{ can be expressed by the polynomials}
$$
\n
$$
\frac{a-1}{a^{n+1}} \quad (n \ge 0), \text{ namely}
$$
\n
$$
= \frac{\psi_n(a^{n+1}t)}{n! \ a^{\frac{1}{2}(n+1)(n-2)}(a-1)^{n+1}} \qquad (1.12)
$$
\n
$$
\text{in } \phi \text{ of } (1.1) \cdot (1.2) \text{ and}
$$

and the properties

$$
\psi_n'(t) = n \psi_{n-1}(t) \tag{1.10}
$$

$$
\psi_n(1-t) = (-1)^n \psi_n(t) \tag{1.11}
$$

In the case of  $a \ge 2$  the solution  $\phi$  of  $(1.1)$  -  $(1.2)$  can be expressed by the polynomials  $\psi_n$  in the intervals  $\frac{1}{a^{n+1}} \le t \le \frac{a-1}{a^{n+1}}$   $(n \ge 0)$ , namely<br>  $\phi(t) = \frac{\psi_n(a^{n+1}t)}{(a^n + b)^n}$ 

$$
\phi(t) = \frac{\psi_n(a^{n+1}t)}{n! \ a^{\frac{1}{2}(n+1)(n-2)}(a-1)^{n+1}} \ . \tag{1.12}
$$

Also, the functions  $\phi_n$   $(n \in \mathbb{N}_0)$  defined by  $\phi_0 = \phi$  from  $(1.1) \cdot (1.2)$  and

$$
\psi_n'(t) = n \psi_{n-1}(t) \qquad (1.10)
$$
\n
$$
n(1-t) = (-1)^n \psi_n(t) \qquad (1.11)
$$
\n
$$
\phi \text{ of } (1.1) \cdot (1.2) \text{ can be expressed by the polynomials}
$$
\n
$$
\frac{\psi_n(a^{n+1}t)}{n! a^{\frac{1}{2}(n+1)(n-2)}(a-1)^{n+1}} \qquad (1.12)
$$
\ndefined by  $\phi_0 = \phi$  from (1.1) \cdot (1.2) and\n
$$
\phi_{n+1}(t) = \int_0^t \phi_n(\tau) d\tau \qquad (1.13)
$$
\ne. We recall for arbitrary  $a > 1$  the following relations\n
$$
\phi_n
$$
, namely\n
$$
1)^n \sum_{\nu_1, \dots, \nu_n \ge 0} \phi\left(\frac{t}{a^n} - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b}\right) \qquad (1.14)
$$
\n
$$
\frac{3^2}{2^n} (a-1)^n \phi(a^{-n}t) \qquad \text{for } t \le a-1 \qquad (1.15)
$$
\n
$$
\frac{1}{(n-1)!} \psi_{n-1}(t) \qquad \text{for } t \ge 1 \qquad (1.16)
$$

for  $n \geq 0$  are needed in this note. We recall for arbitrary  $a > 1$  the following relations between the functions  $\phi$ ,  $\psi_n$  and  $\phi_n$ , namely

$$
\varphi_{n+1}(t) = \int_{0}^{\infty} \varphi_{n}(\tau) d\tau
$$
\n(1.13)\n\nneeded in this note. We recall for arbitrary  $a > 1$  the following relations\n\nfunctions  $\phi$ ,  $\psi_{n}$  and  $\phi_{n}$ , namely\n\n
$$
\phi_{n}(t) = a^{\frac{n(n-3)}{2}} (a-1)^{n} \sum_{\nu_{1}, \dots, \nu_{n} \ge 0} \phi \left( \frac{t}{a^{n}} - \frac{\nu_{1}}{a^{n-1}b} - \dots - \frac{\nu_{n}}{b} \right)
$$
\n(1.14)\n\n
$$
n \in \mathbb{N}, \text{ in particular}
$$
\n
$$
\phi_{n}(t) = a^{\frac{n(n-3)}{2}} (a-1)^{n} \phi(a^{-n}t) \quad \text{for } t \le a-1
$$
\n(1.15)\n\n
$$
\phi_{n}(t) = \frac{1}{(n-1)!} \psi_{n-1}(t) \quad \text{for } t \ge 1
$$
\n(1.16)\n\n
$$
\sum_{\nu_{1} \ge 0} \phi \left( t - \frac{\nu_{1}}{a^{n-1}b} - \dots - \frac{\nu_{n}}{b} \right) = \frac{\psi_{n-1}(a^{n}t)}{(a^{n-3})}
$$
\n(1.17)

for all  $t$  and  $n \in \mathbb{N}$ , in particular

$$
\phi_n(t) = a^{\frac{n(n-3)}{2}} (a-1)^n \phi(a^{-n}t) \quad \text{for } t \leq a-1 \tag{1.15}
$$

as well as

$$
\begin{aligned}\n\text{articular} \\
&= a^{\frac{n(n-3)}{2}} (a-1)^n \phi(a^{-n}t) \qquad \text{for} \ \ t \le a-1 \\
\phi_n(t) &= \frac{1}{(n-1)!} \psi_{n-1}(t) \qquad \text{for} \ \ t \ge 1\n\end{aligned} \tag{1.15}
$$

and

$$
\phi_n(t) = a^{\frac{n(n-3)}{2}} (a-1)^n \sum_{\nu_1, \dots, \nu_n \ge 0} \phi\left(\frac{t}{a^n} - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b}\right) \qquad (1.14)
$$
  
\n1 t and  $n \in \mathbb{N}$ , in particular  
\n
$$
\phi_n(t) = a^{\frac{n(n-3)}{2}} (a-1)^n \phi(a^{-n}t) \qquad \text{for } t \le a-1 \qquad (1.15)
$$
  
\n11 as  
\n
$$
\phi_n(t) = \frac{1}{(n-1)!} \psi_{n-1}(t) \qquad \text{for } t \ge 1 \qquad (1.16)
$$
  
\n
$$
\sum_{\nu_1, \dots, \nu_n \ge 0} \phi\left(t - \frac{\nu_1}{a^{n-1}b} - \dots - \frac{\nu_n}{b}\right) = \frac{\psi_{n-1}(a^n t)}{(n-1)! \ a^{\frac{n(n-3)}{2}} (a-1)^n} \qquad (1.17)
$$
  
\n
$$
\ge \frac{1}{a^n}.
$$
 The solution  $\phi$  of (1.1) - (1.2) satisfies the equation  
\n
$$
\sum_{\nu = -\infty}^{+\infty} \phi\left(t - \frac{\nu}{a^n b}\right) = a^n b
$$

for  $t \geq \frac{1}{a^n}$ .

$$
\sum_{k=-\infty}^{+\infty} \phi\left(t-\frac{\nu}{a^n b}\right) = a^n b
$$

for all  $t \in \mathbb{R}$  and all  $n \in \mathbb{N}_0$ . In [1] this was proved only for  $n = 0$ , but the general form easily follows by means of (1.1) and induction.

The eigenvalue problem

$$
\lambda f(t) = \int_{at-a+1}^{at} f(\tau) d\tau
$$
\n
$$
= \phi = \phi_0 \text{ from (1.1)} - (1.2) \text{ for } \lambda_0 = \frac{1}{4}, \text{ and for the}
$$
\n
$$
\lambda f(t) = \int_{a}^{a} f(\tau) d\tau
$$
\n
$$
= \frac{1}{4} \int_{a}^{b} f(\tau) d\tau
$$

 $\lambda f(t)$ <br>with  $a > 1$  has the solution  $f = \phi$ <br>eigenvalues  $\lambda_n = \frac{a^n}{b}$   $(n \in \mathbb{N})$  the eig  $= \phi_0$  from (1.1) - (1.2) for  $\lambda_0 = \frac{1}{b}$ , and for the eigenvalues  $\lambda_n = \frac{a^n}{b}$  ( $n \in \mathbb{N}$ ) the eigenfunctions  $f = \psi_{n-1}$  and  $f = \phi_n$  (cf. (1.8) and (1.13)), which have non-compact support.

The aim of this paper is to investigate in detail the Cantor intervals for  $a > 2$ , in which the solution  $\phi$  of (1.1) - (1.2) is equal to certain polynomials, and to find these polynomials explicitly, i.e. to generalize (1.12) to the other Cantor intervals. The results are also valid in the limit case  $a = 2$ , where the Cantor intervals degenerate. In this connection we characterize the mapping between corresponding Cantor intervals for different a by Sierpiński-like functional equations. Moreover, for arbitrary  $a > 1$  we find the iterated kernels of the integral equat different a by Sierpiński-like functional equations. Moreover, for arbitrary  $a > 1$  we find the iterated kernels of the integral equation (1.1), as well as new spline approximations for the solution.

Besides of the foregoing results from [1], for the piecewise representation of the solution  $\phi$  of (1.1) - (1.2) by polynomials and for the approximation of  $\phi$  by splines we need an auxiliary sequence  $\gamma_n = \gamma_n(a)$  defined as follows: If *n* has the dyadic representation  $n = d_s \cdots d_1 d_0$  with  $d_s = 1$  and  $d_{\nu} \in \{0, 1\}$ , then

$$
\gamma_n = (a-1) \sum_{\nu=0}^{s} d_{\nu} a^{\nu}.
$$
 (2.1)

The first elements of this sequence are

$$
\gamma_n = (a-1) \sum_{\nu=0} d_{\nu} a^{\nu}.
$$
\n(2.1)

\nThe first elements of this sequence are

\n
$$
\gamma_0 = 0, \quad \gamma_1 = a - 1, \quad \gamma_2 = (a - 1)a, \quad \gamma_3 = (a - 1)(a + 1)
$$
\n
$$
\gamma_4 = (a - 1)a^2, \quad \gamma_5 = (a - 1)(a^2 + 1), \quad \gamma_6 = (a - 1)(a^2 + a)
$$
\n
$$
\gamma_7 = (a - 1)(a^2 + a + 1), \quad \gamma_8 = (a - 1)a^3, \quad \gamma_9 = (a - 1)(a^3 + 1), \dots
$$
\nFor integers  $a \geq 2$  also the numbers  $\gamma_n$  are integers. In particular, for  $a = 2$  we have

\n
$$
\gamma_n = n.
$$
 It is easy to see that the sequence  $\gamma_n$  has the property\n
$$
\gamma_{2n} = a \gamma_n
$$
\n
$$
\gamma_{2n+1} = a \gamma_n + a - 1
$$
\nIn view of  $a \neq 1$  the sequence  $\gamma_n$  can also be defined by (2.2), because the first equation implies  $\gamma_n = 0$  and the next terms of the series.

 $\gamma_n = n$ . It is easy to see that the sequence  $\gamma_n$  has the property

$$
\begin{aligned}\n\gamma_1 + 1) \,, \quad \gamma_8 &= (a - 1)a^3 \,, \quad \gamma_9 = (a - 1)(a^3 + 1) \,, \dots \\
\text{the numbers } \gamma_n \text{ are integers. In particular, for } a = 2 \text{ we have} \\
\text{that the sequence } \gamma_n \text{ has the property} \\
\gamma_{2n} &= a \gamma_n \\
\gamma_{2n+1} &= a \gamma_n + a - 1\n\end{aligned}\n\quad (n \in \mathbb{N}_0).
$$
\n(2.2)

In view of  $a \neq 1$  the sequence  $\gamma_n$  can also be defined by (2.2), because the first equation implies  $\gamma_0 = 0$ , and the next terms of the sequence are determined recursively by (2.2). According to (2.2), the generating function

$$
g(z)=\sum_{n=0}^{\infty}\gamma_nz^n
$$

satisfies the equation

Cantor Sets and Integral-Fu  
\n
$$
g(z) = a(1 + z)g(z2) + \frac{z(a - 1)}{1 - z2}
$$
\nwe find for the solution the series

Defining  $(Tg)(z) = g(z^2)$ , we find for the solution the series

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\n
$$
g(z) = a(1+z)g(z^2) + \frac{z(a-1)}{1-z^2}
$$
\nwhere

\n
$$
g(z) = g(z^2), \text{ we find for the solution, the series
$$
\n
$$
g(z) = (a-1)\sum_{n=0}^{\infty} a^n((1+z)\dot{T})^n \frac{z}{1-z^2} = \frac{a-1}{1-z}\sum_{n=0}^{\infty} \frac{a^nz^{2^n}}{1+z^{2^n}}
$$
\n(2.3)

\nTherefore,  $|z| < 1$ . For  $a = 2$ , we have, of course,  $g(z) = \frac{z}{(1-z)^2}$  (cf. also [7:1]

which is convergent for  $|z| < 1$ . For  $a = 2$  we have, of course,  $g(z) = \frac{z}{(1-z)^2}$  (cf. also [7: p. 451]). For later purpose we list some further properties of  $\gamma_n$ .

**Lemma 2.1.** *The sequence*  $\gamma_n$  *has the following properties:* 

- (i)  $\gamma_{2k+1} = \gamma_{2k} + \gamma_1 \quad (k \geq 0).$
- (ii)  $a^l \gamma_k = \gamma_{2^l k}$  and  $a^l(\gamma_k + 1) = \gamma_{2^l(k+1)-1} + 1$   $(k, l \ge 0)$ .
- (iii)  $\gamma_k + \gamma_l + 1 = a^m$  *if*  $k + l + 1 = 2^m$  *(k,l 2 0).*

**Proof.** Statement (i) and the first equality in (ii) follow immediately from (2.2). The second equality in (ii) can easily be proved by induction with respect to *1,* since it is an identity for  $l = 0$  and the induction step reads in view of  $(2.2)$ 

$$
a^{l+1}(\gamma_k+1)=a\,\gamma_{2^l(k+1)-1}+a=\gamma_{2^{l+1}(k+1)-1}+1.
$$

In order to show statement (iii) we assume without loss of generality that  $k > l$  and that *k* has the representation  $k = d_0 + 2d_1 + \ldots + 2^{m-1}d_{m-1}$  with  $d_{m-1} = 1$  and  $d_{\nu} \in \{0,1\}$ , i.e. the dyadic representation  $k = d_{m-1}d_{m-2} \cdots d_0$ . This implies that *I* has the representation  $l = \bar{d}_0 + 2\bar{d}_1 + ... + 2^{m-2}\bar{d}_{m-2}$  with  $\bar{d}_{\nu} = 1 - d_{\nu}$  since *k* =  $d_0 + 2d_1 + ... + 2^n$ <br>
resentation  $k = d_{m-1}d_{m-2}$ <br>  $+ ... + 2^{m-2}\overline{d}_{m-2}$  with  $k + l = \sum_{\nu=0}^{m-1} 2^{\nu} = 2^m - 1$ .

$$
k+l=\sum_{\nu=0}^{m-1}2^{\nu}=2^m-1.
$$

In view of (2.1) with  $s = m - 1$  and  $n = k$ , resp.  $n = l$  and  $\overline{d}_{\nu}$  instead of  $d_{\nu}$ , we get

$$
\gamma_k + \gamma_l = (a-1) \sum_{\nu=0}^{m-1} a^{\nu} = a^m - 1
$$
.

This completes the proof  $\blacksquare$ 

**Lemma 2.2.** *In the case of*  $a \geq 2$  *we have*  $\gamma_{n+1} \geq \gamma_n + a - 1$   $(n \in \mathbb{N}_0)$ .

**Proof.** For  $n = 2k$  the inequality is even an equality in view of Lemma 2.1/(i). Moreover, it is true also for  $n = 1$ . Assume that  $\gamma_{m+1} \ge \gamma_m + a - 1$  is true for  $m < n = 2k+1$ . Then in view of (2.2) and  $a \geq 2$  we get  $\gamma_{2k+2} = a\gamma_{k+1} \geq a(\gamma_k + a - 1) =$  $\gamma_{2k+1} + a - 1$  and the assertion is proved by induction  $\blacksquare$ 

Moreover, we need the sign sequence  $\varepsilon_n = (-1)^{\nu(n)}$ , where  $\nu(n)$  denotes the number of "1" in the dyadic representation of *n,* i.e. *v(n)* is the binary sum-of-digits function  $(cf. [4]).$ *n*, we need the sign sequence  $\varepsilon_n = (-1)^{\nu(n)}$ , where  $\nu(n)$  denotes the dyadic representation of *n*, i.e.  $\nu(n)$  is the binary sum-of-digits  $\frac{n}{(n)}$  0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 (*n*) 0 1 1 2 1 2 2 3 1 2 2 3 2 where  $\nu(n)$  denotes the number<br>
binary sum-of-digits function<br>  $\frac{11}{3} \begin{vmatrix} 12 & 13 & 14 & 15 \\ 2 & 3 & 3 & 4 \\ 1 & -1 & -1 & 1 \end{vmatrix}$ <br>
and  $\varepsilon_n$ <br>  $\nu(n)$  for  $a \to 1$  the generating<br>  $(|z| < 1)$ . (2.4)<br>
orse sequence (cf. [5]) which

Provevier, we need the sign sequence 
$$
\varepsilon_n = (-1)^{\nu(n)}
$$
, where  $\nu(n)$  denotes the  $\nu(n)$  and  $\nu(n)$  is the binary sum-of-digits.

