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On Restriction Properties of Multiplication Operators

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Abstract. A multiplication operator A acting in a rearrangement-invariant function space E is considered. Infinite dimensional subspaces X of E for which the restriction $A|X$ is an isomorphism are described. Applications to multiplied trigonometric sequences in Banach function spaces are given.

Keywords: *Multiplication operators, rearrangement-invariant functions spaces*

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1. Introduction

Let (T, Σ, μ) be a finite measure space and let E be a rearrangement-invariant function space defined on T (see [4: p. 118]). We consider the multiplication operator by a bounded measurable function $a = a(t)$ $(t \in T)$ given by

$$Ax = ax \quad (x \in E), \qquad (Ax)(t) = a(t)x(t) \quad (t \in T).$$

Obviously, A is a bounded linear operator acting in E. In general A is not invertible and not compact. In order to investigate properties of the operator A we restrict it to some infinite dimensional subspaces of E, where A has a more simple nature. We consider the following two kinds of such subspaces:

1. Infinite dimensional subspaces $X \subset E$ such that the restriction $A|X$ is an isomorphism, i.e. $\inf\{\|Ax\| : x \in X \text{ with } \|x\| = 1\} > 0$.
2. Infinite dimensional subspaces $X \subset E$ such that the restriction $A|X$ is compact.

We give a description of subspaces of both kinds. Clearly, properties of such restrictions are helpful to understand mapping properties of A at all.

In addition, let us make the following observation:

Suppose $X \subset E$ is an infinite dimensional subspace of E such that $A|X$ is an isomorphism. If a sequence $\{x_n\}$ is a basis or unconditional basis of X, then the sequence $\{ax_n\}$ is a basic sequence or unconditional basic sequence in E, respectively. We will

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show that this observation is useful in order to describe properties of the sequences $\{a(t)\cos nt\}$ and $\{a(t)\sin nt\}$ in some Banach function spaces.

The paper is organized as follows.

In Section 2 we explain some notations and formulate a simple statement on general properties of a multiplication operator A by a bounded measurable function. In Section 3 we study subspaces X of E such that $A|X$ is an isomorphism (in the sense explained above). Subspaces of even and of odd functions on a compact symmetric domain $T \subset \mathbb{R}^n$ are of special interest. We find conditions on the function a under which $A|X$ are isomorphisms (Proposition 3.4 is the main result here). In Section 4 we consider multiplied trigonometric sequences, i.e. sequences of the form $\{a(t)\cos nt\}$ and $\{a(t)\sin nt\}$ where a is a continuous function. Using results of Section 3 we answer questions on basic properties of such sequences in spaces $L_p(-\pi, +\pi)$ $(1 < p < +\infty)$. A similar problem is investigated in the multidimensional case. Namely, we find conditions under which a multiplied n-dimensional trigonometric sequence on the cube $K = [-\pi, +\pi]^n$ is an unconditional basic sequence in $L_2(K)$. We also study multiplied lacunary trigonometric sequences. Under some assumptions on a we show that such sequences are unconditonal basic sequences in $L_p(-\pi, +\pi)$ $(1 \le p < +\infty)$. The investigation is based on studying normed sequences $\{x_n\} \subset E$ such that $\|Ax_n\| \to 0$ as $n \to +\infty$. Finally, Section 5 is devoted to the study of subspaces X of E such that $A|X$ is compact. The case $E = L_2(0,1)$ and $a(t) = t$ is considered separately. The question whether $A|X$ is strictly singular is discussed. We give an example of a subspace $X \subset L_p(0,1)$ $(1 < p < 2)$ such that $A|X$, where $Af(t) = tf(t)$, is strictly singular but non-compact. We also discuss spectral properties of a compact selfadjoint operator that corresponds to a compact restriction of a multiplication operator in $L_2(0,1)$.

In the case when $E = L_2(0,1)$ and a is a continuous function, some of the results of this paper were announced in [5].

