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Iterated Integral Operators in Clifford Analysis

H. Begehr

Dedicated to L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. Integral representation formulas of Cauchy-Pompeiu type expressing Clifford-algebra-valued functions in domains of $\mathbb{R}^m$ through its boundary values and its first order derivatives in form of the Dirac operator are iterated in order to get higher order Cauchy-Pompeiu formulas. In the most general representation formulas obtained the Dirac operator is replaced by products of powers of the Dirac and the Laplace operator. Boundary values of lower order operators are involved too. In particular the integral operators provide particular solutions to the inhomogeneous equations $\partial^k w = f$, $\Delta^k w = g$ and $\partial\Delta^k w = h$. The main subject of this paper is to develop the representation formulas. Properties of the integral operators are not studied here.

Keywords: *Cauchy-Pompeiu representations, Dirac operator, Laplace operator, Clifford analysis*

AMS subject classification: 31 B 10, 31 B 30, 30 G 30

1. Introduction

Any point $x \in \mathbb{R}^m$ $(2 \leq m)$ with an orthonormal basis $\{e_k : 1 \leq k \leq m\}$ is represented as $x = \sum_{k=1}^m x_k e_k$. By the convention

$$\left.\begin{aligned} e_1 &= 1 \\ e_j e_k + e_k e_j &= -2\delta_{jk} \quad (2 \leq j,k \leq m) \end{aligned}\right\}$$

a Clifford algebra is introduced (see [6, 9 - 12, 15]) consisting of elements

$$a = \sum_A a_A e_A$$

where the sum is taken over all ordered subsets $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of $\{2, 3, \ldots, m\}$ with $2 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq m$ and

$$e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}.$$

Moreover, in the case $A = \emptyset$ the basis element $e_\emptyset$ is understood to be e_1. If the coefficients a_A are complex rather than real, then the respective algebra is denoted by $\mathbb{C}_m$. One denotes for $a = \sum_A a_A e_A$ the complex conjugate as

$$\overline{a} = \sum_A \overline{a_A}\, \overline{e_A} \qquad \text{where } \begin{cases} \overline{e_1} = e_1 = 1 \\ \overline{e_k} = -e_k \quad (2 \leq k \leq m) \end{cases}$$

H. Begehr: Freie Universität Berlin, I. Math. Inst., Arnimallee 3, D-14195 Berlin

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and $\overline{e_A e_B} = \overline{e_A}\,\overline{e_B}$, and one defines a norm in $\mathbb{C}_m$ by

$$|a| := \left(\sum_A |a_A|^2\right)^{\frac{1}{2}}$$

which via

$$|a|_0 := 2^{\frac{m}{2}}|a|$$

becomes an algebra norm. $\mathbb{R}^m$ is embedded into $\mathbb{C}_m$. In the sequal these elements are denoted by

$$z = \sum_{k=1}^{m} x_k e_k.$$

We remark that then

$$\overline{z} = x_1 - \sum_{k=2}^{m} x_k e_k, \qquad z\overline{z} = \overline{z}z = \sum_{k=1}^{m} x_k^2 = |z|^2$$

$$z^2 = x_1^2 - \sum_{k=2}^{m} x_k^2 + 2x_1 \sum_{k=2}^{m} x_k e_k, \qquad \overline{z}^2 = x_1^2 - \sum_{k=2}^{m} x_k^2 - 2x_1 \sum_{k=2}^{m} x_k e_k.$$

The Dirac operator ∂ and its complex conjugate $\overline{\partial}$ given by

$$\partial = \sum_{k=1}^{m} e_k \frac{\partial}{\partial x_k} \qquad \text{and} \qquad \overline{\partial} = \frac{\partial}{\partial x_1} - \sum_{k=2}^{m} e_k \frac{\partial}{\partial x_k}$$

corresponding to the Cauchy-Riemann and anti-Cauchy-Riemann operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$

in $\mathbb{C}$, respectively, are divisors of the Laplace operator

$$\Delta = \partial\overline{\partial} = \overline{\partial}\partial = \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2}.$$

We note that for any z, or any $z \neq 0$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
&\partial z = z\partial = 2 - m, && \partial\overline{z} = \overline{z}\partial = m \\
&\partial|z|^2 = |z|^2\partial = 2z, && \partial|z|^\alpha = |z|^\alpha\partial = \alpha|z|^{\alpha-2}z \\
&\overline{\partial}|z|^2 = |z|^2\overline{\partial} = 2\overline{z}, && \overline{\partial}|z|^\alpha = |z|^\alpha\overline{\partial} = \alpha|z|^{\alpha-2}\overline{z} \\
&\partial(\overline{z}|z|^{-m}) = (\overline{z}|z|^{-m})\partial = 0.
\end{aligned}$$

Lemma 1. *The Dirac operator acting on polynomials follows the rules*

$$\partial\overline{z}^k = \overline{z}^k\partial = \big(m + 2(k-1)\big)\,\overline{z}^{k-1} + (m-2)z\sum_{\nu=0}^{k-2} \overline{z}^{k-2-\nu}z^\nu \quad \text{for } 2 \le k$$

$$\partial z^k = z^k\partial = (2-m)\sum_{\nu=0}^{k-1} \overline{z}^{k-\nu-1}z^\nu \quad \text{for } 1 \le k.$$

Proof. For the first formula observe

$$\partial\overline{z}^2 = \overline{z}^2\partial = [\overline{z}(\overline{z}+z)]\partial - |z|^2\partial = m(\overline{z}+z) + 2\overline{z} - 2z = (m+2)\overline{z} + (m-2)z$$

and by induction

$$\begin{aligned}
\overline{z}^{k+1}\partial &= [\overline{z}^k(\overline{z}+z)]\partial - [\overline{z}^{k-1}|z|^2]\partial \\
&= \big(m+2(k-1)\big)\overline{z}^{k-1}(\overline{z}+z) + (m-2)z\sum_{\nu=0}^{k-2}\overline{z}^{k-2-\nu}z^\nu(\overline{z}+z) + 2\overline{z}^k \\
&\quad - \big(m+2(k-2)\big)\overline{z}^{k-2}|z|^2 - (m-2)z\sum_{\nu=0}^{k-3}\overline{z}^{k-3-\nu}z^\nu|z|^2 - 2\overline{z}^{k-2}|z|^2 \\
&= (m+2k)\overline{z}^k + (m-2)z\left[\sum_{\nu=0}^{k-2}\overline{z}^{k-1-\nu}z^\nu + \sum_{\nu=0}^{k-2}\overline{z}^{k-2-\nu}z^{\nu+1} - \sum_{\nu=0}^{k-3}\overline{z}^{k-2-\nu}z^{\nu+1}\right] \\
&= (m+2k)\overline{z}^k + (m-2)z\sum_{\nu=0}^{k-1}\overline{z}^{k-1-\nu}z^\nu.
\end{aligned}$$

Similarly, from

$$\partial z = z\partial = 2-m$$

and

$$\begin{aligned}
z^{k+1}\partial &= [z^k(\overline{z}+z)]\partial - [z^{k-1}|z|^2]\partial \\
&= (2-m)\sum_{\nu=0}^{k-1}\overline{z}^{k-\nu-1}z^\nu(\overline{z}+z) + 2z^k - (2-m)\sum_{\nu=0}^{k-2}\overline{z}^{k-\nu-2}z^\nu|z|^2 - 2z^k \\
&= (2-m)\left[\sum_{\nu=0}^{k-1}\overline{z}^{k-\nu}z^\nu + \sum_{\nu=0}^{k-1}\overline{z}^{k-\nu-1}z^{\nu+1} - \sum_{\nu=0}^{k-2}ovz^{k-\nu-1}z^{\nu+1}\right] \\
&= (2-m)\sum_{\nu=0}^{k}\overline{z}^{k-\nu}z^\nu
\end{aligned}$$

the second formula follows ∎

Corollary 1. *The Dirac operator satisfies*

$$\partial(\overline{z}^k + z^k) = (\overline{z}^k + z^k)\partial = 2k\,\overline{z}^{k-1}$$

for $1 \le k$.