\n $\frac{n}{\nu(n)} \begin{vmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \sum_{n=0}^{\infty} \nu(n) z^n$ \n(|z| < 1).

\nSince  $\nu(n)$  mod 2 with values from  $\{0,1\}$  is the Morse sequence (cf. [5])

\nIt is easy to see that the solution is the following. Theorem 1 to  $\varepsilon_n$  by the mapping  $1 \mapsto -1$  and  $0 \mapsto 1$ . It is easy to see that the solution is:

\n $\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}} = \sum_{n=0}^{\infty} \nu(n) z^n$ 

\n $\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}}$ 

Table 1: The first numbers  $\nu(n)$  and  $\varepsilon_n$ 

Considering (2.1) and (2.3) we get in view of  $\frac{\gamma_n}{a-1} \to \nu(n)$  for  $a \to 1$  the generating function

$$
\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{z^{2^n}}{1+z^{2^n}}=\sum_{n=0}^{\infty}\nu(n) z^n \qquad (|z|<1).
$$
 (2.4)

The sequence  $\nu(n)$  mod 2 with values from  $\{0,1\}$  is the Morse sequence (cf. [5]) which is equivalent to  $\varepsilon_n$  by the mapping  $1 \mapsto -1$  and  $0 \mapsto 1$ . It is easy to see that the sequence  $\varepsilon_n$  can be also defined recursively by

Table 1: The first numbers 
$$
\nu(n)
$$
 and  $\varepsilon_n$   
\nand (2.3) we get in view of  $\frac{\gamma_n}{a-1} \to \nu(n)$  for  $a \to 1$  the generating  
\n
$$
\frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}} = \sum_{n=0}^{\infty} \nu(n) z^n
$$
 (|z| < 1). (2.4)  
\n2 with values from {0, 1} is the Morse sequence (cf. [5]) which is  
\nmapping  $1 \to -1$  and  $0 \to 1$ . It is easy to see that the sequence  
\nrecursively by  
\n $\varepsilon_0 = 1$   
\n $\varepsilon_{2n} = \varepsilon_n$  and  $\varepsilon_{2n+1} = -\varepsilon_n$   $(n \ge 0)$ .  
\n $f(z) = \sum_{n=0}^{\infty} \varepsilon_n z^n$ 

According to (2.5) the generating function

$$
f(z) = \sum_{n=0}^{\infty} \varepsilon_n z^n
$$

satisfies the equation  $f(z) = (1 - z)f(z^2)$ . Hence, we get in view of  $f(0) = \varepsilon_0 = 1$  the representation

$$
f(z) = \prod_{n=0}^{\infty} (1 - z^{2^n}) \qquad (|z| < 1) \tag{2.6}
$$

The sequence  $\varepsilon_n$  was already used in [1] for the determination of the signs of the Fourier coefficients of the solution  $\phi$  of (1.1) - (1.2) in the case of  $a = 2$ .

In view of (2.5) it is easy to show by means of induction that the sequence  $\varepsilon_n$  has the properties

renerating function

\n
$$
f(z) = \sum_{n=0}^{\infty} \varepsilon_n z^n
$$
\n
$$
z) = (1 - z)f(z^2).
$$
\nHence, we get in view of  $f(0) = \varepsilon_0 = 1$  the

\n
$$
f(z) = \prod_{n=0}^{\infty} (1 - z^{2^n}) \qquad (|z| < 1).
$$
\n(2.6)

\neady used in [1] for the determination of the signs of the Fourier

\non  $\phi$  of (1.1) - (1.2) in the case of  $a = 2$ .

\neasy to show by means of induction that the sequence  $\varepsilon_n$  has

\n
$$
\sum_{\nu=0}^{2n} \varepsilon_{\nu} = \varepsilon_n \qquad \text{and} \qquad \sum_{\nu=0}^{2n+1} \varepsilon_{\nu} = 0 \qquad (2.7)
$$
\n
$$
= \begin{cases} 0 & \text{for } \nu = 2^{2\mu - 1} \mod 2^{2\mu} \\ 2\varepsilon_{\nu} & \text{else} \end{cases} \qquad (\nu, \mu \in \mathbb{N}) \qquad (2.8)
$$
\non-vanishing differences alternate. Furthermore, we have

\n
$$
\frac{k}{\varepsilon} \qquad \begin{cases} \varepsilon_n a^2 \gamma_n & \text{for } k = 4n \end{cases}
$$

as well as

$$
\varepsilon_{\nu} - \varepsilon_{\nu - 1} = \begin{cases} 0 & \text{for } \nu = 2^{2\mu - 1} \text{ mod } 2^{2\mu} \\ 2\varepsilon_{\nu} & \text{else} \end{cases} \quad (\nu, \mu \in \mathbb{N}) \tag{2.8}
$$

where the signs of the non-vanishing differences alternate. Furthermore, we have

is easy to show by means of induction that  
\n
$$
\sum_{\nu=0}^{2n} \varepsilon_{\nu} = \varepsilon_n \quad \text{and} \quad \sum_{\nu=0}^{2n+1} \varepsilon_{\nu} = 0
$$
\n
$$
e_1 = \begin{cases}\n0 & \text{for } \nu = 2^{2\mu - 1} \mod 2^{2\mu} \\
2\varepsilon_{\nu} & \text{else}\n\end{cases}
$$
\n
$$
e_n = \begin{cases}\n\varepsilon_n a^2 \gamma_n & \text{for } k = 4n \\
-\varepsilon_n \gamma_1 & \text{for } k = 4n + 1 \\
-\varepsilon_n \gamma_4 n + 3 & \text{for } k = 4n + 2 \\
0 & \text{for } k = 4n + 3\n\end{cases}
$$
\n
$$
e_n = \begin{cases}\n\varepsilon_n a^2 \gamma_n & \text{for } k = 4n + 1 \\
-\varepsilon_n \gamma_4 n + 3 & \text{for } k = 4n + 2 \\
0 & \text{for } k = 4n + 3\n\end{cases}
$$
\n
$$
e_n = \begin{cases}\n\varepsilon_n a^2 \gamma_n & \text{for } k = 4n + 1 \\
0 & \text{for } k = 4n + 3\n\end{cases}
$$
\n
$$
e_n = \begin{cases}\n\varepsilon_n a^2 \gamma_n & \text{for } k = 4n + 1 \\
0 & \text{for } k = 4n + 3\n\end{cases}
$$

which follows from (2.2) and (2.5) by induction.

Both sequences  $\gamma_n$  and  $\varepsilon_n$  appear in the following connection.

**Lemma 2.3.** *We have the identity*

Cautor Sets and Integral-Functional Equations

\nLemma 2.3. We have the identity

\n
$$
\prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)}) = \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} e^{-\frac{\gamma_{\nu}}{a^k} p}.
$$
\nProof. This formula is true for  $n = 1$  in view of  $\frac{\gamma_1}{a} = \frac{a-1}{a} = \frac{1}{b}$ . If (2.9) is true for  $n = 1$  in view of  $\frac{\gamma_1}{a} = \frac{a-1}{a} = \frac{1}{b}$ . If (2.9) is true for  $n = 1$ .

a certain *n,* then it follows

2.3. We have the identity

\n
$$
\prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)}) = \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{7}{4}k p}.
$$
\nThis formula is true for  $n = 1$  in view of  $\frac{\gamma_1}{a} = \frac{a-1}{a} = \frac{1}{b}$ . If then it follows

\n
$$
\prod_{k=0}^{n} (1 - e^{-p/(ba^k)}) = (1 - e^{-p/(ba^n)}) \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{7}{4}k p}
$$
\n
$$
= \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{a\gamma_1}{a\gamma_1 + 1}p} - \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{a\gamma_1 + a - 1}{a\gamma_1 + 1}p}
$$
\n
$$
= \sum_{\nu=0}^{2^{n+1} - 1} \varepsilon_{\nu} e^{-\frac{2\gamma}{a\gamma_1 + 1}p}
$$
\nWe used (2.2) and (2.5). Thus, assertion (2.9) is proved by it that

\n
$$
\prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)}) = \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{7}{4}k p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! a^{nm}} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} \gamma_{\nu}^m
$$
\nthe sum

\n
$$
s_n(m) = \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} \gamma_{\nu}^m
$$
\nor  $m = 0, 1, \ldots, n - 1$ .

\ntion 2.1. For  $m \ge n$  we have

where we have used (2.2) and (2.5). Thus, assertion (2.9) *is* proved by induction I

We remark that

$$
= \sum_{\nu=0}^{2^{n+1}-1} \varepsilon_{\nu} e^{-\frac{2\nu}{a^{n+1}} p}
$$
  
we used (2.2) and (2.5). Thus, assertion (2.9) is proved by induction **I**  
ark that  

$$
\prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)}) = \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} e^{-\frac{2\mu}{a^{n}} p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! a^{nm}} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} \gamma_{\nu}^m p^m
$$
(2.10)  
the sum  

$$
s_n(m) = \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} \gamma_{\nu}^m
$$
(2.11)  
or  $m = 0, 1, ..., n - 1$ .  
ition 2.1. For  $m \ge n$  we have  

$$
m) = \frac{(-1)^m m!}{(m-n)!} \frac{a^{\frac{n(2m-n+1)}{2}}}{b^n} \sum_{\mu=0}^{m-n} (-1)^{\mu} {m-n \choose \mu} \frac{\rho_{\mu}(\frac{1}{a}) \rho_{m-n-\mu}(a)}{a^{n\mu}}.
$$

implies that the sum

$$
s_n(m) = \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} \gamma_{\nu}^m \tag{2.11}
$$

equals to 0 for  $m=0,1,\ldots,n-1$ .

**Proposition 2.1.** For  $m \geq n$  we have

hat the sum  
\n
$$
s_n(m) = \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} \gamma_{\nu}^m
$$
\n0 for  $m = 0, 1, ..., n - 1$ .  
\nposition 2.1. For  $m \ge n$  we have  
\n
$$
s_n(m) = \frac{(-1)^m m!}{(m - n)!} \frac{a^{\frac{n(2m - n + 1)}{2}}}{b^n} \sum_{\mu=0}^{m - n} (-1)^{\mu} {m - n \choose \mu} \frac{\rho_{\mu}(\frac{1}{a})\rho_{m - n - \mu}(a)}{a^{n\mu}}
$$
\nof. From (1.3) we get  
\n
$$
\frac{\Phi(p)}{\Phi(\frac{p}{a^n})} = \prod_{k=0}^{n-1} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} = \frac{a^{\frac{n(n-1)}{2}}b^n}{p^n} \prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)})
$$
,  
\new of (2.10) and (2.11) we find  
\n
$$
\phi(p) \frac{1}{\phi(\frac{p}{a^n})} = a^{\frac{n(n-1)}{2}}b^n \sum_{m=0}^{\infty} \frac{(-1)^m s_n(m)}{m! a^{mn}} p^{m-n}
$$
.  
\ne representations (1.4) and (1.6), the last with  $\frac{p}{a^n}$  instead of p, we  
\nby means of the Cauchy product and comparison of coefficients  $\blacksquare$ 

**Proof.** From (1.3) we get

$$
E(t) = \frac{(-1)^{n} m!}{(m-n)!} \frac{a}{b^n} \sum_{\mu=0}^{\infty} (-1)^{\mu} {m-n \choose \mu} \frac{p_{\mu} (\frac{\pi}{a}) p_{m-n-\mu}}{a^{n\mu}}
$$
  
from (1.3) we get  

$$
\frac{\Phi(p)}{\Phi(\frac{p}{a^n})} = \prod_{k=0}^{n-1} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)} = \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \prod_{k=0}^{n-1} (1 - e^{-p/(ba^k)})
$$
  
if (2.10) and (2.11) we find  

$$
\phi(p) \frac{1}{\phi(\frac{p}{a^n})} = a^{\frac{n(n-1)}{2}} b^n \sum_{m=0}^{\infty} \frac{(-1)^m s_n(m)}{m! a^{mn}} p^{m-n}.
$$

and in view of *(2.10)* and *(2.11)* we find

$$
\frac{1}{p} = \prod_{k=0}^{\infty} \frac{1 - e^{-p/(b a^k)}}{p/(b a^k)} = \frac{a^{-\frac{1}{2}} - b^n}{p^n} \prod_{k=0}^{\infty} (1 - e^{-p/(b a^k)})
$$
\n(a) and (2.11) we find

\n
$$
\phi(p) \frac{1}{\phi(\frac{p}{a^n})} = a^{\frac{n(n-1)}{2}} b^n \sum_{m=0}^{\infty} \frac{(-1)^m s_n(m)}{m! a^{mn}} p^{m-n}
$$

Using the representations (1.4) and (1.6), the last with  $\frac{p}{a^n}$  instead of p, we obtain the assertion by means of the Cauchy product and comparison of coefficients  $\blacksquare$ 

In particular, we have

$$
\begin{aligned}\n\text{C,} \text{C,} \text{C
$$

and

$$
\sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} \gamma_{\nu}^{n+1} = \frac{1}{2} (-1)^n (n+1)! (a-1)^n (a^n-1) a^{\frac{n(n-1)}{2}} . \tag{2.13}
$$

### **3. Cantor sets and singular functions**

In this section, we explore Cantor sets which are immediately connected with the solution  $\phi$  of (1.1) - (1.2) in the case of  $a > 2$ . First, we note that in the case of  $a > 2$ Lemma 2.2 implies  $y^{n}(n+1)!(a-1)^{n}(a^{n}-1)a^{\frac{n(n-1)}{2}}$ . (2.13)<br> **Ar functions**<br> **Areold 12.13** . (2.13)<br> **Areold 12.1** . (2.13)<br> **Areold 12.1** . (3.1)<br>  $y_{n} + 1 < y_{n+1}$ . (3.1)<br> **Open intervals**  $G_{kn}$   $(k = 0, 1, ..., 2^{n} - 1; n \in \mathbb{N}_{0})$ **f** singular functions<br>
re Cantor sets which are im<br>
n the case of  $a > 2$ . First,<br>  $\gamma_n + 1 < \gamma_{n+1}$ .<br>
: following open intervals *G*<br>
ion  $G_m$ :<br>  $\left(\frac{\gamma_{2k}+1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right)$ ,  $G_m$ <br>
I  $G_{kn}$  are disiont, we co **g**<br> **g** immediately connected with the so-<br> **g** is  $G_{kn}$  ( $k = 0, 1, ..., 2^n - 1; n \in \mathbb{N}_0$ )<br>  $G_m = \bigcup_{n=0}^{m} \bigcup_{k=0}^{2^n - 1} G_{kn}$  (3.2)<br>
Consider the following closed intervals

$$
\gamma_n + 1 < \gamma_{n+1} \tag{3.1}
$$

Hence, we can define the following open intervals  $G_{kn}$   $(k = 0, 1, \ldots, 2<sup>n</sup> - 1; n \in \mathbb{N}_0)$ and the corresponding union *Gm:* 

es  
\n
$$
\gamma_n + 1 < \gamma_{n+1} \tag{3.1}
$$
\n
$$
\text{effine the following open intervals } G_{kn} \quad (k = 0, 1, \dots, 2^n - 1; \, n \in \mathbb{N}_0)
$$
\n
$$
G_{kn} = \left(\frac{\gamma_{2k} + 1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right), \qquad G_m = \bigcup_{n=0}^m \bigcup_{k=0}^{2^n - 1} G_{kn} \tag{3.2}
$$
\n
$$
\text{that all } G_{kn} \text{ are disjoint, we consider the following closed intervals}
$$