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2. Notation. Simplest properties of multiplication operators

We use the notation $\operatorname{supp} a = \{t \in T : a(t) \neq 0\}$ and $\gamma(a) = \{t \in T : a(t) = 0\} = T \setminus \operatorname{supp} a$. By χ_σ we denote the characteristic function of a set $\sigma \in \Sigma$. For $\delta > 0$ let $\sigma_\delta = \{t : |a(t)| \ge \delta\}$ and $\chi_\delta := \chi_{\sigma_\delta}$. If X is a Banach space, then $S(X)$ denotes the unit sphere of X. By "subspace" we mean a closed infinite dimensional subspace.

Let X, Y be Banach spaces and $T : X \to Y$ a linear bounded operator. Recall that T is called *strictly singular* if, for any subspace $Z \subset X$, $T|Z$ is not an isomorphism, i.e. $\inf\{\|Tx\| : x \in S(X)\} = 0$.

Proposition 2.1. *The following statements are obvious:*

1. *A is injective on E if and only if $\mu(\gamma(a)) = 0$. If $\mu(\gamma(a)) \neq 0$, then* $\dim(\operatorname{Ker} A) = \infty$.

2. *A maps E isomorhically onto E if and only if there exists $\delta > 0$ such that* $\mu(T \setminus \sigma_\delta) = 0$.

3. *A is compact if and only if A is strictly singular, and A is strictly singular if and only if A is the zero operator, i.e.* $\mu(\operatorname{supp} a) = 0$.

4. *A has closed range if and only if there exists* $\delta > 0$ *such that* $\mu(\operatorname{supp} a \setminus \sigma_\delta) = 0$. *If in this case* $\mu(T \setminus \operatorname{supp} a) \neq 0$, *then* $\dim(\operatorname{Im} A) = \infty$.

3. Restrictions of A being isomorphisms

In this section we consider subspaces X of E for which $A|X$ is an isomorphic map. We will use the following lemma.

Lemma 3.1. *Let* $\{x_k\} \subset S(E)$ *and* $\|Ax_k\| \to 0$ *as* $k \to +\infty$. *Then, for every* $\delta > 0$,

$$\|\chi_\delta x_k\| \to 0 \qquad (k \to +\infty). \tag{3.1}$$

Proof. Indeed, $\|\chi_\delta x_k\| \leq \|\frac{1}{\delta} a \chi_\delta x_k\| \leq \|\frac{1}{\delta} a x_k\| = \frac{1}{\delta}\|Ax_k\| \to 0$ as $k \to +\infty$, and the proof is complete ∎

Proposition 3.2. *Let* $X \subset E$ *be a subspace. The following conditions are equivalent:*

1. $A|X$ *is an isomorphism.*
2. *There exist* $\delta, \varepsilon > 0$ *such that* $\|\chi_\delta x\| \geq \varepsilon$ *for every* $x \in S(X)$.
3. *There exists* $\delta > 0$ *such that* $P_\delta|X$ *is an isomorphism where* $P_\delta x = \chi_\delta x$.

Proof. $2 \Leftrightarrow 3$ follows from the definitions and $2 \Rightarrow 1$ follows from Lemma 3.1. To show $1 \Rightarrow 2$ let $A|X$ be an isomorphism. Then there exists $\delta > 0$ such that $\|Ax\| = \|ax\| \geq 2\delta$ for every $x \in S(X)$. Put $c = \sup\{|a(t)| : t \in T\}$. Then

$$2\delta \leq \|ax\| \leq \|ax\chi_\delta\| + \|ax\chi_{T\setminus\sigma_\delta}\| \leq c\|x\chi_\delta\| + \delta\|x\chi_{T\setminus\sigma_\delta}\| \leq c\|x\chi_\delta\| + \delta.$$

Hence $\|x\chi_\delta\| \geq \frac{\delta}{c} = \varepsilon$, and the proof is complete ∎

Proposition 3.3. *Let* T *be a closed domain in* $\mathbb{R}^n$ *with Lebesgue measure* μ *and* X *a subspace of a rearrangement-invariant space* E *defined on* T. *Suppose* $a : T \to \mathbb{R}$ *is a continuous function. Then the following conditions are equivalent:*

1. $A|X$ *is an isomorphism.*
2. *There exist a closed set* $\sigma \subset \operatorname{supp} a$ *and* $\varepsilon > 0$ *such that* $\|\chi_\sigma x\| \geq \varepsilon$ *for every* $x \in S(X)$.
3. *There exists a closed set* $\sigma \subset \operatorname{supp} a$ *such that* $P_\sigma x = \chi_\sigma x$ *is an isomorphism in* X.