For any integral representation the Stokes theorem is a fundamental tool. In Clifford analysis it has the following form (see [10, 12, 14, 16]).

Lemma 2. *Let D be a bounded smooth domain in $\mathbb{R}^m$ and $f, g \in C(\overline{D}; \mathbb{C}_m) \cap C^1(D; \mathbb{C}_m)$. Then*

$$\left.\begin{aligned} \int_D \{(f\partial)g + f(\partial g)\}\, dv &= \int_{\partial D} f\, d\vec{\sigma} g \\ \int_D \{(f\overline{\partial})g + f(\overline{\partial} g)\}\, dv &= \int_{\partial D} f\, d\overline{\vec{\sigma}} g. \end{aligned}\right\} \tag{1.1}$$

Remarks. Here dv is the volume element of D while $d\vec{\sigma} := \vec{n}\, d\sigma$ with $\vec{n} := \sum_{k=1}^m n_k e_k$ is the directed area element of ∂D where $(n_1, \dots, n_m)$ is the outward directed normal vector on ∂D and $d\sigma$ is the area element on ∂D. Moreover, $d\overline{\vec{\sigma}} = \overline{\vec{n}}\, d\sigma$. By the usual method from (1.1) the representation formulae of Cauchy-Pompeiu type

$$\begin{aligned} w(z) &= \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m}\, d\vec{\sigma}(\zeta) w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} \partial w(\zeta)\, dv(\zeta) \\ w(z) &= \frac{1}{\omega_m} \int_{\partial D} \frac{\zeta - z}{|\zeta - z|^m}\, d\overline{\vec{\sigma}(\zeta)} w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\zeta - z}{|\zeta - z|^m} \overline{\partial} w(\zeta)\, dv(\zeta) \end{aligned} \tag{1.2}$$

follow for $w \in C(\overline{D}; \mathbb{C}_m) \cap C^1(D; \mathbb{C}_m)$. Here ω_m is the area of the unit sphere in $\mathbb{R}^m$. Dual formulas are

$$\begin{aligned} w(z) &= \frac{1}{\omega_m} \int_{\partial D} w(\zeta)\, d\vec{\sigma}(\zeta) \frac{\overline{\zeta - z}}{|\zeta - z|^m} - \frac{1}{\omega_m} \int_D (w(\zeta)\partial) \frac{\overline{\zeta - z}}{|\zeta - z|^m}\, dv(\zeta) \\ w(z) &= \frac{1}{\omega_m} \int_{\partial D} w(\zeta)\, d\overline{\vec{\sigma}(\zeta)} \frac{\zeta - z}{|\zeta - z|^m} - \frac{1}{\omega_m} \int_D (w(\zeta)\overline{\partial}) \frac{\zeta - z}{|\zeta - z|^m}\, dv(\zeta). \end{aligned} \tag{1.2$'$}$$

The operator

$$Tf(z) := -\frac{1}{\omega_m} \int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} f(\zeta)\, dv(\zeta) \qquad (f \in L_1(\overline{D}; \mathbb{C}_m))$$

is known to provide a particular weak solution to the inhomogeneous Dirac equation

$$\partial w = f \qquad \text{in } D$$

(see [14]) while the boundary integral in the first formula of (1.2), obviously, is a left-monogenic function, i.e. a solution to the related homogeneous equation

$$\partial w = 0 \qquad \text{in } D \text{ (and in } \mathbb{R}^m \backslash \overline{D} \text{ as well).}$$

Iterating these representation formulas similarly as in [1 - 5, 7, 8] leads to higher order representation formulas. They provide general solutions to equations of the kinds

$$\partial^k w = f, \qquad \Delta^k w = f, \qquad \partial^\ell \Delta^k w = f.$$

The first kind of equation is treated in [16] in the homogeneous case. The second one is the inhomogeneous polyharmonic equation for $\mathbb{C}_m$-valued functions. A representation formula for the third kind of equation seems to be involved in general.

2. Higher order Dirac equation

Iterating the first formula of (1.2) leads to a representation of solutions to the inhomogeneous equation

$$\partial^k w = f \quad \text{in } D \tag{2.1}$$

where D is a bounded smooth domain in $\mathbb{R}^m$ and $k \in \mathbb{N}$.

Lemma 3. *For $z, \tilde{\zeta} \in D$ with $z \neq \tilde{\zeta}$ and $k \in \mathbb{N}_0$*

$$\phi_k(z,\tilde{\zeta}) := \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\bar{\sigma}(\zeta) \, \frac{\overline{(\zeta - \tilde{\zeta})}(\overline{\zeta - \tilde{\zeta}} + \zeta - \tilde{\zeta})^k}{2^k k! \, |\zeta - \tilde{\zeta}|^m} \tag{2.2}$$

satisfies

$$\begin{aligned} \phi_0(z,\tilde{\zeta}) &= 0 \\ \partial_z \phi_k(z,\tilde{\zeta}) &= 0 \\ \phi_k(z,\tilde{\zeta})\partial_{\tilde{\zeta}} &= -\phi_{k-1}(z,\tilde{\zeta}) \quad \textit{for} \quad k \in \mathbb{N} \\ \phi_k(z,\tilde{\zeta})\partial^k_{\tilde{\zeta}} &= (-1)^k \phi_0(z,\tilde{\zeta}) = 0 \quad \textit{for} \quad k \in \mathbb{N}_0 . \end{aligned}$$

Proof. From (1.1) and $\partial \frac{\bar{z}}{|z|^m} = 0$

$$\phi_0(z,\zeta) = \frac{1}{\omega_m} \int_D \left[\left(\frac{\overline{\zeta - z}}{|\zeta - z|^m} \partial_\zeta \right) \frac{\overline{\zeta - \tilde{\zeta}}}{|\zeta - \tilde{\zeta}|^m} + \frac{\overline{\zeta - z}}{|\zeta - z|^m} \left(\partial_\zeta \frac{\overline{\zeta - \tilde{\zeta}}}{|\zeta - \tilde{\zeta}|^m} \right) \right] dv(\zeta) = 0$$

follows in the usual way by first applying (1.1) for $D_\varepsilon = D \backslash \{|\zeta - z| \leq \varepsilon\}$. The second equation establishing the left-monogenicity of ϕ_k is obvious. For the remaining relations observe

$$\partial \frac{\bar{z}(z+\bar{z})^k}{|z|^m} = \partial \frac{(2x_1)^k \bar{z}}{|z|^m} = \frac{\bar{z}(z+\bar{z})^k}{|z|^m} \partial = 2k \frac{\bar{z}(z+\bar{z})^{k-1}}{|z|^m} .$$

