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Duality for Optimal Control-Approximation Problems with Gauges

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Dedicated to L. von Wolfersdorf on his 65th Birthday

Abstract. Looking for m state variables and n control variables such that the sum of the distance functions between the state variables and the control variables becomes minimal is called control-approximation problem. This problem is investigated under constraints. Moreover, the distances between the control variables themselves are taken into account. Powers of several gauges are chosen as distance functions. The considerations happen in Hausdorff locally convex topological real vector spaces.
In particular, location problems of very general type (e.g. so-called multifacility problems) turn out to be special cases of such control-approximation problems.
After the formulation of the primal control-approximation problem some considerations concerning gauges follow. Then a dual problem is given and weak and strong duality assertions are obtained.

Keywords: *Control-approximation problems, location problems, gauges, duality*

AMS subject classification: Primary 49 N 15, 90 B 85, secondary 41 A 65, 90 C 25

1. Introduction

In this paper convex programming problems of the type of so-called control-approximation problems with respect to Hausdorff locally convex topological real vector spaces and with several control and state variables are considered. So m state variables $a_1, \ldots, a_m$ and n control variables $x_1, \ldots, x_n$ will be considered.

The distances between the control and the state variables are measured as typical for control-approximation problems. Here additionally, distances between the control variables themselves are included into the objective function which represents in general a function of these distances and has to be minimized. Location problems (cf. [13]) can be considered as special cases of such problems. In this case the state variables are substituted by fixed location points and the control variables are the wanted location points.

In Section 2 a general control-approximation problem will be formulated. The occurring distances between the images of the control and the state variables are measured by powers of so-called gauges (cf. Section 3 concerning the introduction of gauges). Norms are special symmetric gauges, but gauges open the possibility to consider non-symmetric

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distance measures as it is of interest also from a practical point of view (cf. [15]). Considering gauges instead of norms recently plays an increasing role in approximation and location theory (cf. [3, 4, 14, 15]). Different from location problems, which from their practical background are to be considered in finite (mostly even in two) dimensional spaces, it is reasonable for approximation and control-approximation problems to permit general and in particular infinite dimensional spaces. For example, the control variables (and also the state variables) occurring in boundary control problems for partial differential equations are elements of general function spaces. Moreover, in order to approximate elements in spaces (also in function spaces) which are not equipped with a norm topology it is opportune to exploit seminorms or gauges for the approximation or for the optimal control of the control-approximation process. This is one reason to examine the control-approximation problems in this paper for general Hausdorff locally convex spaces.

Approximation in general spaces has been treated for a long time. The first articles have handled this topic in Hilbert spaces. Approximation in topological vector spaces is studied in [2] and approximation in locally convex topological vector spaces in [9]. For the control-approximation problem considered in this paper the control operators are general linear and continuous operators. Certain boundary control problems for partial differential equations may be considered as special cases of the investigated control-approximation problem. Such problems are analyzed for elliptic differential equations in [6, 7, 21, 25, 27] and for parabolic differential equations in [26]. As concrete realizations of the control operators (cf. operators S_{ji} in the problem formulation (P) in Section 2) turn out in these cases the so-called Green operators mapping the boundary controls into the solution of the boundary value problems. Control-approximation problems in complex normed spaces with linear and continuous operators in the objective functions and in the constraints are explored in [23]. Multiobjective approximation problems in general spaces are considered in [1].

Duality statements for location problems as special cases of the considered control-approximation problem are treated in a variety of papers. It has been given a dual problem for the classical Fermat-Weber problem in [12]. The articles [8, 11] deal with duality for multifacility location problems. Duality for generalized location problems in reflexive Banach spaces with norms as distance functions and with constraints is considered in [19]. For an overview of conjugate duality in location together with geometric programming duality is refered to [16] where a lot of further relevant references concerning location duality can be found. Vector duality for multiobjective control-approximation problems with norms as distance measures has been investigated e.g. in [18, 20, 22, 24]. Duality statements for multiobjective location problems in reflexive Banach spaces with constraints and with gauges as distance functions are investigated in [17]. The scalar location problem as a special case of the considered scalar control-approximation problem without powers of gauges and without considerations of the distances between new location points among themselves is also a special case of the problem treated in [17] for reflexive Banach spaces.

The purpose of this paper is to establish some duality assertions to the general control-approximation problem formulated in Section 2. So in Section 4 a dual problem is derived by means of the Fenchel-Rockafellar concept of conjugate duality. Before that, in Section 3 some basic facts about gauges are presented. Section 5 is devoted to weak and strong duality assertions as well as optimality conditions followed in Section 6 by some concluding remarks.

At the end of the Introduction a few remarks will be made about symbols, notations and definitions.