Proof. It is sufficient to change slightly the proof of Proposition 3.2. Namely, in the proof of $1 \Rightarrow 2$ we note that the set σ_δ is closed and in the proof of $2 \Rightarrow 1$ in view of the compactness of T we have for the closed subset $\sigma \subset \operatorname{supp} a$ that $\delta = \inf\{|a(t)| : t \in \sigma\} > 0$ ∎

Proposition 3.4. *Let T be a compact symmetric domain in $\mathbb{R}^n$ with Lebesgue measure μ and $a : T \to \mathbb{R}$ a continuous function. Suppose that $X \subset E$ is a subspace consisting of even or odd functions. The restriction $A|X$ is an isomorphism if and only if*

$$a(t) = 0 \quad \Longrightarrow \quad a(-t) \neq 0. \tag{3.2}$$

Remark that condition (3.2) implies $a(\theta) \neq 0$ where $\theta = (0, ..., 0) \in \mathbb{R}^n$.

Proof of Proposition 3.4. Suppose that condition (3.2) holds. First we show that there exists $\delta > 0$ such that, for the set $\sigma_1 = \{t \in T : |a(t)| \leq \delta\}$, $\sigma_\delta \supset \sigma = -\sigma_1$. Indeed, suppose the contrary. Then there exists a sequence $\{t_k\} \in T$ such that $a(t_k) \to 0$ and $a(-t_k) \to 0$ as $k \to \infty$. Using the compactness of T and the continuity of a we find a point $t_0 \in T$ such that $a(t_0) = a(-t_0) = 0$. This contradicts (3.2). For every function f from the rearrangement-invariant space E we have $\|f\| = \|f^-\|$ where f^- is defined by $f^-(t) = f(-t)$. Thus for even functions x we have

$$\|x\chi_\sigma\| = \|x^-\chi_\sigma\| = \|x\chi_\sigma^-\| = \|x\chi_{\sigma_1}\|.$$

Similarly, for odd functions x we have

$$\|x\chi_\sigma\| = \|-x^-\chi_\sigma\| = \|x^-\chi_\sigma\| = \|x\chi_{\sigma_1}\|.$$

In view $\|x\| \leq \|x\chi_\delta\| + \|x\chi_{\sigma_1}\| \leq 2\|x\chi_\delta\|$ we have in both cases

$$\|Ax\| = \|ax\| \geq \|ax\chi_\delta\| \geq \delta\|x\chi_\delta\| \geq \frac{\delta}{2}\|x\|.$$

Therefore $A|X$ is an isomorphism.

To prove the "only if" part suppose that (3.2) fails, that is there exists $t_0 \in T$ such that $a(t_0) = a(-t_0)$. We consider two cases:

a) $t_0 \neq 0$. We denote by $Q_k \in \mathbb{R}^n$ the cube with center t_0 and side length $\frac{1}{k}$. Put $\sigma_k = Q_k \cap T$. It is obvious that $\mu(\sigma_k) \neq 0$. Denote $\chi_k := \chi_{\sigma_k}$ and Put $x_k = \chi_k + \chi_k^-$ and $y_k = \chi_k - \chi_k^-$. The function x_k is even and the function y_k is odd. In view of the coninuity of the function a,

$$\left\| A\left(\frac{x_k}{\|x_k\|}\right)\right\| \leq \sup_{t\in\sigma_k} |a(t)| \frac{\|\chi_k\|}{\|x_k\|} + \sup_{t\in-\sigma_k} |a(t)| \frac{\|\chi_k^-\|}{\|x_k\|} \to 0 \qquad (k \to \infty). \tag{3.3}$$

