

# A Multi-Dimensional Hausdorff Moment Problem: Regularization by Finite Moments

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*Dedicated to Prof. Dr. G. Anger on occasion of his 70th birthday*

**Abstract.** We consider the multi-dimensional Hausdorff moment problem over the unit cube: to reconstruct an unknown function from the (inaccurately) given values of the integrals of the unknown function multiplied by all power-products of the independent variables. We describe a regularization scheme using orthogonalization by the tensor product of (shifted) Legendre polynomials and "approximation" of the unknown function by a finite sum, the dimension of the space of approximation playing the role of the regularization parameter. For the case of square integrability of the unknown function we present an estimate of the regularization error that implies convergence if the data error tends to zero.

**Keywords:** *Ill-posed problems, Hausdorff moment problem, polynomial approximation*

**AMS subject classification:** Primary 65 R 30, secondary 41 A 10

## 1. Introduction

We propose to present a regularization of a Hausdorff moment problem, with error estimates given. The literature on the moment problem is impressive, we shall list only a few of them in the bibliography (cf. [2 - 8, 11, 13]). Of particular relevance to the present work are the papers by Ang, Vy and Gorenflo [3] and by Talenti [11] (see also Inglese [7]). In [11], the author considered the one-dimensional case and proved the stability of the finite moment approximation with stability estimates given. But the relationship of the finite moment approximation to the original moment problem was not exhibited as transparently as we feel desirable. In the present paper, we shall consider the multi-dimensional case, specifically the following problem.

**Problem (MP).** Find a function  $u$  in  $L^2(I)$  satisfying the sequence of equations

$$\int_I u(x_1, \dots, x_d) x_1^{k_1} \cdots x_d^{k_d} dx_1 \cdots dx_d = \mu_{k_1 \dots k_d} \quad (k_i \in \mathbb{N}_0; i = 1, \dots, d) \quad (1)$$

where  $d \in \mathbb{N}$ ,  $I = [0, 1]^d \subset \mathbb{R}^d$  and  $\mu = (\mu_{k_1 \dots k_d})$  is a given bounded sequence of real numbers.

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It can be proved that (MP) is an ill-posed problem. Note also that (MP) does not always have a solution for an arbitrary bounded sequence  $(\mu_{k_1, \dots, k_d})$  in the right-hand side of (1). Hence it is realistic to deal also with cases where problem (MP) does not have a solution. We refer to the paper [3] of which the present work can be seen as a continuation. However while [3] deals with a general moment problem, the Hausdorff moment problem has its own peculiarities that we have taken into account, in departure from the general case, especially in the crucial choice of the function  $f(t)$ ,  $[f^{-1}(e^{\frac{1}{2}})]$  being the dimension of our finite approximation.

In our construction of finite-dimensional approximations, we obtain an orthonormalization of the basis functions  $\{x_1^{k_1} \cdots x_d^{k_d}\}$  ( $k_i \in \mathbb{N}_0; i = 1, \dots, d$ ) not through the Gram-Schmidt process but by using the polynomials

$$L_{k_1 \dots k_d}(x_1, \dots, x_d) = L_{k_1}(x_1) \cdots L_{k_d}(x_d), \tag{2}$$

where

$$L_m(x) = \sum_{j=0}^m C_{mj} x^j \quad (m \in \mathbb{N}_0) \tag{3}$$

with

$$C_{mj} = (2m + 1)^{\frac{1}{2}} (-1)^j \frac{(m + j)!}{(j!)^2 (m - j)!} \quad (j = 0, 1, \dots, m). \tag{4}$$

The sequence  $(L_{k_1} \cdots L_{k_d})$  ( $k_i \in \mathbb{N}_0; i = 1, \dots, d$ ) forms a complete orthonormal set in  $L^2(I)$ . In fact,

$$L_m(x) = (2m + 1)^{\frac{1}{2}} P_m(1 - 2x), \tag{5}$$

where  $P_m$  is the Legendre polynomial:

$$P_m(t) = \sum_{j=0}^m \frac{(m + j)!}{(j!)^2 (m - j)! 2^j} (t - 1)^j, \quad \|P_m\|_{L^2(-1,1)}^2 = \frac{2}{2m + 1}$$

(cf. [10: page 227]). Thus  $(L_m)$  is a complete orthonormal basis in  $L^2(0, 1)$  and hence  $(L_{k_1} \cdots L_{k_d})$  is a complete orthonormal set in  $L^2(I)$ . From (3), we have

$$L_{k_1 \dots k_d}(x_1, \dots, x_d) = \sum_{p_1=0}^{k_1} \cdots \sum_{p_d=0}^{k_d} C_{k_1 p_1} \cdots C_{k_d p_d} x_1^{p_1} \cdots x_d^{p_d}. \tag{6}$$