The set of real numbers is abbreviated by $\mathbb{R}$, and the set of non-negative numbers is denoted by $\mathbb{R}_+$. The extended set of real numbers is $\overline{\mathbb{R}}$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. If P is a linear and continuous operator mapping a real topological vector space G into a real topological vector space H, then it is denoted by $P \in L(G \to H)$. The space H' is the algebraic dual space to H, i.e. the space of linear functionals on H. And the space H^* is the topological dual space to H, i.e. the space of linear and continuous functionals on H. Thus $H^* \subseteq H'$. A cone K in a real vector space H is a subset from H, $K \subseteq H$, with the property $\beta k \in K$ for all $k \in K$ and for all $\beta \in \mathbb{R}_+$. For any convex cone K a partial ordering is defined by $x \preceq y$ if $x \leq_K y$, i.e. $y - x \in K$. The dual cone K^* to a cone K is defined by $K^* = \{h^* \in H^* | \langle h^*, h\rangle \geq 0 \text{ for all } h \in K\}$. A subset N of a real vector space H is called *absorbent* if for each $h \in H$ there exists an α such that $[0,\alpha] \cdot h \subseteq N$. If for a subset N of a real vector space H it is valid $\lambda N \subseteq N$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$, then the set N is named *circled.*

2. The control-approximation problem

It is given a certain number of different Hausdorff locally convex topological real vector spaces V_j, X_j, Y_i and Z_{ij} $(i = 1,\dots,m;\ j = 1,\dots,n)$. In each space Y_i an element a_i which can be interpreted as a state variable is considered. In each space X_j an element x_j which can be interpreted as a control variable is searched. Each pair (a_i, x_j) and each pair (x_l, x_j) $(1 \leq l < j \leq n)$ is associated with a distance by means of a corresponding distance function. It is looked for the infimum of the objective function which consists of different distances under certain constraints. So the primal scalar control-approximation problem is given by

$$\inf_{(x,a,v)\in M} S(x,a) \tag{P}$$

with

$$S(x,a) = \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\lambda_{ij}^{\alpha_{ij}} \left[\gamma_{ij}(S_{ji}x_j - a_i)\right]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\right)$$
$$+ \sum_{l=1}^{n-1}\sum_{j=l+1}^{n} \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}} \left[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)\right]^{\tilde{\alpha}_{lj}}.$$

The set M is defined by

$$M = \left\{ (x,a,v) \in (X,Y,V) \;\middle|\; \begin{array}{l} a_i \in W_i,\ x_j \geq_{K_{X_j}} 0,\ v_j \geq_{K_{V_j}} 0 \\ A_{ij}a_i + B_{ij}x_j + C_{ij}v_j + f_{ij} \leq_{K_{Z_{ij}}} 0 \\ (i = 1,\dots,m;\ j = 1,\dots,n) \end{array} \right\}.$$

The following holds:

- $x = (x_1, \ldots, x_n)^{\mathrm{T}}, a = (a_1, \ldots, a_m)^{\mathrm{T}}, v = (v_1, \ldots, v_n)^{\mathrm{T}}$.
- $(X, Y, V) = (X_1, \ldots, X_n; Y_1, \ldots, Y_m; V_1, \ldots, V_n)$.
- γ_{ij} and $\tilde{\gamma}_{lj}$ are gauges in the spaces Y_i and X_j, respectively.
- $\lambda_{ij} \geq 0$ and $\tilde{\lambda}_{lj} \geq 0$.
- $\alpha_{ij} \geq 1$ and $\tilde{\alpha}_{lj} \geq 1$.
- $S_{ji} \in L(X_j \to Y_i)$ and $T_{lj} \in L(X_l \to X_j)$ are linear bounded operators mapping between the indicated spaces.
- $l_{ij}{}^* \in X_j{}^*$ and $\langle \cdot, \cdot \rangle : X_j^* \times X_j \to \mathbb{R}$ is a bilinear form (duality pairing between X_j^* and X_j).
- $W_i \subseteq Y_i$ is closed and convex, $K_{X_j} \subseteq X_j$, $K_{V_j} \subseteq V_j$ and $K_{Z_{ij}} \subseteq Z_{ij}$ are closed and convex cones.
- $A_{ij} \in L(Y_i \to Z_{ij})$, $B_{ij} \in L(X_j \to Z_{ij})$, $C_{ij} \in L(V_j \to Z_{ij})$, $f_{ij} \in Z_{ij}$.

Now, it is easy to see that the following multifacility location problem turns out to be a special case of the stated general control-approximation problem. In particular, the gauges are replaced by norms and the linear operators are the identity operators. Further, the elements a_i are fixed and for the sake of simplicity the constraints are removed:

$$\inf_{x_1, \ldots, x_n \in X} \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \|x_j - a_i\| + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\lambda}_{lj} \|x_l - x_j\|$$

with $\lambda_{ij} > 0$ and $\tilde{\lambda}_{lj} > 0$.

3. Gauges

The distance functions γ_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ and $\tilde{\gamma}_{lj}$ $(1 \leq l < j \leq n)$ in problem (P) are different gauges. Now a few remarks to the introduction of gauges with using assertions from [10] follow.