Hence,

$$\begin{aligned} \partial^k \frac{\bar{z}(z+\bar{z})^k}{|z|^m} &= \frac{\bar{z}(z+\bar{z})^k}{|z|^m} \partial^k = 2^k k! \frac{\bar{z}}{|z|^m} \\ \partial^{k+1} \frac{\bar{z}(z+\bar{z})^k}{|z|^m} &= \frac{\bar{z}(z+\bar{z})^k}{|z|^m} \partial^{k+1} = 0. \end{aligned}$$

Applying the first formula in (1.2) for $z, \tilde{\zeta} \in D$ with $z \neq \tilde{\zeta}$ gives

$$\begin{aligned} &\frac{\overline{(z - \tilde{\zeta})}(\overline{z - \tilde{\zeta}} + z - \tilde{\zeta})^k}{2^k k! \, |z - \tilde{\zeta}|^m} \\ &= \phi_k(z,\tilde{\zeta}) - \frac{1}{\omega_m} \int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} \frac{\overline{(\zeta - \tilde{\zeta})}(\overline{\zeta - \tilde{\zeta}} + \zeta - \tilde{\zeta})^{k-1}}{2^{k-1}(k-1)! \, |\zeta - \tilde{\zeta}|^m} \, dv(\zeta). \end{aligned} \tag{2.3}$$

Using these formulas after having differentiated (2.2) leads to the last two formulas ∎

Definition 1. Let $D \subset \mathbb{R}^m$ be bounded and $f \in L_1(\overline{D};\mathbb{C}_m)$. Then for $k \in \mathbb{N}$

$$T_k f(z) := \frac{(-1)^k}{\omega_m} \int_D \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^{k-1}}{2^{k-1}(k-1)!\,|\zeta - z|^m} f(\zeta)\,dv(\zeta).$$

For $k = 1$ this operator T_1 is the Pompeiu operator T, satisfying $\partial T f = f$ (see [14]). Obviously, if $T_0 f := f$, then $\partial T_k f = T_{k-1} f$ for $k \in \mathbb{N}$.

Theorem 1. *Let $D \subset \mathbb{R}^m$ be bounded and smooth and $w \in C^k(\overline{D};\mathbb{C}_m)$ for $k \in \mathbb{N}$. Then for $z \in D$*

$$\begin{aligned} w(z) = \sum_{\mu=0}^{k-1} \frac{(-1)^\mu}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^\mu}{2^\mu \mu!\,|\zeta - z|^m}\, d\vec{\sigma}(\zeta)\, \partial^\mu w(\zeta) \\ + \frac{(-1)^k}{\omega_m} \int_D \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^{k-1}}{2^{k-1}(k-1)!\,|\zeta - z|^m}\, \partial^k w(\zeta)\,dv(\zeta). \end{aligned} \tag{2.4}$$

Remarks. 1) An analogous formula is

$$\begin{aligned} w(z) = \sum_{\mu=0}^{k-1} \frac{(-1)^\mu}{\omega_m} \int_{\partial D} \frac{(\zeta - z)(\overline{\zeta - z} + \zeta - z)^\mu}{2^\mu \mu!\,|\zeta - z|^m}\, \overline{d\vec{\sigma}(\zeta)}\, \overline{\partial}^\mu w(\zeta) \\ + \frac{(-1)^k}{\omega_m} \int_D \frac{(\zeta - z)(\overline{\zeta - z} + \zeta - z)^{k-1}}{2^{k-1}(k-1)!\,|\zeta - z|^m}\, \overline{\partial}^k w(\zeta)\,dv(\zeta). \end{aligned} \tag{2.4$'$}$$

2) Denoting $\rho_\mu := \partial^\mu w$ and

$$\varphi_\mu(z) := \frac{(-1)^\mu}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^\mu}{2^\mu \mu!\,|\zeta - z|^m}\, d\vec{\sigma}(\zeta)\, \rho_\mu(\zeta)$$

for $0 \le \mu \le k$, then

$$w = \sum_{\mu=0}^{k-1} \varphi_\mu + T_k \rho_k. \tag{2.5}$$

Here the $\partial^\mu \varphi_\mu$ are left-regular and $\partial^k T_k \rho_k = \rho_k$. Thus $T_k \rho$ is a particular solution to $\partial^k w = \rho$ while $\sum_{\mu=0}^{k-1} \varphi_\mu$ is a solution to the related homogeneous equation $\partial^k w = 0$. (2.5) turns out to be the general solution to $\partial^k w = \rho_k$.

In order to express the φ_μ as some power series with left-regular coefficients one observes

$$(\overline{\zeta - z} + \zeta - z)^\mu = \sum_{\tau=0}^{\mu} (-1)^\tau (\overline{\zeta} + \zeta)^{\mu-\tau} (\overline{z} + z)^\tau.$$

Therefore

$$\begin{aligned} \varphi_\mu(z) &= \sum_{\tau=0}^{\mu} \frac{(-1)^{\mu-\tau}}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} (\overline{\zeta} + \zeta)^{\mu-\tau} d\vec{\sigma}(\zeta)\, \rho_\mu(\zeta)\, (\overline{z} + z)^\tau \\ &= \sum_{\tau=0}^{\mu} \alpha_{\tau,\mu}(z)\, (\overline{z} + z)^\tau \end{aligned}$$

where the $\alpha_{\tau,\mu}$, obviously, are left-regular coefficients. So (2.5) has the form

$$w(z) = \sum_{\tau=0}^{k-1} a_\tau(z)(\overline{z}+z)^\tau + T_k\rho \qquad (\rho := \partial^k w)$$

where the coefficients a_τ are left-regular. Observe

$$\left.\begin{aligned}\partial(\overline{z}+z)^\tau &= 2\tau(\overline{z}+z)^{\tau-1}\\ \partial^\tau(\overline{z}+z)^\tau &= 2^\tau\tau!\end{aligned}\right\}.$$

Proof of Theorem 1. For $k = 1$ formula (2.4) coincides with the first Cauchy-Pompeiu formula (1.2). For $k = 2$ use this formula again to get

$$\partial w(z) = \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,d\vec{\sigma}(\zeta)\,\partial w(\zeta) - \frac{1}{\omega_m}\int_D\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,\partial^2 w(\zeta)\,dv(\zeta).$$

Hence,

$$\begin{aligned}w(z) = {}&\frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}d\vec{\sigma}(\zeta)\,w(\zeta)\\ &-\frac{1}{\omega_m}\int_{\partial D}\widetilde{\phi_1}(z,\widetilde{\zeta})\,d\vec{\sigma}(\widetilde{\zeta})\,\partial w(\widetilde{\zeta})\\ &+\frac{1}{\omega_m}\int_D\widetilde{\phi_1}(z,\widetilde{\zeta})\,\partial^2 w(\widetilde{\zeta})\,dv(\widetilde{\zeta})\end{aligned}$$

with

$$\widetilde{\phi}_1(z,\widetilde{\zeta}) = \frac{1}{\omega_m}\int_D\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,\frac{\overline{\widetilde{\zeta} - \zeta}}{|\widetilde{\zeta} - \zeta|^m}\,dv(\zeta).$$