If  $\mu = (\mu_{k_1 \dots k_d})$  is a real sequence, we define the sequence  $\lambda = \lambda(\mu) = (\lambda_{k_1 \dots k_d})$  as follows:

$$\lambda_{k_1 \dots k_d} = \lambda_{k_1 \dots k_d}(\mu) = \sum_{p_1=0}^{k_1} \cdots \sum_{p_d=0}^{k_d} C_{k_1 p_1} \cdots C_{k_d p_d} \mu_{p_1 \dots p_d}.$$

Now put

$$p^n = p^n(\mu) = \sum_{k_1, \dots, k_d=0}^n \lambda_{k_1 \dots k_d}(\mu) L_{k_1 \dots k_d}. \tag{7}$$

## 2. Results

We now give our results in form of three theorems that we shall prove in Section 3.

**Theorem 1.** Let  $\mu = (\mu_{k_1 \dots k_d})$  ( $k_i \in \mathbb{N}_0; i = 1, \dots, d$ ) be a given sequence of real numbers. If  $u$  is the (unique) solution of (1) in  $L^2(I)$ , then

$$\sum_{k_1, \dots, k_d=0}^{\infty} \left( \sum_{p_1, \dots, p_d=0}^{\infty} C_{k_1 p_1} \dots C_{k_d p_d} \mu_{p_1 \dots p_d} \right)^2 < \infty \tag{8}$$

and

$$p^n(\mu) \rightarrow u \quad \text{in } L^2(I) \text{ as } n \rightarrow \infty, \tag{9}$$

where  $C_{ij}$  is defined by (4) if  $j \leq i$  and  $C_{ij} = 0$  if  $j > i$ . Moreover, if the solution  $u$  is in  $H^1(I)$ , then

$$\|p^n(\mu) - u\| \leq \frac{1}{n+1} (F(u))^{\frac{1}{2}} \quad (n \in \mathbb{N}) \tag{10}$$

where  $\|\cdot\|$  is the  $L^2(I)$ -norm and

$$F(u) = \sum_{i=1}^d \int_I x_i(1-x_i) \left| \frac{\partial u}{\partial x_i} \right|^2 dx_1 \dots dx_d.$$

In Theorem 1 above, it is assumed that a solution exists. In the general case that a solution may not exist, Theorem 2 below will be useful.

**Theorem 2.** Let  $u_0 \in L^2(I)$  be the unique solution of (1) corresponding to  $\mu^0 = (\mu_{k_1 \dots k_d}^0)$  ( $k_i \in \mathbb{N}_0; i = 1, \dots, d$ ) in the right-hand side of (1). Let

$$f(t) = \frac{1}{\sqrt{(2\pi)^d}} (3 + 2\sqrt{2})^{dt+d} \tag{11}$$

and for  $0 < \varepsilon < 1$ , put

$$n(\varepsilon) = [f^{-1}(\varepsilon^{-\frac{1}{2}})] \tag{12}$$

where  $[x]$  is the largest integer  $n \leq x$ . Then there exists a function  $\eta(\varepsilon)$  ( $0 < \varepsilon < 1$ ) such that  $\eta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and that for all sequences  $\mu$  satisfying

$$\|\mu - \mu^0\|_{\infty} := \sup_{k_1, \dots, k_d \in \mathbb{N}_0} |\mu_{k_1 \dots k_d} - \mu_{k_1 \dots k_d}^0| \leq \varepsilon$$

we have  $\|p^{n(\varepsilon)}(\mu) - u_0\| \leq \eta(\varepsilon)$ . Moreover, if  $u_0 \in H^1(I)$  and  $0 < \varepsilon < (\frac{\sqrt{2\pi}}{3+2\sqrt{2}})^{2d}$ , then

$$\|p^{n(\varepsilon)}(\mu) - u_0\| \leq \varepsilon^{\frac{1}{2}} + \frac{|F(u_0)|^{\frac{1}{2}}}{C_1 + C_2 \ln(\frac{1}{\varepsilon})}, \tag{13}$$

where

$$C_1 = \frac{\ln(\sqrt{2\pi}(3-2\sqrt{2}))}{\ln(3+2\sqrt{2})} \quad \text{and} \quad C_2 = \frac{1}{2d \ln(3+2\sqrt{2})}. \tag{14}$$

**Remark.** In the first version of this paper (see also [2]), we had derived (13) with  $C_1$  and  $C_2$  replaced by

$$\tilde{C}_1 = \frac{2 \ln \left( \frac{8\sqrt{5}}{27\sqrt{2}} \right)}{2 \ln 3 + \frac{1}{2} \ln 2} \quad \text{and} \quad \tilde{C}_2 = \frac{1}{d(2 \ln 3 + \frac{1}{2} \ln 2)},$$

respectively.