Since $\sigma_k \cap -\sigma_k = \emptyset$ for every $k > k_0$, we have $\|y_k\| \geq \|\chi_k\| > 0$ for $k > k_0$. Therefore the proof of

$$A\left(\frac{y_k}{\|y_k\|}\right) \to 0 \qquad (k \to +\infty) \tag{3.4}$$

is similar to that of (3.3).

b) $t_0 = 0$. Let σ_k be the cube centered at $(\frac{1}{k}, ..., \frac{1}{k})$ with side length $\frac{1}{k}$. In this case the proof of (3.3) - (3.4) is the same as that in the case a). It follows from (3.3) - (3.4) that $A|X$ is not an isomorphism. This completes the proof ∎

4. Multiplied trigonometric sequences

Recall that a sequence $\{x_n\} \subset X$ in a Banach space X is said to be a *basic sequence* if it is a basis of its closed linear span (see [3: p. 1]).

Proposition 4.1. *Let a be a continuous function on $[-\pi, +\pi]$ such that condition (3.2) is fulfilled. Then $\{a(t)\cos nt\}$ and $\{a(t)\sin nt\}$ are basic sequences in the space $L_p(-\pi, +\pi)$ $(1 < p < +\infty)$ and unconditional basic sequences in the space $L_2(-\pi, +\pi)$.*

Proof. It follows immediately from Proposition 3.4 and the well known property that the trigonometric sequence is a basis in the space $L_p(-\pi, +\pi)$ $(1 < p < +\infty, p \neq 2)$ and an unconditonal basis in the space $L_2(-\pi, +\pi)$. We also use the simple statement that each isomorphism maps a basic sequence into a basic sequence and an unconditional basic sequence into an unconditional basic sequence ∎

Remark 4.2. It follows from known results of the theory of rearrangement-invariant spaces that Proposition 4.1 is also valid for rearrangement-invariant spaces on $[-\pi, +\pi]$ with non-trivial Boyd indices (see [4: p. 130]).

Remark 4.3. Proposition 3.4 implies that $\{t\cos nt\}_{n\in\mathbb{N}}$ and $\{t\sin nt\}_{n\in\mathbb{N}}$ are basic sequences in $L_p(0, 2\pi)$ $(1 < p < +\infty, p \neq 2)$ and unconditonal basic sequences in $L_2(0, 2\pi)$. This contrasts to properties of the sequence $\{te_n(t)\}_{n\geq 1}$, where $e_1(t) = 1$, $e_2(t) = \cos t$, $e_3(t) = \sin t$, ... - the trigonometric sequence. It is easy to show that $\{te_n(t)\}_{n\geq 1}$ is not deficiently minimal in $L_2(0, 2\pi)$, i.e. it is not minimal after a deletion of a finite number of elements (see [6: p. 121]).

Let $T = K = [-\pi, +\pi]^n$. Then the set of all possible different n-products

$$\sin(k_1 t_{i_1}) \cdots \sin(k_s t_{i_s}) \cos(k_{s+1} t_{i_{s+1}}) \cdots \cos(k_n t_{i_n}) \tag{4.1}$$

where $0 \leq k_1 \ldots < +\infty$, $1 \leq i_1, \ldots, i_n \leq n$, $0 \leq s \leq n$ and $t_{i_j} \in [-\pi, +\pi]$ forms an orthogonal basis of the space $L_2(K)$. Elements (4.1) with even s generate a subspace of even functions, elements (4.1) with odd s generate a subspace of odd functions.

The proof of the following assertion is similar to that of Proposition 4.1.

Proposition 4.4. *Let a be a continuous function on the cube K such that condition (3.2) is fulfilled. Then the products of the function a by functions (4.1) in the case of even or odd s form unconditional basic sequences in the space $L_2(K)$.*

Let us recall some definitions. A sequence $\{x_k\}$ of elements of a rearrangement-invariant space E is said to be

disjoint if $\mu\{t : x_k(t)x_l(t) \neq 0\} = 0$ $(k \neq l)$

almost disjoint if, for a disjoint (*corresponding*) sequence $\{y_k\} \subset E$, $\frac{\|x_k - y_k\|}{\|x_k\|} \to 0$ as $k \to +\infty$.

