## Some Surprising Results on a One-Dimensional Elliptic Boundary Value Blow-Up Problem

**Y. J. Cheng** 

Abstract. In this paper we consider the one-dimensional elliptic boundary blow-up problem

$$
\Delta_p u = f(u) \quad (a < t < b)
$$
  

$$
u(a) = u(b) = +\infty
$$

where  $\Delta_p u = (|u'(t)|^{p-2}u'(t))'$  is the usual *p*-Laplace operator. We show that the structure of the solutions can be very rich even for a simple function  $f$  which gives a leading that a simliar results might hold also in higher dimensional spaces

**Keywords:** *Boundary blow-up, multiplicity, concave and* convex *nonhinearily*  AMS subject classification: 34 B, 35J

## 1. Introduction and formulation of main results

In the last few years there is a great of interests in the investigation of boundary blowup solutions for elliptic equations [2, 3, 7), which comes originally from differential geometry [6] and electrohydrodynamics [5]. Very recently some existence results of two (one positive and one sign-changing) solutions have been established in [1, 8). The purpose of the present paper is to show through one-dimensional examples that the structure of the solutions can be very rich even for a simple right-hand side. More precisely, we consider the problem **results**<br>vestigation<br>es originall<br>some exist<br>in establish<br>imple righ<br>imple righ<br>and f is

$$
\Delta_p u = \lambda f(u) \quad (a < t < b) \nu(a) = u(b) = +\infty
$$
\n(1)

where

$$
\Delta_p u = (|u'(t)|^{p-2}u'(t))'
$$

is the p-Laplace operator as usual,  $\lambda > 0$  is a parameter, and f is a given continuous function. By a solution  $u = u(t)$  of problem (1) we mean that *u* satisfies the equation in  $(1)$ , i.e.  $(|u'(t)|^{p-2}u'(t))' = \lambda f(u(t))$  for all  $t \in (a, b)$ , and  $\lim_{t \to a+} u(t) = \lim_{t \to b-} u(t) =$  $+\infty$ . By a sign-changing solution  $u(t)$  of (1) we mean that there exist  $t_1, t_2 \in (a, b)$ such that  $u(t_1) > 0$  and  $u(t_2) < 0$ . structure of the solutions can be very rict<br>precisely, we consider the problem<br> $\Delta_p u = \lambda f(u)$ <br> $u(a) = u(b) =$ <br>where<br>is the p-Laplace operator as usual,  $\lambda > 0$  is<br>function. By a solution  $u = u(t)$  of problem<br>(1), i.e.  $(|u'(t)|^{p-2}u$ 

The results for problem (1) in this paper are summarized in the followings three

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**Theorem 1.** For given positive constants  $q, \varepsilon$ ,  $s$  and  $\alpha$ ,  $r$  such that  $q > p-1 > r > s$ . *let f*(*u*) =  $u^q + \varepsilon u^s$  (*u*  $\geq 0$ ) *and*  $f(u) = \alpha |u|^r$  (*u*  $\leq 0$ )<br>*f(u)* =  $u^q + \varepsilon u^s$  (*u*  $\geq 0$ ) *and*  $f(u) = \alpha |u|^r$  (*u*  $\leq 0$ )

$$
f(u) = u^q + \varepsilon u^s \quad (u \ge 0) \qquad \text{and} \qquad f(u) = \alpha |u|^r \quad (u \le 0).
$$

*Then the following statements hold:* 

(i) There exists a constant  $\lambda_1 > 0$  such that problem (1) has at least one sign*changing solution if*  $\lambda > \lambda_1$ , and no such kind of solutions can exist if  $\lambda < \lambda_1$ .

(ii) There exist a constant  $\varepsilon_0 > 0$  such that the solution is unique if  $\varepsilon \geq \varepsilon_0$ .

(iii) If  $\varepsilon \in (0, \varepsilon_0)$ , then there are constants  $\lambda_2 < \lambda_3 \leq \lambda_4$  such that problem (1) *has at least two sign-changing solutions if*  $\lambda \in (\lambda_1, \lambda_4)$ , has at least three sign-changing *solutions if*  $\lambda \in (\lambda_2, \lambda_3)$ , and has a unique sign-changing solution if  $\lambda > \lambda_4$ .





**Theorem 2.** For given positive constants  $q, \varepsilon$ , s and  $\alpha$ ,  $r, \delta$ ,  $\tau$  such that  $q, \tau > p-1$ *r > s let 1*  $\sigma$  *miauth*<br> *niauth*<br> *niauthatq, r*<br> *f(u)* =  $u^q + \varepsilon u^s$  ( $u \ge 0$ )<br> *and*<br> *f(u)* =  $\alpha |u|^r + \delta |u|^r$ 

*Then:* 

(i) There are constants  $\Lambda > 0$  and  $\Lambda > 0$  such that problem (1) has at least *one sign-changing solution if*  $\lambda \leq \Lambda$ , has no sign-changing solutions if  $\lambda > \Lambda$ , and the *solution is unique if*  $\lambda < \Lambda_{-}$ .



Figure 2

(ii) For small  $\varepsilon$  and  $\delta$  satisfying  $\varepsilon << \delta$  there are  $\lambda_1 < \lambda_2$  such that if  $\lambda \in (\lambda_1, \lambda_2)$ , *then problem* (1) has at least four sign-changing solutions.

Next we consider for simplicity the semilinear  $(p = 2)$  problem

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or small 
$$
\varepsilon
$$
 and  $\delta$  satisfying  $\varepsilon \langle \delta$  there are  $\lambda_1 \langle \lambda_2 \rangle$  such that if  $\lambda \in (\lambda_1, \lambda_2)$ ,  
em (1) has at least four sign-changing solutions.  
we consider for simplicity the semilinear  $(p = 2)$  problem  
 $u'' = \lambda f(u) (a \langle t \langle b \rangle)$   
 $u(a) = u(b) = +\infty.$  (2)  
mple problem we have the following, a somehow surprising result.  
rem 3. For given positive constants  $\alpha, q$  and  $\beta$  such that  $q > 3$  and  $\beta < 1$  let  
 $f(u) = \alpha u^q$   $(u \ge 0)$  and  $f(u) = (1 + \beta \sin |u|)|u|$   $(u \le 0)$ 

For this simple problem we have the following, a somehow surprising result.

Theorem 3. For given positive constants  $\alpha$ ,  $q$  and  $\beta$  such that  $q > 3$  and  $\beta < 1$  let

 $(\lambda(2) = (\frac{\pi}{b-a})^2)$ . Then the following results hold:

(i) There are  $\lambda_1, \lambda_2 > 0$  such that problem (2) has at least one sign-changing *solution if*  $\lambda \geq \lambda_1$  and no sign-changing solutions if  $\lambda < \lambda_1$ . The sign-changing solution *is unique, when*  $\lambda > \lambda_2$ .

(ii) For any integer  $n \geq 1$  there exists  $\delta > 0$  such that problem (2) has at least  $n$ *distinct sign-changing solutions when*  $\lambda \in (\lambda(2) - \delta, \lambda(2) + \delta)$ .

(iii) For  $\lambda = \lambda(2)$  problem (2) has infinitely many sign-changing solutions.





**Remark.** Problem (2) has a unique positive solution for all  $\lambda > 0$ . The problems treated in Theorems 1 and 2 have also a unique positive solution if  $\lambda < \lambda_0$  where  $\lambda_0$  is defined by

$$
\int_{0}^{\infty} \frac{du}{\sqrt[4]{\frac{1}{q+1}u^{1+q} + \frac{\varepsilon}{s+1}u^{1+s}}} = \frac{b-a}{2} \sqrt[4]{p'\lambda_0}
$$

and has no positive solution if  $\lambda \geq \lambda_0$ .