Estimate (13) with  $C_1$  and  $C_2$  given by (14) was derived using formula (31) as suggested by one of the referees. Clearly, the constants  $C_1$  and  $C_2$  give a better approximation than do  $\tilde{C}_1$  and  $\tilde{C}_2$ , respectively. While it is not known to us whether  $C_1$  and  $C_2$  are the best constants, we shall show below that the estimate of the order of magnitude of the error for  $\varepsilon \rightarrow 0$  as given by (13) is optimal in the sense of the following theorem.

**Theorem 3.** Assume that there exist two functions  $m : (0, 1] \rightarrow \mathbb{N}$  and  $\alpha : (0, 1] \rightarrow (0, \infty)$  such that  $\lim_{\varepsilon \downarrow 0} m(\varepsilon) = +\infty$  and  $\lim_{\varepsilon \downarrow 0} \alpha(\varepsilon) = 0$ , and

$$\|p^{m(\varepsilon)}(\mu) - u_0\| \leq (1 + (F(u_0))^{\frac{1}{2}}) \alpha(\varepsilon) \tag{15}$$

for all  $u_0 \in H^1(I)$ ,  $I = [0, 1]^d$ , and all  $\mu = (\mu_{k_1 \dots k_d})$  satisfying

$$\|\mu - \mu_0\|_\infty = \sup_{k_1, \dots, k_d \geq 0} |\mu_{k_1 \dots k_d} - \mu_{k_1 \dots k_d}^0| \leq \varepsilon,$$

where

$$\mu_{k_1 \dots k_d}^0 = \int_I u_0(x_1, \dots, x_d) x_1^{k_1} \dots x_d^{k_d} dx_1 \dots dx_d.$$

Then we have

$$\liminf_{\varepsilon \downarrow 0} \alpha(\varepsilon) \ln \left( \frac{1}{\varepsilon} \right) \geq \frac{d}{2} \ln(3 + 2\sqrt{2}). \tag{16}$$

### 3. Proofs

For definiteness, we shall give the proofs of Theorems 1 - 3 in the case  $d = 3$  only. The proofs carry almost *verbatim* to the general case. For  $d = 3$ , we put  $(x_1, x_2, x_3) = (x, y, z)$ ,  $(k_1, k_2, k_3) = (i, j, k)$ ,  $\mu_{k_1 k_2 k_3} = \mu_{ijk}$ ,  $\lambda_{k_1 k_2 k_3} = \lambda_{ijk}$  and  $L_{k_1 k_2 k_3}(x, y, z) = L_i(x)L_j(y)L_k(z) = L_{ijk}(x, y, z)$ .

**Proof of Theorem 1.** We easily get the first result of Theorem 1 using the orthonormality property of the  $L_{ijk}$ 's. In fact, one has in view of (3)

$$\begin{aligned} \int_I u L_{ijk} dx dy dz &= \int_I u(x, y, z) \sum_{p=0}^i \sum_{q=0}^j \sum_{r=0}^k C_{ip} C_{jq} C_{kr} x^p y^q z^r dx dy dz \\ &= \sum_{p=0}^i \sum_{q=0}^j \sum_{r=0}^k C_{ip} C_{jq} C_{kr} \left( \int_I u(x, y, z) x^p y^q z^r dx dy dz \right) \\ &= \sum_{p=0}^i \sum_{q=0}^j \sum_{r=0}^k C_{ip} C_{jq} C_{kr} \mu_{ijk} \\ &= \lambda_{ijk}. \end{aligned}$$

Hence, by completeness and the orthonormality property of the  $L_{ijk}$ 's in  $L^2(I)$  one has

$$u = \sum_{i,j,k=0}^{\infty} \lambda_{ijk}(\mu)L_{ijk} \quad \text{and} \quad \|u\|^2 = \sum_{i,j,k=0}^{\infty} |\lambda_{ijk}(\mu)|^2 < \infty, \quad (17)$$

i.e. (8) holds. To prove (9), we subtract the first equation in (17) from (7) to get

$$p^n(\mu) - u = - \sum_{\max(i,j,k) > n} \lambda_{ijk}(\mu)L_{ijk}.$$

Hence

$$\|p^n(\mu) - u\|^2 = \sum_{\max(i,j,k) > n} |\lambda_{ijk}(\mu)|^2. \quad (18)$$

Combining (8) and (18) gives (9).