In order to show that all  $G_{kn}$  are disjoint, we consider the following closed intervals

lution 
$$
\phi
$$
 of (1.1) - (1.2) in the case of  $a > 2$ . First, we note that in the case of  $a > 2$   
\nLemma 2.2 implies  
\n
$$
\gamma_n + 1 < \gamma_{n+1}
$$
.\n(3.1)  
\nHence, we can define the following open intervals  $G_{kn}$   $(k = 0, 1, ..., 2^n - 1; n \in \mathbb{N}_0)$   
\nand the corresponding union  $G_m$ :  
\n
$$
G_{kn} = \left(\frac{\gamma_{2k} + 1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right), \qquad G_m = \bigcup_{n=0}^{m} \bigcup_{k=0}^{2^n - 1} G_{kn}
$$
.\n(3.2)  
\nIn order to show that all  $G_{kn}$  are disjoint, we consider the following closed intervals  
\n $F_{kn}$   $(k = 0, 1, ..., 2^n - 1; n \in \mathbb{N}_0)$  and the corresponding union  $F_n$ :  
\n
$$
F_{kn} = \left[\frac{\gamma_k}{a^n}, \frac{\gamma_k + 1}{a^n}\right], \qquad F_n = \bigcup_{k=0}^{2^n - 1} F_{kn}
$$
.\n(3.3)  
\nNote that  $F_0 = [0, 1]$  and in view of (3.1), all  $F_{kn}$  with a fixed *n* are disjoint. From

Note that  $F_0 = [0,1]$  and in view of (3.1), all  $F_{kn}$  with a fixed *n* are disjoint. From Lemma 2.1/(ii) we see that  $F_{kn}$  and  $F_{2^ik,n+l}$  have the same left end-points and, analogously,  $F_{kn}$  and  $F_{2^l(k+1)-1,n+l}$  the same right end-points for all  $l \in \mathbb{N}_0$ .  $F_n = \bigcup_{k=0} F_{kn}$ . (3.3)<br>
in view of (3.1), all  $F_{kn}$  with a fixed *n* are disjoint. From<br>  $F_{kn}$  and  $F_{2^ik, n+l}$  have the same left end-points and, analo-<br>  $F_{kn}$  and  $F_{2^ik, n+l}$  have the same left end-points and, analo

**Lemma 3.1.** *In the case of a > 2, for all n*  $\in \mathbb{N}$  *and k* = 0, 1, ...;  $2^{n} - 1$  *we have*  $G_{kn} \subset F_{kn}$  and the disjoint decomposition

$$
F_{kn} = F_{2k, n+1} \cup G_{kn} \cup F_{2k+1, n+1} . \tag{3.4}
$$

**Proof.** In view of (2.2), we have

$$
k+1)-1, n+l
$$
 the same right end-points for all  $l \in$   
\n*n* the case of  $a > 2$ , for all  $n \in \mathbb{N}$  and  $k = 0, 1$   
\ndisjoint decomposition  
\n
$$
F_{kn} = F_{2k, n+1} \cup G_{kn} \cup F_{2k+1, n+1}
$$
\nof (2.2), we have  
\n
$$
F_{kn} = \left[\frac{a\gamma_k}{a^{n+1}}, \frac{a\gamma_k + a}{a^{n+1}}\right] = \left[\frac{\gamma_{2k}}{a^{n+1}}, \frac{\gamma_{2k+1} + 1}{a^{n+1}}\right].
$$

According to (3.2), we see that from the intervals  $G_{\nu n}$  ( $\nu = 0, 1, \ldots, 2^{n} - 1$ ) exactly the interval  $G_{kn}$  lies in  $F_{kn}$ , since  $\gamma_{2k} < \gamma_{2k} + 1 < \gamma_{2k+1} < \gamma_{2k+1} + 1$ . In view of (3.3) this implies the decomposition (3.4) (cf. Figure 1) **1**



Figure 1: The first Cantor intervals

The disjoint composition (3.4) shows that also all  $G_{kn}$  are disjoint and, moreover, that Figure 1: The firs<br>sjoint composition (3.4) shows that a<br>=  $[0, 1] \setminus G_m$ . Since  $\gamma_{2k+1} - \gamma_{2k} =$ <br>=  $\frac{a-2}{2a+1}$ , and for the measure of the op  $a - 1$ , we get for the measure of  $G_{kn}$  that Figure 1: The first Cantor inter<br>The disjoint composition (3.4) shows that also all  $G_{kn}$  are<br> $F_{m+1} = [0,1] \setminus G_m$ . Since  $\gamma_{2k+1} - \gamma_{2k} = a - 1$ , we get<br> $|G_{kn}| = \frac{a-2}{a^{n+1}}$ , and for the measure of the open Cantor set

$$
- \gamma_{2k} = a - 1,
$$
 we  
of the open Canto  

$$
G = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n - 1} G_{kn}
$$

we have

$$
|G| = \sum_{n=0}^{\infty} 2^n \frac{a-2}{a^{n+1}} = 1,
$$

and hence for the perfect Cantor set  $F = [0, 1] \setminus G$  the measure  $|F| = 0$  as in the original construction of Cantor, i.e. in the case of  $a = 3$ . We remark that the Cantor set *G* can construction of Cantor, i.e. in the case of  $a = 3$ . We remark that the Cantor set G can<br>be generated from [0,1] by iteration of the functions  $f_1(x) = \frac{x}{a}$  and  $f_2(x) = \frac{x+a-1}{a}$  (cf. [3: p. 6], [1] or [13]). For  $a = 2$  the intervals  $G_{kn}$  are empty and  $F_n = F = [0, 1]$ .  $G = \bigcup_{n=0}^{{\infty}} \bigcup_{k=0}^{n} G_{kn}$ <br>  $|G| = \sum_{n=0}^{\infty} 2^n \frac{a-2}{a^{n+1}} = 1$ ,<br>  $\therefore$  antor set  $F = [0,1] \setminus G$  the measure  $|F| = 0$  as in the original<br>  $x_i$  in the case of  $a = 3$ . We remark that the Cantor set  $G$  can<br>  $x_i$  iter  $\begin{aligned}\n\mathbf{q} &= \sum_{n=0}^{n} 2^{-n} \frac{1}{a^{n+1}} = 1, \\
\text{et } F &= [0,1] \setminus G \text{ the measure } |F| = 0 \text{ as in the original case of } a = 3. \text{ We remark that the Cantor set } G \text{ can} \\
\text{on of the functions } f_1(x) &= \frac{x}{a} \text{ and } f_2(x) = \frac{x+a-1}{a} \text{ (cf.}\n\end{aligned}$ <br>
re intervals  $G_{kn}$  are empty and  $F_n = F = [0,1].$ <br>
ntroduce n

Next, for arbitrary  $a > 1$  we introduce numbers  $x = x(a)$  of the form

$$
x = (a - 1) \sum_{\nu=1}^{\infty} \frac{\xi_{\nu}}{a^{\nu}} \qquad (\xi_{\nu} \in \{0, 1\})
$$
 (3.5)

which lie in [0,1] in view of

$$
(-1)\sum_{\nu=1}^{\infty} \frac{3\nu}{a^{\nu}} \qquad (\xi_{\nu} \in \{0, 1\}) \tag{3.5}
$$
\n
$$
(a-1)\sum_{\nu=1}^{\infty} \frac{1}{a^{\nu}} = 1 \tag{3.6}
$$
\n1 we write

\n
$$
-1)\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} \qquad (\xi_{\nu} \in \{0, 1\}) \tag{3.7}
$$

In the case of  $\xi_{\nu} = 0$  for  $\nu \ge n + 1$  we write

$$
x_n = (a-1) \sum_{\nu=1}^n \frac{\xi_{\nu}}{a^{\nu}} \qquad (\xi_{\nu} \in \{0,1\})
$$
 (3.7)

for  $n \in \mathbb{N}_0$ . Denoting  $\xi_{\nu} = d_{n-\nu}$  for  $\nu = 1, ..., n$  and  $k = d_0 + 2d_1 + ... + 2^{n-1}d_{n-1}$ , we see from (2.1) that  $x_n = \frac{\gamma_k}{a^n}$  with a certain index  $k \in \{0, 1, ..., 2^n - 1\}$ , i.e.  $x_n$  is for  $n \in \mathbb{N}_0$ . Denoting  $\xi_{\nu} = d_{n-\nu}$  for  $\nu = 1, ..., n$  and  $k = d_0 + 2d_1 + ... + 2^{n-1}d_{n-1}$ ,<br>we see from (2.1) that  $x_n = \frac{\gamma_k}{a^n}$  with a certain index  $k \in \{0, 1, ..., 2^n - 1\}$ , i.e.  $x_n$  is<br>the left end-point of  $F_{kn}$  if w the left end-point of  $F_{kn}$  if we use the notation (3.3) also for  $1 < a \le 2$ . Clearly, in the case of  $a = 2$  these numbers are equal to  $\frac{k}{2^n}$   $(n \in \mathbb{N}; k = 0, 1, ..., 2^n - 1)$  and they lie

densely in [0,1]. The points (3.5) fill the whole interval [0,1] not only for  $a = 2$ , but also for  $1 < a < 2$ . In order to see this we remark that in the case of  $1 < a < 2$  the intervals *F*<sub>2k,n</sub> and *F*<sub>2k+1,n</sub> are overlapping with  $F_{2k,n} \cup F_{2k+1,n} = F_{k,n-1}$ , so that  $F_0 = [0,1]$ implies that  $F_n = [0, 1]$  for all  $n \in \mathbb{N}$  (cf. (3.3)). Hence, the left end-points (3.7) of the intervals  $F_{kn}$   $(k = 0, 1, ..., 2<sup>n</sup> - 1)$  form an  $\varepsilon$ -net  $(\varepsilon = \frac{1}{2a^n})$  for the interval [0,1] since for every fixed *n* each  $x \in [0, 1]$  is contained in at least one  $F_{kn}$ , i.e.  $x_n \leq x \leq x_n + \frac{1}{a^n}$  with  $x_n$  from (3.7). Having already determined  $x_n$  for a given  $x \in [0,1]$ , the next number  $\xi_{n+1}$  in (3.5) reads  $F_n = [0,1]$  for all  $n \in \mathbb{N}$  (cf. (3.3)). Hence, the left end-points (3.<br>  $(k = 0,1,\ldots,2^n - 1)$  form an  $\varepsilon$ -net  $(\varepsilon = \frac{1}{2a^n})$  for the interval [0,1]<br>  $\alpha$  each  $x \in [0,1]$  is contained in at least one  $F_{kn}$ , i.e.  $x_n \$ **Let up the case of a**  $L$  **Let up the interval** [0,1] since for y fixed *n* each  $x \in [0,1]$  is contained in at least one  $F_{kn}$ , i.e.  $x_n \le x \le x_n + \frac{1}{a^n}$  with rom (3.7). Having already determined  $x_n$  for a given  $x \in [0$ 

$$
\xi_{n+1} = \begin{cases} 0 & \text{for } x_n \le x < x_n + \frac{a-1}{a^{n+1}}, \qquad \text{i.e. } x \in F_{2k, n+1} \setminus F_{2k+1, n+1} \\ 1 & \text{for } x_n + \frac{1}{a^{n+1}} < x \le x_n + \frac{1}{a^n}, \text{ i.e. } x \in F_{2k+1, n+1} \setminus F_{2k, n+1} \end{cases}
$$

whereas  $\xi_{n+1}$  can be chosen arbitrarily for  $x_n + \frac{a-1}{a^{n+1}} \leq x \leq x_n + \frac{1}{a^{n+1}}$ , i.e. for  $x \in$  $F_{2k,n+1} \cap F_{2k+1,n+1}$ .

**Lemma 3.2.** *In the case of a* > 2 *the numbers* (3.5) *and*  $y = (a-1) \sum_{\nu=1}^{\infty} \frac{\eta_{\nu}}{a^{\nu}}$  *with*  $\eta_{\nu} \in \{0,1\}$  *have the following properties:* 

(i) The usual order of x and y is equivalent to the lexicographic order of  $(\xi_1, \xi_2, \ldots)$ *and*  $(\eta_1, \eta_2, \ldots)$ *.* 

(ii) The 2<sup>n</sup> intervals  $G_{kn}$  with fixed n are exactly the intervals  $(x, y)$  with  $\xi_{\nu} =$ *i*<sub>2k,n+1</sub>  $\cap$  *i*<sub>2k+1,n+1.<br> **Lemma 3.2.** *In the case of a* > 2 *the numbers* (3.5) and  $y = (a - 1) \sum_{\nu=1}^{\infty} \frac{\eta_{\nu}}{a^{\nu}}$  *i*<sub> $\eta_{\nu} \in \{0,1\}$  *have the following properties:*<br>
(i) *The usual order of x and y is </sub></sub>* 

**Proof.** Let be  $(\xi_1, \xi_2, \ldots) < (\eta_1, \eta_2, \ldots)$  lexicographically, i.e.  $\xi_{\nu} = \eta_{\nu}$  for  $1 \leq \nu \leq \nu$  $m-1$  and  $\xi_m < \eta_m$  for a certain  $m \in \mathbb{N}$ , which is only possible for  $\xi_m = 0$  and  $\eta_m = 1$ . Then we have in view of  $a > 2$  the inequality *n* the case of  $a > 2$  the numbers (3.5) and  $y = ($ <br>following properties:<br>rder of x and y is equivalent to the lexicographic<br>ervals  $G_{kn}$  with fixed n are exactly the interval,<br> $f_{n+1} = 0$ ,  $f_{n+2} = f_{n+3} = ... = 1$  and  $\eta_{n+$ 

$$
y-x \ge \frac{a-1}{a^m} - (a-1) \sum_{\nu=m+1}^{\infty} \frac{1}{a^{\nu}} = \frac{a-2}{a^m} > 0.
$$

Vice versa,  $(\xi_1, \xi_2, \ldots) > (\eta_1, \eta_2, \ldots)$  implies analogously  $x > y$ , so that property (i) is valid.

In order to show property (ii), we first remark that for  $k \leq 2^{n} - 1$  the dyadic representation of k has at most n digits, i.e.  $k = d_0 + 2d_1 + \ldots + 2^{n-1}d_{n-1}$  with  $d_{\mu} \in \{0,1\}$ . Hence, for the left end-point of  $G_{kn}$  we have as in the foregoing case of  $F_{kn}$  and in view of (3.6) the representat  $d_{\mu} \in \{0, 1\}$ . Hence, for the left end-point of  $G_{kn}$  we have as in the foregoing case of  $F_{kn}$ and in view of (3.6) the representation *n*<sub>1</sub> for a certain  $m \in \mathbb{N}$ , which is only possible wo f  $a > 2$  the inequality<br>  $y - x \ge \frac{a-1}{a^m} - (a-1) \sum_{\nu=m+1}^{\infty} \frac{1}{a^{\nu}} = \frac{a-1}{a^m}$ <br>  $\cdots$ ) > ( $\eta_1, \eta_2, \ldots$ ) implies analogously  $x >$ <br>
ow property (ii), we  $\alpha = 2$   $\alpha = 1$   $\alpha = 1$   $\alpha = 1$   $\beta = 1$   $\alpha = 2$   $\alpha = 2$ <br>  $\alpha = 2$   $\alpha = 1$   $\beta = 1$   $\alpha = 2$   $\alpha = 2$ <br>  $\alpha = 1$   $\beta = 1$   $\beta = 2$ <br>  $\alpha = 2$   $\alpha = 1$   $\beta = 1$   $\beta = 1$   $\alpha = 1$   $\beta = 1$ <br>  $\alpha = 1$   $\alpha = 1$   $\beta = 1$   $\alpha = 1$   $\beta = 1$   $\alpha = 1$   $(a-1)$   $\sum_{\nu=m+1}^{\infty} \frac{1}{a^{\nu}} = \frac{a-2}{a^m}$  ><br>implies analogously  $x > y$ , s<br>we first remark that for k<br>iigits, i.e.  $k = d_0 + 2d_1 +$ <br>oint of  $G_{kn}$  we have as in the<br>on<br> $\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} = (a-1) \sum_{\nu=1}^$ (a) implies analogously  $x > y$ <br>
(b), we first remark that for<br>
(a) digits, i.e.  $k = d_0 + 2d_1$ <br>
(b)  $\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} = (a-1) \sum_{\nu=1}^{n}$ <br>
( $\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} = (a-1)$ <br>
( $\sum_{\nu=1}^{n} \frac{\xi$ *an + l* 

$$
\frac{\gamma_{2k}+1}{a^{n+1}} = (a-1)\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} = (a-1)\sum_{\nu=1}^{\infty} \frac{\xi_{\nu}}{a^{\nu}}
$$

with  $\xi_{\nu} = d_{n-\nu}$  for  $\nu = 1, 2, ..., n, \xi_{n+1} = 0$  and  $\xi_{\nu} = 1$  for  $\nu \ge n+2$ . For the right end-point of  $G_{kn}$  we have analogously

$$
\frac{\gamma_{2k+1}}{a^{n+1}} = (a-1)\left(\sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}}\right) = (a-1)\sum_{\nu=1}^{n+1} \frac{\eta_{\nu}}{a^{\nu}}
$$

with  $\eta_{\nu} = d_{n-\nu}$  for  $\nu = 1,2,\ldots,n$  and  $\eta_{n+1} = 1$ , so that property (ii) is proved  $\blacksquare$ 

