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Some Surprising Results on a One-Dimensional Elliptic Boundary Value Blow-Up Problem

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Abstract. In this paper we consider the one-dimensional elliptic boundary blow-up problem

$$\left.\begin{aligned} &\Delta_p u = f(u) \quad (a < t < b) \\ &u(a) = u(b) = +\infty \end{aligned}\right\}$$

where $\Delta_p u = \left(|u'(t)|^{p-2}u'(t)\right)'$ is the usual p-Laplace operator. We show that the structure of the solutions can be very rich even for a simple function f which gives a leading that a simliar results might hold also in higher dimensional spaces

Keywords: *Boundary blow-up, multiplicity, concave and convex nonlinearity*
AMS subject classification: 34 B, 35 J

1. Introduction and formulation of main results

In the last few years there is a great of interests in the investigation of boundary blow-up solutions for elliptic equations [2, 3, 7], which comes originally from differential geometry [6] and electrohydrodynamics [5]. Very recently some existence results of two (one positive and one sign-changing) solutions have been established in [1, 8]. The purpose of the present paper is to show through one-dimensional examples that the structure of the solutions can be very rich even for a simple right-hand side. More precisely, we consider the problem

$$\left.\begin{aligned} &\Delta_p u = \lambda f(u) \quad (a < t < b) \\ &u(a) = u(b) = +\infty \end{aligned}\right\} \tag{1}$$

where

$$\Delta_p u = \left(|u'(t)|^{p-2}u'(t)\right)'$$

is the p-Laplace operator as usual, $\lambda > 0$ is a parameter, and f is a given continuous function. By a solution $u = u(t)$ of problem (1) we mean that u satisfies the equation in (1), i.e. $(|u'(t)|^{p-2}u'(t))' = \lambda f(u(t))$ for all $t \in (a,b)$, and $\lim_{t\to a+} u(t) = \lim_{t\to b-} u(t) = +\infty$. By a sign-changing solution $u(t)$ of (1) we mean that there exist $t_1, t_2 \in (a,b)$ such that $u(t_1) > 0$ and $u(t_2) < 0$.

The results for problem (1) in this paper are summarized in the followings three theorems.

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Theorem 1. *For given positive constants q, ε, s and α, r such that $q > p-1 > r > s$ let*

$$f(u) = u^q + \varepsilon u^s \quad (u \geq 0) \qquad \text{and} \qquad f(u) = \alpha |u|^r \quad (u \leq 0).$$

Then the following statements hold:

(i) *There exists a constant $\lambda_1 > 0$ such that problem (1) has at least one sign-changing solution if $\lambda > \lambda_1$, and no such kind of solutions can exist if $\lambda < \lambda_1$.*

(ii) *There exist a constant $\varepsilon_0 > 0$ such that the solution is unique if $\varepsilon \geq \varepsilon_0$.*

(iii) *If $\varepsilon \in (0, \varepsilon_0)$, then there are constants $\lambda_2 < \lambda_3 \leq \lambda_4$ such that problem (1) has at least two sign-changing solutions if $\lambda \in (\lambda_1, \lambda_4)$, has at least three sign-changing solutions if $\lambda \in (\lambda_2, \lambda_3))$, and has a unique sign-changing solution if $\lambda > \lambda_4$.*

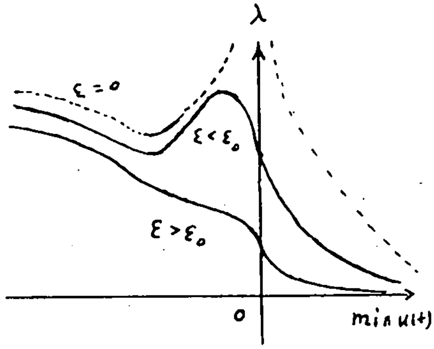

Figure 1

Theorem 2. *For given positive constants q, ε, s and α, r, δ, τ such that $q, \tau > p-1 > r > s$ let*

$$f(u) = u^q + \varepsilon u^s \quad (u \geq 0) \qquad \text{and} \qquad f(u) = \alpha |u|^r + \delta |u|^\tau \quad (u \leq 0).$$

Then:

(i) *There are constants $\Lambda > 0$ and $\Lambda_- > 0$ such that problem (1) has at least one sign-changing solution if $\lambda \leq \Lambda$, has no sign-changing solutions if $\lambda > \Lambda$, and the solution is unique if $\lambda < \Lambda_-$.*