For the proof of (10) we rely, with  $\alpha_k = \int_0^1 v(x)L_k(x) dx$ , on the identity (cf. [11])

$$\int_0^1 x(1-x)|v'(x)|^2 dx = \sum_{k=0}^{\infty} k(k+1)\alpha_k^2 \quad \forall v \in H^1(0,1). \quad (19)$$

In view of (17) and (19), we have

$$\sum_{i,j,k=1}^{\infty} i^2 \lambda_{ijk}^2 \leq \int_I x(1-x) \left| \frac{\partial u}{\partial x}(x,y,z) \right|^2 dx dy dz \quad (20)$$

$$\sum_{i,j,k=1}^{\infty} j^2 \lambda_{ijk}^2 \leq \int_I y(1-y) \left| \frac{\partial u}{\partial y}(x,y,z) \right|^2 dx dy dz \quad (21)$$

$$\sum_{i,j,k=1}^{\infty} k^2 \lambda_{ijk}^2 \leq \int_I z(1-z) \left| \frac{\partial u}{\partial z}(x,y,z) \right|^2 dx dy dz. \quad (22)$$

Adding (20) - (22) together, we get

$$\sum_{i,j,k=1}^{\infty} (i^2 + j^2 + k^2) \lambda_{ijk}^2 \leq F(u), \quad (23)$$

where, we recall,

$$F(u) = \int_I \left( x(1-x) \left| \frac{\partial u}{\partial x}(x,y,z) \right|^2 + y(1-y) \left| \frac{\partial u}{\partial y}(x,y,z) \right|^2 + z(1-z) \left| \frac{\partial u}{\partial z}(x,y,z) \right|^2 \right) dx dy dz.$$

Note that

$$i^2 + j^2 + k^2 \geq (n+1)^2 \quad \text{for } \max(i,j,k) > n. \quad (24)$$

We get in view of (18), (23) and (24) that

$$(n+1)^2 \|p^n(\mu) - u\|^2 \leq \sum_{\max(i,j,k) > n} (i^2 + j^2 + k^2) \lambda_{ijk}^2(\mu) \leq F(u),$$

which gives the desired inequality. The proof of Theorem 1 is completed ■

**Proof of Theorem 2.** We have

$$\|p^n(\mu) - u_0\| \leq \|p^n(\mu) - p^n(\mu^0)\| + \|p^n(\mu^0) - u_0\| \quad (n \in \mathbb{N}). \quad (25)$$

We shall estimate the two terms in the right-hand side of (25). Using (18) gives

$$\|p^n(\mu^0) - u_0\|^2 = \sum_{\max(i,j,k) > n} \left( \sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}\mu_{pqr}^0 \right)^2. \quad (26)$$

For the first term, we get in view of (7)

$$p^n(\mu) - p^n(\mu^0) = \sum_{i,j,k=0}^n \sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}(\mu_{pqr} - \mu_{pqr}^0)L_{ijk}. \quad (27)$$

Hence

$$\begin{aligned} \|p^n(\mu) - p^n(\mu^0)\|^2 &= \sum_{i,j,k=0}^n \left( \sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}(\mu_{pqr} - \mu_{pqr}^0) \right)^2 \\ &\leq \|\mu - \mu^0\|_{\infty}^2 \sum_{i,j,k=0}^n \left( \sum_{p,q,r=0}^{\infty} |C_{ip}C_{jq}C_{kr}| \right)^2. \end{aligned} \quad (28)$$

Noting that  $C_{mj} = 0$  for  $j > m$ , we have

$$\sum_{p,q,r=0}^{\infty} |C_{ip}C_{jq}C_{kr}| = \sum_{p=0}^i |C_{ip}| \sum_{q=0}^j |C_{jq}| \sum_{r=0}^k |C_{kr}|. \quad (29)$$

Substituting (29) into (28) gives

$$\|p^n(\mu) - p^n(\mu^0)\|^2 \leq \|\mu - \mu^0\|_{\infty}^2 \left( \sum_{i=0}^n \left( \sum_{p=0}^i |C_{ip}| \right)^2 \right)^3. \quad (30)$$

From (3) - (5), one gets

$$\sum_{p=0}^i |C_{ip}| = L_i(-1) = (2i + 1)^{\frac{1}{2}} P_i(3). \quad (31)$$

On the other hand, one has (cf. [9: page 48], or [1: Equation (22.10.10)])

$$P_i(3) = \frac{1}{\pi} \int_0^{\pi} (3 + 2\sqrt{2} \cos \phi)^i d\phi \quad (i \in \mathbb{N}_0).$$