**Remark.** In the case of  $a=2$ ,  $(\xi_1,\xi_2,...)<(\eta_1,\eta_2,...)$  implies only  $x\leq y$ .

Lemma 3.2 shows once more that all *Gkn* are disjoint. Property (ii) from Lemma Lemma 3.2 shows once more that all  $G_{kn}$  are disjoint. Property (ii) from Lemma 3.2 means that the left end-points  $x^-$  of  $G_{kn}$  and the corresponding right end-points  $x^+$  can be written in the form Cantor Se<br>
= 2,  $(\xi_1, \xi_2, ...)$ <br>
ce that all *G*<br>
ints  $x^-$  of *G*<br>  $\frac{1}{x+1}$  and<br>
and-points be *e* case of  $a = 2$ ,  $(\xi_1, \xi_2)$ <br>*ys* once more that a<br>left end-points  $x^-$  of<br>*i* the form<br> $\sum_{\nu=1}^n \frac{\xi_\nu}{a^\nu} + \frac{1}{a^{n+1}}$ or Sets and Integral-Functional 1<br>  $\xi_2, \ldots$   $\leq (\eta_1, \eta_2, \ldots)$  implies<br>
Il  $G_{kn}$  are disjoint. Propert<br>
of  $G_{kn}$  and the correspondit<br>
and  $x^+ = (a-1) \left( \sum_{\nu=1}^n x^+ \right)$ <br>
s belong to the closed set F ightarrow  $2, (\xi_1, \xi_2, \ldots) < (\eta_1, \eta_2, \ldots)$  implies only  $x \leq y$ .<br>
that all  $G_{kn}$  are disjoint. Property (ii) from Lemma<br>
tts  $x^-$  of  $G_{kn}$  and the corresponding right end-points<br>  $\frac{1}{\sqrt{1-x^2}}$  and  $x^+ = (a-1)\left(\sum_{\nu=1}$ 

$$
x^{-} = (a - 1) \sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} \quad \text{and} \quad x^{+} = (a - 1) \left( \sum_{\nu=1}^{n} \frac{\xi_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}} \right)
$$

with  $\xi_{\nu} \in \{0,1\}$ . Since these end-points belong to the closed set F, also all points of the form (3.5) belong to *F.* 

Now, for a fixed  $a > 2$  and a fixed  $c \geq 2$  we define a function  $g_0 : F \mapsto [0,1]$  by

$$
g_0(x) = (c-1) \sum_{\nu=1}^{\infty} \frac{\xi_{\nu}}{c^{\nu}}
$$
 (3.8)

with  $x = x(a)$  from (3.5), i.e.  $g_0: x(a) \mapsto x(c)$ . According to property (i) from Lemma 2.3, this function is strictly increasing and, obviously, it is also continuous. We extend  $g_0$  to the whole interval  $[0,1]$  by the definition 1)  $\sum_{\nu=1} \frac{\zeta_{\nu}}{a^{\nu}} + \frac{1}{a^{n+1}}$  and  $x^+$ <br>
Since these end-points belong to<br>
elong to *F*.<br>
xed  $a > 2$  and a fixed  $c \ge 2$  we defi<br>  $g_0(x) = (c-1) \sum_{\nu=1}^{\infty}$ <br>
om (3.5), i.e.  $g_0 : x(a) \mapsto x(c)$ . Acco<br>
i is strictly  $a^{n+1}$  and  $x = (a^{n+1}) \left( \sum_{\nu=1}^{\infty} a^{\nu} \right) a^{n+1}$ <br>
esee end-points belong to the closed set F, also all points of<br>  $g_0(x) = (c-1) \sum_{\nu=1}^{\infty} \frac{\xi_{\nu}}{c^{\nu}}$  (3.8)<br>
i.e.  $g_0 : x(a) \mapsto x(c)$ . According to property (i) from

$$
g_0\left(x^- + \frac{a-2}{a^{n+1}}t\right) = g_0(x^-) + \frac{c-2}{c^{n+1}}t \qquad (0 < t < 1), \tag{3.9}
$$

i.e. in view of  $x^- + \frac{a-2}{a+1} = x^+$  we extend the function  $g_0$  linearly on the intervals  $G_{kn}$ , so that it remains continuous and increasing (but only for  $c > 2$  strictly increasing). Moreover, replacing t by  $1 - t$  in (3.9) we get<br>  $q_0\left(x^+ - \frac{a-2}{a-1}t\right) = q_0(x^+) - \frac{c-1}{a-1}$ 

$$
g_0\left(x^+ - \frac{a-2}{a^{n+1}}t\right) = g_0(x^+) - \frac{c-2}{c^{n+1}}t \qquad (0 < t < 1) \ . \tag{3.10}
$$

Next, we show that the function  $g = g_0$  satisfies for  $0 \le t \le 1$  the following system of functional equations: we show<br>ional equa<br>(i)  $g(\frac{1}{a} +$ 

- *a-2*
- (ii)  $g(\frac{t}{a}) = \frac{1}{a}g(t)$ .
- (iii)  $g(t) + g(1 t) = 1$ .

The general solution of (ii) alone reads  $g(t) = t^{\alpha} Q(\frac{\ln t}{\ln a})$ , where  $Q(x + 1) = Q(x)$  is an arbitrary 1-periodic function and  $\alpha = \frac{\ln c}{\ln a}$ .

**Proposition 3.1.** The function  $g = g_0$  is the unique bounded solution of the func*tional equations (i) - (iii) in [0, 1].* 

**Proof.** 1. First, we show that the function  $g_0$  satisfies equations (i) - (iii). Clearly,  $g_0$  satisfies (i) in view of (3.9) with  $n = 0$ , and (ii) follows immediately from (3.5), (3.8) and (3.9). In order to show that  $g_0$  satisfies also equation (iii), we assume first that  $x \in F$ , i.e. x is of the form (3.5). Then in view of (3.6) we have

$$
1 - x = (a - 1) \sum_{\nu=1}^{\infty} \frac{\overline{\xi}_{\nu}}{a^{\nu}}
$$

with  $\overline{\xi}_{\nu} = 1 - \xi_{\nu}$ , and in view of (3.8) we get

L. Berg and M. Krüppel  
\n
$$
= 1 - \xi_{\nu}
$$
, and in view of (3.8) we get  
\n
$$
g_0(x) + g_0(1 - x) = (c - 1) \sum_{\nu=1}^{\infty} \frac{\xi_{\nu}}{c^{\nu}} + (c - 1) \sum_{\nu=1}^{\infty} \frac{\overline{\xi}_{\nu}}{c^{\nu}} = (c - 1) \sum_{\nu=1}^{\infty} \frac{1}{c^{\nu}} = 1
$$
\ncase of  $x \notin F$ , i.e.  $x \in G_{kn}$ ,  $x$  has the representation  $x = x - \frac{a-2}{a+1}$ .  
\n2. 1. so that  $g_0(x)$  is given by (3.9) In view of  $1 - x = 1 - x = -\frac{a-2}{a+1}$ .

In the case of  $x \notin F$ , i.e.  $x \in G_{kn}$ ,  $x$  has the representation  $x = x^- + \frac{a-2}{a^{n+1}}t$  with  $0 < t < 1$ , so that  $g_0(x)$  is given by (3.9). In view of  $1 - x = 1 - x^- - \frac{a-2}{a^{n+1}}t$ , where  $1 - x^- = y^+$  for a certain right end-poi  $0 < t < 1$ , so that  $g_0(x)$  is given by (3.9). In view of  $1 - x = 1 - x - \frac{a-2}{a-1}t$ , where  $g_0(x) + g_0(1-x) = (c-1) \sum_{\nu=1} \frac{1}{c^{\nu}} + (c-1) \sum_{\nu=1} \frac{1}{c^{\nu}} = (c-1) \sum_{\nu=1} \frac{1}{c^{\nu}} =$ <br>In the case of  $x \notin F$ , i.e.  $x \in G_{kn}$ ,  $x$  has the representation  $x = x^- + \frac{a}{a}$ ,<br> $0 < t < 1$ , so that  $g_0(x)$  is given by (3.9). In vi

$$
g_0(1-x)=g_0(1-x^-)-\frac{c-2}{c^{n+1}}t.
$$

This together with (3.9) implies that

$$
g_0(x) + g_0(1-x) = g_0(x^-) + g_0(1-x^-) = 1
$$

since  $x^- \in F$ .

This together with (3.9) implies that<br>  $g_0(x) + g_0(1 - x) = g_0(x^-) + g_0(1 - x^-) = 1$ <br>
since  $x^- \in F$ .<br>
2. Let g be a further solution of equations (i) - (iii). For  $0 \le t \le 1$  we put<br>  $d(t) = |g_0(t) - g(t)|$ . In view of (i) we have  $d(t) = 0$ exists a point  $t_0 \in [0,1]$  with  $d(t_0) > 0$ , then for  $t_1 = a \min\{t_0, 1-t_0\}$  we have  $t_1 \in [0,1]$ . We show that  $d(t_1) = c d(t_0)$ . In the case of  $t_0 < \frac{1}{a}$  this follows immediately from (ii). In the case of  $t_0 > 1-\frac{1}{a}$  we first get from (iii) that  $d(1-t_0) = d(t_0)$  and afterwards from In the case of  $t_0 > 1 - \frac{1}{a}$  we first get from (ii) that  $d(1 - t_0) = d(t_0)$  and atterwards from<br>
(ii) that  $d(t_1) = c d(t_0)$ . Thus for the sequence  $t_n = a \min\{t_{n-1}, 1 - t_{n-1}\}$  we obtain<br>  $d(t_n) = c^n d(t_0)$  and in view of  $c \ge 2$  $d(t_n) = c^n d(t_0)$  and in view of  $c \geq 2$  a contradiction to the boundedness of g *g(1 - x)* =  $g_0(x^-) + g_0(1 - x^-) = 1$ <br>
solution of equations (i) - (iii). For  $0 \le t \le 1$  we put<br>
v of (i) we have  $d(t) = 0$  for  $\frac{1}{a} \le t \le 1 - \frac{1}{a}$ . Hence, if there<br>  $d(t_0) > 0$ , then for  $t_1 = a \min\{t_0, 1 - t_0\}$  we have

**Proposition 3.2.** *Suppose that g satisfies properties* (ii) *and (*iii). *Then we have* 

$$
g\left(x_n + \frac{t}{a^n}\right) = g_0(x_n) + \frac{1}{c^n} g(t)
$$
\n(3.11)

*for*  $0 \le t \le 1$ , *with*  $x_n$  *from* (3.7). Moreover,  $g(x_n) = g_0(x_n)$ .

 $g\left(x_n + \frac{t}{a^n}\right) = g_0(x_n) + \frac{1}{c^n} g(t)$  (3.11)<br>
for  $0 \le t \le 1$ , with  $x_n$  from (3.7). Moreover,  $g(x_n) = g_0(x_n)$ .<br> **Proof.** Equation (ii) for  $t = 0$  implies  $g(0) = 0$ , hence in view of  $g_0(0) = 0$  we<br>
have an identity for  $n =$  $g\left(x_n + \frac{t}{a^n}\right) = g_0(x_n) + \frac{1}{c^n} g(t)$ <br>for  $0 \le t \le 1$ , with  $x_n$  from (3.7). Moreover,  $g(x_n) = g_0(x_n)$ .<br>**Proof.** Equation (ii) for  $t = 0$  implies  $g(0) = 0$ , hence in view of  $g_0$ <br>have an identity for  $n = 0$ . Assume that the and we get from (ii) and (3.8) that *c*  $d(t_0)$ . Thus for the sequence  $t_n = a$  mind<br>in view of  $c \ge 2$  a contradiction to the<br>3.2. Suppose that g satisfies properties (i<br> $g\left(x_n + \frac{t}{a^n}\right) = g_0(x_n) + \frac{1}{c^n} g(t)$ <br>h  $x_n$  from (3.7). Moreover,  $g(x_n) = g_0(x_n)$ <br>tion (i ation (ii) for  $t = 0$  implies  $g(0) = 0$ , hence in view of<br>
for  $n = 0$ . Assume that the assertion is true for a cert<br>
e either  $x_n \leq \frac{1}{a}$  or  $x_n \geq 1 - \frac{1}{a}$ . In the first case  $x_n =$ <br>
(8) that<br>  $g\left(x_n + \frac{t}{a^n}\right) = \frac{1}{$  $g(u_n + a_n) = g_0(u_n) + \frac{1}{C^n} g(t)$ <br>
with  $x_n$  from (3.7). Moreover,  $g(x_n) = g_0(x_n)$ .<br>
Quation (ii) for  $t = 0$  implies  $g(0) = 0$ , hence in view<br>
ity for  $n = 0$ . Assume that the assertion is true for a cer<br>
ave either  $x_n \leq \frac{1}{a}$ 

$$
g\left(x_n + \frac{t}{a^n}\right) = \frac{1}{c}g_0(x_{n-1}) + \frac{1}{c^n}g(t) = g_0(x_n) + \frac{1}{c^n}g(t)
$$

for  $0 \le t \le 1$ . In the case of  $x_n \ge 1 - \frac{1}{a}$  we have in view of  $\xi_n = 1$  the representation

$$
x_n \in F
$$
, we have either  $x_n \leq \frac{1}{a}$  or  $x_n \geq 1 - \frac{1}{a}$ . In the first case  $x_n = \frac{x_n}{a}$   
from (ii) and (3.8) that  

$$
g\left(x_n + \frac{t}{a^n}\right) = \frac{1}{c}g_0(x_{n-1}) + \frac{1}{c^n}g(t) = g_0(x_n) + \frac{1}{c^n}g(t)
$$
for  $0 \leq t \leq 1$ . In the case of  $x_n \geq 1 - \frac{1}{a}$  we have in view of  $\xi_n = 1$  the re
$$
1 - x_n = (a - 1)\sum_{\nu=1}^{n-1} \frac{1 - \xi_{\nu}}{a^{\nu}} + (a - 1)\sum_{\nu=n+1}^{\infty} \frac{1}{a^{\nu}} = y_{n-1} + \frac{1}{a^n}
$$
with  $y_{n-1} \leq \frac{1}{a}$  and in view of (iii) and  $g(1) = 1 - g(0) = 1$  the relation  

$$
1 - g(x_n) = g(1 - x_n) = g\left(y_{n-1} + \frac{1}{a^n}\right) = g_0(y_{n-1}) + \frac{1}{c^n}
$$

$$
1 - g(x_n) = g(1 - x_n) = g\left(y_{n-1} + \frac{1}{a^n}\right) = g_0(y_{n-1}) + \frac{1}{c^n}
$$

which implies that

Cantor Sets and Integral-Funct  
\n
$$
g(x_n) = 1 - g_0(y_{n-1}) - \frac{1}{c^n} = g_0(x_n).
$$

Now, we get by application of (iii) the relation

$$
g\left(x_n + \frac{t}{a^n}\right) = g\left(1 - y_{n-1} - \frac{1}{a^n} + \frac{t}{a^n}\right) = 1 - g\left(y_{n-1} + \frac{1-t}{a^n}\right)
$$
\n
$$
= 1 - g_0(y_{n-1}) - \frac{g(1-t)}{c^n} = g_0(x_n) + \frac{1}{c^n}g(t)
$$
\n
$$
0 \le t \le 1
$$
, which proves (3.11) by induction. The second assertion of the proposition was from (3.11) for  $t = 0$  **Remarks.** 1. For  $g = g_0$  and  $t = (a - 1) \sum_{\nu=1}^{\infty} \frac{\xi_{n+\nu}}{a^{\nu}}$  equation (3.11) easily follows  
\n $x = x_n + \frac{t}{a^n}$  and (3.8).  
\n2. Equations (iii) and (3.11) imply

for  $0 \le t \le 1$ , which proves (3.11) by induction. The second assertion of the proposition follows from  $(3.11)$  for  $t = 0$ 

from  $x = x_n + \frac{t}{a^n}$  and (3.8). **Remarks.** 1. For  $g = g_0$  and  $t = (a-1)\sum_{\nu=1}^{\infty} \frac{\xi_{n+\nu}}{a^{\nu}}$  equation (3.11) easily follows

2. Equations (iii) and (3.11) imply

$$
g\left(z_n-\frac{t}{a^n}\right)=g_0(z_n)-\frac{1}{c^n}g(t)
$$

for  $0 \le t \le 1$  with  $z_n = 1 - x_n$  and  $g(z_n) = g_0(z_n)$ .