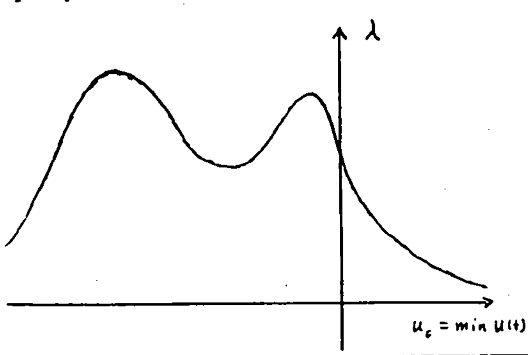

Figure 2

(ii) *For small ε and δ satisfying $\varepsilon << \delta$ there are $\lambda_1 < \lambda_2$ such that if $\lambda \in (\lambda_1, \lambda_2)$, then problem* (1) *has at least four sign-changing solutions.*

Next we consider for simplicity the semilinear ($p = 2$) problem

$$\left.\begin{aligned} &u'' = \lambda f(u) \quad (a < t < b) \\ &u(a) = u(b) = +\infty. \end{aligned}\right\} \tag{2}$$

For this simple problem we have the following, a somehow surprising result.

Theorem 3. *For given positive constants α, q and β such that $q > 3$ and $\beta < 1$ let*

$$f(u) = \alpha u^q \quad (u \geq 0) \qquad \textit{and} \qquad f(u) = (1 + \beta \sin|u|)|u| \quad (u \leq 0)$$

$(\lambda(2) = (\frac{\pi}{b-a})^2)$. *Then the following results hold:*

(i) *There are $\lambda_1, \lambda_2 > 0$ such that problem* (2) *has at least one sign-changing solution if $\lambda \geq \lambda_1$ and no sign-changing solutions if $\lambda < \lambda_1$. The sign-changing solution is unique, when $\lambda > \lambda_2$.*

(ii) *For any integer $n \geq 1$ there exists $\delta > 0$ such that problem* (2) *has at least n distinct sign-changing solutions when $\lambda \in (\lambda(2) - \delta, \lambda(2) + \delta)$.*

(iii) *For $\lambda = \lambda(2)$ problem* (2) *has infinitely many sign-changing solutions.*

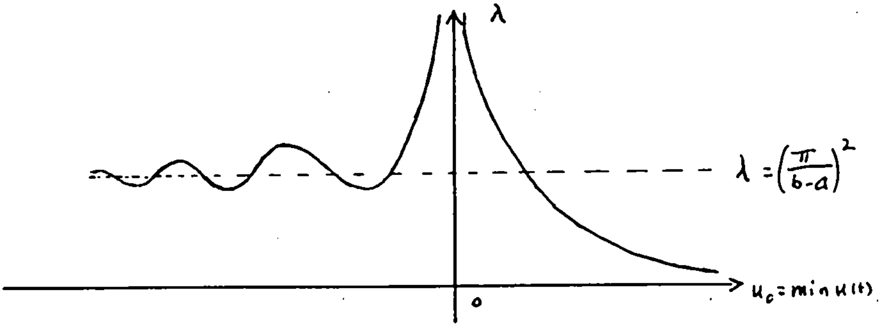

Figure 3

Remark. Problem (2) has a unique positive solution for all $\lambda > 0$. The problems treated in Theorems 1 and 2 have also a unique positive solution if $\lambda < \lambda_0$ where λ_0 is defined by

$$\int_0^\infty \frac{du}{\sqrt[p]{\frac{1}{q+1}\,u^{1+q} + \frac{\varepsilon}{s+1}\,u^{1+s}}} = \frac{b-a}{2}\sqrt[p]{p'\lambda_0}$$

and has no positive solution if $\lambda \geq \lambda_0$.

2. Some basic analysis

It is easy to see that the equation in (1) has an first integral

$$\frac{1}{p'}|u'(t)|^p = \lambda F(u) + C \qquad \text{with } F(u) = \int_0^u f(x)\,dx \tag{3}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $t_0 \in (a,b)$ be a minimum point of $u(t)$, which exists by the boundary condition. Then $u'(t_0) = 0$ and $C = -\lambda F(u_0), u_0 = \min u(t)$, and

$$|u'(t)|^p = \lambda p'\big(F(u) - F(u_0)\big) \qquad (a < t < b).$$

If $f = f(u)$ is non-negative, then we see that $u = u(t)$ is convex and the minimum point $t = t_0$ is unique. Consequetly, $u'(t) \le 0$ for $t \in (a, t_0)$ and $u'(t) \ge 0$ for $t \in (t_0, b)$. Moreover,

$$u'(t) = \sqrt[p]{\lambda p'\big(F(u) - F(u_0)\big)}\,\mathrm{sign}\,(t - t_0) \qquad (a < t < b).$$

Direct integration yields

$$(b - t_0)\sqrt[p]{p'\lambda} = \int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = (t_0 - a)\sqrt[p]{p'\lambda}$$

which implies $t_0 = \frac{a+b}{2}$ and thus $u = u(t)$ must be symmetric.