Let H be a real vector space and G a non-empty subset of H, $G \subseteq H$. The functional γ_G: $H \longrightarrow \mathbb{R}_+$ with

$$\gamma_G(h) = \begin{cases} \infty & \text{for } \{\lambda > 0 | h \in \lambda G\} = \emptyset \\ \inf\{\lambda > 0 | \, h \in \lambda G\} & \text{else} \end{cases}$$

is called *Minkowski functional* of the set G. Now define $G_E = [0,1] \cdot G$ (then $G = G_E$ if G is circled or G is convex with $0 \in G$) and assume G to be absorbent. Then the functional

$$\gamma_G(h) = \inf \left\{\lambda > 0 | \, h \in \lambda G_E\right\} \tag{3.1}$$

is well-defined, that means $\mathrm{dom}(\gamma_G) = H$.

Henceforth, the subset G is specified. The set G is assumed to be absorbent and convex, i.e. $G = G_E$. And the space H is even a topological real vector space for the rest of the section. Then, according to definition (3.1), for the functional γ_G

$$\gamma_G(h) \geq 0 \quad \text{for all } h \in H \tag{3.2}$$

$$\gamma_G(0) = 0 \tag{3.3}$$

$$\mu\gamma_G(h) = \gamma_G(\mu h) \quad \text{for all } \mu \in \mathbb{R}_+ \text{ and } h \in H \tag{3.4}$$

$$\gamma_G(h_1 + h_2) \leq \gamma_G(h_1) + \gamma_G(h_2) \text{ for all } h_1, h_2 \in H \tag{3.5}$$

holds. Here γ_G is said to be a *gauge*. If additionally the property

$$\gamma_G(\mu\, h) = |\mu|\, \gamma_G(h) \qquad \text{for all } \mu \in \mathbb{R} \text{ and } h \in H$$

is fulfilled, that happens if the set G is circled, then the gauge is named *seminorm*. Finally, the seminorm becomes a *norm* if $\gamma_G(h) = 0$ is sufficient for $h = 0$.

An example for a proper gauge γ_G that is a gauge which is not a norm and not a seminorm is given by the set $G = \{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 |\, x \in [-1,2] \text{ and } y \in \mathbb{R}\}$. The gauge γ_G in $H = \mathbb{R}^2$ is then defined by (3.1) with the help of G.

For the set G

$$\{h \in H |\, \gamma_G(h) < 1\} \subseteq G \subseteq \{h \in H |\, \gamma_G(h) \leq 1\} \tag{3.6}$$

holds. If the set G is even closed, then (3.6) becomes

$$G = \{h \in H |\, \gamma_G(h) \leq 1\}.$$

Now the dual gauge $\gamma_G{}^*$ to the gauge γ_G in the algebraic dual space H' is introduced by means of the polar set G° of the set G. With the bilinear form $\langle\cdot,\cdot\rangle : H' \times H \longrightarrow \mathbb{R}$ the definition of G° is given by

$$G^\circ = \left\{h^* \in H' \,\middle|\, \sup_{h \in G} \langle h^*, h\rangle \leq 1\right\}.$$

And the dual gauge is given by

$$\gamma_G^*(h^*) = \sup_{h \in G} \langle h^*, h\rangle. \tag{3.7}$$

The generalized Cauchy-Schwarz inequality holds in the following manner for gauges in the subspace H^* of H':

$$\langle h^*, h\rangle \leq \gamma_G(h) \cdot \gamma_G^*(h^*) \qquad \text{for all } h \in H \text{ and } h^* \in H^*. \tag{3.8}$$

So the gauges γ_G and γ_G^* which are dual to each other can also be given by

$$\gamma_G(h) = \sup_{h^* \in G^\circ} \langle h^*, h\rangle$$

$$\gamma_G^*(h^*) = \inf\{\lambda > 0 |\, h^* \in \lambda\, G^\circ\}.$$

The different gauges γ_{ij} in the space Y_i in problem (P) are defined by different absorbent, closed and convex sets $G_{ij} \subset Y_i$. The dual gauges $\gamma_{ij}{}^*$ are built with definition (3.7). The same assertions hold for the gauges $\tilde{\gamma}_{lj}$ $(1 \leq l < j \leq n)$ within the definition of problem (P).

4. The dual problem

Similar to the investigations in [19] and following the Fenchel-Rockafellar approach of duality by means of the perturbation of the given optimization problem a dual control-approximation problem can be assigned to problem (P). Thereby the derivation in [19] must be changed in the following way.