We claim that

$$P_i(3) \leq 2\sqrt{\frac{3 + 2\sqrt{2}}{\pi\sqrt{2}}} \cdot \frac{1}{\sqrt{2i + 1}} (3 + 2\sqrt{2})^i \quad (i \in \mathbb{N}_0) \quad (32)$$

the proof of which will be given later. Using (31) and (32) one gets after some rearrangements

$$\sum_{p=0}^i |C_{ip}| \leq 2\sqrt{\frac{3+2\sqrt{2}}{\pi\sqrt{2}}}(3+2\sqrt{2})^i \quad (i \in \mathbb{N}_0)$$

and

$$\sum_{i=0}^n \left( \sum_{p=0}^i |C_{ip}| \right)^2 \leq \frac{4(3+2\sqrt{2})}{\pi\sqrt{2}} \sum_{i=0}^n (3+2\sqrt{2})^{2i} \leq \frac{1}{2\pi} (3+2\sqrt{2})^{2n+2}.$$

Substituting the latter inequality into (30) gives

$$\|p^n(\mu) - p^n(\mu^0)\| \leq \|\mu - \mu^0\|_\infty \frac{1}{\sqrt{8\pi^3}} (3+2\sqrt{2})^{3n+3}. \quad (33)$$

Put

$$f(t) = \frac{1}{\sqrt{8\pi^3}} (3+2\sqrt{2})^{3t+3} \quad \text{and} \quad n(\varepsilon) = \lceil f^{-1}(\varepsilon^{-\frac{1}{2}}) \rceil.$$

Then we have in view of (33)

$$\|p^n(\mu) - p^n(\mu^0)\| \leq \varepsilon^{\frac{1}{2}}. \quad (34)$$

If we let

$$\eta(\varepsilon) = \varepsilon^{\frac{1}{2}} + \sum_{\max(i,j,k) > n(\varepsilon)} \left( \sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \mu_{pqr}^0 \right)^2,$$

then, in view of (25), (26) and (34), we have  $\|p^{n(\varepsilon)}(\mu) - u_0\| \leq \eta(\varepsilon)$ . As  $\varepsilon \rightarrow 0$ , we have  $f^{-1}(\varepsilon^{-\frac{1}{2}}) \rightarrow \infty$  and  $n(\varepsilon) \rightarrow \infty$ . By (8)

$$\sum_{\max(i,j,k) > n(\varepsilon)} \left( \sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \mu_{pqr}^0 \right)^2 \rightarrow 0 \quad \text{as } n(\varepsilon) \rightarrow \infty.$$

Hence  $\eta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Now, let  $u_0 \in H^1(I)$ . Then by Theorem 1 one has

$$\|p^{n(\varepsilon)}(\mu^0) - u_0\| \leq \frac{(F(u_0))^{\frac{1}{2}}}{n(\varepsilon) + 1}. \quad (35)$$

The definition of  $n(\varepsilon)$  implies  $n(\varepsilon) + 1 > f^{-1}(\varepsilon^{-\frac{1}{2}})$ , and  $f$  being a monotone increasing function,  $f(n(\varepsilon) + 1) > \varepsilon^{-\frac{1}{2}}$  or, equivalently,

$$\frac{1}{\sqrt{8\pi^3}} (3+2\sqrt{2})^{3(n(\varepsilon)+1)+3} > \varepsilon^{-\frac{1}{2}}.$$

It follows that

$$(3+2\sqrt{2})^{n(\varepsilon)+2} > \sqrt{2\pi}\varepsilon^{-\frac{1}{4}},$$

i.e.,

$$n(\varepsilon) + 2 > \frac{1}{\ln(3 + 2\sqrt{2})} \ln\left(\frac{\sqrt{2\pi}}{\varepsilon^{1/6}}\right).$$

Thus we get after some calculations

$$n(\varepsilon) + 1 > \frac{\ln(\sqrt{2\pi}(3 - 2\sqrt{2}))}{\ln(3 + 2\sqrt{2})} + \frac{1}{6\ln(3 + 2\sqrt{2})} \ln\left(\frac{1}{\varepsilon}\right).$$