To establish the existence and the structure of solutions of problem (1) it suffices to study the nonlinear integral equation

$$\int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = \frac{b-a}{2}\sqrt[p]{p'\lambda}. \tag{4}$$

Obviously, a necessary condition for the existence of solutions for problem (1) is

$$\int^{+\infty} \frac{du}{\sqrt[p]{F(u)}} < +\infty \tag{5}$$

and so throughout this paper we shall assume that this condition holds.

Rewriting the integral in (4) gives that it is equivalent to

$$\int_0^{+\infty} \frac{du}{\sqrt[p]{F(u + u_0) - F(u_0)}} = \frac{b-a}{2}\sqrt[p]{p'\lambda}. \tag{6}$$

It follows from here that problem (1) has at most one (positive) solution under the condition that $f(u)$ is non-decreasing on $\mathbb{R}$ (or on $\mathbb{R}_+$). On the other hand, if $f(u) > 0$ for $u > 0$, then it has at least one positive solution for all $\lambda < \lambda_0$, where $\lambda_0 \in (0, +\infty]$ is defined by

$$\int_0^{+\infty} \frac{du}{\sqrt[p]{F(u)}} = \frac{b-a}{2}\sqrt[p]{p'\lambda_0}. \tag{7}$$

By summarying, let λ_0 be as above. Then we have the following

Theorem 4. *If $f(u) > 0$ $(u > 0)$, then problem* (1) *has at least one positive solution for all $\lambda < \lambda_0$. Moreover, if f is also non-decreasing on $\mathbb{R}_+$, then the solution is unique.*

Example. Consider the problem

$$\left.\begin{aligned} &\Delta_p u = \lambda(u^q + \varepsilon u^s) \quad (a < t < b) \\ &u(a) = u(b) = +\infty \end{aligned}\right\} \tag{8}$$

where $0 < r < q$, $\varepsilon > 0$ and $q > p-1$. Then condition (5) is satisfied, and further $\lambda_0 = +\infty$ if $s \geq p-1$ and $\lambda_0 < +\infty$ if $s \in (0, p-1)$. Hence problem (8) has a unique positive solution for all $\lambda > 0$ if $q > r \geq p-1$, and for all $\lambda < \lambda_0$ if $0 < s < p-1$, where λ_0 satisfies

$$\int_0^{+\infty} \frac{du}{\sqrt[p]{\frac{1}{q+1} u^{1+q} + \frac{\varepsilon}{s+1} u^{1+s}}} = \frac{b-a}{2} \sqrt[p]{p'\lambda_0}.$$

3. Proofs

To investigate solutions of problem (1) which change its sign, we define

$$f_+(u) = f(u) \quad (u \geq 0) \qquad \text{and} \qquad f_-(u) = f(-u) \quad (u \leq 0).$$

Then we have

$$F(u) = \begin{cases} F_+(u) = \int_0^u f_+(x)\,dx & \text{for } u \geq 0 \\ -F_-(-u) = -\int_0^{-u} f_-(x)\,dx & \text{for } u < 0. \end{cases}$$

Now equation (6) becomes

$$\int_0^{+\infty} \frac{du}{\sqrt[p]{F_+(u) + F_-(v_0)}} + \int_0^{v_0} \frac{du}{\sqrt[p]{F_-(v_0) - F_-(u)}} = \frac{b-a}{2} \sqrt[p]{p'\lambda} \tag{9}$$

where $v_0 = -u_0 > 0$. First observe that the first integral in (9) is strictly decreasing in v_0 if $f_-(u)$ is non-negative. For the second integral in (9), using

$$\int_0^{v_0} \frac{du}{\sqrt[p]{F_-(v_0) - F_-(u)}} = \int_0^1 \frac{v_0\,ds}{\sqrt[p]{F_-(v_0) - F_-(sv_0)}}$$

and

$$F_-(v_0) - F_-(sv_0) = \int_0^1 f_-\big((s + t - st)v_0\big)(1-s)v_0\,dt,$$

we see that it is decreasing or increasing in v_0 if $f_-(u)u^{1-p}$ is increasing or decreasing, respectively. Therefore, if $f_-(u)$ is non-negative and $f_-(u)u^{1-p}$ is increasing, then (9) has at most one solution, which gives the uniqueness of sign-changing solutions of problem (1). On the other hand, if $f_-(u)u^{1-p}$ is decreasing, then the first and the second integrals in (9) will compete to each other and thus the existence of a multiple solution is possible. In particular, if