A perturbation function Φ is introduced by

$$\Phi(x,a,v,p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\lambda_{ij}^{\alpha_{ij}}\left[\gamma_{ij}(S_{ji}x_j - a_i + p_{ij})\right]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\right) + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}}\left[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j + \tilde{p}_{lj})\right]^{\tilde{\alpha}_{lj}}$$

for $A_{ij}a_i + B_{ij}x_j + C_{ij}v_j + f_{ij} \leq_{K_{Z_{ij}}} q_{ij}$ where $a_i \in W_i$, $x_j \geq_{K_{X_j}} 0$ and $v_j \geq_{K_{V_j}} 0$, and

$$\Phi(x,a,v,p,q) = \infty \qquad \text{else.}$$

Further,

$$N(p,q) = \inf_{(x,a,v)\in(X,Y,V)} \Phi(x,a,v,p,q),$$

where

$$p = \left(p_{11}, p_{12}, p_{13}, \ldots, p_{mn}, \tilde{p}_{12}, \tilde{p}_{13}, \ldots, \tilde{p}_{n-1,n}\right)$$
$$q = \left(q_{11}, q_{12}, \ldots, q_{mn}\right)$$

are the perturbation variables with $p_{ij} \in Y_i$, $q_{ij} \in Z_{ij}$ and $\tilde{p}_{lj} \in X_j$. The product space $([\hat{Y}, \tilde{X}], Z)$ is defined by

$$(p,q) \in ([\hat{Y}, \tilde{X}], Z).$$

Then $N(0,0) = \inf_{(x,a,v)\in M} S(x,a) = \inf(P)$ holds.

The dual problem (P^*) to the primal probem (P) is defined by (cf. [5])

$$\sup_{(p^*,q^*)\in([\hat{Y}^*,\tilde{X}^*],Z^*)} [-\Phi^*(0,0,0,p^*,q^*)]$$

where the conjugate function to

$$\Phi : (x,a,v,p,q) \in (X,Y,V,[\hat{Y},\tilde{X}],Z) \longmapsto \Phi(x,a,v,p,q) \in \overline{\mathbb{R}}$$

is denoted by

$$\Phi^* : (x^*,a^*,v^*,p^*,q^*) \in (X^*,Y^*,V^*,[\hat{Y}^*,\tilde{X}^*],Z^*) \longmapsto \Phi^*(x^*,a^*,v^*,p^*,q^*) \in \overline{\mathbb{R}}$$

and is defined by

$$\begin{aligned}
\Phi^*&(x^*,a^*,v^*,p^*,q^*) \\
&= \sup_{\substack{(x,a,v,p,q)\in\\(X,Y,V,[\hat{Y},\tilde{X}],Z)}} \Bigg\{ \sum_{j=1}^{n}\left(\langle x_j^*, x_j\rangle + \langle v_j^*, v_j\rangle\right) + \sum_{i=1}^{m}\langle a_i^*, a_i\rangle \\
&\quad + \sum_{l=1}^{m}\sum_{j=1}^{n}\left(\langle p_{ij}^*, p_{ij}\rangle + \langle q_{ij}^*, q_{ij}\rangle\right) + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\langle\tilde{p}_{lj}^*, \tilde{p}_{lj}\rangle - \Phi(x,a,v,p,q)\Bigg\}.
\end{aligned}$$

The result of the calculation of Φ^* (cf. [19] for analogous calculations with norms as distances) yields the dual problem

$$\sup_{(p^*,q^*)\in M^*} R(p^*,q^*) \qquad (P^*)$$

with

$$R(p^*,q^*) = \sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n}(1-\alpha_{ij})[\gamma_{ij}^*(p_{ij}^*)]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \sum_{\substack{l=1\\ \tilde{\alpha}_{lj}>1}}^{n-1}\sum_{j=l+1}^{n}(1-\tilde{\alpha}_{lj})[\tilde{\gamma}_{lj}^*(\tilde{p}_{lj}^*)]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}$$
$$-\sum_{i=1}^{m}\sup_{a_i\in W_i}\left\langle \sum_{j=1}^{n}\left(\alpha_{ij}\lambda_{ij}p_{ij}^* + A_{ij}^*q_{ij}^*\right), a_i\right\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle q_{ij}^*, f_{ij}\rangle$$

and

$$M^* = \left\{ (p^*,q^*)\in([\hat{Y}^*,\tilde{X}^*],Z^*) \;\middle|\; \begin{array}{l} \gamma_{ij}^*(p_{ij}^*)\le 1 \text{ for } \alpha_{ij}=1 \\ \tilde{\gamma}_{lj}^*(\tilde{p}_{lj}^*)\le 1 \text{ for } \tilde{\alpha}_{lj}=1 \\ q_{ij}^* \le_{K_{Z_{ij}}^*} 0, \quad \sum_{i=1}^{m} C_{ij}^*q_{ij}^* \le_{K_{V_j}^*} 0 \\ \sum_{i=1}^{m}(B_{ij}^*q_{ij}^* - \alpha_{ij}\lambda_{ij}S_{ji}^*p_{ij}^* - l_{ij}^*) \quad + \\ \sum_{l=1}^{j-1}\tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\tilde{p}_{lj}^* - \sum_{l=j+1}^{n}\tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}^*\tilde{p}_{jl}^* \le_{K_{X_j}^*} 0 \end{array} \right\}.$$

The following notations hold:

- $(p^*,q^*) = \left(p_{11}^*,p_{12}^*,p_{13}^*,\ldots,p_{mn}^*,\tilde{p}_{12}^*,\tilde{p}_{13}^*,\ldots,\tilde{p}_{n-1,n}^*,q_{11}^*,q_{12}^*,\ldots,q_{mn}^*\right)^{\mathrm{T}}$, $p_{ij}^*\in Y_i^*$, $q_{ij}^*\in Z_{ij}^*$, $\tilde{p}_{lj}^*\in X_j^*$.
- The dual product space to $([\hat{Y},\tilde{X}],Z)$ is defined by

$$([\hat{Y}^*,\tilde{X}^*],Z^*) = \Big(\underbrace{Y_1^*,\ldots,Y_1^*}_{n\times},\ldots,\underbrace{Y_i^*,\ldots,Y_i^*}_{n\times},\ldots,\underbrace{Y_m^*,\ldots,Y_m^*}_{n\times},$$
$$X_2^*,X_3^*,\ldots,X_n^*,X_3^*,X_4^*,\ldots,X_n^*,\ldots,X_{n-1}^*,X_n^*,X_n^*,$$
$$Z_{11}^*,Z_{12}^*,\ldots,Z_{1n}^*,Z_{21}^*,Z_{22}^*,\ldots,Z_{2n}^*,\ldots,Z_{m1}^*,\ldots,Z_{mn}^*\Big)$$

  where Y_i^*, X_j^*, Z_{ij}^* are the topological dual spaces to Y_i, X_j, Z_{ij}.
- γ_{ij}^* and $\tilde{\gamma}_{lj}^*$ are the dual gauges to γ_{ij} and $\tilde{\gamma}_{lj}$ in the spaces Y_i^* and X_j^*, respectively; at that γ_{ij}^* and $\tilde{\gamma}_{lj}^*$ are defined by (3.7), depending on the definition of γ_{ij} and $\tilde{\gamma}_{lj}$, respectively.

- $S_{ji}^* \in L(Y_i^* \to X_j^*)$ and $T_{jl}^* \in L(X_l^* \to X_j^*)$ are the adjoint operators to S_{ji} and T_{jl}, respectively.
- $K_{X_j}^* \subset X_j^*$, $K_{V_j}^* \subset V_j^*$ and $K_{Z_{ij}}^* \subset Z_{ij}^*$ are the dual cones to K_{X_j}, K_{V_j} and $K_{Z_{ij}}$, respectively.
- $A_{ij}^* \in L(Z_{ij}^* \to Y_i^*)$, $B_{ij}^* \in L(Z_{ij}^* \to X_j^*)$ and $C_{ij}^* \in L(Z_{ij}^* \to V_j^*)$ are the adjoint operators to A_{ij}, B_{ij} and C_{ij}, respectively.

The other occurring symbols were explained after the definition of M.

5. Duality assertions

This section is devoted to duality assertions. At first, a weak duality assertion is given. Although the weak duality turns out to be a consequence of the considered conjugate duality approach to construct the dual problem, it seems to be interesting to find a more direct proof which does not make use of conjugate functions and of the general results of conjugate duality by means of the perturbation approach. Such a direct derivation of the weak duality is given in the following considerations. Moreover, this direct proof is shorter than the extensive evaluations necessary for the construction of the dual problem by means of perturbation and which therefore have been omitted above. Further, this direct proof gives a close and clear connection between problem (P) and its dual problem (P^*) by means of a chain of estimations in order to come from the primal objective function to the dual objective function.

Theorem 1. *For the objective functions S and R and the constraint sets M and M^* from problems (P) and (P^*)*

$$S(x,a) \geq R(p^*,q^*) \qquad \text{for all } (x,a,v) \in M \text{ and } (p^*,q^*) \in M^* \tag{5.1}$$

holds.

Proof. Let be given $(x,a,v) \in M$ and $(p^*,q^*) \in M^*$. Because of the statement of problem (P),

$$\begin{aligned} S(x,a) = &\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\lambda_{ij}^{\alpha_{ij}}\left[\gamma_{ij}(S_{ji}x_j - a_i)\right]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\Big) \\ &+\sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}}\left[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)\right]^{\tilde{\alpha}_{lj}} \end{aligned} \tag{5.2}$$

is valid. For $a, b \in \mathbb{R}_+$ the Young inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ $(\frac{1}{p} + \frac{1}{q} = 1)$ is fulfilled. If $\alpha_{ij} > 1$, then it is appointed $p := \alpha_{ij}$, $a := \lambda_{ij}\,\gamma_{ij}(S_{ji}x_j - a_i)$ and $b := \gamma_{ij}^*(p_{ij}^*)$ and with inequality (3.8)

$$\alpha_{ij}\,\lambda_{ij}\,\langle p_{ij}^*, S_{ji}x_j - a_i\rangle + (1-\alpha_{ij})\left[\gamma_{ij}^*(p_{ij}^*)\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} \leq \lambda_{ij}^{\alpha_{ij}}\left[\gamma_{ij}(S_{ji}x_j - a_i)\right]^{\alpha_{ij}} \tag{5.3}$$

follows. If $\alpha_{ij} = 1$, then from inequality (3.8) and because of $(p^*, q^*) \in M^*$, i.e. $\gamma_{ij}^*(p_{ij}^*) \leq 1$, the inequality

$$\lambda_{ij}\langle p_{ij}^*, S_{ji}x_j - a_i\rangle \leq \lambda_{ij}\gamma_{ij}(S_{ji}x_j - a_i) \tag{5.4}$$

follows. For the gauge $\tilde{\gamma}_{lj}$ and its dual gauge $\tilde{\gamma}_{lj}^*$ there are analogous inequalities as (5.3) and (5.4). They can be produced also by using the Young inequality and inequality (3.8).