It is easy to see that the equation in (1) has an first integral

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\n2. Some basic analysis  
\nIt is easy to see that the equation in (1) has an first integral  
\n
$$
\frac{1}{p'}|u'(t)|^p = \lambda F(u) + C \quad \text{with } F(u) = \int_0^u f(x) dx
$$
\n(3)  
\nwhere  $\frac{1}{p} + \frac{1}{p'} = 1$ . Let  $t_0 \in (a, b)$  be a minimum point of  $u(t)$ , which exists by the  
\nboundary condition. Then  $u'(t_0) = 0$  and  $C = -\lambda F(u_0), u_0 = \min u(t)$ , and  
\n
$$
|u'(t)|^p = \lambda p'(F(u) - F(u_0)) \quad (a < t < b).
$$
\nIf  $f = f(u)$  is non-negative, then we see that  $u = u(t)$  is convex and the minimum point

boundary condition. Then  $u'(t_0) = 0$  and  $C = -\lambda F(u_0), u_0 = \min u(t)$ , and

$$
|u'(t)|^p = \lambda p'(F(u) - F(u_0)) \qquad (a < t < b).
$$

If  $f = f(u)$  is non-negative, then we see that  $u = u(t)$  is convex and the minimum point *t* = *t*<sub>0</sub> is unique. Consequetly,  $u'(t) \le 0$  for  $t \in (a, t_0)$  and  $u'(t) \ge 0$  for  $t \in (t_0, b)$ .<br>
Moreover,<br>  $u'(t) = \sqrt{\lambda p'(F(u) - F(u_0))} \operatorname{sign}(t - t_0)$   $(a < t < b)$ .<br>
Direct integration yields<br>  $(b - t_0) \sqrt[n]{p'\lambda} = \int_{u_0}^{+\infty} \frac{du}{\sqrt[n]{$ Moreover,  $|u'(t)|^p = \lambda p'(F(u) - F(u_0))$   $(a < t < b)$ .<br>
ion-negative, then we see that  $u = u(t)$  is convex and the<br>
ie. Consequetly,  $u'(t) \leq 0$  for  $t \in (a, t_0)$  and  $u'(t) \geq 0$ <br>  $u'(t) = \sqrt[p]{\lambda p'(F(u) - F(u_0))} \operatorname{sign}(t - t_0)$   $(a < t < b)$ <br>
ion vields

$$
u'(t) = \sqrt[n]{\lambda p'(F(u) - F(u_0))} \operatorname{sign}(t - t_0) \qquad (a < t < b).
$$

Direct integration yields

$$
(t) = \sqrt[p]{\lambda p'(F(u) - F(u_0))} \operatorname{sign}(t - t_0) \qquad (a < t < b)
$$
  
\n
$$
u \text{ yields}
$$
  
\n
$$
(b - t_0) \sqrt[p]{p' \lambda} = \int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = (t_0 - a) \sqrt[p]{p' \lambda}
$$
  
\n
$$
= \frac{a + b}{2} \text{ and thus } u = u(t) \text{ must be symmetric.}
$$
  
\nthe existence and the structure of solutions of problem  
\nlinear integral equation  
\n
$$
+ \infty
$$
  
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$$
\int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = \frac{b - a}{2} \sqrt[p]{p' \lambda}.
$$
  
\nassary condition for the existence of solutions for prob-  
\n
$$
f^{+\infty} - du
$$

which implies  $t_0 = \frac{a+b}{2}$  and thus  $u = u(t)$  must be symmetric.

To establish the existence and the structure of solutions of problem (1) it suffices to study the nonlinear integral equation

$$
\sqrt[p]{p'\lambda} = \int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = (t_0 - a) \sqrt[p]{p'\lambda}
$$
  
and thus  $u = u(t)$  must be symmetric.  
stence and the structure of solutions of problem (1) it suffices  
integral equation  

$$
\int_{u_0}^{+\infty} \frac{du}{\sqrt[p]{F(u) - F(u_0)}} = \frac{b - a}{2} \sqrt[p]{p'\lambda}.
$$
(4)  
condition for the existence of solutions for problem (1) is  

$$
\int_{-\infty}^{+\infty} \frac{du}{\sqrt[p]{F(u)}} < +\infty
$$
(5)  
paper we shall assume that this condition holds.  
cal in (4) gives that it is equivalent to  

$$
\int_{0}^{+\infty} \frac{du}{\sqrt[p]{F(u + u_0) - F(u_0)}} = \frac{b - a}{2} \sqrt[p]{p'\lambda}.
$$
(6)  
at problem (1) has at most one (positive) solution under the  
condecreasing on  $\mathbb{R}$  (or on  $\mathbb{R}$ .) On the other hand if  $f(x) > 0$ 

Obviously, a necessary condition for the existence of solutions for problem (1) is

$$
\int^{+\infty} \frac{du}{\sqrt[n]{F(u)}} < +\infty \tag{5}
$$

and so throughout this paper we shall assume that this condition holds.

Rewriting the integral in (4) gives that it is equivalent to

$$
\int_{0}^{+\infty} \frac{du}{\sqrt[n]{F(u+u_0)-F(u_0)}} = \frac{b-a}{2}\sqrt[n]{p'\lambda}.
$$
 (6)

It follows from here that problem (1) has at most one (positive) solution under the condition that  $f(u)$  is non-decreasing on  $\mathbb{R}$  (or on  $\mathbb{R}_+$ ). On the other hand, if  $f(u) > 0$ for  $u > 0$ , then it has at least one positive solution for all  $\lambda < \lambda_0$ , where  $\lambda_0 \in (0, +\infty]$ is defined by

$$
\int_{0}^{+\infty} \frac{du}{\sqrt[n]{F(u)}} = \frac{b-a}{2} \sqrt[n]{p'\lambda_0}.
$$
\n(7)

By summarying, let  $\lambda_0$  be as above. Then we have the following

**Theorem 4.** If  $f(u) > 0$   $(u > 0)$ , then problem (1) has at least one positive *solution for all*  $\lambda < \lambda_0$ . Moreover, if f is also non-decreasing on  $\mathbb{R}_+$ , then the solution *is unique. (a)*  $\begin{aligned} \text{for all } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable, } \text{if } x \text{ is a random variable$ 

Example. Consider the problem

$$
\Delta_p u = \lambda (u^q + \varepsilon u^s) \quad (a < t < b) u(a) = u(b) = +\infty
$$
 (8)

where  $0 < r < q$ ,  $\varepsilon > 0$  and  $q > p - 1$ . Then condition (5) is satisfied, and further  $\lambda_0 = +\infty$  if  $s \geq p - 1$  and  $\lambda_0 < +\infty$  if  $s \in (0, p - 1)$ . Hence problem (8) has a unique positive solution for all  $\lambda > 0$  if  $q > r \geq p-1$ , and for all  $\lambda < \lambda_0$  if  $0 < s < p-1$ , where  $\lambda_0$  satisfies

$$
\int_{0}^{+\infty} \frac{du}{\sqrt[3]{\frac{1}{q+1}u^{1+q} + \frac{\epsilon}{s+1}u^{1+s}}} = \frac{b-a}{2} \sqrt{p'\lambda_0}.
$$

## **3. Proofs**

To investigate solutions of problem (1) which change its sign, we define

*f*+(*u*) = *f*(*u*) (*u*  $\geq$  0) and *f*<sub>-</sub>(*u*) = *f*(-*u*) (*u*  $\leq$  0).