$$\int_0 \frac{du}{\sqrt[p]{F_+(u)}} = +\infty \qquad \text{and} \qquad \lim_{u\to+\infty} f_-(u)u^{1-p} = 0,$$

then the left side in (9) goes to infinity, when v_0 goes either to zero or to infinity. Hence there is $\lambda_- > 0$ such that problem (1) has at least two sign-changing solutions for $\lambda > \lambda_-$, and no sign-changing solution if $\lambda < \lambda_-$. A typical example is

$$f_+(u) = \sum \alpha_i u^{q_i} \qquad \text{and} \qquad f_-(u) = \sum \beta_j u^{r_j}$$

where $\alpha_i, \beta_j > 0$ and $q_i > p-1 > r_j$. When f_+ and f_- are simply given by αu^q and βu^r $(q > p-1 > r)$, respectively, then we have a complete charaterization of signchanging solutions, namely, two solutions if $\lambda > \lambda_-$, a unique solution if $\lambda = \lambda_-$, and no solutions if $\lambda < \lambda_-$. In other situations the multiplicity of sign-changing solutions can be very complicated and three representive examples are as in Theorems 1 - 3.

Proof of Theorem 1. To this end we study the function in the left-hand side of (9). In this case we have

$$F_+(u) = \frac{1}{q+1}\,u^{q+1} + \frac{\varepsilon}{1+s}\,u^{1+s} \qquad \text{and} \qquad F_-(u) = \frac{\alpha}{r+1}\,u^{r+1}$$

and

$$\int_0^1 \frac{v_0\,ds}{\sqrt[p]{F_-(v_0) - F_-(sv_0)}} = c_2\,v_0^{1-\frac{1+r}{p}} \qquad \text{with } c_2 = \int_0^1 \frac{ds}{\sqrt[p]{\alpha(1-s^{r+1})/(1+r)}}$$

and the function in (9) is

$$\int_0^\infty \frac{du}{\sqrt[p]{\frac{1}{q+1}\,u^{q+1} + \frac{\varepsilon}{1+s}\,u^{1+s} + \frac{\alpha}{1+r}\,v_0^{r+1}}} + c_2\,v_0^{1-\frac{r+1}{p}} =: F_\varepsilon(v_0). \tag{10}$$

The first conclusion of Theorem 1 follows from that $F_\varepsilon(v_0) > 0$ is continuous on $[0, +\infty)$ and goes to infinity, as $v_0 \to \infty$. To get a complete picture for the existence of sign-changing solutions we let

$$k_1 = (1+r)\Big(\frac{1}{s+1} - \frac{1}{p}\Big), \qquad k_2 = \frac{(1+r)(q-s)}{1+s}, \qquad k_3 = 1 - \frac{r+1}{p}$$

and

$$g(v_0) = \int_0^{+\infty} \Big(\frac{1}{q+1}\,u^{q+1} + \frac{\varepsilon}{1+s}\,u^{1+s} + \frac{\alpha}{1+r}\,v_0^{r+1}\Big)^{-\frac{1}{p}} du.$$

Then $F_\varepsilon(v_0) = g(v_0) + c_2 v_0^{k_3}$. We first study the property of $g(v_0)$ in a neighbourhood of the origin.

By the change of variable $u = x\, v_0^{\frac{1+r}{1+s}}$ we get for $v_0 > 0$

$$g(v_0) = \int_0^{+\infty} \frac{v_0^{k_1}\, dx}{\sqrt[p]{\frac{1}{q+1}\, x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s}\, x^{1+s} + \frac{\alpha}{1+r}}}$$

$$g(0) = \int_0^{+\infty} \frac{v_0^{k_1}\, dx}{\sqrt[p]{\frac{1}{q+1}\, x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s}\, x^{1+s}}}.$$

Consequently,

$$\frac{g(v_0) - g(0)}{v_0^{k_1}} = - \int_0^{+\infty} G(v_0, x)\, dx$$

where

$$G(v_0, x) = \frac{1}{\sqrt[p]{\frac{1}{q+1}\, x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s}\, x^{1+s}}} - \frac{1}{\sqrt[p]{\frac{1}{q+1}\, x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s}\, x^{1+s} + \frac{\alpha}{1+r}}}.$$