Inequalities (5.3) and (5.4) allow to estimate the primal objective function $S(x, a)$ as

$$\begin{aligned} S(x,a) \geq & \sum_{\substack{i=1 \\ \alpha_{ij}>1}}^{m} \sum_{j=1}^{n} (1-\alpha_{ij})[\gamma_{ij}^*(p_{ij}^*)]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} \\ & + \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\alpha_{ij}\lambda_{ij}\langle p_{ij}^*, S_{ji}x_j - a_i\rangle + \langle l_{ij}^*, x_j\rangle\Big) \\ & + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\langle \tilde{p}_{lj}^*, T_{lj}x_l - x_j\rangle \\ & + \sum_{\substack{l=1 \\ \tilde{\alpha}_{lj}>1}}^{n-1} \sum_{j=l+1}^{n} (1-\tilde{\alpha}_{lj})[\tilde{\gamma}_{lj}^*(\tilde{p}_{lj}^*)]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}. \end{aligned} \tag{5.5}$$

The assumptions $(x, a, v) \in M$ and $(p^*, q^*) \in M^*$ induce the inequalities

$$\big\langle q_{ij}^*, A_{ij}a_i + B_{ij}x_j + C_{ij}v_j + f_{ij}\big\rangle \geq 0. \tag{5.6}$$

By means of some technical calculations the following identity can be pointed out:

$$\sum_{j=1}^{n}\left(\left\langle \sum_{l=1}^{j-1} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\tilde{p}_{lj}^* - \sum_{l=j+1}^{n} \tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}^*\tilde{p}_{jl}^*, x_j\right\rangle\right) = \sum_{l=1}^{n-1}\sum_{j=l+1}^{n} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\langle \tilde{p}_{lj}^*, x_j - T_{lj}x_l\rangle.$$

Inequalities (5.5) and (5.6) and this identity imply

$$\begin{aligned} S(x,a) \geq & \sum_{\substack{i=1 \\ \alpha_{ij}>1}}^{m} \sum_{j=1}^{n} (1-\alpha_{ij})[\gamma_{ij}^*(p_{ij}^*)]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} \\ & - \sum_{i=1}^{m} \left\langle \sum_{j=1}^{n} A_{ij}^*q_{ij}^* + \alpha_{ij}\lambda_{ij}p_{ij}^*, a_i \right\rangle \\ & - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle q_{ij}^*, f_{ij}\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle C_{ij}^*q_{ij}^*, v_j\rangle \end{aligned}$$

$$+\sum_{j=1}^{n}\left\langle \sum_{i=1}^{m}(\alpha_{ij}\lambda_{ij}S_{ji}^{*}p_{ij}^{*}+l_{ij}^{*}-B_{ij}^{*}q_{ij}^{*}) + \sum_{l=1+j}^{n}\tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}^{*}\tilde{p}_{jl}^{*}-\sum_{l=1}^{j-1}\tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\tilde{p}_{lj}^{*},x_j\right\rangle + \sum_{l=1}^{n-1}\sum_{\substack{j=l+1\\ \tilde{\alpha}_{lj}>1}}^{n}(1-\tilde{\alpha}_{lj})[\tilde{\gamma}_{lj}^{*}(\tilde{p}_{lj}^{*})]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}. \tag{5.7}$$

Finally,

$$\left\langle \sum_{i=1}^{m} C_{ij}^{*}\, q_{ij}^{*}, v_j\right\rangle \le 0$$

and

$$\left\langle \sum_{i=1}^{m}(B_{ij}^{*}q_{ij}^{*}-\alpha_{ij}\lambda_{ij}S_{ji}^{*}p_{ij}^{*}-l_{ij}^{*})+\sum_{l=1}^{j-1}\tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\tilde{p}_{lj}^{*}-\sum_{l=j+1}^{n}\tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}^{*}\tilde{p}_{jl}^{*},x_j\right\rangle \le 0$$

holds because of $(x,a,v)\in M$ and $(p^*,q^*)\in M^*$. So inequality (5.7) becomes

$$\begin{aligned} S(x,a) \ge & \sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n}(1-\alpha_{ij})[\gamma_{ij}^{*}(p_{ij}^{*})]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \sum_{l=1}^{n-1}\sum_{\substack{j=l+1\\ \tilde{\alpha}_{lj}>1}}^{n}(1-\tilde{\alpha}_{lj})[\tilde{\gamma}_{lj}^{*}(\tilde{p}_{lj}^{*})]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}} \\ & -\sum_{i=1}^{m}\sup_{a_i\in W_i}\left\langle \sum_{j=1}^{n}(\alpha_{ij}\lambda_{ij}p_{ij}^{*}+A_{ij}^{*}q_{ij}^{*}),a_i\right\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle q_{ij}^{*},f_{ij}\rangle \\ = & R(p^*,q^*). \end{aligned}$$

The proof of Theorem 1 is completed ∎

The next theorem makes a strong duality assertion.