Then we have

$$
\oint_{0}^{1} \sqrt[4]{\frac{1}{q+1}} u^{1+q} + \frac{e}{s+1} u^{1+s}
$$
  
\n
$$
u = f(u) \quad (u \ge 0)
$$
 and 
$$
f_{-}(u) = f(-u) \quad (u
$$
  
\n
$$
F(u) = \begin{cases} F_{+}(u) = \int_{0}^{u} f_{+}(x) dx & \text{for } u \ge 0 \\ -F_{-}(-u) = -\int_{0}^{-u} f_{-}(x) dx & \text{for } u < 0. \end{cases}
$$
  
\n(b) becomes

Now equation (6) becomes

$$
x_1 \text{ solutions of problem (1) which change its sign, we define}
$$
\n
$$
f_{+}(u) = f(u) \quad (u \ge 0) \qquad \text{and} \qquad f_{-}(u) = f(-u) \quad (u \le 0).
$$
\n
$$
F(u) = \begin{cases} F_{+}(u) = \int_0^u f_{+}(x) \, dx & \text{for } u \ge 0 \\ -F_{-}(-u) = -\int_0^{-u} f_{-}(x) \, dx & \text{for } u < 0. \end{cases}
$$
\n
$$
f_{+}(0) = \int_0^u f_{+}(u) \, du
$$
\n
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f_{+}(u) = \int_0^u f_{+}(u) \, du
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f_{+}(u) + \int_0^u f_{+}(u) \, du
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$$
f_{+}(u) = \int_0^u f_{+}(u) \, du
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$$
f_{+}(u) = \int_0^u f_{+}(u) \, du
$$
\n<

where  $v_0 = -u_0 > 0$ . First observe that the first integral in (9) is strictly decreasing in  $v_0$  if  $f_-(u)$  is non-negative. For the second integral in (9), using Now equation (<br>  $\int_{0}^{+\infty}$ <br>
where  $v_0 = -u_0$ <br>  $v_0$  if  $f_-(u)$  is not

$$
f'(u) = \begin{cases} F_+(u) = \int_0^u f_+(x) dx & \text{for } u \ge 0 \\ -F_-(-u) = -\int_0^{-u} f_-(x) dx & \text{for } u < 0. \end{cases}
$$
  
becomes  

$$
\frac{du}{F_+(u) + F_-(v_0)} + \int_0^{v_0} \frac{du}{\sqrt[r]{F_-(v_0) - F_-(u)}} = \frac{b-a}{2}
$$
  
0. First observe that the first integral in (9) is st  
negative. For the second integral in (9), using  

$$
\int_0^{v_0} \frac{du}{\sqrt[r]{F_-(v_0) - F_-(u)}} = \int_0^1 \frac{v_0 ds}{\sqrt[r]{F_-(v_0) - F_-(sv_0)}}
$$

$$
F_{-}(v_0)-F_{-}(sv_0)=\int\limits_{0}^{1}f_{-}((s+t-st)v_0)(1-s)v_0 dt,
$$

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we see that it is decreasing or increasing in  $v_0$  if  $f_-(u)u^{1-p}$  is increasing or decreasing, respectively. Therefore, if  $f_-(u)$  is non-negative and  $f_-(u)u^{1-p}$  is increasing, then (9) has at most one solution, which gives the uniqueness of sign-changing solutions of problem (1). On the other hand, if  $f_-(u)u^{1-p}$  is decreasing, then the first and the second integrals in (9) will compete to each other and thus the existence of a multiple solution is possible. In particular, if decreasing or increasing in  $v_0$  if  $f_-(u)u^{1-p}$  is increasin<br>nerefore, if  $f_-(u)$  is non-negative and  $f_-(u)u^{1-p}$  is<br>one solution, which gives the uniqueness of sign-chan<br>n the other hand, if  $f_-(u)u^{1-p}$  is decreasing or increase  $f_-(u)$  is<br> *n*, which  $\infty$ <br> **chand**, if<br>
icular, if<br>  $\infty = +\infty$ <br>
to infinit, problem

$$
\int_0^{\infty} \frac{du}{\sqrt[n]{F_+(u)}} = +\infty \quad \text{and} \quad \lim_{u \to +\infty} f_-(u)u^{1-p} = 0,
$$

then the left side in (9) goes to infinity, when  $v_0$  goes either to zero or to infinity. Hence there is  $\lambda_{-} > 0$  such that problem (1) has at least two sign-changing solutions for  $\lambda > \lambda_{-}$ , and no sign-changing solution if  $\lambda < \lambda_{-}$ . A typical example is  $\frac{du}{\sqrt[n]{F_+(u)}} = +\infty$  and  $u^{-1}$ <br>
(9) goes to infinity, when  $v_0$  goes<br>
such that problem (1) has at legn-changing solution if  $\lambda < \lambda_{-1}$ .<br>  $f_+(u) = \sum \alpha_i u^{q_i}$  and<br>
and  $g_+ > n - 1 > n_1$ . When the

$$
f_+(u) = \sum \alpha_i u^{q_i}
$$
 and  $f_-(u) = \sum \beta_j u^{r_j}$ 

solution is possible. In particular, if<br>  $\int_0^1 \frac{du}{\sqrt[4]{F_+(u)}} = +\infty$  and  $\lim_{u \to +\infty} f_-(u)u^{1-p} = 0$ ,<br>
then the left side in (9) goes to infinity, when  $v_0$  goes either to zero or to infinity. Hence<br>
there is  $\lambda \to 0$  su where  $\alpha_i, \beta_j > 0$  and  $q_i > p - 1 > r_j$ . When  $f_+$  and  $f_-$  are simply given by  $\alpha u^q$  and  $\beta u^r$  ( $q > p - 1 > r$ ), respectively, then we have a complete charaterization of where  $\alpha_i, \beta_j > 0$  and  $q_i > p - 1 > r_j$ . When  $f_+$  and  $f_-$  are simply given by  $\alpha u^q$ signchanging solutions, namely, two solutions if  $\lambda > \lambda_{-}$ , a unique solution if  $\lambda = \lambda_{-}$ , and no solutions if  $\lambda < \lambda_{-}$ . In other situations the multiplicity of sign-changing solutions  $f_+ (u) = \sum_{q \text{ odd}}$ <br>  $f_+ (u) = \frac{1}{q+1} u^{q+1}$ <br>  $f_+ (u) = \frac{1}{q+1} u^{q+1}$ 

can be very complicated and three representive examples are as in Theorems 1 - 3.<br> **Proof of Theorem 1.** To this end we study the function in the left-hand sid<br>
(9). In this case we have<br>  $F_+(u) = \frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s}$ **Proof of Theorem 1.** To this end we study the function in the left-hand side of In this case we have  $F_{+}(u) = \frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s} u^{1+s}$  and  $F_{-}(u) = \frac{\alpha}{r+1} u^{r+1}$ (9). In this case we have

$$
F_{+}(u) = \frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s} u^{1+s} \quad \text{and} \quad F_{-}(u) = \frac{\alpha}{r+1} u^{r+1}
$$

and

signchanging solutions, namely, two solutions if 
$$
\lambda > \lambda_{-}
$$
, a unique solution if  $\lambda =$   
and no solutions if  $\lambda < \lambda_{-}$ . In other situations the multiplicity of sign-changing solution  
can be very complicated and three representative examples are as in Theorems 1 - 3  
**Proof of Theorem 1.** To this end we study the function in the left-hand side  
(9). In this case we have  

$$
F_{+}(u) = \frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s} u^{1+s} \quad \text{and} \quad F_{-}(u) = \frac{\alpha}{r+1} u^{r+1}
$$
and  
and  

$$
\int_{0}^{1} \frac{v_{0} ds}{\sqrt[r]{F_{-}(v_{0}) - F_{-}(s v_{0})}} = c_{2} v_{0}^{1-\frac{1+r}{p}} \quad \text{with } c_{2} = \int_{0}^{1} \frac{ds}{\sqrt[r]{\alpha(1 - s^{r+1})/(1+r)}}
$$
and the function in (9) is  

$$
\int_{0}^{\infty} \frac{du}{\sqrt[r]{\frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s} u^{1+s} + \frac{\alpha}{1+r} v_{0}^{r+1}}} + c_{2} v_{0}^{1-\frac{r+1}{p}} =: F_{\varepsilon}(v_{0}).
$$
The first conclusion of Theorem 1 follows from that  $F_{\varepsilon}(v_{0}) > 0$  is continuous on [0,