By the Pompeiu formula (1.2)

$$\frac{(\overline{z - \widetilde{\zeta}})(\overline{z - \widetilde{\zeta}} + z - \widetilde{\zeta})}{2\,|z - \widetilde{\zeta}|^m} = \phi_1(z_1,\widetilde{\zeta}) + \widetilde{\phi}_1(z,\widetilde{\zeta})$$

and from the Green formula (1.1) and Lemma 3

$$\begin{aligned}\int_{\partial D}\phi_1(z,\widetilde{\zeta})d\vec{\sigma}(\widetilde{\zeta})\,\partial w(\widetilde{\zeta}) &= \int_D\left[(\phi_1(z,\widetilde{\zeta})\partial_{\widetilde{\zeta}})\,\partial w(\widetilde{\zeta}) + \phi_1(z,\widetilde{\zeta})\partial^2 w(\widetilde{\zeta})\right]dv(\widetilde{\zeta})\\ &= \int_D\phi_1(z,\widetilde{\zeta})\partial^2 w(\widetilde{\zeta})dv(\widetilde{\zeta}).\end{aligned}$$

This shows

$$\begin{aligned}w(z) = {}&\frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}d\vec{\sigma}(\zeta)w(\zeta)\\ &-\frac{1}{\omega_m}\int_{\partial D}\frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)}{2\,|\zeta - z|^m}\,d\vec{\sigma}(\zeta)\,\partial w(\zeta)\\ &+\frac{1}{\omega_m}\int_D\frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)}{2\,|\zeta - z|^m}\,\partial^2 w(\zeta)\,dv(\zeta)\\ ={}&\varphi_0 + \varphi_1 + T_2\partial^2 w\end{aligned}$$

where φ_0 is left-regular, i.e. $\partial\varphi_0 = 0$, and φ_1 satisfies $\partial^2\varphi_1 = 0$.

Assume now $w \in C^{k-1}(\overline{D}; \mathbb{C}_m)$ is represented as

$$w = \sum_{\mu=0}^{k-2} \varphi_\mu + T_{k-1}\partial^{k-1}w$$

where

$$\varphi_\mu(z) = \frac{(-1)^\mu}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^\mu}{2^\mu \mu! \, |\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, \partial^\mu w(\zeta).$$

Then applying this representation to ∂w for $w \in C^k(\overline{D}; \mathbb{C}_m)$

$$\partial w = \sum_{\mu=0}^{k-2} \widetilde{\varphi}_\mu + T_{k-1}\partial^k w$$

where

$$\widetilde{\varphi}_\mu(z) = \frac{(-1)^\mu}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})(\overline{\zeta - z} + \zeta - z)^\mu}{2^\mu \mu! \, |\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, \partial^{\mu+1} w(\zeta)$$

and the first formula from (1.2)

$$w = \varphi_0 + T_1 \partial w$$

where

$$\varphi_0(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, w(\zeta)$$

gives

$$w = \varphi_0 + \sum_{\mu=0}^{k-2} T_1 \widetilde{\varphi}_\mu + T_1 T_{k-1} \partial^k w.$$

From (2.3)

$$\begin{aligned}
&T_1 \widetilde{\varphi}_\mu(z) \\
&= \frac{(-1)^{\mu+1}}{\omega_m} \int_{\partial D} \frac{1}{\omega_m} \int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, \frac{(\overline{\widetilde{\zeta} - \zeta})(\overline{\widetilde{\zeta} - \zeta} + \widetilde{\zeta} - \zeta)^\mu}{2^\mu \mu! \, |\widetilde{\zeta} - \zeta|^m} \, dv(\zeta) \, d\vec{\sigma}(\widetilde{\zeta}) \, \partial^{\mu+1} w(\widetilde{\zeta}) \\
&= -\frac{1}{\omega_m} \int_{\partial D} \left[\frac{(\overline{z - \widetilde{\zeta}})(\overline{z - \widetilde{\zeta}} + z - \widetilde{\zeta})^{\mu+1}}{2^{\mu+1}(\mu+1)! \, |\widetilde{\zeta} - z|^m} - \phi_{\mu+1}(z, \widetilde{\zeta}) \right] d\vec{\sigma}(\widetilde{\zeta}) \, \partial^{\mu+1} w(\widetilde{\zeta}),
\end{aligned}$$

and from the first formula (1.1) and Lemma 3

$$\begin{aligned}
&\int_{\partial D} \phi_{\mu+1}(z, \zeta) \, d\vec{\sigma}(\zeta) \, \partial^{\mu+1} w(\zeta) \\
&\quad = \int_D \left[(\phi_{\mu+1}(z, \zeta) \, \partial_\zeta) \, \partial^{\mu+1} w(\zeta) + \phi_{\mu+1}(z, \zeta) \, \partial^{\mu+2} w(\zeta) \right] dv(\zeta) \\
&\quad = \int_D \left[\phi_{\mu+1}(z, \zeta) \, \partial^{\mu+2} w(\zeta) - \phi_\mu(z, \zeta) \, \partial^{\mu+1} w(\zeta) \right] dv(\zeta).
\end{aligned}$$

Hence,

$$\sum_{\mu=0}^{k-2} T_1\widetilde{\varphi}_\mu = \sum_{\mu=1}^{k-1} \varphi_\mu + \frac{1}{\omega_m}\int_D \phi_{k-1}(z,\zeta)\,\partial^k w(\zeta)\,dv(\zeta).$$

Moreover, from (2.3)

$$\begin{aligned}
&T_1 T_{k-1}\partial^k w(z)\\
&= \frac{(-1)^k}{\omega_m}\int\limits_D \frac{1}{\omega_m}\int\limits_D \frac{\overline{\zeta - z}}{|\zeta - z|^m}\,\frac{(\overline{\widetilde{\zeta}-\zeta})(\overline{\widetilde{\zeta}-\zeta}+\widetilde{\zeta}-\zeta)^{k-2}}{2^{k-2}(k-2)!\,|\widetilde{\zeta}-\zeta|^m}\,dv(\zeta)\,\partial^k w(\widetilde{\zeta})\,dv(\widetilde{\zeta})\\
&= -\frac{1}{\omega_m}\int_D \left[\phi_{k-1}(z,\widetilde{\zeta}) - \frac{(\overline{z-\widetilde{\zeta}})(\overline{z-\widetilde{\zeta}}+z-\widetilde{\zeta})^{k-1}}{2^{k-1}(k-1)!\,|z-\widetilde{\zeta}|^m}\right]\partial^k w(\widetilde{\zeta})\,dv(\widetilde{\zeta}).
\end{aligned}$$

This shows

$$w = \sum_{\mu=0}^{k-1}\varphi_\mu + T_k\partial^k w,$$

i.e. (2.4) ∎

3. Integral representations in terms of powers of the Laplacian

As the Laplace operator is the product of ∂ and $\overline{\partial}$ a representation formula in terms of the Laplacian can be obtained by iterating both formulas (1.2). In this section $2 < m$ is assumed. The case $m = 2$ is well-known.