Theorem 2. *Let be* $-\infty<\inf(P)=\inf_{(x,a,v)\in M}S(x,a)<+\infty$ *and there exists an admissible element* $(\bar{x},\bar{a},\bar{v})\in M$ *with*

$$A_{ij}\,\bar{a}_i+B_{ij}\,\bar{x}_j+C_{ij}\,\bar{v}_j+f_{ij}\in \operatorname{int}(-K_{Z_{ij}}) \qquad \begin{pmatrix} i=1,\dots,m\\ j=1,\dots,n\end{pmatrix}. \tag{5.8}$$

Then there is a solution $(\mathring{p}^*,\mathring{q}^*)\in M^*$ *of the dual problem* (P^*) *satisfying the strong duality assertion*

$$\inf_{(x,a,v)\in M}S(x,a)=\max_{(p^*,q^*)\in M^*}R(p^*,q^*)=R(\mathring{p}^*,\mathring{q}^*). \tag{5.9}$$

Remark 1. Condition (5.8) is a regularity condition, the so-called Slater condition.

Proof of Theorem 2. According to the assumption inf(P) is finite and

$$\begin{aligned}\inf_{(x,a,v)\in M} S(x,a) &= \inf_{(x,a,v)\in M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\lambda_{ij}^{\alpha_{ij}}\left[\gamma_{ij}(S_{ji}x_j - a_i)\right]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\right)\right.\\ &\quad\left. + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}}\left[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)\right]^{\tilde{\alpha}_{lj}}\right\}\\ &= \inf(P)\end{aligned}$$

holds. If problem (P) is stable, then there is a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ of problem (P^*) according to the Fenchel-Rockafellar duality theory fulfilling (cf. [5])

$$\inf_{(x,a,v)\in M} S(x,a) = \max_{(p^*,q^*)\in M^*} R(p^*,q^*) = R(\mathring{p}^*, \mathring{q}^*).$$

Indeed, because of the Slater condition (5.8) the stability of problem (P) can be proved. For this the fulfilment of two criteria for the stability of problem (P) will be shown:

1. $-\infty < \inf_{(x,a,v)\in M} S(x,a) < +\infty$.
2. The subdifferential of the function N at the point $(p,q) = (0,0)$ is non-empty, $\partial N(0,0) \neq \emptyset$ (N is here the infimum function of the perturbation function Φ from Section 4, $N(p,q) = \inf_{(x,a,v)\in(X,Y,V)} \Phi(x,a,v,p,q)$).

The first condition is an assumption of Theorem 2. The second condition is a conclusion from the fact that the function N is convex and continuous in $(p,q) = (0,0)$. The convexity of N is easy to show because it is built by convex functions, constraints and perturbations. The continuity of N at $(p,q) = (0,0)$ is implied by the Slater condition (5.8). So problem (P) is stable and a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ exists satisfying

$$\begin{aligned}\inf_{(x,a,v)\in M} S(x,a) &= \inf_{(x,a,v)\in M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\lambda_{ij}^{\alpha_{ij}}\left[\gamma_{ij}(S_{ji}x_j - a_i)\right]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\right)\right.\\ &\quad\left. + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}}\left[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)\right]^{\tilde{\alpha}_{lj}}\right\}\\ &= \sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n}(1-\alpha_{ij})\left[\gamma_{ij}^*(\mathring{p}_{ij}^*)\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}}\\ &\quad - \sum_{\substack{l=1\\ \tilde{\alpha}_{lj}>1}}^{n-1}\sum_{j=l+1}^{n}(1-\tilde{\alpha}_{lj})\left[\tilde{\gamma}_{lj}^*(\mathring{\tilde{p}}_{lj}^*)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}\\ &\quad - \sum_{i=1}^{m}\sup_{a_i\in W_i}\left\langle \sum_{j=1}^{n}(\alpha_{ij}\lambda_{ij}\mathring{p}_{ij}^* + A_{ij}^*\mathring{q}_{ij}^*), a_i\right\rangle\\ &\quad - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle \mathring{q}_{ij}^*, f_{ij}\rangle\\ &= R(\mathring{p}^*, \mathring{q}^*).\end{aligned} \tag{5.10}$$