Writing the difference as an integral we see

$$G(v_0, x) = \frac{\alpha}{p(r+1)} \int_0^1 \left(\frac{1}{q+1}\, x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s}\, x^{1+s} + \frac{\alpha}{1+r}\, \theta \right)^{-\frac{p+1}{p}} d\theta$$

which implies that $G(v_0, x)$ is decreasing in v_0. Since

$$G(0, x) = \frac{\alpha}{p(r+1)} \int_0^1 \left(\frac{\varepsilon}{1+s}\, x^{1+s} + \frac{\alpha}{1+r}\, \theta \right)^{-\frac{p+1}{p}} d\theta$$

is integrable over $(0, +\infty)$, due to $s \in (0, p-1)$, we deduce by the dominate convergence theorem that

$$\lim_{v_0 \to 0} \frac{g(v_0) - g(0)}{v_0^{k_1}} = - \int_0^{+\infty} G(0, x)\, dx < 0.$$

Thus we obtain that near the origin F_ε is increasing if $k_1 > 1 - \frac{r+1}{p}$, which is equivalent to $r > s$, and is decreasing when $r < s$. Similarly, we get that

$$\lim_{v_0 \to \infty} g'(v_0)\, v_0^{1+(r+1)(\frac{1}{p} - \frac{1}{q+1})} = -\frac{\alpha}{p} \int_0^1 \left(\frac{1}{q+1} x^{q+1} + \frac{\alpha}{1+r} \right)^{-\frac{p+1}{p}} dx$$

which implies that F_ε is increasing for large $v_0 > 0$, because $q > p-1$. Since $\frac{dF_\varepsilon}{dv_0}$ is increasing in ε and tends to $(1-\frac{r+1}{p})c_2 v_0^{-\frac{r+1}{p}}$ uniformly in v_0 as $\varepsilon \to \infty$, we deduce in the case $r > s$ that there is an $\varepsilon_0 \geq 0$ such that $\min_{v_0} \frac{dF_\varepsilon}{dv_0} > 0$ for $\varepsilon > \varepsilon_0$. Moreover, when $\varepsilon = 0$,

$$F_0(v_0) = c_1 v_0^{1-\frac{q+1}{p}} + c_2 v_0^{1-\frac{r+1}{p}}$$

for $1-\frac{q+1}{p} < 0 < 1-\frac{r+1}{p}$ and thus $\frac{dF_0}{dv_0} < 0$ for small $v_0 > 0$. Further, $F_\varepsilon(v_0) \to F_0(v_0)$ on compacta in $C^1(0,+\infty)$ as $\varepsilon \to 0$ and we assert that $\varepsilon_0 > 0$ and $\min_{v_0} \frac{dF_\varepsilon}{dv_0} < 0$ for $\varepsilon \in (0,\varepsilon_0)$. Whence, we obtain that the sign-changing solution is unique when $\varepsilon \geq \varepsilon_0$. If $\varepsilon \in (0,\varepsilon_0)$, using the fact that F_ε is increasing for both small and large v_0, we deduce that there are $0 < v_1 < v_2 \leq v_3 < +\infty$ such that F_ε is increasing on $(0,v_1)$ and $(v_3,+\infty)$ and is decreasing on (v_1,v_2). Thus let

$$\lambda_1 = \min F_\varepsilon(v_0), \quad \lambda_2 = F_\varepsilon(v_1), \quad \lambda_3 = F_\varepsilon(v_2), \quad \lambda_4 = \max_{F_\varepsilon'(v_0)=0} F_\varepsilon(v_0).$$

Then problem (1) has no sign-changing solutions if $\lambda < \lambda_1$ and has at least two sign-changing solutions $\lambda \in (\lambda_1,\lambda_4)$. In particular, it has at least three sign-changing solutions if $\lambda \in (\lambda_2,\lambda_3)$ and has a unique sign-changing solution as $\lambda > \lambda_4$. The proof is complete ∎

Proof of Theorem 2. The idea is the same as in the proof of Theorem 1, so we will be sketch in many places. In view of

$$F_+(u) = \frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} \qquad \text{and} \qquad F_-(u) = \frac{\alpha}{r+1}u^{r+1} + \frac{\delta}{\tau+1}u^{\tau+1}$$

it follows that equation (9) takes the form

$$\begin{aligned} &\int_0^{+\infty} \frac{du}{\sqrt[p]{\frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} + F_-(v_0)}} \\ &\qquad + \int_0^1 \frac{v_0\,dx}{\sqrt[p]{r(x)\,v_0^{s+1} + \tau(x)\,v_0^{\tau+1}}} = \frac{b-a}{2}\sqrt[p]{\alpha p'\lambda} \end{aligned} \tag{11}$$

where

$$r(x) = \frac{\alpha}{r+1}(1-x^{r+1}) \qquad \text{and} \qquad \tau(x) = \frac{\delta}{\tau+1}(1-x^{\tau+1}).$$