3. The statement of Proposition 3.1 is also valid if we replace (iii) by

$$
g\left(z_n - \frac{1}{a^n}\right) = g_0(z_n) - \frac{1}{c^n} g(t)
$$
\n
$$
z_n = 1 - x_n \text{ and } g(z_n) = g_0(z_n).
$$
\nent of Proposition 3.1 is also valid if we replace (i)

\n
$$
g\left(\frac{a-1}{a} + \frac{t}{a}\right) = \frac{c-1}{c} + \frac{1}{c} g(t) \qquad (0 \le t \le 1),
$$
\n
$$
n = 1. \text{ Proposition 3.2 implies that } g = g_0 \text{ sati}
$$

i.e. by (3.11) with  $n = 1$ . Proposition 3.2 implies that  $g = g_0$  satisfies this equation. The proof of the uniqueness can be carried out analogously as in the second part of the proof of Proposition 3.1, however, with the sequence  $g(z_n - \frac{1}{a^n}) = g_0(z_n) - \frac{1}{c^n} g(t_n)$ <br>  $f(z_n) = g_0(z_n)$ .<br>  $f(z_n) = g_0(z_n)$ .<br>  $f(z_n) = \frac{1}{c} + \frac{1}{c} g(t)$  (0<br>  $f(z_n) = \frac{1}{c} + \frac{1}{c} g(t)$  (1)<br>  $f(z_n) = \begin{cases} a^t e^{-t} & \text{if } t_{n-1} < \frac{1}{a} \\ a^t e^{-t} > 1 \end{cases}$ <br>  $f(z_n) = \begin{cases} a^t e^{-t} & \text{if } t_{n-1} &$ 

$$
t_n = \begin{cases} a \, t_{n-1} & \text{if } t_{n-1} < \frac{1}{a} \\ a \, t_{n-1} - a + 1 & \text{if } t_{n-1} > 1 - \frac{1}{a} \end{cases}
$$

Thus we have a generalization of a result of W. Sierpiński [10] concerning the case of  $a = 3$  and  $c = 2$ , where  $g_0$  is Cantor's singular function (cf. also [9: p. 241]). A nonconstant  $g : [0,1] \mapsto [0,1]$  is called *(strictly) singular*, if it is continuous and *(strictly)*<br>increasing with  $g'(t) = 0$  a.e. *(cf.* [6*]*, where also some examples of strictly singular<br>functions are given). In the case increasing with  $g'(t) = 0$  a.e. (cf. [6], where also some examples of strictly singular functions are given). In the case of  $c = 2$ ,  $g_0$  is a singular function which is constant on the closed intervals  $\overline{G}_{kn}$ , more precisely, (3.10) implies in view of  $\gamma_k(2) = k$  that if  $t_{n-1} < \frac{1}{a}$ <br>if  $t_{n-1} > 1 - \frac{1}{a}$ .<br>V. Sierpiński [10] concerning the case of<br>r function (cf. also [9: p. 241]). A non-<br>ingular, if it is continuous and (strictly)<br>also some examples of strictly singular<br>a singula *g* ( $\frac{1}{a}$ ) =  $\frac{1}{2}g(t)$  and  $g(t)$  +  $g(t)$  =  $\frac{1}{2}g(t)$  =  $\frac{1}{2}g(t)$  =  $\frac{1}{2}g(t)$  =  $\frac{1}{2}g(t)$  =  $\frac{1}{2}g(t)$  =  $\frac{1}{2}g(t)$  = 0 a.e. (cf. ly) *singular*, if it is continuous and (strictly)  $\frac{1}{2}$ ) = 0 a.

$$
g_0(t) = \frac{2k+1}{2^{n+1}} \qquad \text{for } t \in \overline{G}_{kn} . \tag{3.12}
$$

**Proposition 3.3.** In the case of  $c = 2$ ,  $g = g_0$  is the unique function of bounded *variation on* [0, 1] *satisfying only* 

$$
g\left(\frac{t}{a}\right) = \frac{1}{2}g(t) \qquad \text{and} \qquad g(t) + g(1-t) = 1. \tag{3.13}
$$

Proof. We show that every function *g of* bounded variation on [0,1] satisfying (3.13) has the property

upper

\nvery function 
$$
g
$$
 of bounded variation on  $[0,1]$  satisfying  $(3.13)$ 

\n $g(t) = \frac{1}{2}$  for  $\frac{1}{a} \leq t \leq \frac{a-1}{a}$ .

\n(3.14)

\nlation of  $g$  in the interval  $G_0 = G_{00}$ . In view of  $(3.11)$  with  $a^{1/2}$ .

Let *D* denote the total variation of *g* in the interval  $G_0 = G_{00}$ . In view of (3.11) with  $c = 2$  and Lemma 3.1 we have

$$
\text{for } \frac{1}{a} \leq t
$$
\n
$$
g \text{ in the interval}
$$
\n
$$
\bigvee_{G_{kn}} (g) = \frac{1}{2^n} D
$$
\n
$$
U
$$
\nSince the int

for  $k = 0, 1, \ldots, 2<sup>n</sup> - 1$  and all  $n \in \mathbb{N}$ . Since the intervals  $G_{kn}$  are disjoint, for the total variation of g on the set  $G_m$  defined by (3.2) we get

$$
g(t) = \frac{1}{2} \quad \text{for} \quad \frac{1}{a} \le t \le \frac{a-1}{a}.
$$
  
variation of g in the interval  $G_0 = G_0$   
 $\Rightarrow$  have  

$$
\bigvee_{G_{kn}} (g) = \frac{1}{2^n} D
$$
  
and all  $n \in \mathbb{N}$ . Since the intervals  $G_{kn}$   
 $G_m$  defined by (3.2) we get  

$$
\bigvee_{G_m} (g) = \sum_{n=0}^{m} \sum_{k=0}^{2^n - 1} \frac{1}{2^n} D = (m+1) D.
$$
  
 $D = 0$  since g has bounded variation,

For  $m \to \infty$  this implies  $D = 0$  since g has bounded variation, i.e. (3.14) is valid. Now the statement follows from Proposition 3.1

## 4. Properties of the eigenfunctions

In order to obtain relations between eigenfunctions of the integral equation (1.18), we first remember that a solution  $f$  of (1.18) with  $a > 1$  is infinitely often differentiable and that we get by differentiation

$$
\lambda f^{(n)}(t) = a^n \int_{at-a+1}^{at} f^{(n)}(\tau) d\tau.
$$

Hence, the *n*-th derivative  $f^{(n)}$  is also an eigenfunction of (1.18) to the eigenvalue  $\lambda a^{-n}$ , Hence, the *n*-th derivative  $f^{(n)}$  is also an eigenfunction of (1.18) to the eigenvalue  $\lambda a^{-n}$ , so far as  $f^{(n)}$  does not vanish identically (cf. [1: Formula (6.6) for  $\lambda = \frac{1}{b}$ ]). Next, we shall see that each derivative of *f* can be expressed as a linear combination of *f* with different arguments. For the first derivative *f'* we have ftration<br>  $\lambda f^{(n)}(t) = a^n \int_{at-a+1}^{at} f^{(n)}(\tau) d\tau$ .<br>  $f^{(n)}$  is also an eigenfunction of (1.18) to the eigenvalue  $\lambda a^{-n}$ ,<br>
sish identically (cf. [1: Formula (6.6) for  $\lambda = \frac{1}{b}$ ]). Next, we<br>
we of f can be expressed as a

$$
f'(t) = \frac{a}{\lambda} [f(at) - f(at - a + 1)].
$$
 (4.1)

In order to obtain a representation for the higher derivatives, we need the former sequences  $\gamma_n$  and  $\varepsilon_n$ .

Lemma 4.1. Suppose that f is an eigenfunction of (1.18) with the eigenvalue  $\lambda$ and  $n \in \mathbb{N}_0$ . Then we have

ivative of 
$$
f
$$
 can be expressed as a linear combination of  $f$  with  
or the first derivative  $f'$  we have

\n
$$
f'(t) = \frac{a}{\lambda} [f(at) - f(at - a + 1)] .
$$
\n(4.1)

\nrepresentation for the higher derivatives, we need the former se-  
pose that  $f$  is an eigenfunction of (1.18) with the eigenvalue  $\lambda$   
have

\n
$$
f^{(n)}(t) = \lambda^{-n} a^{\frac{n(n+1)}{2}} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} f(a^n t - \gamma_{\nu}) .
$$
\n(4.2)

**Proof.** For  $n = 0$  this equation is an identity. If (4.2) is true for an integer *n*, then we have in view of  $(4.1)$ ,  $(2.2)$  and  $(2.5)$  that

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\nProof. For 
$$
n = 0
$$
 this equation is an identity. If (4.2) is true for an integer n, t

\nhave in view of (4.1), (2.2) and (2.5) that

\n
$$
f^{(n+1)}(t) = \lambda^{-n} a^{\frac{n(n+1)}{2} + n} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} f'(a^n t - \gamma_{\nu})
$$
\n
$$
= \lambda^{-n-1} a^{\frac{n(n+1)}{2} + n+1} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} \left( f(a^{n+1} t - a \gamma_{\nu}) - f(a^{n+1} t - a \gamma_{\nu} - \gamma_1) \right)
$$
\n
$$
= \lambda^{-n-1} a^{\frac{(n+1)(n+2)}{2}} \sum_{\nu=0}^{2^{n+1}-1} \varepsilon_{\nu} f(a^{n+1} t - \gamma_{\nu}),
$$
\nthat (4.2) is proved by induction

such that (4.2) is proved by induction  $\blacksquare$ 

Taking into account that  $f = \phi_n$  is an eigenfunction of (1.18) to the eigenvalue  $\lambda = \frac{a^n}{b}$ , and considering

$$
\lambda^{-n} a^{\frac{n(n+1)}{2}} = \frac{a^{\frac{n(n+1)}{2}} b^n}{a^{n^2}} = \frac{b^n}{a^{\frac{n(n-1)}{2}}}
$$

as well as  $\phi(t) = \phi_n^{(n)}(t)$  for all *t*, we get the following inversion of (1.14).

*the representation*

Taking into account that 
$$
f = \phi_n
$$
 is an eigenfunction of (1.18) to the eigenvalue  $\frac{a^n}{b}$ , and considering  
\n
$$
\lambda^{-n} a^{\frac{n(n+1)}{2}} = \frac{a^{\frac{n(n+1)}{2}}b^n}{a^{n^2}} = \frac{b^n}{a^{\frac{n(n-1)}{2}}}
$$
\nwell as  $\phi(t) = \phi_n^{(n)}(t)$  for all  $t$ , we get the following inversion of (1.14).  
\nCorollary 4.1. For all  $t \in \mathbb{R}$  and for all  $n \in \mathbb{N}_0$ , the solution  $\phi$  of (1.1) – (1.2) has  
\nrepresentation  
\n
$$
\phi(t) = \frac{b^n}{a^{\frac{n(n-1)}{2}}} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} \phi_n(a^n t - \gamma_{\nu})
$$
\n(4.3)  
\nProposition 4.1. The polynomials  $\psi_n$  have the property

**Proposition 4.1.** *The polynomials*  $\psi_n$  *have the property* 

2"-1 .(2-- n+1) *rn! a 2 vbm(t 'v) = (M - n)! b' bm\_n () (4.4) V=0*  **Proof.** We apply Lemma 4.1 with *I* = *,bm* and *A = bm\_n(t)* and *A"a 2 = a*

*for arbitrary*  $m \ge n \ge 0$ .

and use that

*rbitrary* 
$$
m \ge n \ge 0
$$
.  
\n**Proof.** We apply Lemma 4.1 with  $f = \psi_m$  and  $\lambda = \frac{a^{m+1}}{b}$  and use that  
\n
$$
\psi_m^{(n)}(t) = \frac{m!}{(m-n)!} \psi_{m-n}(t) \quad \text{and} \quad \lambda^{-n} a^{\frac{n(n+1)}{2}} = \frac{a^{\frac{n(n+1)}{2}b^n}}{a^{n(m+1)}} = \frac{b^n}{a^{\frac{n(2m-n+1)}{2}}}
$$

Relation (4.4) is proved after replacing t by  $\frac{t}{a^n}$ 

We remark that for  $n > m \geq 0$  the left-hand side of (4.4) vanishes, since the sums (2.11) vanish for these m and n. This is also the reason why for  $m > n$  the degree of the polynomials (4.4) reduces from *m* to  $m - n$ . In particular, for  $m \ge n = 1$  we have se that<br>  $\frac{b^n}{a^{\frac{n(2m-n+1)}{2}}}$ <br>
es, since the dependence of  $n \geq n = 1$  where  $n \geq 2$ 

$$
\psi_m(t) - \psi_m(t - a + 1) = ma^{m-1}(a - 1)\psi_{m-1}\left(\frac{t}{a}\right).
$$
 (4.5)

By analytic continuation this equation is even valid for all *a* different from the poles of as a function of *a* (these poles lie on the circle  $|a| = 1$ ). For  $t = \frac{a}{2}$  (4.5) simplifies in view of  $(1.1)$  to equation is even valued<br>boles lie on the circle<br> $\left(\frac{a}{2}\right) = \frac{m}{2}a^{m-1}(a - a)$ <br>on derive from (1.9) to *flatteriata* is even valid for all *a*<br> *f* se poles lie on the circle  $|a| = 1$ ). F<br>  $\psi_m\left(\frac{a}{2}\right) = \frac{m}{2}a^{m-1}(a-1)\psi_{m-1}\left(\frac{1}{2}\right)$ <br> *f*, considering (1.3) and the general *i* different fr<br>
For  $t = \frac{a}{2}$  (4)<br>
<br> **)**<br>
<br> **ating function**  $\nu$  (these poles<br>  $\nu_m\left(\frac{a}{2}\right)$ <br>  $\nu \in \mathbb{N}_0$ , consi<br>  $\mu$ , we can de<br>  $\nu_m(t) = \sum_{\mu=0}^m$ <br>
ains (4.6) as 1012 L. Ber<br>By analytic cor<br> $\psi_m$  as a function<br>view of (1.1) to<br>for m odd. For<br>Bernoulli numb<br>which for  $t = \frac{6}{2}$ <br>and<br>imply by compaint

$$
\psi_m\left(\frac{a}{2}\right) = \frac{m}{2}a^{m-1}(a-1)\psi_{m-1}\left(\frac{1}{2}\right)
$$
\n(4.6)

for *m* odd. For all  $m \in \mathbb{N}_0$ , considering (1.3) and the generating function  $\frac{p}{e^p-1}$  for the Bernoulli numbers  $B_{\mu}$ , we can derive from (1.9) the representation

$$
\psi_m(t) = \sum_{\mu=0}^m \binom{m}{\mu} B_\mu (1-a)^\mu a^{m-\mu} \psi_{m-\mu} \left( \frac{t}{a} \right) ,
$$

 $\psi_m(t) = \sum_{\mu=0}^m \binom{m}{\mu} B_{\mu} (1-a)^{\mu} a^{m-\mu} \psi_{m-\mu} \left(\frac{t}{a}\right)$ ,<br>which for  $t = \frac{a}{2}$  contains (4.6) as a special case. Moreover, for all  $m \in \mathbb{N}$ , (1.7), (1.9)<br>and

$$
\frac{\partial}{\partial p}e^{tp}\Phi(p) = e^{tp}\Phi(p)\Big(t + \frac{d}{dp}\ln \Phi(p)\Big)
$$

imply by comparison of coefficients the recursion formula

$$
\psi_m(t) = \sum_{\mu=0} {m \choose \mu} B_{\mu} (1-a)^{\mu} a^{m-\mu} \psi_{m-\mu} \left(\frac{t}{a}\right),
$$
  
\n
$$
= \frac{a}{2} \text{ contains (4.6) as a special case. Moreover, for all } m \in \mathbb{N},
$$
  
\n
$$
\frac{\partial}{\partial p} e^{tp} \Phi(p) = e^{tp} \Phi(p) \left(t + \frac{d}{dp} \ln \Phi(p)\right)
$$
  
\nmparison of coefficients the recursion formula  
\n
$$
\psi_m(t) = \left(t - \frac{1}{2}\right) \psi_{m-1}(t) + \frac{1}{m} \sum_{\mu=2}^{m} {m \choose \mu} B_{\mu} \frac{(a-1)^{\mu}}{a^{\mu}-1} \psi_{m-\mu}(t),
$$

which for  $t = 0$  is already known from [1].