Theorem 2. *Let $D \subset \mathbb{R}^m$ be a bounded and smooth domain and $w \in C^2(\overline{D};\mathbb{C}_m)$. Then for $z \in D$*

$$\begin{aligned}
w(z) &= \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,d\bar{\sigma}(\zeta)\,w(\zeta)\\
&\quad - \frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,\overline{d\bar{\sigma}(\zeta)}\,\partial w(\zeta)\\
&\quad + \frac{1}{\omega_m}\int_D \frac{|\zeta - z|^{2-m}}{2-m}\,\Delta w(\zeta)\,dv(\zeta).
\end{aligned} \tag{3.1}$$

Remark. Formula (3.1) has the form $w = \varphi_0 + \varphi_1 + S_1\Delta w$ with a left-regular function φ_0 and φ_1 with left anti-regular ∂-derivative $\partial\varphi_1$, i.e. $\overline{\partial}(\partial\varphi_1) = 0$. S_1 is the well-known potential operator

$$S_1\rho(z) = \frac{1}{\omega_m}\int_D \frac{|\zeta - z|^{2-m}}{2-m}\,\rho(\zeta)\,dv(\zeta) \qquad (\rho \in L_1(\overline{D};\mathbb{C}_m)).$$

Proof of Theorem 2. Applying the second formula (1.2) to ∂w gives

$$\partial w(z) = \frac{1}{\omega_m}\int_{\partial D}\frac{\zeta - z}{|\zeta - z|^m}\,\overline{d\bar{\sigma}(\zeta)}\,\partial w(\zeta) - \frac{1}{\omega_m}\int_D \frac{\zeta - z}{|\zeta - z|^m}\,\Delta w(\zeta)\,dv(\zeta).$$

Thus together with the first formula (1.2)

$$\begin{aligned} w(z) = {} & \frac{1}{\omega_m}\int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\vec{\sigma}(\zeta)\, w(\zeta) \\ & - \frac{1}{\omega_m}\int_{\partial D} \tilde{\psi}_1(z,\tilde{\zeta})\, \overline{d\vec{\sigma}(\tilde{\zeta})}\, \partial w(\tilde{\zeta}) \\ & + \frac{1}{\omega_m}\int_{D} \tilde{\psi}_1(z,\tilde{\zeta})\, \Delta w(\tilde{\zeta})\, dv(\tilde{\zeta}) \end{aligned}$$

with

$$\tilde{\psi}_1(z,\tilde{\zeta}) := \frac{1}{\omega_m}\int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} \frac{\tilde{\zeta} - \zeta}{|\tilde{\zeta} - \zeta|^m}\, dv(\zeta)$$

follows. Setting

$$\psi_1(z,\tilde{\zeta}) := \frac{1}{\omega_m}\int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\vec{\sigma}(\zeta)\, \frac{|\zeta - \tilde{\zeta}|^{2-m}}{2-m},$$

then for $z, \tilde{\zeta} \in D$ with $z \neq \tilde{\zeta}$ from (1.2)

$$\frac{|z - \tilde{\zeta}|^{2-m}}{2-m} = \psi_1(z,\tilde{\zeta}) + \tilde{\psi}_1(z,\tilde{\zeta})$$

is seen. (1.1) then leads to

$$\int_{\partial D} \psi_1(z,\tilde{\zeta})\, \overline{d\vec{\sigma}(\tilde{\zeta})}\, \partial w(\tilde{\zeta}) = \int_D \left[(\psi_1(z,\tilde{\zeta})\, \overline{\partial}_{\tilde{\zeta}})\, \partial w(\tilde{\zeta}) + \psi_1(z,\tilde{\zeta})\, \Delta w(\tilde{\zeta}) \right] dv(\tilde{\zeta}).$$

As $\psi_1(z,\tilde{\zeta})\,\overline{\partial}_{\tilde{\zeta}} = \phi_0(z,\tilde{\zeta}) = 0$ (see Lemma 3) this proves (3.1) ∎

In order to generalize (3.1) the next lemma will be used.

Lemma 4. *For* $1 \le k$ *and* $z \neq 0$

$$\left.\begin{aligned} \Delta^{k-1}|z|^{2k-m} &= 2^{k-1}(k-1)!\prod_{\nu=2}^{k}(2\nu - m)\,|z|^{2-m} \\ \Delta^{k}|z|^{2k-m} &= 0. \end{aligned}\right\}$$

Proof. Obviously, the formula holds for $k = 1$. Then arguing inductively

$$\begin{aligned} \Delta^k |z|^{2(k+1)-m} &= \Delta^{k-1}\overline{\partial}(2(k+1) - m)\,|z|^{2k-m} z \\ &= \Delta^{k-1}(2(k+1) - m)\, 2k\, |z|^{2k-m} \\ &= 2^k k! \prod_{\nu=2}^{k+1}(2\nu - m)\,|z|^{2-m}. \end{aligned}$$

This is the first formula for $k+1$ rather than for k. Because of harmonicity of the right-hand side of the first formula the second follows immediately ∎

Theorem 3. *Let $D \subset \mathbb{R}^m$ be a bounded smooth domain and $w \in C^{2k}(\overline{D}; \mathbb{C}_m)$ for $1 \leq k$. Then*

$$\begin{aligned} w(z) = \sum_{\mu=1}^{k} \Bigg\{ & \frac{1}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})|\zeta - z|^{2(\mu-1)-m}}{2^{\mu-1}(\mu-1)! \prod_{\nu=1}^{\mu-1}(2\nu - m)} d\vec{\sigma}(\zeta)\, \Delta^{\mu-1} w(\zeta) \\ & - \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2\mu-m}}{2^{\mu-1}(\mu-1)! \prod_{\nu=1}^{\mu}(2\nu - m)} d\overline{\vec{\sigma}(\zeta)}\, \partial \Delta^{\mu-1} w(\zeta) \Bigg\} \\ & + \frac{1}{\omega_m} \int_{D} \frac{|\zeta - z|^{2k-m}}{2^{k-1}(k-1)! \prod_{\nu=1}^{k}(2\nu - m)} \Delta^k w(\zeta)\, dv(\zeta). \end{aligned} \tag{3.2}$$

Proof. For $k = 1$ this is just formula (3.1). Let now $w \in C^4(\overline{D}; \mathbb{C}_m)$. Then from (3.1) applied as well to Δw as to w one has

$$\begin{aligned} w(z) = {} & \varphi_0(z) + \varphi_1(z) \\ & + \frac{1}{\omega_m} \int_{\partial D} \Phi_1(z, \tilde{\zeta})\, d\vec{\sigma}(\tilde{\zeta})\, \Delta w(\tilde{\zeta}) \\ & - \frac{1}{\omega_m} \int_{\partial D} \Psi_1(z, \tilde{\zeta})\, d\overline{\vec{\sigma}(\zeta)}\, \partial \Delta\, w(\tilde{\zeta}) \\ & + \frac{1}{\omega_m} \int_{D} \Psi_1(z, \tilde{\zeta})\, \Delta^2 w(\tilde{\zeta})\, dv(\tilde{\zeta}) \end{aligned}$$

with

$$\begin{aligned} \varphi_0(z) &:= \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\vec{\sigma}(\zeta)\, w(\zeta) \\ \varphi_1(z) &:= -\frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2-m}}{2 - m} \overline{d\vec{\sigma}(\zeta)}\, \partial w(\zeta) \\ \Phi_1(z, \tilde{\zeta}) &:= \frac{1}{\omega_m} \int_{D} \frac{|\zeta - z|^{2-m}}{2 - m} \frac{\overline{\tilde{\zeta} - \zeta}}{|\tilde{\zeta} - \zeta|^m} dv(\zeta) \\ \Psi_1(z, \tilde{\zeta}) &:= \frac{1}{\omega_m} \int_{D} \frac{|\zeta - z|^{2-m}}{2 - m} \frac{|\zeta - \tilde{\zeta}|^{2-m}}{2 - m} dv(\zeta) \end{aligned}$$