(20) - 
$$
F_{-}(sv_{0})
$$
  
\n(3)  $\sqrt[4]{\alpha(1-s^{r+1})/(1+r)}$   
\n(4)  $\int_{0}^{\infty} \frac{du}{\sqrt[4]{\frac{1}{q+1}u^{q+1} + \frac{\epsilon}{1+s}u^{1+s} + \frac{\alpha}{1+r}v_{0}^{r+1}}} + c_{2}v_{0}^{1-\frac{r+1}{p}} =: F_{\epsilon}(v_{0}).$  (10)  
\n(11)  $\int_{0}^{\infty} \frac{du}{\sqrt[4]{\frac{1}{q+1}u^{q+1} + \frac{\epsilon}{1+s}u^{1+s} + \frac{\alpha}{1+r}v_{0}^{r+1}}} + c_{2}v_{0}^{1-\frac{r+1}{p}} =: F_{\epsilon}(v_{0}).$  (11)  
\n(12)  $\int_{0}^{\infty} \frac{du}{\sqrt[4]{\frac{1}{q+1}u^{q+1} + \frac{\alpha}{1+r}v_{0}^{r+1}}} + \frac{1}{\int_{0}^{\infty} \frac{u^{1+s}}{1+s} + \frac{\alpha}{1+r}v_{0}^{r+1}} = \frac{r+1}{p}$   
\n(1 +  $r \Big(\frac{1}{s+1} - \frac{1}{p}\Big), \qquad k_{2} = \frac{(1+r)(q-s)}{1+s}, \qquad k_{3} = 1 - \frac{r+1}{p}$   
\n(1 +  $r \Big(\frac{1}{s+1} - \frac{1}{p}\Big), \qquad k_{4} = \frac{(1+r)(q-s)}{1+s} + \frac{\alpha}{1+r}v_{0}^{r+1} - \frac{1}{r+1}du$   
\n(1 +  $r \Big(\frac{1}{s+1} - \frac{1}{r}\Big), \qquad k_{5} = \frac{(1+r)(q-s)}{1+r} - \frac{1}{r+1}du$   
\n(1 +  $r \Big(\frac{1}{s+1} - \frac{1}{r}\Big), \qquad k_{6} = \frac{(1+r)(q-s)}{1+r} - \frac{1}{r+1}du$   
\n(1 +  $r \Big(\frac{1}{s+1} - \frac{1}{r}\Big), \qquad k_{7} = \frac{(1+r)(q-s)}{1+s} - \frac{1}{1+r}v_{0}^{r+1} - \frac{1}{r+1}du$   
\

The first conclusion of Theorem 1 follows from that  $F_e(v_0) > 0$  is continuous on  $[0, +\infty)$ and goes to infinity, as  $v_0 \to \infty$ . To get a complete picture for the existence of sign-<br>changing solutions we let<br> $k_1 = (1+r)\left(\frac{1}{s+1} - \frac{1}{p}\right), \qquad k_2 = \frac{(1+r)(q-s)}{1+s}, \qquad k_3 = 1 - \frac{r+1}{p}$ changing solutions we let  $\frac{1}{r}v_0^{r+1} + c_2v_0^{r+1} =: F_{\epsilon}(v_0).$ <br>
In that  $F_{\epsilon}(v_0) > 0$  is continuous on<br>
complete picture for the existence<br>  $\frac{(1+r)(q-s)}{1+s}, \qquad k_3 = 1 - \frac{r+1}{p}$ 

g solutions we let  

$$
k_1 = (1+r)\left(\frac{1}{s+1} - \frac{1}{p}\right)
$$
,  $k_2 = \frac{(1+r)(q-s)}{1+s}$ ,  $k_3 = 1 - \frac{r+1}{p}$ 

and

$$
g(v_0) = \int_{0}^{+\infty} \left(\frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} + \frac{\alpha}{1+r}v_0^{r+1}\right)^{-\frac{1}{p}} dt
$$

Then  $F_{\epsilon}(v_0) = g(v_0) + c_2 v_0^{k_3}$ . We first study the property of  $g(v_0)$  in a neighbourhood of the origin.

By the change of variable  $u = x v_0^{\frac{1+r}{1+r}}$  we get for  $v_0 > 0$ 

Some Results on an Elliptic Block  
+ 
$$
c_2v_0^{k_3}
$$
. We first study the property of  $g(v_0)$   
variable  $u = x v_0^{\frac{1+r}{1+s}}$  we get for  $v_0 > 0$   

$$
g(v_0) = \int_0^{+\infty} \frac{v_0^{k_1} dx}{\sqrt[n]{\frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\epsilon}{1+s} x^{1+s} + \frac{\alpha}{1+r}}}
$$

$$
g(0) = \int_0^{+\infty} \frac{v_0^{k_1} dx}{\sqrt[n]{\frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\epsilon}{1+s} x^{1+s}}}.
$$

Consequently,

$$
0 \quad \sqrt{q+1} \quad \sqrt{y} \quad 0 \quad 1 \quad 1+s \quad \sqrt{y}
$$
\n
$$
\frac{g(v_0) - g(0)}{v_0^{k_1}} = - \int_{0}^{+\infty} G(v_0, x) \, dx
$$

where

$$
\frac{g(v_0) - g(0)}{v_0^{k_1}} = -\int_0^{+\infty} G(v_0, x) dx
$$
  

$$
G(v_0, x) = \frac{1}{\sqrt[3]{\frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\epsilon}{1+s} x^{1+s}}} - \frac{1}{\sqrt[3]{\frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\epsilon}{1+s} x^{1+s} + \frac{\alpha}{1+r}}}.
$$
  
ng the difference as an integral we see  

$$
G(v_0, x) = \frac{\alpha}{p(r+1)} \int_0^1 \left(\frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\epsilon}{1+s} x^{1+s} + \frac{\alpha}{1+r} \theta\right)^{-\frac{p+1}{p}} d\theta
$$
  
implies that  $G(v_0, x)$  is decreasing in  $v_0$ . Since

Writing the difference as an integral we see

$$
G(v_0,x)=\frac{\alpha}{p(r+1)}\int\limits_{0}^{1}\Big(\frac{1}{q+1}\,x^{q+1}v_0^{k_2}+\frac{\varepsilon}{1+s}\,x^{1+s}+\frac{\alpha}{1+r}\,\theta\Big)^{-\frac{p+1}{p}}d\theta
$$

which implies that  $G(v_0, x)$  is decreasing in  $v_0$ . Since

rence as an integral we see  
\n
$$
= \frac{\alpha}{p(r+1)} \int_{0}^{1} \left( \frac{1}{q+1} x^{q+1} v_0^{k_2} + \frac{\varepsilon}{1+s} x^{1+s} + \frac{\alpha}{1+r} \theta \right)
$$
\nat  $G(v_0, x)$  is decreasing in  $v_0$ . Since  
\n
$$
G(0, x) = \frac{\alpha}{p(r+1)} \int_{0}^{1} \left( \frac{\varepsilon}{1+s} x^{1+s} + \frac{\alpha}{1+r} \theta \right)^{-\frac{p+1}{p}} d\theta
$$
\n
$$
G(0, +\infty), \text{ due to } s \in (0, p-1), \text{ we deduce by the domi}
$$

is integrable over  $(0,+\infty),$  due to  $s\in (0,p-1),$  we deduce by the dominate convergence theorem that

$$
f(x) = \frac{1}{p(r+1)} \int_{0}^{x} \left( \frac{1}{1+s} x^{1+s} + \frac{1}{1+r} \theta \right)
$$
  
\n
$$
\int_{0}^{\infty} \int_{0}^{\infty} \cos \theta \, dx \, dx \, dy
$$
  
\n
$$
\lim_{v_0 \to 0} \frac{g(v_0) - g(0)}{v_0^{k_1}} = -\int_{0}^{+\infty} G(0, x) \, dx < 0.
$$

is integrable over  $(0, +\infty)$ , due to  $s \in (0, p-1)$ , we deduce by the dominate convergence<br>theorem that<br> $\lim_{v_0 \to 0} \frac{g(v_0) - g(0)}{v_0^{k_1}} = -\int_0^{+\infty} G(0, x) dx < 0$ .<br>Thus we obtain that near the origin  $F_{\epsilon}$  is increasing if to  $r > s$ , and is decreasing when  $r < s$ . Similarly, we get that

$$
\lim_{v_0 \to 0} \frac{g(v_0) - g(0)}{v_0^{k_1}} = -\int_0^{+\infty} G(0, x) dx < 0.
$$
\nstation that near the origin  $F_e$  is increasing if  $k_1 > 1 - \frac{r+1}{p}$ , which is  $\epsilon$  and is decreasing when  $r < s$ . Similarly, we get that