Likely as in the proof of Theorem 1 we define

$$L(v_0) = \ell_1(v_0) + \ell_2(v_0)$$

where

$$\ell_1(v_0) = \int_0^{+\infty} \frac{du}{\sqrt[p]{\frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} + F_-(v_0)}}$$

$$\ell_2(v_0) = \int_0^1 \frac{v_0\,dx}{\sqrt[p]{r(x)\,v_0^{r+1} + \tau(x)\,v_0^{\tau+1}}}.$$

The key points to prove that the function $L(v_0)$ have a graph as shown in Figure 2 are

$$L(v_0) = F_0(v_0) = c_1\, v_0^{\frac{p-q-1}{p}} + c_2\, v_0^{\frac{p-r-1}{p}} \qquad \text{when } \varepsilon = \delta = 0$$

which has the property $F_0'(v_0)(v_0 - v) > 0$ for some $v > 0$ and L converges to F_0 on compacta of $(0, +\infty)$ in C^1 as $\varepsilon, \delta \to 0$ and $L \to 0$ as $v_0 \to +\infty$.

To complete the proof we need to show two more things: Monotonicity of L near the origin and $L(0) \le L(v_2)$ where v_2 is the second local maximum of L. But

$$L(0) = c_1 \varepsilon^{\frac{q-p-1}{p(q-s)}} \qquad \text{and} \qquad L(\delta^{\frac{1}{r-\tau}}) \ge \ell_2(\delta^{\frac{1}{r-\tau}}) = c_2 \delta^{\frac{r+1-p}{p(r-\tau)}}$$

for some constants $c_1 > 0$ and $c_2 > 0$ from which it follows that if ε and δ satisfy $c_1 \varepsilon^{\frac{q-p-1}{p(q-s)}} \le c_2 \delta^{\frac{r+1-p}{p(r-\tau)}}$, then we are done.

To get the monotonicity of L in the nearby of the origin, we exploit the technique in the proof of Theorem 1 and can show that

$$\lim_{v_0 \to 0} \frac{\ell_1(v_0) - \ell_1(0)}{v_0^{k_1}} = -\frac{\alpha}{p(r+1)} \int_0^{+\infty} dx \int_0^1 \left(\frac{\varepsilon}{1+s}\, x^{s+1} + \frac{\alpha}{r+1}\, \theta \right)^{-\frac{p+1}{p}} d\theta < 0$$

where we have used that $r > \tau$ and $F_-(v_0) = \frac{\alpha}{r+1}\, v_0^{r+1}(1 + o(1))$ as $v_0 \to 0$,

$$\lim_{v_0 \to 0} \frac{\ell_2(v_0) - \ell_2(0)}{v_0^{k_3}} = \int_0^1 \frac{dx}{\sqrt[p]{r(x)}} > 0 \qquad \text{and} \qquad \lim_{v_0 \to 0} \frac{L(v_0) - L(0)}{v_0^{k_1}} < 0.$$

Finally, the uniqueness follows from the results in [4] that the function ℓ_2 has the following property: there is $v > 0$ such that $\ell_2'(v_0) > 0$ for $v_0 \in (0, v)$ and $\ell_2'(v_0) < 0$ for $v_0 \in (v, +\infty)$. Then $L'(v_0) < 0$ for $v_0 \in [v, +\infty)$ since clearly $\ell_1'(v_0) < 0$ for $v_0 > 0$. The proof is complete ∎

Proof of Theorem 3. By virtue of $p = 2$ and

$$F_+(u) = \frac{\alpha}{q+1}\, u^{q+1} \qquad \text{and} \qquad F_-(u) = \frac{1}{2}\, u^2 + \beta(\sin u - u \cos u)$$

we have that

$$\int_0^{+\infty} \frac{du}{\sqrt{F_+(u) + F_-(v_0)}} = c(q)\Big(\frac{1}{2} v_0^2 + \beta(\sin v_0 - v_0 \cos v_0)\Big)^{\frac{1}{q+1} - \frac{1}{2}} =: g_1(v_0)$$

$$\int_0^{v_0} \frac{du}{\sqrt{F_-(v_0) - F_-(u)}} = \int_0^1 \frac{v_0\, dx}{\sqrt{h(x, v_0)}} =: g_2(v_0)$$

where

$$h(x,v_0) = e(x)\,v_0^2 + \beta\,\varepsilon(x,v_0)$$

$$\varepsilon(x,v_0) = \sin v_0 - \sin(xv_0) - v_0\big(\cos v_0 - x\cos(xv_0)\big)$$

$$c(q) = \int_0^{+\infty}\Big(1+\frac{\alpha}{q+1}\,u^{q+1}\Big)^{-\frac{1}{2}}du$$