where $\Phi_1(z, \tilde{\zeta}) = \overline{\partial}_{\tilde{\zeta}} \Psi_1(z, \tilde{\zeta})$. From (1.2) observing

$$\Delta\, |z|^{4-m} = \overline{\partial}[(4 - m)\, z |z|^{2-m}] = 2(4 - m)\, |z|^{2-m}$$

it follows for $z \neq \tilde{\zeta}$

$$\begin{aligned} \frac{|z - \tilde{\zeta}|^{4-m}}{2(4 - m)(2 - m)} = {} & \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\vec{\sigma}(\zeta) \frac{|\zeta - \tilde{\zeta}|^{4-m}}{2(4 - m)(2 - m)} \\ & - \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2-m}}{2 - m} \overline{d\vec{\sigma}(\zeta)} \frac{(\zeta - \tilde{\zeta})\, |\zeta - \tilde{\zeta}|^{2-m}}{2(2 - m)} \\ & + \Psi_1(z, \tilde{\zeta}) \end{aligned}$$

and by differentiating

$$\frac{\overline{(\tilde\zeta - z)}|\tilde\zeta - z|^{2-m}}{2(2-m)} = -\tilde\Phi_1(z,\tilde\zeta)\,\overline{\partial_{\tilde\zeta}} + \tilde\Psi_1(z,\tilde\zeta)\,\overline{\partial_{\tilde\zeta}} + \Phi_1(z,\tilde\zeta)$$

where

$$\tilde\Phi_1(z,\tilde\zeta) := \frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,\overline{d\vec\sigma(\zeta)}\,\frac{(\zeta-\tilde\zeta)|\zeta-\tilde\zeta|^{2-m}}{2(2-m)}$$

$$\tilde\Psi_1(z,\tilde\zeta) := \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^{m}}\,d\vec\sigma(\zeta)\,\frac{|\zeta-\tilde\zeta|^{4-m}}{2(4-m)(2-m)}.$$

Thus

$$\begin{aligned}
w(z) &= \varphi_0(z) + \varphi_1(z)\\
&\quad + \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{(\tilde\zeta - z)}\,|\tilde\zeta - z|^{2-m}}{2(2-m)}\,d\vec\sigma(\tilde\zeta)\,\Delta w(\tilde\zeta)\\
&\quad - \frac{1}{\omega_m}\int_{\partial D}\frac{|\tilde\zeta - z|^{4-m}}{2(4-m)(2-m)}\,\overline{d\vec\sigma(\tilde\zeta)}\,\partial\Delta w(\tilde\zeta)\\
&\quad + \frac{1}{\omega_m}\int_{D}\frac{|\tilde\zeta - z|^{4-m}}{2(4-m)(2-m)}\,\Delta^2 w(\tilde\zeta)\,dv(\tilde\zeta)
\end{aligned}$$

because by (1.1)

$$\begin{aligned}
&\int_{\partial D}\Big[\Big(\tilde\Phi_1(z,\zeta) - \tilde\Psi_1(z,\zeta)\Big)\,\overline{\partial_\zeta}\Big]\,d\vec\sigma(\zeta)\,\Delta w(\zeta)\\
&\quad - \int_{\partial D}\Big(\tilde\Phi_1(z,\zeta) - \tilde\Psi_1(z,\zeta)\Big)\,\overline{d\vec\sigma(\zeta)}\,\partial\Delta w(\zeta)\\
&\quad + \int_{D}\Big(\tilde\Phi_1(z,\zeta) - \tilde\Psi_1(z,\zeta)\Big)\,\Delta^2 w(\zeta)\,dv(\zeta)\\
&= \int_D\Big[\Big(\tilde\Phi_1(z,\zeta) - \tilde\Psi_1(z,\zeta)\Big)\,\partial_\zeta\overline{\partial_\zeta}\Big]\,\Delta w(\zeta)\,dv(\zeta)\\
&= 0
\end{aligned}$$

as again using (1.1)

$$\begin{aligned}
&\Big(\tilde\Phi_1(z,\tilde\zeta) - \tilde\Psi_1(z,\tilde\zeta)\Big)\Delta_{\tilde\zeta}\\
&= -\frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,\overline{d\vec\sigma(\zeta)}\,\frac{\tilde\zeta - \zeta}{|\tilde\zeta - \zeta|^m} - \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,d\vec\sigma(\zeta)\,\frac{|\zeta - \tilde\zeta|^{2-m}}{2-m}\\
&= \frac{1}{\omega_m}\int_D\left[\overline{\partial_\zeta}\frac{|\zeta - z|^{2-m}}{2-m}\,\frac{\zeta - \tilde\zeta}{|\zeta - \tilde\zeta|^m} - \frac{\overline{\zeta - z}}{|\zeta - z|^m}\,\partial_\zeta\frac{|\zeta - \tilde\zeta|^{2-m}}{2-m}\right]dv(\zeta)\\
&= 0.
\end{aligned}$$

This proves (3.2) in the case $k = 2$.

In order to prove (3.2) for any $k > 1$ assume it holds for $k-1$. Applying this formula for Δw leads to

$$\begin{aligned}\Delta w(z) &= \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,d\vec{\sigma}(\zeta)\,\Delta w(\zeta)\\ &+\sum_{\mu=1}^{k-2}\frac{1}{\omega_m}\int_{\partial D}\frac{(\overline{\zeta - z})\,|\zeta - z|^{2\mu-n}}{2^\mu\mu!\prod_{\nu=1}^{\mu}(2\nu-m)}\,d\vec{\sigma}(\zeta)\,\Delta^{\mu+1}w(\zeta)\\ &-\sum_{\mu=1}^{k-1}\frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2\mu-m}}{2^{\mu-1}(\mu-1)!\prod_{\nu=1}^{\mu}(2\nu-m)}\,\overline{d\vec{\sigma}(\zeta)}\,\partial\Delta^\mu w(\zeta)\\ &+\frac{1}{\omega_m}\int_D\frac{|\zeta - z|^{2(k-1)-m}}{2^{k-2}(k-2)!\prod_{\nu=1}^{k-1}(2\nu-m)}\,\Delta^k w(\zeta)\,dv(\zeta).\end{aligned}$$