\n
$$
\lim_{v_0 \to \infty} g'(v_0) v_0^{1 + (r+1)(\frac{1}{p} - \frac{1}{q+1})} = -\frac{\alpha}{p} \int_0^1 \left(\frac{1}{q+1} x^{q+1} + \frac{\alpha}{1+r}\right)^{-\frac{p+1}{p}} dx
$$

532 Y. J. Cheng<br>which implies that  $F_{\epsilon}$  is increasing for large  $v_0 > 0$ , because  $q > p - 1$ . Since  $\frac{dF_{\epsilon}}{dv_0}$  is<br>increasing in  $\varepsilon$  and tends to  $(1 - \frac{r+1}{p})c_2 v_0^{-\frac{r+1}{p}}$  uniformly in  $v_0$  as  $\varepsilon \to \infty$ , we  $\begin{array}{l} \text{vec }v_0>0,\text{ b} \ \text{uniform} \ \text{function} \ \text{function} \end{array}$ which implies that  $F_e$  is increasing for large  $v_0 > 0$ , because  $q > p - 1$ . Since  $\frac{dF_e}{dv_0}$  is increasing in  $\varepsilon$  and tends to  $(1 - \frac{r+1}{p})c_2 v_0^{-\frac{r+1}{p}}$  uniformly in  $v_0$  as  $\varepsilon \to \infty$ , we deduce in which implies that  $F_{\epsilon}$  is increasing for large  $v_0 > 0$ , becalincreasing in  $\epsilon$  and tends to  $\left(1 - \frac{r+1}{p}\right)c_2 v_0^{-\frac{r+1}{p}}$  uniformly in the case  $r > s$  that there is an  $\epsilon_0 \ge 0$  such that  $\min_{v_0} \frac{dF_{\epsilon}}{dv_0}$ the case  $r > s$  that there is an  $\varepsilon_0 \geq 0$  such that  $\min_{v_0} \frac{dF_e}{dv_0} > 0$  for  $\varepsilon > \varepsilon_0$ . Moreover, when  $\varepsilon = 0$ , *Follarge*  $v_0 > 0$ ,<br>  $(1 - \frac{r+1}{p}) c_2 v_0^{-\frac{r+1}{p}}$  uniform<br>  $\sin \epsilon_0 \ge 0$  such that min<br>  $F_0(v_0) = c_1 v_0^{1-\frac{q+1}{p}} + c_2 v_0^{1}$ <br>  $\frac{d}{dx}$  thus  $\frac{dF_0}{dx} < 0$  for small increasing in  $\varepsilon$  and tends to  $(1 - \frac{r+1}{p})c_2 v_0^{-\frac{r+1}{p}}$  uniformly in  $v_0$  as  $\varepsilon \to \infty$ , we deduce in<br>the case  $r > s$  that there is an  $\varepsilon_0 \ge 0$  such that  $\min_{v_0} \frac{dF_t}{dv_0} > 0$  for  $\varepsilon > \varepsilon_0$ . Moreover,<br>when

$$
F_0(v_0) = c_1 v_0^{1-\frac{q+1}{p}} + c_2 v_0^{1-\frac{r+1}{p}}
$$

 $\epsilon \in (0,\epsilon_0)$ . Whence, we obtain that the sign-changing solution is unique when  $\epsilon \geq \epsilon_0$ . If  $\varepsilon \in (0,\varepsilon_0)$ , using the fact that  $F_{\varepsilon}$  is increasing for both small and large  $v_0$ , we deduce that there are  $0 < v_1 < v_2 \le v_3 < +\infty$  such that  $F_{\epsilon}$  is increasing on  $(0, v_1)$  and  $(v_3, +\infty)$ and is decreasing on  $(v_1, v_2)$ . Thus let

$$
\lambda_1 = \min F_{\epsilon}(v_0), \quad \lambda_2 = F_{\epsilon}(v_1), \quad \lambda_3 = F_{\epsilon}(v_2), \quad \lambda_4 = \max_{F'_{\epsilon}(v_0) = 0} F_{\epsilon}(v_0).
$$

Then problem (1) has no sign-changing solutions if  $\lambda < \lambda_1$  and has at least two signchanging solutions  $\lambda \in (\lambda_1, \lambda_4)$ . In particular, it has at least three sign-changing solutions if  $\lambda \in (\lambda_2, \lambda_3)$  and has a unique sign-changing solution as  $\lambda > \lambda_4$ . The proof is complete I really solutions  $\lambda \in (\lambda_1, \lambda_4)$ . In particular, it has at least three sign-changing<br>
s if  $\lambda \in (\lambda_2, \lambda_3)$  and has a unique sign-changing solution as  $\lambda > \lambda_4$ . The p<br>
plete **I**<br> **Proof of Theorem 2.** The idea is the sa Thus let  $\lambda_2 = F_{\epsilon}(v_1)$ <br>gn-changin<br>gn-changin<br> $\lambda_4$ ). In particular is a unique<br>The idea is In view<br> $\frac{\epsilon}{+s}$   $u^{1+s}$ <br>akes the fo *r*<sub>9</sub>  $\leq$  *v*<sub>2</sub>  $\leq$  *v*<sub>3</sub>  $\leq$  *r*<sub>2</sub>  $\leq$  *v*<sub>3</sub>  $\leq$  *r*<sub>2</sub>  $\leq$  *v*<sub>3</sub>  $\leq$  *F<sub>c</sub>*(*v*<sub>1</sub>),  $\lambda_3 = F_e(v_2), \lambda_4 = \max_{F_e(v_0)=0} F_e(v_0).$ <br> *r*<sub>(1</sub>) has no sign-changing solutions if  $\lambda \leq \lambda_1$  and has at least two s

**Proof of Theorem** *2.* The idea is the same asin the proof of Theorem 1, so we will be sketch in many places. In view of

$$
F_{+}(u) = \frac{1}{q+1} u^{q+1} + \frac{\varepsilon}{1+s} u^{1+s} \quad \text{and} \quad F_{-}(u) = \frac{\alpha}{r+1} u^{r+1} + \frac{\delta}{r+1} u^{r+1}
$$

it follows that equation (9) takes the form

$$
(\lambda_2, \lambda_3) \text{ and has a unique sign-changing solution as } \lambda > \lambda_4. \text{ The proof is}
$$
\n
$$
\text{of Theorem 2. The idea is the same as in the proof of Theorem 1, so we}
$$
\n
$$
\frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} \qquad \text{and} \qquad F_{-}(u) = \frac{\alpha}{r+1}u^{r+1} + \frac{\delta}{r+1}u^{r+1}
$$
\n
$$
\text{at equation (9) takes the form}
$$
\n
$$
\int_{0}^{+\infty} \frac{du}{\sqrt[n]{\frac{1}{q+1}u^{q+1} + \frac{\varepsilon}{1+s}u^{1+s} + F_{-}(v_{0})}} \qquad (11)
$$
\n
$$
+ \int_{0}^{1} \frac{v_{0}dx}{\sqrt[n]{r(x)v_{0}^{s+1} + r(x)v_{0}^{r+1}}} = \frac{b-a}{2} \sqrt[n]{\alpha p' \lambda}
$$
\n
$$
r(x) = \frac{\alpha}{r+1} (1-x^{r+1}) \qquad \text{and} \qquad r(x) = \frac{\delta}{r+1} (1-x^{r+1}).
$$
\n
$$
\text{the proof of Theorem 1, we define}
$$
\n
$$
\int_{0}^{+\infty} \frac{1}{\sqrt[n]{r(x)v_{0}^{s+1} + r(x)v_{0}^{r+1}}} = \frac{\delta}{n} (1-x^{r+1}).
$$

where

$$
r(x) = \frac{\alpha}{r+1} (1 - x^{r+1}) \quad \text{and} \quad \tau(x) = \frac{\delta}{r+1} (1 - x^{r+1}).
$$
  
as in the proof of Theorem 1 we define  

$$
L(v_0) = \ell_1(v_0) + \ell_2(v_0)
$$

$$
\ell_1(v_0) = \int_0^{+\infty} \frac{du}{\sqrt[n]{\frac{1}{q+1} u^{q+1} + \frac{\epsilon}{1+s} u^{1+s} + F_-(v_0)}}
$$

$$
\ell_2(v_0) = \int_0^1 \frac{v_0 dx}{\sqrt[n]{r(x) v_0^{r+1} + r(x) v_0^{r+1}}} \dots
$$