$$e(x) = \frac{1-x^2}{2}$$

and the function in the left-hand side of (9) takes the form

$$g_1(v_0) + g_2(v_0) := H(v_0).$$

Clearly, H is well defined and continuous on $(0,+\infty)$. Since $q>3$, we see that

$$g_1(v_0) \to +\infty \qquad \text{as } v_0 \to 0+$$

and

$$\left.\begin{aligned} &g_1(v_0) \to 0 \\ &g_2(v_0) \to \int_0^1 \frac{dx}{\sqrt{e(x)}} = \frac{\pi}{\sqrt{2}} \end{aligned}\right\} \qquad \text{as } v_0 \to +\infty$$

and therefore $H(0+) = +\infty$ and $H(+\infty) = \frac{\pi}{\sqrt{2}}$.

To show the results (ii) and (iii) we need to prove that H oscillates around the line $H = \frac{\pi}{\sqrt{2}}$, and first show that g_1 oscillates around $H = \frac{\pi}{\sqrt{2}}$.

Choosing $v_0 = n\pi$ ($n \in \mathbb{N}$ is even) we get

$$\begin{aligned} h(x,n\pi) &= (n\pi)^2 e(x) - \beta\Big(\sin(n\pi x) + n\pi\big(1 - x\cos(n\pi x)\big)\Big) \\ &\le (n\pi)^2 e(x) - \beta\Big(n\pi(1-x) - \sin\big(n\pi(1-x)\big)\Big) \\ &\le (n\pi)^2 e(x) \end{aligned}$$

thereafter

$$g_2(n\pi) > \int_0^1 \frac{n\pi\,dx}{\sqrt{e(x)(n\pi)^2}} = \frac{\pi}{\sqrt{2}}.$$

Analogously, for odd integer n, $g_2(n\pi) < \frac{\pi}{\sqrt{2}}$.

To show the function H is also oscillating, it suffices to show that, for odd n, $H(n\pi) < \frac{\pi}{\sqrt{2}}$ since $g_1 > 0$ and $g_2(n\pi) > \frac{\pi}{\sqrt{2}}$ for even n. For this purpose, first we have

$$\begin{aligned} \frac{\pi}{\sqrt{2}} - g_2(n\pi) &= \int_0^1 \left(\frac{n\pi}{\sqrt{e(x)(n\pi)^2}} - \frac{n\pi}{\sqrt{e(x)(n\pi)^2 + \beta\,\varepsilon(x,n\pi)}}\right) dx \\ &= \frac{1}{2}\int_0^1 dx \int_0^1 \frac{n\pi\beta\,\varepsilon(x,n\pi)}{\big(e(x)(n\pi)^2 + \theta\beta\,\varepsilon(x,n\pi)\big)^{\frac{3}{2}}}\,d\theta \\ &\ge \frac{1}{2}\int_0^1 \frac{n\pi\beta\,\varepsilon(x,n\pi)}{\big(e(x)(n\pi)^2 + \beta\,\varepsilon(x,n\pi)\big)^{\frac{3}{2}}}\,dx, \end{aligned}$$

due to $\varepsilon(x, n\pi) > 0$ for all $x \in (0,1)$. Using $f_-(u) \geq (1-\beta)u$, we deduce that

$$\begin{aligned} e(x)(n\pi)^2 + \beta\,\varepsilon(x, n\pi) &= n\pi(1-x)\int_0^1 f_-\big((x+t-tx)n\pi\big)\,dt \\ &\geq n\pi(1-x)\int_0^1 (1-\beta)(x+t-tx)n\pi\,dt \\ &\geq \frac{1}{2}(1-\beta)(1-x)(n\pi)^2 \end{aligned}$$

for all $x \in (0,1)$. Thereafter

$$\frac{\pi}{\sqrt{2}} - g_2(n\pi) \geq \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}}(n\pi)^{-2}\int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}}\,dx,$$

and by a change of variable $z = \frac{1}{\sqrt{1-x}}$ (then $x = 1 - \frac{1}{z^2}$)

$$\begin{aligned} \int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}}\,dx &= 2\int_1^{+\infty} \varepsilon\Big(1 - \frac{1}{z^2}, n\pi\Big)dz \\ &= 2\int_1^{+\infty} \Big(-\sin\Big(\frac{n\pi}{z^2}\Big) + n\pi\Big(1 - \Big(1-\frac{1}{z^2}\Big)\cos\frac{n\pi}{z^2}\Big)\Big)dz \quad cr \\ &= 2\int_1^{+\infty} \Big(n\pi(1-\cos\frac{n\pi}{z^2}) + (-\sin\frac{n\pi}{z^2} + \frac{n\pi}{z^2}\cos\frac{n\pi}{z^2})\Big)dz \end{aligned}$$