Inserting this into (3.1) gives

$$\begin{aligned}w(z) &= \frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,d\vec{\sigma}(\zeta)\,w(\zeta) - \frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,\overline{d\vec{\sigma}(\zeta)}\,\partial w(\zeta)\\ &+\sum_{\mu=1}^{k-1}\left\{\frac{1}{\omega_m}\int_{\partial D}\Phi_\mu(z,\tilde{\zeta})\,d\vec{\sigma}(\tilde{\zeta})\,\Delta^\mu w(\tilde{\zeta}) - \frac{1}{\omega_m}\int_{\partial D}\Psi_\mu(z,\tilde{\zeta})\,\overline{d\vec{\sigma}(\tilde{\zeta})}\,\partial\Delta^\mu w(\tilde{\zeta})\right\}\\ &+\frac{1}{\omega_m}\int_D\Psi_{k-1}(z,\tilde{\zeta})\,\Delta^k w(\tilde{\zeta})\,dv(\tilde{\zeta})\end{aligned}$$

with

$$\Phi_{\mu+1}(z,\tilde{\zeta}) := \frac{1}{\omega_m}\int_D\frac{|\zeta - z|^{2-m}}{2-m}\,\frac{(\overline{\tilde{\zeta}-\zeta})\,|\tilde{\zeta}-\zeta|^{2\mu-m}}{2^\mu\mu!\prod_{\nu=1}^{\mu}(2\nu-m)}.dv(\zeta)$$
$$(0 \le \mu \le k-2)$$
$$\Psi_\mu(z,\tilde{\zeta}) := \frac{1}{\omega_m}\int_D\frac{|\zeta - z|^{2-m}}{2-m}\,\frac{|\tilde{\zeta}-\zeta|^{2\mu-m}}{2^{\mu-1}(\mu-1)!\prod_{\nu=1}^{\mu}(2\nu-m)}\,dv(\zeta)$$
$$(1 \le \mu \le k-1)$$

satisfying $\Psi_\mu(z,\tilde{\zeta})\,\overline{\partial_{\tilde{\zeta}}} = \Phi_\mu(z,\tilde{\zeta})$ for $1 \le \mu \le k-1$. Formula (3.1) shows for $\mu \in \mathbb{N}_0$

$$\frac{|\tilde{\zeta} - z|^{2(\mu+1)-m}}{2^\mu\mu!\prod_{\nu=1}^{\mu+1}(2\nu-n)} = \tilde{\Psi}_\mu(z,\tilde{\zeta}) + \tilde{\Phi}_\mu(z,\tilde{\zeta}) + \Psi_\mu(z,\tilde{\zeta})$$

with

$$\tilde{\Psi}_\mu(z,\zeta) := \frac{1}{\omega_m}\int_{\partial D}\frac{\overline{\zeta - z}}{|\zeta - z|^m}\,d\vec{\sigma}(\zeta)\,\frac{|\tilde{\zeta}-\zeta|^{2(\mu+1)-m}}{2^\mu\mu!\prod_{\nu=1}^{\mu+1}(2\nu-m)}$$
$$\tilde{\Phi}_\mu(z,\zeta) := \frac{1}{\omega_m}\int_{\partial D}\frac{|\zeta - z|^{2-m}}{2-m}\,\overline{d\vec{\sigma}(\zeta)}\,\frac{(\overline{\tilde{\zeta}-\zeta})|\tilde{\zeta}-\zeta|^{2\mu-m}}{2^\mu\mu!\prod_{\nu=1}^{\mu}(2\nu-m)}.$$

By differentiation for $1 \le \mu \le k-1$

$$\frac{(\overline{\tilde{\zeta}-z})\,|\tilde{\zeta}-z|^{2\mu-m}}{2^\mu\mu!\prod_{\nu=1}^{\mu}(2\nu-m)} = \tilde{\Psi}_\mu(z,\tilde{\zeta})\overline{\partial_{\tilde{\zeta}}} + \tilde{\Phi}_\mu(z,\tilde{\zeta})\,\overline{\partial_{\tilde{\zeta}}} + \Phi_\mu(z,\tilde{\zeta})$$

follows. Thus

$$\begin{aligned} w(z) = \; & \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\vec{\sigma}(\zeta)\, w(\zeta) \\ & - \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2-m}}{2-m} \overline{d\vec{\sigma}(\zeta)}\, \partial w(\zeta) \\ & + \sum_{\mu=1}^{k-1} \frac{1}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})\, |\zeta - z|^{2\mu - m}}{2^\mu \mu! \prod_{\nu=1}^{\mu} (2\nu - m)} d\vec{\sigma}(\zeta)\, \Delta^\mu w(\zeta) \\ & - \sum_{\mu=1}^{k-1} \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2(\mu+1)-m}}{2^\mu \mu! \prod_{\nu=1}^{\mu+1} (2\nu - m)} \overline{d\vec{\sigma}(\zeta)}\, \partial \Delta^\mu w(\zeta) \\ & + \frac{1}{\omega_m} \int_D \frac{|\zeta - z|^{2k-m}}{2^{k-1}(k-1)! \prod_{\nu=1}^{k} (2\nu - m)} \Delta^k w(\zeta)\, dv(\zeta), \end{aligned}$$

i.e. (3.2) follows observing

$$\begin{aligned} & \sum_{\mu=1}^{k-1} \Big\{ \int_{\partial D} [(\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta)\, \overline{\partial_\zeta}\,]\, d\vec{\sigma}(\zeta)\, \Delta^\mu w(\zeta) \\ & \qquad - \int_{\partial D} (\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta)\, \overline{d\vec{\sigma}(\zeta)}\, \partial \Delta^\mu w(\zeta) \Big\} \\ & \qquad + \int_D (\widetilde{\Phi}_{k-1} + \widetilde{\Psi}_{k-1})(z,\zeta)\, \Delta^k w(\zeta) dv(\zeta) \\ & \quad = \sum_{\mu=1}^{k-1} \int_D \Big\{ \Big((\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta)\, \Delta_\zeta \Big) \Delta^\mu w(\zeta) \\ & \qquad + \Big((\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta) \overline{\partial_\zeta} \Big) \partial \Delta^\mu w(\zeta) \\ & \qquad - \Big((\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta) \overline{\partial_\zeta} \Big) \partial \Delta^\mu w(\zeta) \\ & \qquad - (\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta) \Delta^{\mu+1} w(\zeta) \Big\} dv(\zeta) \\ & \qquad + \int_D (\widetilde{\Phi}_{k-1} + \Psi_{k-1})(z,\zeta)\, \Delta^k w(\zeta) dv(\zeta) \\ & \quad = \int_D (\widetilde{\Phi}_0 + \widetilde{\Psi}_0)(z,\zeta)\, w(\zeta)\, dv(\zeta) \\ & \quad = 0. \end{aligned}$$

Here

$$(\widetilde{\Phi}_\mu + \widetilde{\Psi}_\mu)(z,\zeta)\, \Delta_\zeta = (\widetilde{\Phi}_{\mu-1} + \widetilde{\Psi}_{\mu-1})(z,\zeta)$$

and

$$
\begin{aligned}
(\widetilde{\Phi}_0 + \widetilde{\Psi}_0)(z,\widetilde{\zeta})
&= \frac{1}{\omega_m}\int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} d\bar{\sigma}(\zeta)\,\frac{|\widetilde{\zeta}-\zeta|^{2-m}}{2-m} + \frac{1}{\omega_m}\int_{\partial D} \frac{|\zeta - z|^{2-m}}{2-m} \overline{d\bar{\sigma}(\zeta)}\,\frac{\overline{\widetilde{\zeta}-\zeta}}{|\widetilde{\zeta}-\zeta|^m} \\
&= 0
\end{aligned}
$$

is used and (1.1) applied ∎

4. General representation

In the same way as (2.4) and (3.2) are proved by iteration these two representations could be used to develop a general formula. Because this seems to be involved when done at once one prefers a step by step procedure. We are not able to give the general formula but progressing as indicated one can get any kind of representation desired.