Likely as in the proof of Theorem 1 we define

$$
L(v_0) = \ell_1(v_0) + \ell_2(v_0)
$$

where

$$
+\int_{0}^{1} \frac{v_{0}dx}{\sqrt[4]{r(x)v_{0}^{s+1} + r(x)v_{0}^{r+1}}} = \frac{b-a}{2}\sqrt[4]{\alpha p'\lambda}
$$
\n
$$
=\frac{\alpha}{r+1}(1-x^{r+1}) \quad \text{and} \quad \tau(x) = \frac{\delta}{r+1}(1-x^{r+1}).
$$
\n
$$
\text{coof of Theorem 1 we define}
$$
\n
$$
L(v_{0}) = \ell_{1}(v_{0}) + \ell_{2}(v_{0})
$$
\n
$$
\ell_{1}(v_{0}) = \int_{0}^{+\infty} \frac{du}{\sqrt[4]{\frac{1}{q+1}u^{q+1} + \frac{\epsilon}{1+s}u^{1+s} + F_{-}(v_{0})}}
$$
\n
$$
\frac{\ell_{2}(v_{0})}{\sqrt[4]{r(x)v_{0}^{r+1} + r(x)v_{0}^{r+1}}} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots
$$

The key points to prove that the function  $L(v_0)$  have a graph as shown in Figure 2 are

Some Results on an Elliptic Flow-Up Prot-  
uts to prove that the function 
$$
L(v_0)
$$
 have a graph as shown in  

$$
L(v_0) = F_0(v_0) = c_1 v_0^{\frac{p-q-1}{p}} + c_2 v_0^{\frac{p-r-1}{p}}
$$
when  $\varepsilon = \delta = 0$   
the property  $F'_n(v_0)(v_0 - v) > 0$  for some  $v > 0$  and L converge

which has the property  $F_0'(v_0)(v_0 - v) > 0$  for some  $v > 0$  and L converges to  $F_0$  on compacta of  $(0, +\infty)$  in  $C^1$  as  $\varepsilon, \delta \to 0$  and  $L \to 0$  as  $v_0 \to +\infty$ .  $L(v_0) = F_0(v_0) = c_1 v_0^{\frac{p-q-1}{p}} + c_2 v_0^{\frac{p-r-1}{p}}$  when  $\varepsilon = \delta$ <br>
i.e property  $F'_0(v_0)(v_0 - v) > 0$  for some  $v > 0$  and  $L \in (0, +\infty)$  in  $C^1$  as  $\varepsilon, \delta \to 0$  and  $L \to 0$  as  $v_0 \to +\infty$ .<br>
lete the proof we need to show two

To complete the proof we need to show two more things: Monotonicity of *L* near the origin and  $L(0) \leq L(v_2)$  where  $v_2$  is the second local maximum of *L*. But

$$
L(0) = c_1 \varepsilon^{\frac{q-p-1}{p(q-r)}} \qquad \text{and} \qquad L(\delta^{\frac{1}{r-r}}) \geq \ell_2(\delta^{\frac{1}{r-r}}) = c_2 \delta^{\frac{r+1-p}{p(r-r)}}
$$

for some constants  $c_1 > 0$  and  $c_2 > 0$  from which it follows that if  $\varepsilon$  and  $\delta$  satisfy

To get the monotonicity of *L* in the nearby of the origin, we exploit the technique in the proof of Theorem 1 and can show that

trigin and 
$$
L(0) \leq L(v_2)
$$
 where  $v_2$  is the second local maximum of  $L$ . But

\n
$$
L(0) = c_1 \varepsilon^{\frac{q-p-1}{p(q-1)}}
$$
\nand

\n
$$
L(\delta^{\frac{1}{r-r}}) \geq \ell_2(\delta^{\frac{1}{r-r}}) = c_2 \delta^{\frac{r+1-p}{p(r-r)}}
$$
\nwhere constants  $c_1 > 0$  and  $c_2 > 0$  from which it follows that if  $\varepsilon$  and  $\delta$  is  $\frac{p-1}{p-1} \leq c_2 \delta^{\frac{r+1-p}{p(r-r)}}$ , then we are done.

\nSo get the monotonicity of  $L$  in the nearby of the origin, we exploit the technic property.

\nFrom  $\ell_1(v_0) - \ell_1(0) = -\frac{\alpha}{p(r+1)} \int_0^{+\infty} dx \int_0^1 \left( \frac{\varepsilon}{1+s} x^{s+1} + \frac{\alpha}{r+1} \theta \right)^{-\frac{p+1}{p}} d\theta < 0$ 

\nwe have used that  $r > \tau$  and  $F_{-}(v_0) = \frac{\alpha}{r+1} v_0^{r+1} (1 + o(1))$  as  $v_0 \to 0$ ,

\n
$$
\lim_{v_0 \to 0} \frac{\ell_2(v_0) - \ell_2(0)}{v_0^{k_3}} = \int_0^1 \frac{dx}{\sqrt[3]{r(x)}} > 0
$$
\nand

\n
$$
\lim_{v_0 \to 0} \frac{L(v_0) - L(0)}{v_0^{k_1}} < 0
$$
\nby, the uniqueness follows from the results in [4] that the function  $\ell_2$  has a constant, the function  $\ell_1$  is a constant.

where we have used that  $r > \tau$  and  $F_-(v_0) = \frac{\alpha}{r+1} v_0^{r+1} (1 + o(1))$  as  $v_0 \to 0$ ,

$$
\lim_{\delta \to 0} \frac{\ell_1(v_0) - \ell_1(v)}{v_0^{k_1}} = -\frac{\alpha}{p(r+1)} \int_0^r dx \int_0^r \left( \frac{\epsilon}{1+s} x^{s+1} + \frac{\alpha}{r+1} \theta \right)^{-\frac{1}{p}} d\theta <
$$
\nwe have used that  $r > \tau$  and  $F_{-}(v_0) = \frac{\alpha}{r+1} v_0^{r+1} (1 + o(1))$  as  $v_0 \to 0$ ,  
\n
$$
\lim_{v_0 \to 0} \frac{\ell_2(v_0) - \ell_2(0)}{v_0^{k_3}} = \int_0^1 \frac{dx}{\sqrt[3]{r(x)}} > 0 \quad \text{and} \quad \lim_{v_0 \to 0} \frac{L(v_0) - L(0)}{v_0^{k_1}} < 0.
$$

Finally, the uniqueness follows from the results in [4] that the function  $\ell_2$  has the following property: there is  $v > 0$  such that  $\ell_2^{\gamma}(v_0) > 0$  for  $v_0 \in (0, v)$  and  $\ell_2^{\gamma}(v_0) < 0$ for  $v_0 \in (v, +\infty)$ . Then  $L'(v_0) < 0$  for  $v_0 \in [v, +\infty)$  since clearly  $\ell'_1(v_0) < 0$  for  $v_0 > 0$ . The proof is complete  $\blacksquare$  $\frac{1}{2} = \int_{0}^{1} \frac{d}{\sqrt[3]{r}}$ <br>follows from  $L'(v_0) < 0$ <br>*L'*( $v_0$ ) < 0<br>**1** 3. By vir the reduction  $v_0 \in$ <br>of  $p =$ 