A rearrangement-invariant space E is said to have an *absolutely continuous norm* if, for every decreasing sequence of measurable sets $\{\sigma_k\}$ such that $\cap\sigma_k = \emptyset$ and for every $x \in E$, $\|\chi_{\sigma_k} x\| \to 0$ as $k \to +\infty$.

Lemma 4.5. *Let E be a rearrangement-invariant space with absolutely continuous norm and $\mu(\gamma(a)) = 0$. Assume that a sequence $\{x_k\} \subset S(E)$ satisfies* (3.1) *for some $\delta > 0$. Then $\{x_k\}$ contains an almost disjoint subsequence $\{x_{k_i}\}$ such that for a corresponding disjoint sequence $\{y_i\}$*

$$\sup\{|a(t)| : t \in \operatorname{supp} y_i\} \to 0 \qquad (i \to +\infty). \tag{4.2}$$

Proof. First we show that for every $\varepsilon, \delta > 0$ and every k there exist $0 < \delta' < \frac{\delta}{2}$ and $k' > k$ such that for the characteristic function $\chi_{\delta',\delta}$ of the set $\sigma_\delta \cup (T \setminus \sigma_{\delta'})$

$$\|\chi_{\delta',\delta}\, x_{k'}\| < \varepsilon. \tag{4.3}$$

Indeed, using (3.1) we choose $k' > k$ such that $\|\chi_\delta\, x_{k'}\| < \frac{\varepsilon}{2}$. Since the norm is absolutely continuous we can choose $0 < \delta' < \frac{\delta}{2}$ such that $\|\chi_{T\setminus\delta'}\, x_{k'}\| < \frac{\varepsilon}{2}$. Now we have

$$\|\chi_{\delta',\delta}\, x_{k'}\| \le \|\chi_\delta\, x_{k'}\| + \|\chi_{T\setminus\delta'}\, x_{k'}\| < \varepsilon.$$

We construct a required almost disjoint sequence using an inductive process. In the first step we put $k = 1$ and, using the absolute continuity of the norm, choose δ_1 such that $\|\chi_{T\setminus\sigma_{\delta_1}}\, x_1\| < 1$. Suppose that the $(i-1)$st step is done. Using (4.3) we choose $k_i > k_{i-1}$ and $0 < \delta_i < \frac{\delta_{i-1}}{2}$ such that $\|\chi_{\delta_i,\delta_{i-1}}\, \chi_{k_i}\| < \frac{1}{i}$. As a correspondent disjoint sequence $\{y_i\}$ we take $y_i = \chi_{\{t:\delta_i \le |a(t)| \le \delta_{i-1}\}}\, x_{k_i}$. By construction, $\{y_i\}$ is disjoint and, since $\delta_i \to 0$ if $i \to \infty$, condition (4.2) holds. Futhermore,

$$\frac{\|x_{k_i} - y_i\|}{\|x_{k_i}\|} = \|x_{k_i} - \chi_{\{t:\,\delta_i \le |a(t)| < \delta_{i-1}\}}\, x_{k_i}\| = \|\chi_{\delta_i,\delta_{i-1}}\, x_{k_i}\| < \frac{1}{i} \to 0$$

as $i \to +\infty$, and the proof is complete ∎

The following proposition is a simple corollary of Lemmas 3.1 and 4.5.

Proposition 4.6. *Let E be a rearrangement-invariant space with absolutely continuous norm and $\mu(\gamma(a)) = 0$. Suppose also that $\|Ax_k\| \to 0 \ \ (k \to +\infty)$ for some $\{x_k\} \subset S(E)$. Then $\{x_k\}$ contains an almost disjoint subsequence such that a corresponding disjoint sequence satisfies condition* (4.2).