The proof of Theorem 2 is completed ∎

Remark 2. Let there exists a solution $(\mathring{x}, \mathring{a}, \mathring{v}) \in M$ of problem (P). Under the Slater condition (5.8) the strong duality assertion is fulfilled. So there is a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ for problem (P^*) and

$$S(\mathring{x}, \mathring{a}) = \min_{(x,a,v)\in M} S(x,a) = \max_{(p^*,q^*)\in M^*} R(p^*,q^*) = R(\mathring{p}^*, \mathring{q}^*). \tag{5.11}$$

With the Young inequality and inequality (3.8) the following necessary optimality conditions can be derived from (5.11) in a similar way as in [23]:

(i) $\left\langle \sum_{i=1}^{m} C_{ij}^* \mathring{q}_{ij}^*, \mathring{v}_j \right\rangle = 0.$

(ii) $\langle \mathring{q}_{ij}^*, A_{ij}\mathring{a}_i + B_{ij}\mathring{x}_j + C_{ij}\mathring{v}_j + f_{ij} \rangle = 0.$

(iii) $\langle \mathring{p}_{ij}^*, S_{ji}\mathring{x}_j - \mathring{a}_i \rangle = \begin{cases} \lambda_{ij}^{\alpha_{ij}-1} [\gamma_{ij}(S_{ji}\mathring{x}_j - \mathring{a}_i)]^{\alpha_{ij}} & \text{if } \alpha_{ij} > 1 \\ \gamma_{ij}(S_{ji}\mathring{x}_j - \mathring{a}_i) & \text{if } \alpha_{ij} = 1. \end{cases}$

(iv) $\gamma_{ij}^*(\mathring{p}_{ij}^*) = \begin{cases} \lambda_{ij}^{\alpha_{ij}-1} [\gamma_{ij}(S_{ji}\mathring{x}_j - \mathring{a}_i)]^{\alpha_{ij}-1} & \text{if } \alpha_{ij} > 1 \\ 1 & \text{if } \alpha_{ij} = 1. \end{cases}$

(v) $\langle \mathring{\tilde{p}}_{lj}^*, T_{lj}\mathring{x}_l - \mathring{x}_j \rangle = \begin{cases} \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}-1} [\tilde{\gamma}_{lj}(T_{lj}\mathring{x}_l - \mathring{x}_j)]^{\tilde{\alpha}_{lj}} & \text{if } \tilde{\alpha}_{lj} > 1 \\ \tilde{\gamma}_{lj}(T_{lj}\mathring{x}_l - \mathring{x}_j) & \text{if } \tilde{\alpha}_{lj} = 1. \end{cases}$

(vi) $\tilde{\gamma}_{lj}^*(\mathring{\tilde{p}}_{lj}^*) = \begin{cases} \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}-1} [\tilde{\gamma}_{lj}(T_{lj}\mathring{x}_l - \mathring{x}_j)]^{\tilde{\alpha}_{lj}-1} & \text{if } \tilde{\alpha}_{lj} > 1 \\ 1 & \text{if } \tilde{\alpha}_{lj} = 1. \end{cases}$

(vii) $\left\langle \left[\sum_{i=1}^{m} (B_{ij}^* \mathring{q}_{ij}^* - \alpha_{ij}\lambda_{ij} S_{ij}^* \mathring{p}_{ij}^* - l_{ij}^*) + \sum_{l=1}^{j-1} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj} \mathring{\tilde{p}}_{lj}^* - \sum_{l=j+1}^{n} \tilde{\alpha}_{jl}\tilde{\lambda}_{jl} T_{jl}^* \mathring{\tilde{p}}_{lj}^* \right], \mathring{x}_j \right\rangle = 0.$

(viii) $\sup_{a_i \in W_i} \left\langle \sum_{j=1}^{n} (\alpha_{ij}\lambda_{ij}\mathring{p}_{ij}^* + A_{ji}^* \mathring{q}_{ij}^*),\ a_i \right\rangle = \left\langle \sum_{j=1}^{n} (\alpha_{ij}\, \lambda_{ij}\, \mathring{p}_{ij}^* + A_{ji}^* \mathring{q}_{ij}^*), \mathring{a}_i \right\rangle.$

These necessary optimality conditions can be interpreted as a mixture and generalization of the classical Kolmogorov condition in best approximation theory and of the maximum principle in optimal control theory, and finally, of the complementary slackness conditions in linear programming.

6. Conclusions and summary

A primal control-approximation problem (P) was formulated. As distance functions powers of gauges were used. For problem (P) a dual problem (P^*) was established. By means of the Fenchel-Rockafellar theory of duality and former obtained results weak and strong duality assertions were derived. So the following results were deduced:

1. For all elements $(x, a, v) \in M$ and $(p^*, q^*) \in M^*$ the weak duality assertion (5.1) holds.

2. If $\inf(P)$ is finite and the Slater condition (5.8) is fulfilled, then there is a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ of the dual problem (P^*) such that the strong duality assertion (5.9) is satisfied. If the infimum of the objective function in problem (P) is attained, then the strong duality assertion (5.11) and the optimality conditions (i) - (viii) are fulfilled.

In a forthcoming paper we will apply the derived scalar duality results to the investigation of multiobjective control-approximation problems concerning vector duality.

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