**Proof of Theorem 3.** By virtue of  $p = 2$  and

$$
F_{+}(u) = \frac{\alpha}{q+1} u^{q+1}
$$
 and  $F_{-}(u) = \frac{1}{2} u^{2} + \beta (\sin u - u \cos u)$ 

we have that

$$
p(r+1) \int_{0}^{r} \int_{0}^{r} (1+s)^{r} f(t) dt
$$
\nwe have used that  $r > r$  and  $F_{-}(v_{0}) = \frac{\alpha}{r+1} v_{0}^{r+1} (1+o(1))$  as  $v_{0} \to 0$ ,  
\n
$$
\lim_{v_{0} \to 0} \frac{\ell_{2}(v_{0}) - \ell_{2}(0)}{v_{0}^{k_{3}}} = \int_{0}^{1} \frac{dx}{\sqrt[3]{r(x)}} > 0 \quad \text{and} \quad \lim_{v_{0} \to 0} \frac{L(v_{0}) - L(0)}{v_{0}^{k_{1}}} < 0.
$$
\nthe uniqueness follows from the results in [4] that the function  $\ell_{2}$  has  
\n $v_{0} \to 0$  for  $v_{0} \in (0, v)$  and  $\ell_{2}(v_{0})$  for  
\n $v_{0} \in (0, v)$  and  $\ell_{2}(v_{0}) \in (v, +\infty)$ . Then  $L'(v_{0}) < 0$  for  $v_{0} \in [v, +\infty)$  since clearly  $\ell_{1}'(v_{0}) < 0$  for  $v$  of is complete  $\blacksquare$   
\n
$$
\text{soof of Theorem 3. By virtue of } p = 2 \text{ and}
$$
\n
$$
F_{+}(u) = \frac{\alpha}{q+1} u^{q+1} \quad \text{and} \quad F_{-}(u) = \frac{1}{2} u^{2} + \beta(\sin u - u \cos u)
$$
\n
$$
\text{the that}
$$
\n
$$
\int_{0}^{+\infty} \frac{du}{\sqrt{F_{+}(u) + F_{-}(v_{0})}} = c(q) \left(\frac{1}{2} v_{0}^{2} + \beta(\sin v_{0} - v_{0} \cos v_{0})\right)^{\frac{1}{r+1} - \frac{1}{2}} =: g_{1}(v_{0})
$$
\n
$$
\int_{0}^{v_{0}} \frac{du}{\sqrt{F_{-}(v_{0}) - F_{-}(u)}} = \int_{0}^{1} \frac{v_{0} dx}{\sqrt{h(x, v_{0})}} =: g_{2}(v_{0})
$$

where

 $\label{eq:2.1} \frac{1}{\sqrt{2\pi}}\int_{0}^{\pi} \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)^{2}d\mu$ 

ng  
\n
$$
h(x, v_0) = e(x) v_0^2 + \beta \varepsilon(x, v_0)
$$
\n
$$
\varepsilon(x, v_0) = \sin v_0 - \sin(xv_0) - v_0 (\cos v_0 - x \cos(xv_0))
$$
\n
$$
c(q) = \int_0^{+\infty} \left(1 + \frac{\alpha}{q+1} u^{q+1}\right)^{-\frac{1}{2}} du
$$
\n
$$
e(x) = \frac{1-x^2}{2}
$$

and the function in the left-hand side of (9) takes the form

$$
g_1(v_0)+g_2(v_0):=H(v_0).
$$

Clearly, *H* is well defined and continuous on  $(0, +\infty)$ . Since  $q > 3$ , we see that

$$
g_1(v_0) \to +\infty \quad \text{as} \quad v_0 \to 0+
$$

and

$$
e(x) = \frac{1 - x^2}{2}
$$
  
the left-hand side of (9) takes the form  
 $g_1(v_0) + g_2(v_0) := H(v_0).$   
effined and continuous on  $(0, +\infty)$ . Since  $q > 3$ ,  
 $g_1(v_0) \rightarrow +\infty$  as  $v_0 \rightarrow 0+$   
 $g_1(v_0) \rightarrow 0$   
 $g_2(v_0) \rightarrow \int_0^1 \frac{dx}{\sqrt{e(x)}} = \frac{\pi}{\sqrt{2}}$  as  $v_0 \rightarrow +\infty$   
 $\Rightarrow v_0 \rightarrow +\infty$  and  $H(+\infty) = \frac{\pi}{2}$ .

and therefore  $H(0+) = +\infty$  and  $H(+\infty) = \frac{\pi}{\sqrt{2}}$ .

To show the results (ii) and (iii) we need to prove that *H* oscillates around the line  $H=\frac{\pi}{\sqrt{2}}$ , and first show that  $g_1$  oscillates around  $H=\frac{\pi}{\sqrt{2}}$ .

Choosing  $v_0 = n\pi$  ( $n \in \mathbb{N}$  is even) we get

$$
h(x,n\pi) = (n\pi)^2 e(x) - \beta \Big( \sin(n\pi x) + n\pi (1 - x \cos(n\pi x)) \Big)
$$
  
\n
$$
\leq (n\pi)^2 e(x) - \beta \Big( n\pi (1 - x) - \sin (n\pi (1 - x)) \Big)
$$
  
\n
$$
\leq (n\pi)^2 e(x)
$$

thereafter

$$
\pi)^{-}e(x)
$$
\n
$$
g_2(n\pi) > \int\limits_0^1 \frac{n\pi dx}{\sqrt{e(x)(n\pi)^2}} = \frac{\pi}{\sqrt{2}}.
$$

Analogously, for odd integer *n*,  $g_2(n\pi) < \frac{\pi}{\sqrt{2}}$ .

To show the function  $H$  is also oscillating, it suffices to show that, for odd  $n$ , To show the function *H* is also oscillating, it suffices to show that, for odd *n*  $H(n\pi) < \frac{\pi}{\sqrt{2}}$  since  $g_1 > 0$  and  $g_2(n\pi) > \frac{\pi}{\sqrt{2}}$  for even *n*. For this purpose, first we have

$$
\frac{1}{\sqrt{2}} \sqrt{e(x)/(n\pi)} = \sqrt{2}
$$
  
\nby, for odd integer *n*,  $g_2(n\pi) < \frac{\pi}{\sqrt{2}}$ .  
\nby the function *H* is also oscillating, it suffices to show that, for  $\frac{\pi}{\sqrt{2}}$  since  $g_1 > 0$  and  $g_2(n\pi) > \frac{\pi}{\sqrt{2}}$  for even *n*. For this purpose, first 
$$
\frac{\pi}{\sqrt{2}} - g_2(n\pi) = \int_0^1 \left( \frac{n\pi}{\sqrt{e(x)(n\pi)^2}} - \frac{n\pi}{\sqrt{e(x)(n\pi)^2 + \beta \epsilon(x, n\pi)}} \right) dx
$$

$$
= \frac{1}{2} \int_0^1 dx \int_0^1 \frac{n\pi \beta \epsilon(x, n\pi)}{(e(x)(n\pi)^2 + \beta \beta \epsilon(x, n\pi))^{\frac{3}{2}}} d\theta
$$

$$
\geq \frac{1}{2} \int_0^1 \frac{n\pi \beta \epsilon(x, n\pi)}{(e(x)(n\pi)^2 + \beta \epsilon(x, n\pi))^{\frac{3}{2}}} dx, \quad \dots
$$

due to  $\varepsilon(x, n\pi) > 0$  for all  $x \in (0,1)$ . Using  $f_-(u) \ge (1 - \beta)u$ , we deduce that

$$
e(x)(n\pi)^2 + \beta \varepsilon(x, n\pi) = n\pi(1-x) \int_0^1 f_{-}((x+t-tx)n\pi) dt
$$
  
\n
$$
\ge n\pi(1-x) \int_0^1 (1-\beta)(x+t-tx)n\pi dt
$$
  
\n
$$
\ge \frac{1}{2}(1-\beta)(1-x)(n\pi)^2
$$
  
\n0,1). Therefore  
\n
$$
\frac{\pi}{\sqrt{2}} - g_2(n\pi) \ge \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}}(n\pi)^{-2} \int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}} dx,
$$
  