It can be checked directly that the right-hand side of the latter inequality is positive if and only if

$$0 < \varepsilon < \left(\frac{\sqrt{2\pi}}{3 + 2\sqrt{2}}\right)^6.$$

Let  $\varepsilon$  satisfy this inequality. By the preceding arguments, (35) gives

$$\|p^{n(\varepsilon)}(\mu^0) - u_0\| \leq \frac{(F(u_0))^{\frac{1}{2}}}{C_1 + C_2 \ln\left(\frac{1}{\varepsilon}\right)} \quad (36)$$

with

$$C_1 = \frac{\ln(\sqrt{2\pi}(3 - 2\sqrt{2}))}{\ln(3 + 2\sqrt{2})} \quad \text{and} \quad C_2 = \frac{1}{6\ln(3 + 2\sqrt{2})}.$$

Combining (34) and (36) gives

$$\|p^{n(\varepsilon)}(\mu) - u_0\| \leq \varepsilon^{\frac{1}{2}} + \frac{(F(u_0))^{\frac{1}{2}}}{C_1 + C_2 \ln\left(\frac{1}{\varepsilon}\right)}.$$

Now the proof of Theorem 2 will be completed once (32) is proved. In fact, one has

$$\begin{aligned} P_i(3) &= \frac{1}{\pi} \int_0^\pi (3 + 2\sqrt{2} \cos \phi)^i d\phi \\ &= \frac{1}{\pi} \left( \int_0^{\pi/2} + \int_{\pi/2}^\pi \right) (3 + 2\sqrt{2} \cos \phi)^i d\phi \\ &= \frac{1}{\pi} \int_0^{\pi/2} \left( (3 + 2\sqrt{2} \cos \phi)^i + (3 - 2\sqrt{2} \cos \phi)^i \right) d\phi. \end{aligned} \quad (37)$$

From (37), with  $t = \cos \phi$ , it follows that

$$P_i(3) \leq \frac{2}{\pi} \int_0^{\pi/2} (3 + 2\sqrt{2} \cos \phi)^i d\phi \leq \frac{2}{\pi} \int_0^1 \frac{(3 + 2\sqrt{2}t)^i}{\sqrt{1-t^2}} dt \leq \frac{2}{\pi} \int_0^1 \frac{(3 + 2\sqrt{2}t)^i}{\sqrt{1-t}} dt.$$

Putting  $s = \sqrt{1-t}$  and  $\zeta = s\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}$  we get, after some computations,

$$P_i(3) \leq \frac{4}{\pi} (3 + 2\sqrt{2})^i \sqrt{\frac{3 + 2\sqrt{2}}{2\sqrt{2}}} \int_0^1 (1 - \zeta^2)^i d\zeta.$$



Letting  $\zeta = \cos \theta$ , we have

$$P_i(3) \leq \frac{4}{\pi} (3 + 2\sqrt{2})^i \sqrt{\frac{3 + 2\sqrt{2}}{2\sqrt{2}}} \int_0^{\pi/2} \sin^{2i+1} \theta d\theta. \tag{38}$$

Putting  $I_n = \int_0^{\pi/2} \sin^n \theta d\theta$  we have (cf. [12: p. 681])

$$I_{2k} = \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)} \cdot \frac{\pi}{2} \quad \text{and} \quad I_{2k-1} = \frac{2 \cdot 4 \cdots (2k-2)}{1 \cdot 3 \cdots (2k-1)}$$

It follows that

$$I_m I_{m-1} = \frac{\pi}{2m} \quad (m \in \mathbb{N}). \tag{39}$$

But

$$I_{m-1} = \int_0^{\pi/2} \sin^{m-1} \theta d\theta > \int_0^{\pi/2} \sin^m \theta d\theta = I_m.$$

Hence (39) gives

$$I_m^2 \leq I_m I_{m-1} = \frac{\pi}{2m}. \tag{40}$$

Substituting (40) into (38), we get

$$P_i(3) \leq \frac{4}{\pi} (3 + 2\sqrt{2})^i \sqrt{\frac{3 + 2\sqrt{2}}{2\sqrt{2}}} \cdot \sqrt{\frac{\pi}{2(2i+1)}}.$$

This establishes (32) and completes the proof of Theorem 2 ■

**Proof of Theorem 3.** We choose

$$u_0(x, y, z) = \frac{1}{m(\varepsilon)} L_{m(\varepsilon)+1}(x) = \frac{1}{m(\varepsilon)} L_{m(\varepsilon)+1,0,0}(x, y, z),$$

$$\mu_{pqr}^0 = \int_I u_0(x, y, z) x^p y^q z^r dx dy dz.$$

Since  $\int_0^1 L_i(x) x^j dx = 0$  ( $0 \leq j < i$ ) one has

$$\mu_{pqr}^0 = 0 \quad \text{for } p \leq m(\varepsilon), \tag{41}$$

i.e.,  $u_0$  is orthogonal for all the  $L_{ijk}$ 's,  $0 \leq i \leq m(\varepsilon)$ . Put  $\mu^0 = (\mu_{pqr}^0)$  and  $\mu = (\mu_{pqr})$  with