In the following, we once more restrict ourselves to  $a \geq 2$  and apply Lemma 4.1 to the solution  $\phi$  of (1.1) - (1.2), i.e. we consider  $f = \phi$  and  $\lambda = \frac{1}{b}$ . For  $t \in F_{kn}$ , i.e. according to (3.3) for [1].<br>strict<br>we con<br> $\leq t \leq$ which for  $t = 0$  is already known from [1].<br>
In the following, we once more restrict ourselves to  $a \ge 2$  and apply Lemma 4.1<br>
to the solution  $\phi$  of (1.1) - (1.2), i.e. we consider  $f = \phi$  and  $\lambda = \frac{1}{b}$ . For  $t \in F_{kn}$ 

$$
\frac{\gamma_k}{a^n} \leq t \leq \frac{\gamma_k+1}{a^n} ,
$$

the solution  $\phi$  of (1.1) - (1.2) with  $a \ge 2$ , which vanishes outside of (0,1), we get from<br> *Lemma* 4.1  $\phi^{(n)}(t) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^n \phi(a^n t - \gamma_k)$  for  $t \in F_{kn}$ , (4.7) Lemma 4.1

$$
\phi^{(n)}(t) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^n \phi(a^n t - \gamma_k) \qquad \text{for } t \in F_{kn} , \qquad (4.7)
$$

and otherwise we have  $\phi^{(n)}(t) = 0$ , namely for  $t \in G_{n-1}$  with  $n \ge 1$ . In view of  $\phi(t) > 0$ for  $t \in (0,1)$  this result implies [13: Proposition 4.1] that  $F_n$  is the support of  $\phi^{(n)}$  and, moreover, for  $n = 2$  that  $\phi(t)$  is strictly convex for t in  $F_{02}$  or  $F_{32}$ , and strictly concave for *t* in  $F_{12}$  or  $F_{22}$ . In the case of  $a = 2$  where  $\gamma_k = k$  formula (4.7) reduces to (1.2), i.e. we consider  $f = \phi$  and  $\lambda = \frac{1}{b}$ . For  $t \in F_{kn}$ , i.e.<br>  $\frac{\gamma_k}{a^n} \le t \le \frac{\gamma_k + 1}{a^n}$ ,<br>
but in view of (3.1)  $a^n t - \gamma_\nu \notin (0, 1)$  for  $\nu \ne k$ . Hence, for<br>
2) with  $a \ge 2$ , which vanishes outside of (0,1), we ge we have  $0 \le a^n t - \gamma_k \le 1$ , but in view of  $(3.1) a^n t - \gamma_{\nu} \notin (0,1)$  for  $\nu \ne k$ . Hence, for<br>the solution  $\phi$  of  $(1.1)$  -  $(1.2)$  with  $a \ge 2$ , which vanishes outside of  $(0,1)$ , we get from<br>Lemma 4.1<br> $\phi^{(n)}(t) = \varepsilon_k a^{\frac{n$  $\psi_m(t) = (t - \frac{1}{2})\psi_{m-1}(t)$ <br>  $t = 0$  is already known from<br>
a following, we once more restrict to  $\phi$  of (1.1) - (1.2), i.e.<br>  $\frac{\gamma_k}{a^n}$ :<br>  $0 \le a^n t - \gamma_k \le 1$ , but in view<br>
on  $\phi$  of (1.1) - (1.2) with  $a \ge$ <br>  $\phi^{(n)}(t) = \v$ 

$$
\phi^{(n)}(t) = \varepsilon_k 2^{\frac{n(n+3)}{2}} \phi(2^n t - k) \qquad (k = [2^n t]) , \qquad (4.8)
$$

of the  $L_2$ -norms of  $\phi^{(n)}$ , namely in particular to  $\phi^{(n)}(\frac{k}{2^n}) = 0$  (cf. [11]). Formula (4.7) is very useful for the calculation

$$
\|\phi^{(n)}\|^2 = a^{n(n+1)}b^{2n} \sum_{k=0}^{2^n-1} \int_{-\gamma_k/a^n}^{(\gamma_k+1)/a^n} \phi^2(a^nt-\gamma_k) dt = 2^n a^{n^2}b^{2n} \|\phi\|^2
$$

Moreover, we find for the corresponding scalar product by *m* partial integrations

$$
(\phi^{(n)}, \phi^{(n+2m)}) = (-1)^m (\phi^{(n+m)}, \phi^{(n+m)}) = (-1)^m 2^{n+m} a^{(n+m)^2} b^{2(n+m)} ||\phi||^2
$$
  
whereas  $(\phi^{(n)}, \phi^{(n+2m+1)}) = 0$  in view of the symmetry  $\phi(t) = \phi(1-t)$ .

### 5. Relations with polynomials

Cantor Sets and Integral-Functional Equations<br>
5. Relations with polynomials<br>
For  $a > 2$  and  $t \in \overline{G}_{kn}$  given by (3.2) we have the inequality  $\frac{1}{a} \le a^n t - \gamma_k \le$ <br>
Hence, we get in view of  $\overline{G}_{kn} \subset F_{kn}$ , (4.7) and For  $a > 2$  and  $t \in \overline{G}_{kn}$  given by (3.2) we have the inequality  $\frac{1}{a} \leq a^n t - \gamma_k \leq 1 - \frac{1}{a}$ .

$$
\phi^{(n)}(t)=\varepsilon_k a^{\frac{n(n+1)}{2}}b^n\phi(a^nt-\gamma_k)=\varepsilon_k a^{\frac{n(n+1)}{2}}b^{n+1}\qquad(t\in\overline{G}_{kn})
$$

Thus for  $t \in \overline{G}_{kn}$ ,  $\phi(t)$  is a polynomial of degree *n*, a fact which is already known from [1], but now we also know the main term of this polynomial:

if the polynomials

\n
$$
\overline{S}_{kn}
$$
 given by (3.2) we have the inequality  $\frac{1}{a} \leq a^n t - \gamma_k \leq 1 - \frac{1}{a}$ .\nor of  $\overline{G}_{kn} \subset F_{kn}$ , (4.7) and  $\phi(\tau) = b$  for  $\frac{1}{a} \leq \tau \leq 1 - \frac{1}{a}$  that

\n
$$
\varepsilon_k a^{\frac{n(n+1)}{2}} b^n \phi(a^n t - \gamma_k) = \varepsilon_k a^{\frac{n(n+1)}{2}} b^{n+1} \qquad (t \in \overline{G}_{kn})
$$
.\nif, it is a polynomial of degree n, a fact which is already known from the main term of this polynomial:

\n
$$
\phi(t) = \varepsilon_k \frac{a^{\frac{n(n+1)}{2}} b^{n+1}}{n!} t^n + \dots \qquad (t \in \overline{G}_{kn})
$$
.\n(5.1)

\nThen determine the complete polynomials and include the limit case

\nexists  $\overline{G}$ ,  $\overline{G}$  represents to single points  $\frac{2k+1}{2}$ . Since  $G$  lies densely.

Moreover, we can even determine the complete polynomials and include the limit case  $a = 2$ , where the intervals  $\overline{G}_{kn}$  degenerate to single points  $\frac{2k+1}{2n+1}$ . Since *G* lies densely in [0, 1], the function  $\phi$  is uniquely determined by means of these polynomials and continuity.  $f \in \overline{G}_{kn}$ .<br>
ials and include the<br>
its  $\frac{2k+1}{2n+1}$ . Since *G* lians of these polync<br>
of the closed interva<br>
(2) has the represen<br>
( $t \in \overline{G}_{kn}$ )

Theorem 5.1. *In the case of a*  $\geq$  2 and t in one of the closed intervals  $\overline{G}_{kn}$  for  $k = 0, 1, \ldots, 2^n - 1$  ( $n \in \mathbb{N}$ ), the solution  $\phi$  of  $(1.1) - (1.2)$  has the representation

$$
\begin{aligned}\n\phi & \text{is uniquely determined by means of these polynomials and} \\
\hline\n\end{aligned}\n\begin{aligned}\nI_n \text{ the case of } a \geq 2 \text{ and } t \text{ in one of the closed intervals } \overline{G}_{kn} \text{ for } (n \in \mathbb{N}), \text{ the solution } \phi \text{ of } (1.1) - (1.2) \text{ has the representation} \\
\phi(t) &= c_n \sum_{\nu=0}^{2k} \varepsilon_{\nu} \psi_n(a^{n+1}t - \gamma_{\nu}) \qquad (t \in \overline{G}_{kn}) \qquad (5.2) \\
\mathbf{c}_n &= \frac{b^{n+1}}{a^{\frac{n(n+1)}{2}} n!} = \frac{1}{a^{\frac{(n+1)(n-2)}{2}} (a-1)^{n+1} n!} \qquad (5.3) \\
\text{the representation (4.3) with } n+1 \text{ instead of } n. \text{ For } t \in \overline{G}_{kn}, \text{ i.e.}\n\end{aligned}
$$

*where*  $c_n$  *is given by* 

$$
c_n = \frac{b^{n+1}}{a^{\frac{n(n+1)}{2}} n!} = \frac{1}{a^{\frac{(n+1)(n-2)}{2}} (a-1)^{n+1} n!}
$$
(5.3)

**Proof.** We use the representation (4.3) with  $n + 1$  instead of *n*. For  $t \in \overline{G}_{kn}$ , i.e.

$$
\frac{\gamma_{2k}+1}{a^{n+1}}\leq t\leq \frac{\gamma_{2k+1}}{a^{n+1}},
$$

we have the inequalities  $a^{n+1}t - \gamma_{2k+1} \leq 0$  and  $a^{n+1}t - \gamma_{2k} \geq 1$ . According to (3.1) and  $\phi_{n+1}(\tau) = 0$  for  $\tau \leq 0$ , the terms  $\phi_{n+1}(a^{n+1}t - \gamma_{\nu})$  vanish for  $\nu \geq 2k + 1$ , but for and  $\phi_{n+1}(\tau) = 0$  for  $\tau \le 0$ , the terms  $\phi_{n+1}(a^{n+1}t - \gamma_{\nu})$  vanish for  $\nu \ge 2k + 1$ , but for  $\nu \le 2k$ , in view of (1.16) with  $n + 1$  instead of *n*, we have the representations

$$
\phi_{n+1}(a^{n+1}t-\gamma_\nu)=\frac{1}{n!}\psi_n(a^{n+1}t-\gamma_\nu)
$$

for  $\nu = 0, 1, \ldots, 2k$ . This altogether implies the assertion  $\blacksquare$ 

We remark that, for  $k = 0$ , formula (5.2) reduces to (1.12).

Next, we are going to extend (5.2) to the larger interval  $F_{kn} \supset G_{kn}$ .

**Proposition 5.1.** For  $a \geq 2$  and  $t \in F_{kn}$ , i.e.  $t = \frac{\gamma_k + r}{a^n}$  with  $0 \leq \tau \leq 1$ , the *solution*  $\phi$  of (1.1) – (1.2) has the property

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\nProposition 5.1. For 
$$
a \geq 2
$$
 and  $t \in F_{kn}$ , i.e.  $t = \frac{\gamma_k + \tau}{a^n}$  with  $0 \leq \tau \leq 1$ , the *tion*  $\phi$  of (1.1) – (1.2) has the property

\n
$$
\phi\left(\frac{\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) = c_{n-1} \sum_{\nu=0}^{k-1} \varepsilon_{\nu} \psi_{n-1}(\gamma_k + \tau - \gamma_{\nu}) \qquad (0 \leq \tau \leq 1) \qquad (5.4)
$$
\nre  $c_n$  is given by (5.3).

\nProof. According to  $0 \leq a^n t - \gamma_k = \tau \leq 1$  for  $t \in F_{kn}$  and  $\gamma_{\nu} + 1 \leq \gamma_{\nu+1}$  for  $\nu \geq 0$ 

\nLemma 2.2) we have the relations

\n
$$
a^n t - \gamma_{\nu} = \gamma_k + \tau - \gamma_{\nu} \begin{cases} \geq 1 & \text{for } \nu < k \\ \in [0,1] & \text{for } \nu = k \\ \leq 0 & \text{for } \nu > k. \end{cases}
$$
\nce, (5.4) follows from (4.3) in view of  $\phi_n(t) = 0$  for  $t \leq 0$ , as well as (1.15) and

*where*  $c_n$  *is given by* (5.3).

(cf. Lemma 2.2) we have the relations

$$
a^{n}t - \gamma_{\nu} = \gamma_{k} + \tau - \gamma_{\nu} \begin{cases} \geq 1 & \text{for } \nu < k \\ \in [0,1] & \text{for } \nu = k \\ \leq 0 & \text{for } \nu > k. \end{cases}
$$

Hence, (5.4) follows from (4.3) in view of  $\phi_n(t) = 0$  for  $t \leq 0$ , as well as (1.15) and  $(1.16)$ 

*By m* differentiations of (5.4) we get in view of  $\phi^{(m)}(0) = 0$  and  $\psi'_m = m\psi_{m-1}$  the *Corollary 5.1. In the case of*  $a \geq 2$  *and*  $n > m \geq 0$  *the derivatives*  $\phi^{(m)}$  *of the solution of*  $(1.1) - (1.2)$  *have the values* we have the relations<br>  $a^n t - \gamma_\nu = \gamma_k + \tau$ .<br>
ws from (4.3) in view<br>
tiations of (5.4) we ge<br>
tiations of (5.4) we ge<br>
1. In the case of  $a \ge$ <br>
(m)  $\left(\frac{\gamma_k}{a^n}\right) = \frac{a^{mn} c_{n-1}}{(n-m)}$ <br>
(m)  $\left(\frac{\gamma_k}{a^n}\right) = \frac{a^{mn} c_{n-1}}{(n-m)}$ as well as  $(1.15)$ <br>
and  $\psi'_m = m\psi_{m-1}$ <br> *generivatives*  $\phi^{(m)}$ <br>  $(\psi_m)$  $f \phi^{(m)}(0) = 0$  and  $\psi'_m = m\psi_{m-1}$  the<br>  $> m \ge 0$  the derivatives  $\phi^{(m)}$  of the<br>  $\phi^{(m)} = \phi^{(m)}$  (5.5)<br>  $\phi^{(m)} = \phi^{(m)}$  (5.5)<br>  $\phi^{(m)}(k - \psi)$  (5.6)<br>  $\phi^{(m)}(k - \psi) = \phi^{(m)}(k - \psi)$  (5.6)

$$
\phi^{(m)}\left(\frac{\gamma_k}{a^n}\right) = \frac{a^{mn} c_{n-1} (n-1)!}{(n-m-1)!} \sum_{\nu=0}^{k-1} \varepsilon_{\nu} \psi_{n-m-1}(\gamma_k - \gamma_{\nu})
$$
(5.5)  
0 \le k \le 2<sup>n</sup> - 1.  
In particular, in the case of  $a = 2$  where  $\gamma_{\nu} = \nu$  the values (5.5) with  $m = 0$  simplify

*with*  $0 \leq k \leq 2^n - 1$ .

to

In the case of 
$$
a \ge 2
$$
 and  $n > m \ge 0$  the derivatives  $\phi^{(m)}$  of the  
\n1.2) have the values  
\n
$$
\left(\frac{\gamma_k}{a^n}\right) = \frac{a^{mn} c_{n-1} (n-1)!}{(n-m-1)!} \sum_{\nu=0}^{k-1} \epsilon_{\nu} \psi_{n-m-1}(\gamma_k - \gamma_{\nu})
$$
\n
$$
\text{the case of } a = 2 \text{ where } \gamma_{\nu} = \nu \text{ the values (5.5) with } m = 0 \text{ simplify}
$$
\n
$$
\phi\left(\frac{k}{2^n}\right) = \frac{1}{2^{\frac{n(n-3)}{2}} (n-1)!} \sum_{\nu=0}^{k-1} \epsilon_{\nu} \psi_{n-1}(k-\nu) \,. \tag{5.6}
$$
\n
$$
a = 2 \text{ we obtain, for example,}
$$