Corollary 4.7. *Let E be a rearrangement-invariant space with absolutely continuous norm and $\mu(\gamma(a)) = 0$. Suppose also that a subspace $X \subset E$ does not contain any almost disjoint sequence. Then $A|X$ is an isomorphism.*

Corollary 4.8. *Let X be a subspace included into all spaces $L_p(\mu)$ $(1 \le p < +\infty)$. Suppose that the L_p-norms are equivalent on X and $\mu(\gamma(a)) = 0$. Then the restriction $A|X$ is an isomorphism in every space $L_p(\mu)$ $(1 \le p < +\infty)$.*

Proof. Assume the contrary. Proposition 4.6 yields that X contains an almost disjoint sequence $\{x_k\}$. By virtue of well known stability properties of basic sequences, $\{x_k\}$ contains a subsequense $\{x_{k_i}\}$ such that it is equivalent to the standard basis of l_p in the space L_p and to the standard basis of l_q in the space L_q. But it is a classical result that the standard bases of l_p and l_q are not equivalent ∎

Let us remind that a sequence $\{n_k\}$ of positive integers is said to be *lacunary* if $\inf_k \frac{n_{k+1}}{n_k} = \lambda > 1$.

Corollary 4.9. *Suppose $T = [-\pi, +\pi]$, $1 \leq p < +\infty$, $\mu(\gamma(a)) = 0$ and that the sequence $\{n_k\}$ is lacunary. Then $\{ae_{n_k}\}$ with $\{e_n\}$ being the trigonometric sequence is an unconditional basic sequence in $L_p(-\pi, +\pi)$.*

Proof. It follows immediately from a known property of lacunary sequences in L_p $(1 \leq p < +\infty)$: the L_p-norms on the linear span of a lacunary sequence are equivalent ∎

5. Restrictions of A with the compact mapping property

In this section we study subspaces X of a rearrangement-invariant space E such that $A|X$ are compact mappings. The symbol $x_n \xrightarrow{w} x$ means weak convergence.

Proposition 5.1. *Let X be a subspace of a reflexive rearrangement-invariant space E and $\mu(\gamma(a)) = 0$. The following conditions are equivalent:*

1. *$A|X$ is compact.*

2. *For every $\{x_k\} \in S(X)$ such that $x_k \xrightarrow{w} 0$ and for every $\delta > 0$ we have $\|\chi_\delta x_k\| \to 0$ as $k \to +\infty$.*

3. *Every weakly zero sequence $\{x_k\} \subset S(X)$ contains an almost disjoint subsequence such that for a corresponding disjoint sequence $\{y_i\}$ condition (4.2) holds.*

Proof. Since in a reflexive Banach space a compact operator maps weakly zero convergent sequences to sequences converging strongly to zero, $1 \Rightarrow 2$ follows from Lemma 3.1. The proof of $2 \Rightarrow 3$ follows from Lemma 4.5. $3 \Rightarrow 1$: Let $\{x_k\} \subset S(X)$ and $x_k \xrightarrow{w} 0$ as $k \to +\infty$. Choose an almost disjoint subsequence $\{x_{k_i}\}$ of the sequence $\{x_k\}$ such that for a correspondent disjoint sequence $\{y_i\}$ (4.2) holds. Then $\|Ay_i\| \to 0$. Therefore $\|Ax_{k_i}\| \to 0$ as $i \to +\infty$. This means that $A|X$ is compact ∎

Let us formulate an alternative version of Proposition 5.1 adapted to the Hilbert space $L_2(0,1)$ and the multiplication by the independent variable t.

