Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 17 (1998), No. 4, 1021–1024

Domain Identification for a Nonlinear Elliptic Equation

D. D. Trong

Abstract. It is proposed to identify the domain $\Omega \subset \mathbb{R}^n$ of a nonlinear elliptic equation subject to given Cauchy data on part of the known outer boundary Γ and to the zero condition on the unknown inner boundary γ. It is proved that under the assumption $\overline{\overset{\circ}{\Omega}} = \Omega$, the domain Ω is uniquely determined.

Keywords: *Domain identification, nonlinear elliptic equations, zero Dirichlet condition*

AMS subject classification: 35 R 30, 35 J 60

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain limited by an outer boundary Γ and an inner boundary γ, where Γ is known, but γ is unknown. Let

$$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

be a continuously differentiable function. We consider the nonlinear partial differential equation

$$F(x, u, Du, D^2u) = 0 \qquad (x \in \Omega), \tag{1}$$

where $u = u(x)$, $Du = \left(\frac{\partial u}{\partial x_i}\right)_{1 \le i \le n}$ and $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$, subject to the boundary conditions

$$u|_{\Gamma_0} = f, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma_0} = g, \qquad u|_\gamma = 0 \tag{2}$$

where Γ_0 is an open subset of Γ.

In the present paper, we consider domains $\Omega \subset \mathbb{R}^n$ satisfying

$$\overline{\overset{\circ}{\Omega}} = \Omega \tag{3}$$

where $\mathring{A}$ is the interior of the set A. Our problem is to determine a pair (Ω, u) satisfying (1) - (2). The case that u is a harmonic function and the interior boundary γ is a star-shaped Jordan curve was considered in [1]. The present paper extends [1] in two ways. First, our equation is a fully nonlinear elliptic one (satisfying the maximum principle)

D. D. Trong: Hochiminh City Nat. Univ., Dept. Math. & Comp. Sci., 227 Nguyen Van Cu str., Q5 Hochiminh City, Vietnam

ISSN 0232-2064 / \$ 2.50 © Heldermann Verlag Berlin

curves that are the boundaries of simply connected subdomains with mutually disjoint closures. For physical applications of the problem, the reader is referred to, e.g., [1] and the references therein.

We assume that

$$d(\Gamma,\gamma) > 0 \tag{4}$$

where $d(A,B)$ is the distance between two subsets $A, B \subset \mathbb{R}^n$. We say that equation (1) is *elliptic* if

$$\sum_{i,j=1}^{n} \frac{\partial F}{\partial q_{ij}}(x,u,p,q)\,\xi_i\xi_j \geq C\,|\xi|^2 \tag{5}$$

with a constant where $C > 0$, for all $\xi = (\xi_1,..,\xi_n) \in \mathbb{R}^n$, $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$, $x \in \Omega$, $(u,p,q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ and $q = (q_{ij})_{1\leq i,j\leq n}$. We also assume that

$$\frac{\partial F}{\partial u}(x,u,p,q) \leq 0 \qquad \text{for all } (x,u,p,q) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \tag{6}$$

$$F(x,0,0,0) = 0 \qquad \text{for all } x \in \mathbb{R}. \tag{7}$$

We note here that if $F(x,u,p,q) = q_{11} + q_{22} + \ldots + q_{nn}$, then $F(x,u,Du,D^2u) = \Delta u$.

We have the following result.

Theorem. *If $f \not\equiv 0$ or if $g \not\equiv 0$ on Γ_0, then there exists at most one pair (Ω,u) with $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0) \cap C(\overline{\Omega})$ satisfying (1) – (2) provided (3) – (7) hold.*

Proof. Suppose (Ω^i, u^i) satisfy (1) - (2) with

$$u^i \in C^2(\Omega^i) \cap C^1(\Omega^i \cup \Gamma_0) \cap C(\overline{\Omega^i})$$

and let γ^i be the inner boundaries of Ω^i $(i = 1,2)$. Put

$$\omega = \Big\{x \in \Omega^1 : d(x,\Gamma) < \min\{d(\Gamma,\gamma^1), d(\Gamma,\gamma^2)\}\Big\}. \tag{8}$$

By (4), ω is connected and $\omega \neq \emptyset$. By the properties of Ω^i we have $\omega \subset \Omega^1 \cap \Omega^2$. Let W be the connected component of $\Omega^1 \cap \Omega^2$ such that $\omega \subset W$. We shall prove that

$$u^1 = u^2 \qquad \text{on } \; W \subset \Omega^1 \cap \Omega^2. \tag{9}$$

Putting $v = u^2 - u^1$ and

$$\begin{aligned} u^0(x,t) &= u^1(x) + t\big(u^2(x) - u^1(x)\big) \\ p^0(x,t) &= Du^1(x) + t\big(Du^2(x) - Du^1(x)\big) \\ q^0(x,t) &= D^2u^1(x) + t\big(D^2u^2(x) - D^2u^1(x)\big) \end{aligned}$$

we have

$$g(x,1) - g(x,0) = \int_0^1 \frac{\partial g}{\partial t}(x,t)dt$$

where

$$g(x,t) = F\big(x, u^0(x,t), p^0(x,t), q^0(x,t)\big).$$

This gives

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial v}{\partial x_i} + c(x)\,v = 0 \tag{10}$$

where

$$\left.\begin{aligned} a_{ij}(x) &= \int_0^1 \frac{\partial F}{\partial q_{ij}}(x, u^0, p^0, q^0)\,dt \\ b_i(x) &= \int_0^1 \frac{\partial F}{\partial p_i}(x, u^0, p^0, q^0)\,dt \\ c(x) &= \int_0^1 \frac{\partial F}{\partial u}(x, u^0, p^0, q^0)\,dt. \end{aligned}\right\} \tag{11}$$