Thus in the case of  $a = 2$  we obtain, for example,

Prolary 3.1. In the case of 
$$
a \geq 2
$$
 and  $n > m \geq 0$  the derivatives  $\phi^{(m)}$ 

\n
$$
\phi^{(m)}\left(\frac{\gamma_k}{a^n}\right) = \frac{a^{mn}c_{n-1}(n-1)!}{(n-m-1)!} \sum_{\nu=0}^{k-1} \varepsilon_{\nu} \psi_{n-m-1}(\gamma_k - \gamma_{\nu})
$$
\n
$$
\leq k \leq 2^n - 1.
$$
\nparticular, in the case of  $a = 2$  where  $\gamma_{\nu} = \nu$  the values (5.5) with  $m = 0$  s

\n
$$
\phi\left(\frac{k}{2^n}\right) = \frac{1}{2^{\frac{n(n-3)}{2}(n-1)!}} \sum_{\nu=0}^{k-1} \varepsilon_{\nu} \psi_{n-1}(k - \nu).
$$
\nIn the case of  $a = 2$  we obtain, for example,

\n
$$
\phi\left(\frac{1}{2}\right) = 2, \quad \phi\left(\frac{1}{4}\right) = 1, \quad \phi\left(\frac{1}{8}\right) = \frac{1}{9}, \quad \phi\left(\frac{3}{8}\right) = \frac{17}{9},
$$
\n
$$
\phi\left(\frac{1}{16}\right) = \frac{1}{24}, \quad \phi\left(\frac{3}{16}\right) = \frac{145}{288}, \quad \phi\left(\frac{5}{16}\right) = \frac{431}{288}, \quad \phi\left(\frac{7}{16}\right) = \frac{575}{288}, \dots
$$
\nmark that the particular formula (5.6) can also be derived from (1.17). No case of  $a = 2$  the left-hand side from (1.17) can be written in the form

\n
$$
\sum_{\nu=0}^{\infty} \frac{\nu_1}{\nu_2} \psi_{\nu}(\nu_1, \nu_2)
$$

We remark that the particular formula (5.6) can also be derived from (1.17). Namely, in the case of  $a = 2$  the left-hand side from  $(1.17)$  can be written in the form

$$
\phi\left(\frac{1}{2}\right) = 2, \quad \phi\left(\frac{1}{4}\right) = 1, \quad \phi\left(\frac{1}{8}\right) = \frac{1}{9}, \quad \phi\left(\frac{3}{8}\right) = \frac{17}{9},
$$
\n
$$
\frac{1}{6} = \frac{1}{24}, \quad \phi\left(\frac{3}{16}\right) = \frac{145}{288}, \quad \phi\left(\frac{5}{16}\right) = \frac{431}{288}, \quad \phi\left(\frac{7}{16}\right) = \frac{575}{288},
$$
\nthat the particular formula (5.6) can also be derived from (1.1)

\nof  $a = 2$  the left-hand side from (1.17) can be written in the for

\n
$$
\sum_{\nu_i \geq 0} \phi\left(t - \frac{\nu_1}{2^n} - \ldots - \frac{\nu_n}{2}\right)
$$
\n
$$
= \phi(t) + \phi\left(t - \frac{1}{2^n}\right)
$$
\n
$$
+ 2\sum_{\nu=0}^{\infty} (\nu^2 + \nu + 1) \left(\phi\left(t - \frac{4\nu + 2}{2^n}\right) + \phi\left(t - \frac{4\nu + 3}{2^n}\right)\right)
$$
\n
$$
+ 2\sum_{\nu=1}^{\infty} (\nu^2 + 1) \left(\phi\left(t - \frac{4\nu}{2^n}\right) + \phi\left(t - \frac{4\nu + 1}{2^n}\right)\right).
$$

Putting in  $(1.17)$  with  $a = 2$  successively

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\n
$$
t = 2
$$
 successively  
\n $t = \frac{1}{2^n}$ ,  $t = \frac{2}{2^n}$ ,  $t = \frac{3}{2^n}$ , ...  
\n $\Rightarrow \phi(\frac{k}{2^n})$   $(k = 0, 1, 2, ...)$  a linear sys

we obtain for the values  $\phi(\frac{k}{2^n})$   $(k = 0, 1, 2, ...)$  a linear system of equations with a Toeplitz matrix *T*, which is the inverse of the Toeplitz matrix  $(\varepsilon_{i-1})$   $(\varepsilon_i = 0 \text{ for } i < 0)$ 

1 1 11 —1 1 *2 1 1 T = 2 2 1 1 , T' = 1 —1 —1 1 4 2 2 1 1 4 4 2 2 1 1 1 —1 1 —1 —1* 

Since the right-hand side of (1.17) with  $a = 2$  is  $c_{n-1}\psi_{n-1}(2^n t)$  (cf. (5.3)), we obtain (5.6) alter simple calculations.

### 6. Reduced representations

The polynomial relation  $(5.4)$  reads for  $k = 1$ 

*71+T (* a ) <sup>+</sup>*g5 <sup>1</sup> -* <sup>=</sup>e\_*ibn\_i(7i + r)* (0 < *T <* 1), (6.1)

where  $c_n$  is given by (5.3). For large  $k$ , (5.4) is rather redundant so that we want to derive a reduced representation. For convenience, the first parameters  $l_{\nu}$ ,  $\varepsilon_{l_{\nu}}$  and  $k_{\nu}$ appearing in the later formula  $(6.2)$  are shown in Table 2 for the interesting indices  $\nu$ with  $d_{\nu} \neq 0$ .

**Proposition 6.1.** Assume that  $a \geq 2$  and that the number  $k \in \mathbb{N}$  has the dyadic *representation*  $k = d_0 + d_1 2 + d_2 2^2 + \ldots + d_s 2^s$ ,  $d_s = 1$  and  $d_\sigma \in \{0, 1\}$ . Then with *the notations*  $k_{\nu} = d_0 + d_1 2 + \ldots + d_{\nu} 2^{\nu}$  and  $l_{\nu} = d_{\nu+1} + d_{\nu+2} 2 + \ldots + d_s 2^{s-\nu-1}$  for  $0 \leq \nu \leq s$  *we have the relation*  $\left(\frac{\gamma_1 + \tau}{a^n}\right) + \phi\left(\frac{\tau}{a^n}\right) = c_{n-1}\psi_{n-1}(\gamma_1 - \rho_2)$ <br>
2 **3** and representation. For convenience,<br>  $\alpha$  leads representation. For convenience,<br>  $\alpha$  leads formula (6.2) are shown in T<br>
2 **3** and that<br>  $k = d_0 + d_1 2 + d_2$  $\leq \tau \leq 1$ ),<br>
ant so that we war<br>
rameters  $l_{\nu}$ ,  $\varepsilon_{l_{\nu}}$  an<br>
he interesting indic<br>  $r \ k \in \mathbb{N}$  has the dy<br>  $d_{\sigma} \in \{0,1\}$ . Then<br>  $2^2 + \ldots + d_s 2^{s-\nu-1}$ <br>  $\left(\frac{\gamma_{k_{\nu}} + \tau}{a^{\nu}}\right)$ 

have the relation  
\n
$$
\phi\left(\frac{\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) = \sum_{\nu=0}^s \varepsilon_{l_\nu} d_\nu c_{n-\nu-1} \psi_{n-\nu-1} \left(\frac{\gamma_{k_\nu} + \tau}{a^\nu}\right)
$$
\n(6.2)

*for*  $0 \le \tau \le a^{\sigma}$ , whenever  $d_0 = d_1 = \ldots = d_{\sigma-1} = 0$  and  $d_{\sigma} \ne 0$ .

**Proof.** Equation (6.2) can be derived by successive application of (4.5) to (5.4). But an inductive proof is more lucid. For  $k = 1$ , the representation is true in view of (6.1). In order to prove the assertion by induction, we assume that (6.2) *is* valid for a fixed *k* and take into consideration that the parameters  $d_{\nu}$ ,  $k_{\nu}$ ,  $l_{\nu}$ , *s* and  $\sigma$  depend on **1016** L. Berg and M. Krüppel<br>
fixed *k* and take into consideration that the parameters  $d_{\nu}$ ,  $k_{\nu}$ ,  $l_{\nu}$ , *s* and *σ* depend or<br> *k*. Moreover, we recognize that  $k = k_{\nu} + l_{\nu}2^{\nu+1}$ , i.e.  $k \equiv k_{\nu} \mod 2^{\nu+1}$ 

k	dyadic	$\varepsilon_k$	$\mathbf{I}$ $l_0$	$\varepsilon_{l_0}$	$k_0$	l <sub>1</sub>	$\varepsilon_{l_1}$	k <sub>1</sub>	l <sub>2</sub>	$\varepsilon_{l_2}$	$k_2$	$l_3$	$\varepsilon_{l_2}$	$k_3$
0	0	ı												
$\mathbf{I}$	1	$-1$	0	1	ı									
$\overline{2}$	10	-1				0	1	$\overline{2}$						
3	11	ı	ı	-1	1	0	ı	3						
$\boldsymbol{4}$	100	$-1$							0	ı	4			
$\overline{5}$	101	ı	$\mathbf{2}$	$-1$	$\mathbf{1}$				0	1	5			
6	110	$\mathbf{l}$				1	$-1$	$\mathbf{2}$	0	ı	6			
$\overline{7}$	111	-1	3	ı	1	I	$-1$	3	0	1	$\overline{7}$			
8	1000	$-1$										$\boldsymbol{0}$		8
9	1001	1	4	$-1$	ı							0	ı	9
10	1010	ı				$\overline{2}$	-1	$\mathbf{2}$				0	1	10
11	1011	-1	5	ı	1	$\overline{2}$	-1	3				0	ı	11
12	1100	1							1	-1	4	$\bf{0}$		12
13	1101	-1	6	ı	ı				1	-1	5	0	ı	13
14	1110	-1				3	$\mathbf 1$	$\boldsymbol{2}$	1	-1	6	$\mathbf 0$	1	14
15	1111	1	7	-1	1	3	1	3	1	-1	7	0	1	15

Table 2: The first parameters  $l_{\nu}$ ,  $\varepsilon_{l_{\nu}}$  and  $k_{\nu}$ 

**1.** *Induction from k to 2k:* In view of  $2k = d_0 2 + d_1 2^2 + \ldots + d_s 2^{s+1} = 2k_{\nu} + 2^{\nu+2}l_{\nu}$ , the parameters of *2k* depend on the parameters of *k* in the following way:  $k = \frac{d}{d}$  **c**  $\frac{1}{2}$  **c c**  $\frac{1}{2}$  **c c**  $\frac{1}{2}$  **c c**  $\frac{1}{2}$  **c**  $\frac{1}{2}$ 

$$
\begin{array}{c|cccccc}\nk & d_{\nu} & k_{\nu} & l_{\nu} & s & \sigma \\
\hline\n2k & d_{\nu-1} & 2k_{\nu-1} & l_{\nu-1} & s+1 & \sigma+1\n\end{array}
$$

Table *3:* The parameters of *2k* expressed by those of *<sup>k</sup>*

where  $d_{-1} = k_{-1} = l_s = 0$  and  $l_{-1} = k_s = k$ . Making in (6.2) the substitution  $n \mapsto n-1$ , Table 3: The parameters of 2k expressed by those<br>where  $d_{-1} = k_{-1} = l_s = 0$  and  $l_{-1} = k_s = k$ . Making in (6.2) the so<br> $\nu \mapsto \nu - 1$ ,  $\tau \mapsto \frac{\tau}{a}$ , so that  $0 \le \tau \le a^{\sigma+1}$  for the new  $\tau$ , we obtain

Induction from k to 2k: In view of 
$$
2k = d_0 2 + d_1 2^2 + ... + d_s 2^{s+1} = 2k_{\nu}
$$
  
\nameters of 2k depend on the parameters of k in the following way:  
\n
$$
\frac{k}{2k} \frac{d_{\nu}}{d_{\nu-1}} \frac{k_{\nu}}{2k_{\nu-1}} \frac{1}{l_{\nu-1}} \frac{s}{s+1} \frac{\sigma}{\sigma+1}
$$
\nTable 3: The parameters of 2k expressed by those of k  
\n
$$
l_{-1} = k_{-1} = l_s = 0 \text{ and } l_{-1} = k_s = k.
$$
 Making in (6.2) the substitution  $n = 1, \tau \mapsto \frac{\tau}{a}$ , so that  $0 \le \tau \le a^{\sigma+1}$  for the new  $\tau$ , we obtain  
\n
$$
\phi\left(\frac{a\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) = \sum_{\nu=0}^{s+1} \varepsilon_{l_{\nu-1}} d_{\nu-1} c_{n-\nu-1} \psi_{n-\nu-1} \left(\frac{a\gamma_{k_{\nu-1}} + \tau}{a^{\nu}}\right),
$$
\nview of  $a\gamma_k = \gamma_{2k}$ ,  $\varepsilon_k = \varepsilon_{2k}$  and Table 3 this is nothing else than (6.2) of k.  
\nInduction from 2k to 2k + 1: Formula (6.2) reads for 2k instead of k and of  $\tau$   
\n
$$
\left(\frac{\gamma_{2k} + \gamma_1 + \tau}{a^n}\right) - \varepsilon_{2k} \phi\left(\frac{\gamma_1 + \tau}{a^n}\right) = \sum_{\nu=0}^{s} \varepsilon_{l_{\nu}} d_{\nu} c_{n-\nu-1} \psi_{n-\nu-1} \left(\frac{\gamma_{k_{\nu}} + \gamma_1 + \gamma_{k_{\nu-1}}}{a^{\nu}}\right)
$$
\nthe parameters are those belonging to 2k. According to  $d_0 = 0$  we have

and in view of  $a\gamma_k = \gamma_{2k}$ ,  $\varepsilon_k = \varepsilon_{2k}$  and Table 3 this is nothing else than (6.2) with  $2k$ instead of *k.* 

*2. Induction from 2k to 2k* + 1: Formula (6.2) reads for 2k, instead of *k* and  $\gamma_1 + \tau$ <br>  $\phi\left(\frac{\gamma_{2k} + \gamma_1 + \tau}{\sigma^n}\right) - \epsilon_{2k}\phi\left(\frac{\gamma_1 + \tau}{\sigma^n}\right) = \sum_{k=1}^{s} \epsilon_{l_k} d_{k} c_{n-k-1} \psi_{n-k-1}\left(\frac{\gamma_{k_k} + \gamma_1 + \tau}{\sigma^n}\right)$ instead of *<sup>r</sup>*

instead of k.  
\n2. *Induction from 2k to 2k + 1*: Formula (6.2) reads for 2k instead of k and 
$$
\gamma_1 + \tau
$$
  
\ninstead of  $\tau$   
\n
$$
\phi\left(\frac{\gamma_{2k} + \gamma_1 + \tau}{a^n}\right) - \epsilon_{2k}\phi\left(\frac{\gamma_1 + \tau}{a^n}\right) = \sum_{\nu=0}^s \epsilon_{l_{\nu}} d_{\nu} c_{n-\nu-1} \psi_{n-\nu-1} \left(\frac{\gamma_{k_{\nu}} + \gamma_1 + \tau}{a^{\nu}}\right)
$$
\nwhere the parameters are those belonging to 2k. According to  $d_0 = 0$  we have  $\sigma \ge 1$ ,  
\nso that the last equation is valid at least for  $0 \le \gamma_1 + \tau \le a = \gamma_1 + 1$ , i.e. at least for

 $0 \leq \tau \leq 1$ . Multiplying (6.1) by  $\varepsilon_{2k}$  and adding the result to the foregoing equation, we obtain

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\n1. Multiplying (6.1) by 
$$
\varepsilon_{2k}
$$
 and adding the result to the foregoing equa

\n
$$
\phi\left(\frac{\gamma_{2k} + \gamma_1 + \tau}{a^n}\right) + \varepsilon_{2k}\phi\left(\frac{\tau}{a^n}\right)
$$
\n
$$
= \varepsilon_{2k}c_{n-1}\psi_n(\gamma_1 + \tau) + \sum_{\nu=1}^s \varepsilon_{l_{\nu}}d_{\nu}c_{n-\nu-1}\psi_{n-\nu-1}\left(\frac{\gamma_{k_{\nu}} + \gamma_1 + \tau}{a^{\nu}}\right).
$$
\nis nothing else than (6.2) with  $2k + 1$  instead of  $2k$ , since  $\gamma_{2k} + \gamma_1$ 