$$\mu_{pqr} = \varepsilon(-1)^{p+q+r} + \mu_{pqr}^0. \tag{42}$$

It is clear that  $\|\mu - \mu^0\|_\infty = \sup_{p,q,r} |\mu_{pqr} - \mu_{pqr}^0| = \varepsilon$ . Hence, from assumption (15) of Theorem 3, one has

$$\|p^{m(\varepsilon)}(\mu) - u_0\| \leq \alpha(\varepsilon)(1 + (F(u_0))^{\frac{1}{2}}). \tag{43}$$

We estimate the left-hand side of (43). The orthonormality of the sequence  $(L_{ijk})$  implies

$$\|p^{m(\varepsilon)}(\mu) - u_0\|^2 = \|p^{m(\varepsilon)}(\mu)\|^2 + \|u_0\|^2 = \|p^{m(\varepsilon)}(\mu)\|^2 + \frac{1}{m^2(\varepsilon)}. \quad (44)$$

On the other hand, one has in view of (42), (29) and (31)

$$\|p^{m(\varepsilon)}(\mu)\|^2 = \varepsilon^2 \left( \sum_{i=0}^{m(\varepsilon)} \left( \sum_{p=0}^i |C_{ip}| \right)^2 \right)^3 = \varepsilon^2 \left( \sum_{i=0}^{m(\varepsilon)} (2i+1) |P_i(3)|^2 \right)^3. \quad (45)$$

From (37) we conclude

$$P_i(3) \geq \frac{1}{\pi} \int_0^{\pi/2} (3 + 2\sqrt{2})^i d\phi. \quad (46)$$

We shall make use of the inequality

$$(2i+1) |P_i(3)|^2 \geq \frac{(3 + 2\sqrt{2})^{2i}}{2\pi^2}, \quad (47)$$

the proof of which will be given later. Combining (47) with (45) gives

$$\begin{aligned} \|p^{m(\varepsilon)}(\mu)\|^2 &\geq \varepsilon^2 \left( \sum_{i=0}^{m(\varepsilon)} \frac{(3 + 2\sqrt{2})^{2i}}{2\pi^2} \right)^3 \\ &= \frac{\varepsilon^2}{8\pi^6} \left( \frac{(3 + 2\sqrt{2})^{2(m(\varepsilon)+1)} - 1}{(3 + 2\sqrt{2})^2 - 1} \right)^3 \\ &\geq \frac{\varepsilon^2}{64\pi^6} (3 + 2\sqrt{2})^{6m(\varepsilon)}. \end{aligned} \quad (48)$$

From (43), (45) and (48) we get

$$(1 + (F(u_0))^{\frac{1}{2}})^2 \alpha^2(\varepsilon) \geq \frac{\varepsilon^2}{64\pi^6} (3 + 2\sqrt{2})^{6m(\varepsilon)} + \frac{1}{m^2(\varepsilon)} \geq \min_{t>0} g(t), \quad (49)$$

where

$$g(t) = \frac{\varepsilon^2}{64\pi^6} (3 + 2\sqrt{2})^{6t} + \frac{1}{t^2}.$$

It is clear that  $g$  attains its minimum at  $t_\varepsilon > 0$  satisfying

$$g'(t_\varepsilon) = \frac{3\varepsilon^2}{32\pi^6} (3 + 2\sqrt{2})^{6t_\varepsilon} \ln(3 + 2\sqrt{2}) - \frac{2}{t_\varepsilon^3} = 0. \quad (50)$$

From (49) one has

$$(1 + (F(u_0))^{\frac{1}{2}})^2 \alpha^2(\varepsilon) \geq g(t_\varepsilon) \geq t_\varepsilon^{-2},$$

which implies

$$\alpha(\varepsilon) \ln \left( \frac{1}{\varepsilon} \right) \geq \frac{1}{1 + (F(u_0))^{\frac{1}{2}}} \cdot \frac{\ln \left( \frac{1}{\varepsilon} \right)}{t_\varepsilon}. \tag{51}$$

From (50) one has

$$t_\varepsilon^3 (3 + 2\sqrt{2})^{6t_\varepsilon} = \frac{64\pi^6}{3\varepsilon^2 \ln(3 + 2\sqrt{2})}.$$