Since $\Gamma \subset \partial W$, we get from (2)

$$v|_{\Gamma_0} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial n}\Big|_{\Gamma_0} = 0. \tag{12}$$

In view of (10), (12) and (6), we can use the uniqueness theorem for the Cauchy problem for elliptic equations (see, e.g., [3]) to get $v = 0$ on W, i.e. (9) holds. To continue with the proof, we suppose by contradiction that $\Omega^1 \neq \Omega^2$. Without loss of generality, we can assume that $\Omega^1 \setminus \overline{\Omega^2} \neq \emptyset$ (in fact, if $\Omega^1 \setminus \overline{\Omega^2} = \Omega^2 \setminus \overline{\Omega^1} = \emptyset$, then $\overline{\Omega^1} = \overline{\Omega^2}$, and by (3), $\Omega^1 = \mathring{\overline{\Omega^1}} = \mathring{\overline{\Omega^2}} = \Omega^2$). Since $\Omega^1 \setminus \overline{\Omega^2} \subset \Omega^1 \setminus \overline{W}$ we have

$$\Omega^1 \setminus \overline{W} \neq \emptyset. \tag{13}$$

Let U be a connected component of $\Omega^1 \setminus \overline{W}$. In view of (8), we have

$$U \subset \Big\{x \in \Omega^1 : d(x,\Gamma) \geq \min\{d(\Gamma,\gamma^1), d(\Gamma,\gamma^2)\}\Big\}. \tag{14}$$

Hence

$$\partial U \cap \Gamma = \emptyset. \tag{15}$$

Note that

$$\partial U \subset \partial(\Omega^1 \setminus \overline{W}) = \partial(\Omega^1 \cap (\mathbb{R}^n \setminus \overline{W})) \subset \partial\Omega^1 \cup \partial W.$$

We can combine the above inclusion with (15) to get

$$\partial U \subset \gamma^1 \cup \partial W. \tag{16}$$

We claim that

$$u^1|_{\partial U} = 0. \tag{17}$$

In fact, for $x \in \partial U$, there are two cases:

(a) $x \in \gamma^1$. In this case, (2) gives

$$u^1(x) = 0. \tag{18}$$

(b) $x \notin \gamma^1$. By (16), $x \in \partial W \setminus \gamma^1$. But $\partial W \subset \partial(\Omega^1 \cap \Omega^2) \subset \gamma^1 \cup \gamma^2$. Hence $x \in \gamma^2$. We prove that $x \in \Omega^1$. Indeed, $x \in \partial U \subset \overline{\Omega^1}$. From (15), we get

$$x \in \overline{\Omega^1} \setminus (\Gamma \cup \gamma^1) = \overline{\Omega^1} \setminus \partial\Omega^1 = \Omega^1.$$

In this case, one has in view of (9)

$$u^1(x) = u^2(x) = 0. \tag{19}$$

In either case, from (18) and (19) we have that (17) holds.

Similarly as for (10), we get in view of (7)

$$\sum_{i,j=1}^{n} a^1_{ij}(x) \frac{\partial^2 u^1}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^1_i(x) \frac{\partial u^1}{\partial x_i} + c^1(x)u^1 = 0 \tag{20}$$

for $x \in \Omega^1$, where a^1_{ij}, b^1_i, c^1 have the same forms as in (11). We do not write out their explicit forms but only note that (20) is an elliptic equation with

$$c^1(x) \leq 0. \tag{21}$$

Using the maximum principle for elliptic equations (see [2: Chapter 2/p. 53]), we get in view of (17) and (21)

$$u^1(x) = 0 \qquad \text{on } U. \tag{22}$$

Now, using the uniqueness result for elliptic continuation [3], we get from (22) $u^1(x) = 0$ on Ω^1. Hence

$$u^1|_{\Gamma_0} = \frac{\partial u^1}{\partial n}\Big|_{\Gamma_0} = 0.$$

This is a contradiction and the proof of the theorem is completed ∎

References

[1] Ang, D. D. and L. K. Vy: *Domain identification for harmonic functions.* Acta Appl. Math. 38 (1995), 217 - 238.

[2] Friedman, A.: *Partial Diferential Equations of Parabolic Type.* Englewood Cliffs (N.J.): Prentice Hall 1964.

[3] Pederson, R. N.: *On the unique continuation theorem for certain second and fourth order elliptic equations.* Comm. Pure Appl. Math. 11 (1958), 67 - 80.

Received 17.02.1998