where we have used n being an odd integer. Change variables once more, we obtain

$$\int_1^{+\infty} \Big(1 - \cos\frac{n\pi}{z^2}\Big)dz = \int_{\frac{1}{\sqrt{n\pi}}}^{+\infty} \Big(1 - \cos\frac{1}{w^2}\Big)\sqrt{n\pi}\,dw \geq C\sqrt{n\pi}$$

where $C = \int_1^{+\infty}(1 - \cos\frac{1}{z^2})\,dz$. On the other hand,

$$\int_1^{+\infty} \Big(-\sin\frac{n\pi}{z^2} + \frac{n\pi}{z^2}\cos\frac{n\pi}{z^2}\Big)dz \leq \int_1^{+\infty} \frac{2n\pi}{z^2}\,dz = 2n\pi.$$

Whence

$$\int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}}\,dx \geq 2\big(Cn\pi\sqrt{n\pi} - 2n\pi\big)$$

and consequetly for large n the estimate

$$\frac{\pi}{\sqrt{2}} - g_2(n\pi) \geq \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}} C(n\pi)^{-\frac{1}{2}}$$

holds. Combing the above estimates, we obtain by noting $g_1(n\pi) \leq C_1(n\pi)^{\frac{2}{q+1}-1}$ that

$$\frac{\pi}{\sqrt{2}} - H(n\pi) = \frac{\pi}{\sqrt{2}} - g_2(n\pi) - g_1(n\pi) \geq C_2(n\pi)^{-\frac{1}{2}} - C_1(n\pi)^{\frac{2}{q+1}-1} > 0$$

since $-\frac{1}{2} > \frac{2}{q+1} - 1$ (this is the source for the condition on q).

Assertions (ii) and (iii) follow from the oscillatory property of H. Using

$$\lim_{v_0 \to \infty} H(v_0) = \frac{\pi}{\sqrt{2}}, \qquad \lim_{v_0 \to 0+} H(v_0) = +\infty, \qquad H(n\pi) < \frac{\pi}{\sqrt{2}}$$

we deduce from the continuity of H that the minimum of $H(v_0)$ on $(0, +\infty)$ achives and thus the equation $H(v_0) = \mu$ is solvable if and only if $\mu \geq \min H(v_0)$ which complete the proof of assertion (i) ∎

4. Final remarks

In this note we have only carried out some basic calculations to exhibit the rich structure for boundary blow-up problems, even it is very elementary (just calculus), but it is certainly not easy to give a complete bifurcation picture for all involved parameters, for instance $q, r, s, \tau, \varepsilon, \delta, \alpha$ in Theorem 2. From our one-dimensional examples we can see that there is a big difference between Dirichlet boundary value problems and boundary bolw-up problems. If the boundary is Dirichlet, then there are infinitely many sign solutions in the superlinear case, but it can have only finite number of sign solutions for boundary bolw-up problems (note, the function in (2) is not superlinear at $-\infty$). As our examples are of one-dimensional character, one may say that it would not be representive, so it will be interesting to study those equations in two-dimensional or higher dimensional domains.

References

[1] Aftalion, A. and W. Reichel: *Existence of two boundary blow-up solutions for semilinear elliptic equations.* J. Diff. Equ. 141 (1997), 400 - 421.

[2] Bandle, C. and M. Essén: *On the solutions of quasilinear elliptic problems with boundary blow-up.* In: Partial Differential Equations of Elliptic Type (Symposia Mathematica: Vol. 35; eds.: A. Alvino at al.). Univ. Press, Cambridge (1994), pp. 93 - 111.

[3] Bandle, C. and M. Marcus: *Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with boundary blow-up.* Ann. Inst. Henri Poicaré 12 (1995), 155 - 171.

[4] Cheng, Y. J.: *On an open problem of Ambrosetti, Brezis and Cerami*. Diff. Int. Equ. (to appear).

[5] Keller, J. B.: *Electrohydrodynamics*. Part I: *The equilibrium of a charged gas in a container*. J. Rat. Mech. Anal. 5 (1956), 715 - 724.

[6] Loewner, C. and L. Nirenberg: *Partial differential equations invariant under conformal or projective transformations*. In: Contribution to Analysis (ed.: L. Ahlfors). Acad. Press, New York (1974), pp. 245 - 272.

[7] Matero, J.: *Nonlinear Elliptic Problems with Boundary Blow-Up*. Diss. in Math. 6. Uppsala: Univ. 1997.

[8] Mckenna, P. J., Reichel, W. and W. Walter: *Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up*. Nonlin. Anal. Theory, Methods and Appl. 28 (1997), 1213 - 1225.

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