Hence

$$3 \ln t_\varepsilon + 6t_\varepsilon \ln(3 + 2\sqrt{2}) = \ln \left( \frac{64\pi^6}{3 \ln(3 + 2\sqrt{2})} \right) + 2 \ln \left( \frac{1}{\varepsilon} \right).$$

Multiplying this equality by  $t_\varepsilon^{-1}$ , letting  $\varepsilon \rightarrow 0$  and noting that  $t_\varepsilon \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ , we get

$$\lim_{\varepsilon \downarrow 0} \frac{\ln \left( \frac{1}{\varepsilon} \right)}{t_\varepsilon} = 3 \ln(3 + 2\sqrt{2}). \tag{52}$$

On the other hand, in view of (19) and the definitions of  $F(u_0)$  and  $u_0$  we have

$$\begin{aligned} F(u_0) &= \int_I x(1-x) \left| \frac{\partial u_0}{\partial x}(x, y, z) \right|^2 dx dy dz \\ &\quad + \int_I y(1-y) \left| \frac{\partial u_0}{\partial y}(x, y, z) \right|^2 dx dy dz \\ &\quad + \int_I z(1-z) \left| \frac{\partial u_0}{\partial z}(x, y, z) \right|^2 dx dy dz \\ &= \frac{1}{m^2(\varepsilon)} (m(\varepsilon) + 1)(m(\varepsilon) + 2). \end{aligned} \tag{53}$$

Hence, in view of (52) and (53), inequality (51) implies the desired inequality (16).

To complete the proof of Theorem 3, we shall prove inequality (47). Putting  $t = \cos \phi$  and  $s = \sqrt{1-t}$ , and then letting  $\zeta = s \sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}$ , we get the inequality

$$P_i(3) \geq \frac{1}{\pi} (3 + 2\sqrt{2})^i \sqrt{\frac{3 + 2\sqrt{2}}{\sqrt{2}}} J, \tag{54}$$

where

$$J = \int_0^{\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}} (1 - \zeta^2)^i d\zeta.$$

We shall estimate  $J$ . One has

$$J = I_{2i+1} - K_i,$$

where

$$I_{2i+1} = \int_0^{\pi/2} \sin^{2i+1} \theta d\theta \quad \text{and} \quad K_i = \int_0^1 \sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}} (1 - \zeta^2)^i d\zeta.$$

Using (39) one has  $I_{2i+1}I_{2i+2} = \frac{\pi}{2(2i+2)}$ . Hence

$$I_{2i+1}^2 \geq I_{2i+1}I_{2i+2} = \frac{\pi}{2(2i+2)}. \tag{55}$$

On the other hand,

$$\begin{aligned} K_i &= \int_{\frac{2\sqrt{2}}{3+2\sqrt{2}}}^1 (1-\zeta)^i \frac{d\zeta}{2\sqrt{\zeta}} \\ &\leq \frac{1}{2} \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \int_{\frac{2\sqrt{2}}{3+2\sqrt{2}}}^1 (1-\zeta)^i d\zeta \\ &= \frac{1}{2(i+1)} \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1}. \end{aligned} \tag{56}$$

From (55) and (56) we deduce

$$\begin{aligned} J &\geq \sqrt{\frac{\pi}{2(2i+2)}} - \frac{1}{2(i+1)} \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1} \\ &= \frac{1}{\sqrt{i+1}} \left( \sqrt{\frac{\pi}{4}} - \frac{1}{2\sqrt{i+1}} \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1} \right) \\ &\geq \frac{1}{\sqrt{i+1}} \left( \sqrt{\frac{\pi}{4}} - \frac{1}{2} \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \cdot \frac{3}{3+2\sqrt{2}} \right) \\ &> \frac{1}{2\sqrt{i+1}}. \end{aligned}$$

In view of the latter inequality, (54) gives

$$(2i+1)|P_i(3)|^2 \geq \frac{2i+1}{4(i+1)\pi^2\sqrt{2}} (3+2\sqrt{2})^{2i+1} \geq \frac{(3+2\sqrt{2})^{2i}}{2\pi^2}.$$

This establishes (47) and completes the proof of Theorem 3 ■

**Acknowledgement.** The work of the first author was completed with a financial support from the National Program of Basic Research of Vietnam (project "Nondestructive Evaluation"). The second author appreciates financial support from the Research Commission of Free University of Berlin (project "Convolutions") and travel grants from DFG (Deutsche Forschungsgemeinschaft) in Bonn. All three authors would like to thank the referees for their valuable comments and constructive criticisms. Our special thanks go to one of the referees for his (or her) suggestions leading to a sharpening of our estimates in Theorem 2.

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Received 23.12.1997; in revised form 14.12.1998