Proposition 5.2. *Let $E = L_2(0,1)$ and $a(t) = t$. Suppose X is a subspace of $L_2(0,1)$. The following conditions are equivalent:*

1. *$A|X$ is compact.*

2. *For every othonormal sequence $\{x_n\} \subset X$ and for every $\delta \in (0,1)$, $\int_\delta^1 |x_k(t)|^2\,dt \to 0$ as $k \to +\infty$.*

3. *Every orthonormal sequence $\{x_k\} \subset X$ contains an almost disjoint subsequence $\{x_{k_i}\}$ such that, for a corresponding disjoint sequence $\{y_i\}$, $\operatorname{supp} y_i \subset [\delta_i, \delta_{i-1}]$ where $\{\delta_i\}$ is a decreasing sequence of real numbers with $\delta_i \to 0$ as $i \to +\infty$.*

Remark 5.3. Proposition 5.2 fails if its conditions are only fulfilled for some orthonormal sequences. Namely, it follows from the results of [2] that there exists an orthonormal basis $\{x_k\}$ of $L_2(0,1)$ such that $\|tx_k(t)\| \to 0$ as $k \to +\infty$. But the multiplication operator by the independent variable t is non-compact in $L_2(0,1)$.

It is well known that every strictly singular operator in a Hilbert space is compact (see [1]). We have noted (Proposition 2.1) that a strictly singular multiplication operator in a rearrangement-invariant space E is compact. What about restrictions of a multiplication operator A acting in a rearrangement-invariant space E? Is every strictly singular restriction compact? In general the answer is "no".

Example 5.4. Put $E = L_p(-1,+1)$ $(1 < p < 2)$. By $\{r_n\}$ we denote the sequence of Rademacher functions defined on $[0,1]$ and extended by zero on $[-1,0)$. Let χ_n be the characteristic function of the interval $(-\frac{1}{2^n}, -\frac{1}{2^{n+1}})$ and $\tilde{\chi_n}$ the corresponding normalized function. We consider the sequence $\{x_n\}$ given by $x_n = r_n + \tilde{\chi_n}$. It is easy to show that $\{x_n\}$ in the space $L_p(-1,+1)$ $(1 < p < 2)$ is equivalent to the standard basis of l_p. Let X be the subspace of $L_p(-1,+1)$ spanned by $\{x_n\}$. Now we consider the multiplication operator by the function $a(t) = t$ acting in $L_p(-1,+1)$ $(1 < p < 2)$. It is also easy to see that the sequence $\{Ax_n\}$ is equivalent to the standard basis of the space l_2. But it is well known that the natural imbedding of l_p into l_2 is strictly singular and non-compact.

To close this section, we consider spectral properties of some compact operator connected with a multiplication operator in $L_2(0,1)$.

Let a be a continuous function such that $a(t_0) = 0$ for some $t_0 \in [0,1]$. Given a subspace $X \subset L_2(0,1)$, we denote by P_X the orthogonal projection onto X. Suppose that $A|X$ is compact. We denote by $B_X = P_X A P_X$ the compact selfadjoint operator acting in $L_2(0,1)$.

Proposition 5.5. *Let $\{\lambda_n\}$ be a sequence of real numbers such that*

$$\max_{t\in[0,1]} a(t) > \lambda_1 > \lambda_2 > \dots \qquad \textit{and} \qquad \lim_{k\to\infty} \lambda_n = 0.$$

Then there exists a subspace $X \subset L_2[0,1]$ such that λ_n $(n \in \mathbb{N})$ are eigenvalues of the compact selfajoint operator B_X.

Proof. It is obvious that for the characteristic function χ_σ of $\sigma \in \Sigma$ and for the orthogonal projection P_σ corresponding to the one-dimensional subspace generated by χ_σ

$$P_\sigma A P_\sigma \chi_\sigma = \frac{\langle a(t), \chi_\sigma(t)\rangle}{\|\chi_\sigma\|} = \int_\sigma a(t)\,dt \Big/ \sqrt{\int_\sigma dt}$$

Using this observation, the continuity of the function a and an induction process it is easy to construct a sequence of disjoint sets σ_n such that, for the orthogonal projection P_X onto the subspace $X = \operatorname{cl\,span}\{\chi_{\sigma_n}\}$, $P_X A P_X(\chi_{\sigma_n}) = \lambda_n \chi_{\sigma_n}$. This means that $B_X(\chi_{\sigma_n}) = \lambda_n \chi_{\sigma_n}$ ∎

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