\n
$$
\text{angle of variable } z = \frac{1}{\sqrt{1-x}} \text{ (then } x = 1 - \frac{1}{z^2})
$$

for all  $x \in (0,1)$ . Thereafter

$$
\frac{\pi}{\sqrt{2}}-g_2(n\pi)\geq \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}}(n\pi)^{-2}\int\limits_{0}^{1}\frac{\varepsilon(x,n\pi)}{(1-x)^{\frac{3}{2}}}dx,
$$

and by a change of variable  $z = \frac{1}{\sqrt{1-x}} (\text{then } x = 1 - \frac{1}{z^2})$ 

$$
\frac{\pi}{\sqrt{2}} - g_2(n\pi) \ge \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}}(n\pi)^{-2} \int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}} dx,
$$
\na change of variable  $z = \frac{1}{\sqrt{1-x}}$  (then  $x = 1 - \frac{1}{z^2}$ )  
\n
$$
\int_0^1 \frac{\varepsilon(x, n\pi)}{(1-x)^{\frac{3}{2}}} dx = 2 \int_1^{+\infty} \varepsilon \left(1 - \frac{1}{z^2}, n\pi\right) dz
$$
\n
$$
= 2 \int_1^{+\infty} \left(-\sin\left(\frac{n\pi}{z^2}\right) + n\pi \left(1 - \left(1 - \frac{1}{z^2}\right)\cos\frac{n\pi}{z^2}\right)\right) dz \text{ or}
$$
\n
$$
= 2 \int_1^{+\infty} \left(n\pi(1 - \cos\frac{n\pi}{z^2}) + (-\sin\frac{n\pi}{z^2} + \frac{n\pi}{z^2}\cos\frac{n\pi}{z^2})\right) dz
$$
\nwe have used *n* being an odd integer. Change variables once more, we obtain

where we have used *n* being an odd integer. Change variables once more, we obtain

$$
-\frac{1}{2}\int_{1}^{1} \frac{\tan(1 - \cos z)}{z^{2}} dz - \frac{1}{2} \int_{2}^{2} \cos z^{2}
$$
\nused *n* being an odd integer. Change variables once more  
\n
$$
\int_{1}^{+\infty} \left(1 - \cos \frac{n\pi}{z^{2}}\right) dz = \int_{\frac{1}{\sqrt{n\pi}}}^{+\infty} \left(1 - \cos \frac{1}{w^{2}}\right) \sqrt{n\pi} dw \ge C\sqrt{n\pi}
$$
\n
$$
\int_{1}^{+\infty} \left(-\sin \frac{n\pi}{z^{2}} + \frac{n\pi}{z^{2}} \cos \frac{n\pi}{z^{2}}\right) dz \le \int_{1}^{+\infty} \frac{2n\pi}{z^{2}} dz = 2n\pi.
$$

where  $C = \int_1^{+\infty} (1 - \cos \frac{1}{z^2}) dz$ . On the other hand, where we have used<br>  $+\infty$ <br>  $\int_{1}^{+\infty}$ <br>
where  $C = \int_{1}^{+\infty} (1 - \frac{1}{1})$ <br>
Whence

$$
\left(1 - \cos\frac{n\pi}{z^2}\right)dz = \int\limits_{\frac{1}{\sqrt{n\pi}}} \left(1 - \cos\frac{1}{w^2}\right)\sqrt{n\pi} \, dw \ge C\sqrt{r}
$$
\n
$$
\left(1 - \cos\frac{1}{z^2}\right)dz. \text{ On the other hand,}
$$
\n
$$
\int\limits_{1}^{+\infty} \left(-\sin\frac{n\pi}{z^2} + \frac{n\pi}{z^2}\cos\frac{n\pi}{z^2}\right)dz \le \int\limits_{1}^{+\infty} \frac{2n\pi}{z^2} \, dz = 2n\pi.
$$
\n
$$
\int\limits_{0}^{1} \frac{\varepsilon(x, n\pi)}{(1 - x)^{\frac{3}{2}}} \, dx \ge 2(Cn\pi\sqrt{n\pi} - 2n\pi)
$$

$$
\int\limits_{0}^{1}\frac{\varepsilon(x,n\pi)}{(1-x)^{\frac{3}{2}}}dx\geq2(Cn\pi\sqrt{n\pi}-2n\pi)
$$

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and consequetly for large *n* the estimate

$$
a \text{ the estimate}
$$
  

$$
\frac{\pi}{\sqrt{2}} - g_2(n\pi) \ge \frac{2^{\frac{3}{2}}\beta}{(1-\beta)^{\frac{3}{2}}}C(n\pi)^{-\frac{1}{2}}
$$

holds. Combing the above estimates, we obtain by noting  $g_1(n\pi) \leq C_1(n\pi)^{\frac{2}{q+1}-1}$  that

$$
\frac{\pi}{\sqrt{2}} - g_2(n\pi) \ge \frac{2^{\frac{n}{2}}\beta}{(1-\beta)^{\frac{3}{2}}} C(n\pi)^{-\frac{1}{2}}
$$
  
Combing the above estimates, we obtain by noting  $g_1(n\pi) \le C_1(n\pi)^{\frac{2}{q+1}}$ .  

$$
\frac{\pi}{\sqrt{2}} - H(n\pi) = \frac{\pi}{\sqrt{2}} - g_2(n\pi) - g_1(n\pi) \ge C_2(n\pi)^{-\frac{1}{2}} - C_1(n\pi)^{\frac{2}{q+1}-1} > 0
$$
  
 $\frac{1}{2} > \frac{2}{q+1} - 1$  (this is the source for the condition on *q*).  
eritions (ii) and (iii) follow from the oscillatory property of *H*. Using  

$$
\lim_{v_0 \to \infty} H(v_0) = \frac{\pi}{\sqrt{2}}, \qquad \lim_{v_0 \to 0+} H(v_0) = +\infty, \qquad H(n\pi) < \frac{\pi}{\sqrt{2}}
$$
  
ce from the continuity of *H* that the minimum of  $H(v_0)$  on  $(0, +\infty)$  achi

since  $-\frac{1}{2}$  >  $\frac{2}{q+1}$  - 1 (this is the source for the condition on *q*).

Assertions (ii) and (iii) follow from the oscillatory property of *H.* Using

$$
\sqrt{2}
$$
  
\n $\frac{2}{q+1} - 1$  (this is the source for the condition on *q*).  
\n $\ln S$  (ii) and (iii) follow from the oscillatory property of *H*. Using  
\n
$$
\lim_{v_0 \to \infty} H(v_0) = \frac{\pi}{\sqrt{2}}, \qquad \lim_{v_0 \to 0+} H(v_0) = +\infty, \qquad H(n\pi) < \frac{\pi}{\sqrt{2}}
$$

we deduce from the continuity of *H* that the minimum of  $H(v_0)$  on  $(0, +\infty)$  achives and thus the equation  $H(v_0) = \mu$  is solvable if and only if  $\mu \ge \min H(v_0)$  which complete the proof of assertion (i)  $\blacksquare$ 

## 4. Final remarks

In this note we have only carried out some basic calculations to exhibit the rich structure for boundary blow-up problems, even it is very elementary (just calculus), but it is certainly not easy to give a complete bifurcation picture for all involved parameters, for instance  $q, r, s, \tau, \varepsilon, \delta, \alpha$  in Theorem 2. From our one-dimensional examples we can see that there is a big difference between Dirichlet boundary value problems and boundary bolw-up problems. If the boundary is Dirichlet, then there are infinitely many sign solutions in the superlinear case, but it can have only finite number of sign solutions for boundary bolw-up problems (note, the function in (2) is not superlinear at  $-\infty$ ). As our examples are of one-dimensional character, one may say that it would not be representive, so it will be interesting to study those equations in two-dimensional or **4. Final remarks**<br>In this note we have only carried out some basic calculation<br>for boundary blow-up problems, even it is very element:<br>certainly not easy to give a complete bifurcation picture fo<br>instance  $q, r, s, \tau, \varepsilon,$ 

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