On the other hand such a representation formula is not of too much of interest. Rather than representing a function w from $C^{2k+\ell}(D;\mathbb{C}_m)$ through its derivatives of the kind $\Delta^k\partial^\ell w$ in D and its lower order derivatives $\Delta^\kappa\partial^\lambda w$ $(0 \le \kappa < k, 0 \le \lambda < \ell)$ on ∂D one could just use $\Delta^{k+\frac{\ell}{2}} w$ in D and $\Delta^\kappa w$ $(0 \le \kappa < k + \frac{\ell}{2})$ on ∂D for ℓ even and $\Delta^{k+\frac{\ell-1}{2}}\partial w$ in D and $\Delta^\kappa w$, $\Delta^\kappa \partial w$ $(0 \le \kappa < k + \frac{\ell-1}{2})$ on ∂D for ℓ odd, respectively.

Consider the area integral

$$
I_k(z) = \frac{1}{\omega_m}\int_D \frac{|\zeta - z|^{2k-m}}{2^{k-1}(k-1)!\prod_{\nu=1}^{k}(2\nu - m)} \Delta^k w(\zeta)\, dv(\zeta)
$$

in (3.2) and assume $w \in C^{2k+1}(D;\mathbb{C}_m)$. By (1.2) then

$$
\begin{aligned}
I_k(z) = {} & \frac{1}{\omega_m}\int_{\partial D} \frac{(\overline{\zeta - z})\,|\zeta - z|^{2k-m}}{2^k k!\prod_{\nu=1}^{k}(2\nu - m)}\, d\bar{\sigma}(\zeta)\,\Delta^k w(\zeta) \\
& - \frac{1}{\omega_m}\int_D \frac{(\overline{\zeta - z})\,|\zeta - z|^{2k-m}}{2^k k!\prod_{\nu=1}^{k}(2\nu - m)} \partial\Delta^k w(\zeta)\, dv(\zeta)
\end{aligned}
$$

follows when as usual first the domain $D_\varepsilon = \{\zeta \in D : \varepsilon < |\zeta - z|\}$ is considered. Thus

$$
\begin{aligned}
w(z) = {} & \sum_{\mu=0}^{k} \frac{1}{\omega_m}\int_{\partial D} \frac{(\overline{\zeta - z})\,|\zeta - z|^{2\mu-m}}{2^\mu \mu!\prod_{\nu=1}^{\mu}(2\nu - m)}\, d\bar{\sigma}(\zeta)\,\Delta^\mu w(\zeta) \\
& - \sum_{\mu=1}^{k} \frac{1}{\omega_m}\int_{\partial D} \frac{|\zeta - z|^{2\mu-m}}{2^{\mu-1}(\mu-1)!\prod_{\nu=1}^{\mu}(2\nu - m)}\, \overline{d\bar{\sigma}(\zeta)}\,\partial\Delta^{\mu-1} w(\zeta) \qquad (4.1)\\
& - \frac{1}{\omega_m}\int_D \frac{(\overline{\zeta - z})\,|\zeta - z|^{2k-m}}{2^k k!\prod_{\nu=1}^{k}(2\nu - m)}\, \partial\Delta^k w(\zeta)\, dv(\zeta).
\end{aligned}
$$

Denote the last area integral by $I_{k1}(z)$. Then by (1.2) for $w \in C^{2(k+1)}(D;\mathbb{C}_m) \cap C^{2k}(\overline{D};\mathbb{C}_m)$

$$\begin{aligned} I_{k1}(z) = \frac{1}{\omega_m}\int_{\partial D}\Bigg[&\frac{(\overline{\zeta - z})^2|\zeta - z|^{2k-m}}{2^{k+1}(k+1)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{|z|^{2(k+1)-m}}{2^{k+1}(k+1)!\prod_{\nu=2}^{k+1}(2\nu - m)}\Bigg] d\vec{\sigma}(\zeta)\,\partial\Delta^k w(\zeta) \\ - \frac{1}{\omega_m}\int_{D}\Bigg[&\frac{(\overline{\zeta - z})^2|\zeta - z|^{2k-m}}{2^{k+1}(k+1)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{|z|^{2(k+1)-m}}{2^{k+1}(k+1)!\prod_{\nu=2}^{k+1}(2\nu - m)}\Bigg] \partial^2\Delta^k w(\zeta)\,dv(\zeta). \end{aligned}$$

This area integral $I_{k2}(z)$ is for $w \in C^{2(k+1)+1}(D;\mathbb{C}_m) \cap C^{2(k+1)}(\overline{D};\mathbb{C}_m)$

$$\begin{aligned} I_{k2}(z) = \frac{1}{\omega_m}\int_{\partial D}\Bigg[&\frac{(\overline{\zeta - z})^3|\zeta - z|^{2k-m}}{2^{k+2}(k+2)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{[2(\overline{\zeta - z}) + (\zeta + z)]\,|\zeta - z|^{2(k+1)-m}}{2^{k+2}(k+2)!\prod_{\nu=2}^{k+1}(2\nu - m)}\Bigg] d\vec{\sigma}(\zeta)\,\partial^2\Delta^k w(\zeta) \\ - \frac{1}{\omega_m}\int_{D}\Bigg[&\frac{(\overline{\zeta - z})^3|\zeta - z|^{2k-m}}{2^{k+2}(k+2)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{[2(\overline{\zeta - z}) + (\zeta - z)]\,|\zeta - z|^{2(k+1)-m}}{2^{k+2}(k+2)!\prod_{\nu=2}^{k+1}(2\nu - m)}\Bigg] \partial^3\Delta^k w(\zeta)\,dv(\zeta). \end{aligned}$$

Assuming $w \in C^{2(k+2)}(D;\mathbb{C}_m) \cap C^{2(k+1)+1}(\overline{D};\mathbb{C}_m)$ the area integral $I_{k3}(z)$ here is

$$\begin{aligned} I_{k3}(z) = \frac{1}{\omega_m}\int_{\partial D}\Bigg[&\frac{(\overline{\zeta - z})^4|\zeta - z|^{2k-m}}{2^{k+3}(k+3)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{[3(\overline{\zeta - z})^2 + 2|\zeta - z|^2 + (\zeta - z)^2]\,|\zeta - z|^{2(k+1)-m}}{2^{k+3}(k+3)!\prod_{\nu=2}^{k+1}(2\nu - m)} \\ &+ \frac{|\zeta - z|^{2(k+2)-m}}{2^{k+3}(k+3)!\prod_{\nu=3}^{k+2}(2\nu - m)}\Bigg] d\vec{\sigma}(\zeta)\,\partial^3\Delta^k w(\zeta) \\ - \frac{1}{\omega_m}\int_{D}\Bigg[&\frac{(\overline{\zeta - z})^4|\zeta - z|^{2k-m}}{2^{k+3}(k+3)!\prod_{\nu=1}^{k}(2\nu - m)} \\ &+ \frac{[3(\overline{\zeta - z})^2 + 2|\zeta - z|^2 + (\zeta - z)^2]\,|\zeta - z|^{2(k+1)-m}}{2^{k+3}(k+3)!\prod_{\nu=2}^{k+1}(2\nu - m)} \\ &+ \frac{|\zeta - z|^{2(k+2)-m}}{2^{k+3}(k+3)!\prod_{\nu=3}^{k+2}(2\nu - m)}\Bigg] \partial^4\Delta^k w(\zeta)\,dv(\zeta). \end{aligned}$$

For proving these formulas Lemma 1 and Corollary 1 are useful.

A dual formula to (4.1) as analogously one to (2.4) can be given where ∂ is replaced by $\overline{\partial}$ and $\zeta - z$ by $\overline{\zeta - z}$.

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