But this is nothing else than (6.2) with  $2k + 1$  instead of  $2k$ , since  $\gamma_{2k} + \gamma_1 = \gamma_{2k+1}$ ,  $\varepsilon_{2k} = -\varepsilon_{2k+1}$ , and  $k_{\nu}$  of  $2k$  is even so that  $\gamma_{k_{\nu}} + \gamma_1 = \gamma_{k_{\nu}+1}$  for  $\nu \geq 1$ , and the parameters of  $2k + 1$  depend on the parameters of  $2k$  in the following way:<br>  $\frac{2k}{2k+1} \frac{d_{\nu}}{d_{\nu}} \frac{k_{\nu}}{k_{\nu}+1} \frac{l_{\nu}}{l_{\nu}} \frac{s}{s}$  or (6.2) with  $2k + 1$  instear<br>is even so that  $\gamma_{k_v}$  +<br>on the parameters of  $2k$ <br> $2k$   $d_v$   $k_v$   $l_v$   $s$ <br> $i+1$   $d_v$   $k_v + 1$   $l_v$   $s$ 

$$
\begin{array}{c|cccc}\n2k & d_{\nu} & k_{\nu} & l_{\nu} & s & \sigma \\
\hline\n2k+1 & d_{\nu} & k_{\nu}+1 & l_{\nu} & s & 0\n\end{array}
$$

Table 4: The parameters of *2k + 1* expressed by those of *2k* 

for  $\nu \ge 1$ , whereas  $d_0 = 1$  for  $2k + 1$  and  $\varepsilon_{l_0} = \varepsilon_{2k}$  for the parameter  $l_0$  of  $2k + 1$ 

Remark. For large *k* formula (6.2) has the advantage that the sum on the righthand side consists of O(ln *k)* terms only compared to the *k* terms in the sum of (5.4). Moreover, many  $d_{\nu}$  in (6.2) can vanish. If the terms with  $d_{\nu} = 0$  are cancelled, then the remaining terms have alternating signs ending with  $\varepsilon_{l_n} = 1$  in view of  $l_s = 0$ . Hence, (6.2) implies<br>  $\phi\left(\frac{\gamma_k + \tau}{a^n}\right$ remaining terms have alternating signs ending with  $\varepsilon_{l_a} = 1$  in view of  $l_a = 0$ . Hence, (6.2) implies Table 4: The parameters of  $2k + 1$  *l<sub>v</sub>* s 0<br>
Table 4: The parameters of  $2k + 1$  expressed by those of  $2k$ <br>
whereas  $d_0 = 1$  for  $2k + 1$  and  $\varepsilon_{l_0} = \varepsilon_{2k}$  for the parameter  $l_0$  of  $2k +$ <br>
rk. For large k formula for  $2k + 1$  and  $\varepsilon_{l_0} = \varepsilon_{2k}$  for th<br> *k* formula (6.2) has the advants<br> *i k*) terms only compared to th<br>
2) can vanish. If the terms with<br> *ernating signs ending with*  $\varepsilon_{l_s}$ <br>  $\frac{\gamma_m + \tau}{a^n}$  =  $c_{n-s-1}\psi_{n-s-1}$ 

$$
\phi\left(\frac{\gamma_k+\tau}{a^n}\right)+\phi\left(\frac{\gamma_m+\tau}{a^n}\right)=c_{n-s-1}\psi_{n-s-1}\left(\frac{\gamma_k+\tau}{a^s}\right) \qquad (0\leq \tau \leq a^{\sigma})
$$

with  $m = k_{s-1}$ , i.e.  $k = m + 2^s$  and  $\gamma_k = \gamma_m + a^s \gamma_1$ .

For  $t \in \overline{G}_{kn}$ , from  $\overline{G}_{kn} \subset F_{kn}$ , (6.2), (5.3) and (1.12) we obtain instead of (5.2) the reduced polynomial representation

$$
\frac{1+\tau}{n} + \phi\left(\frac{\gamma_m + \tau}{a^n}\right) = c_{n-s-1}\psi_{n-s-1}\left(\frac{\gamma_k + \tau}{a^s}\right) \qquad (0 \le \tau \le a^{\sigma})
$$
  
, i.e.  $k = m + 2^s$  and  $\gamma_k = \gamma_m + a^s \gamma_1$ .  
  
n, from  $\overline{G}_{kn} \subset F_{kn}$ , (6.2), (5.3) and (1.12) we obtain instead of (5.2) the  
omial representation  

$$
\phi\left(\frac{\gamma_{2k} + \tau}{a^{n+1}}\right) = \varepsilon_k c_n \psi_n(\tau) + \sum_{\nu=1}^s \varepsilon_{l_\nu} d_\nu c_{n-\nu} \psi_{n-\nu} \left(\frac{\gamma_{k_\nu} + \tau}{a^\nu}\right) \qquad (6.3)
$$
  
 $\le a - 1$ , and the parameters  $d_\nu$ ,  $k_\nu$ ,  $l_\nu$  and s are those of 2k. The first

where  $1 \leq r \leq a-1$ , and the parameters  $d_{\nu}$ ,  $k_{\nu}$ ,  $l_{\nu}$  and *s* are those of 2k. The first term of (6.3) cannot be included into the sum with  $\nu = 0$  in view of  $d_0 = 0$ .

### 7. Approximation by splines

Finally, we return to the general case  $a > 1$ . From (1.3) we observe that the Laplace transform  $\Phi$  of the solution  $\phi$  of (1.1) - (1.2) is the limit of

e parameters 
$$
d_{\nu}
$$
,  $k_{\nu}$ ,  $l_{\nu}$  and s are those of 2k. The first  
ded into the sum with  $\nu = 0$  in view of  $d_0 = 0$ .  
**splines**  
eral case  $a > 1$ . From (1.3) we observe that the Laplace  
of (1.1) - (1.2) is the limit of  

$$
G_n(p) = \prod_{k=0}^{n-1} \frac{1 - e^{-p/(ba^k)}}{p/(ba^k)}
$$
(7.1)

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for  $n \to \infty$ . On account of Lemma 2.3 we have for  $n \geq 1$ 

üppel  
\nLemma 2.3 we have for 
$$
n \ge 1
$$
  
\n
$$
G_n(p) = \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} e^{-\frac{1}{a} \kappa p}.
$$
\n(7.2)

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for  $n \to \infty$ . On account of Lemma 2.3 we have for  $n \ge 1$ <br>  $G_n(p) = \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{2\kappa}{a} p}$ . (7.2)<br>
According to  $\mathcal{L}^{-1} \{p^{-n}\} = \frac{t^{n-1}}{(n-1)!}$  and the shi obtain for the original function  $g_n$  of  $G_n$  the representation

$$
G_n(p) = \frac{a^{\frac{n(n-1)}{2}}b^n}{p^n} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} e^{-\frac{7\kappa}{4}\mu}.
$$
 (7.2)  

$$
\frac{t^{n-1}}{(n-1)!}
$$
 and the shift property of the Laplace transform, we  
ction  $g_n$  of  $G_n$  the representation  

$$
g_n(t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} (a^n t - \gamma_{\nu})_+^{n-1}
$$
 (7.3)

where  $c_n$  is given by (5.3) and  $t_+ = t$  for  $t \ge 0$  and  $t_+ = 0$  elsewhere. We see that the functions  $g_n$  are splines consisting of piecewise polynomials of degree at most  $n-1$ . Moreover,  $g_n(t) = 0$  for  $t \notin (0,1)$  since the sums (2.11) vanish for  $m < n$ , and according to  $G_n(0) = 1$  we have  $\int_0^1 g_n(t) dt = 1$ . In view of  $G_n(p) \to \Phi(p)$  we get *Theory*  $\epsilon_{n-1}$   $\sum_{\nu=0} \epsilon_{\nu} (a^n t)$ <br> *Theory*  $t \geq 0$  *i*<br> *Theory Theory I* is increase 1) since the sums (2.<br> *Theory Z Z Theory Theory Theory Z Z Z Theory Z Z Z Z Z Z Z Z Z*  $\binom{n}{n} = \frac{t^{n-1}}{(n-1)!}$  and the shift property of the L<br>
l function  $g_n$  of  $G_n$  the representation<br>  $g_n(t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} (a^n t - \gamma_{\nu})_+^{n-1}$ <br>
(5.3) and  $t_+ = t$  for  $t \ge 0$  and  $t_+ = 0$  else<br>
ollines consis  $\begin{aligned}\n\sigma^n\end{aligned} = \frac{t^{n-1}}{(n-1)!}$  and the shift property<br>
al function  $g_n$  of  $G_n$  the representation<br>  $g_n(t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu (a^n t - \gamma_\nu)!$ <br>
(5.3) and  $t_+ = t$  for  $t \ge 0$  and  $t_+$ <br>
plines consisting of piecewise po *es* consisting of piecewise polynomials of degree at most  $n - 1$ .<br>  $\notin (0, 1)$  since the sums (2.11) vanish for  $m < n$ , and according<br>  $g_n(t) dt = 1$ . In view of  $G_n(p) \to \Phi(p)$  we get<br>  $\frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n - 1} \varepsilon_{$ 

$$
\lim_{n \to \infty} \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{2\mu}{a} p} = \int_{0}^{1} e^{-pt} \phi(t) dt
$$

and, moreover, from the proof of [1: Theorem 3.1] we know that  $g_n$  is uniformly convergent to the solution  $\phi$  of (1.1) - (1.2), i.e.

$$
\lim_{n \to \infty} \frac{a^{\frac{n(n-1)}{2}} b^n}{p^n} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} e^{-\frac{2\mu}{a} p} = \int_0^1 e^{-pt} \phi(t) dt
$$
\nthe proof of [1: Theorem 3.1] we know that  $g_n$  is uniformly con-  
\n $\phi$  of (1.1) - (1.2), i.e.  
\n
$$
\phi(t) = \lim_{n \to \infty} c_{n-1} \sum_{\nu=0}^{2^n - 1} \varepsilon_{\nu} (a^n t - \gamma_{\nu})_+^{n-1}.
$$
\n(7.4)  
\nthe kernel  
\n
$$
k_1(s, t) = \begin{cases} b & \text{for } \frac{s}{a} \leq t \leq \frac{s + a - 1}{a} \\ 0 & \text{elsewhere,} \end{cases}
$$
\n(7.5)  
\nan be written as Fredholm integral equation

If we introduce the kernel

 $\bullet$ 

$$
k_1(s,t) = \begin{cases} b & \text{for } \frac{s}{a} \le t \le \frac{s+a-1}{a} \\ 0 & \text{elsewhere,} \end{cases} \tag{7.5}
$$

then equation  $(1.1)$  can be written as Fredholm integral equation

$$
\phi(t)=\int\limits_0^1 k_1(s,t)\phi(s)\,ds\,.
$$

It is possible to calculate also the iterated kernels  $k_n$  defined by

$$
k_{n+1}(s,t) = \int_{0}^{1} k_1(s,\tau) k_n(\tau,t) d\tau
$$

**Proposition 7.1.** For the iterated kernels  $k_n$   $(n \geq 1)$  we have the representation

$$
k_n(s,t)=g_n\left(t-\frac{s}{a^n}\right)
$$

where the splines  $g_n$  are given by  $(7.3)$ , i.e.

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re given by (7.3), i.e.  

$$
k_n(s,t) = c_{n-1} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} (a^n t - s - \gamma_{\nu})_+^{n-1} . \qquad (7.6)
$$
  
7.6) is true for  $n = 1$ . Assume that (7.6) is valid for a fixed  $n \ge 1$ .

**Proof.** Formula (7.6) is true for  $n = 1$ . Assume that (7.6) is valid for a fixed  $n \geq 1$ . In view of (7.5) we have

Formula (7.5) is true for 
$$
n = 1
$$
. Assume that (7.6) is valid for a fixe  
\n(7.5) we have  
\n
$$
\varepsilon_{\nu} \int_{0}^{1} k_{1}(s,\tau)(a^{n}t - \tau - \gamma_{\nu})_{+}^{n-1} d\tau
$$
\n
$$
= b \varepsilon_{\nu} \int_{s/a}^{(s+a-1)/a} (a^{n}t - \tau - \gamma_{\nu})_{+}^{n-1} d\tau
$$
\n
$$
= \frac{b \varepsilon_{2\nu+1}}{a^{n}n} (a^{n+1}t - s - \gamma_{2\nu+1})_{+}^{n} + \frac{b \varepsilon_{2\nu}}{a^{n}n} (a^{n+1}t - s - \gamma_{2\nu})_{+}^{n},
$$
\nhave used (2.2) and (2.5). Hence (7.6) follows by  $c_{n} = \frac{b}{a^{n}n} c_{n-1}$  and in

where we have used (2.2) and (2.5). Hence (7.6) follows by  $c_n = \frac{b}{a^n n} c_{n-1}$  and induction

 $f_n(t) = g_{n+1}(t)$ , and (7.4) follows once more from [1: Theorem 3.1].

have the similar representations

Starting with 
$$
f_0(t) = k_1(0, t)
$$
 and calculating the iterates  $f_n = Lf_{n-1}$ , we find  
\n $f_0(t) = g_{n+1}(t)$ , and (7.4) follows once more from [1: Theorem 3.1].  
\nThe iterates  $f_n$  of the function  $f_0$ ,  $f_0(t) = 1$  for  $t \in [0, 1]$  and  $f_0(t) = 0$  elsewhere,  
\nthe similar representations  
\n
$$
f_n(t) = \int_0^1 k_n(s,t) ds = \frac{c_{n-1}}{n} \sum_{\nu=0}^{2^n-1} \varepsilon_\nu ((a^n t - \gamma_\nu)_+^n - (a^n t - \gamma_\nu - 1)_+^n)
$$

with  $t_+$  defined as before, and they also converge to the solution  $\phi$  of (1.1) - (1.2). In the case of  $a = 2$  where  $\gamma_{\nu} = \nu$  the last representations reduce to

(7.4) follows once more from [1: Theorem 3.1].  
\nIf the function 
$$
f_0
$$
,  $f_0(t) = 1$  for  $t \in [0, 1]$  and  $f_0(t) = 0$  elsewhere,  
\nsentations  
\n
$$
f_n(s,t) ds = \frac{c_{n-1}}{n} \sum_{\nu=0}^{2^n-1} \varepsilon_{\nu} ((a^n t - \gamma_{\nu})_+^n - (a^n t - \gamma_{\nu} - 1)_+^n)
$$
\nbefore, and they also converge to the solution  $\phi$  of (1.1) - (1.2). In  
\n
$$
f_n(t) = \frac{1}{2^{\frac{n(n-3)}{2}n!}} \sum_{\nu=0}^{2^n-1} (\varepsilon_{\nu} - \varepsilon_{\nu-1})(2^n t - \nu)_+^n
$$
\n(7.7)  
\nthe coefficients  $\varepsilon_{\nu} - \varepsilon_{\nu-1}$  for  $\nu \ge 1$  were calculated by (2.8). Let

with  $\varepsilon_{-1} = 0$ , where the coefficients  $\varepsilon_{\nu} - \varepsilon_{\nu-1}$  for  $\nu \ge 1$  were calculated by (2.8). Let us mention that the function  $f = f_n$  of (7.7) is the (unique up to a constant factor) non-vanishing L-integrable solution of a particular two-scale difference equation, which arises from  $(1.1)$  with  $a = 2$  by means of the trapezoidal rule (cf. [2]).

Corrections. Unfortunately, [1] contains some misprints. On p. 164<sup>1</sup> replace  $\Phi(0, p)$  by  $\Phi(0, a)$ . On p. 164<sup>9</sup> cancel: *quad*. On p. 165<sub>3</sub> replace *n* at the top of the product by  $n-1$ . On p. 176<sup>7</sup> replace (6.8) by (6.7). Moreover, the proof of the corollary on p. 176 becomes more lucid, if one recognizes that the first relation in (8.1) is also valid for  $t < 0$ .  $f_n(t) = \frac{1}{2^{\frac{n(n-3)}{2}} n!} \sum_{\nu=0}^{2^n-1} (\varepsilon_{\nu} - \varepsilon_{\nu-1}) (2^n t - \nu)$ <br>with  $\varepsilon_{-1} = 0$ , where the coefficients  $\varepsilon_{\nu} - \varepsilon_{\nu-1}$  for  $\nu \ge 1$  were us mention that the function  $f = f_n$  of (7.7) is the (unique non-vanishi

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