Estimates for Quasiconformal Mappings
onto
Canonical Domains

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Abstract. In this paper we establish estimates for K-quasiconformal mappings $z = g(w)$ of a domain bounded by two circles $|w| = 1, |w| = q$ and n continua situated in $q < |w| < 1$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along n arcs on the circles $|z| = R_j(g)$ $(j = 1, \ldots, n)$ such that $|z| = 1$ and $|z| = Q$ correspond to $|w| = 1$ and $|w| = q$, respectively. The bounds in the estimates for Q, R_j and $|g(w)|$ are explicitly given, most of them are optimal. They are deduced mainly from [17].

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1. Introduction and notations

The generalization of Carleman's (see [1: p. 212], [2: p. 177], [12: p. 15]) area inequality for doubly-connected domains to multiply-connected domains in [15] improves many Grötzsch's [4, 6, 8] and Rengel's [13] significant circular slits theorems for conformal mappings. In [16] we establish further area inequalities for *K-quasiconformal mappings* (see the definition in [10: p. 16]). Combining this with Grötzsch's [4, 5, 7] inequalities yields in [17] sharp or asymptotic sharp estimates for K-quasiconformal mappings of the circular ring $Q < |z| < 1$ with concentric circular slits onto domains lying in $q < |w| < 1$. In this paper we establish estimates for the inverse mappings of those studied in [17]. Here the consideration is partly similar to the case of conformal mappings ($K = 1$, see [14: pp. 121 - 124]) using two auxiliary functions introduced in Section 4.

Let now B be any domain given in the w-plane, bounded by two circles $|w| = 1$, $|w| = q$ and pn $(p, n \in \mathbb{N})$ boundary components $\sigma_1, \ldots, \sigma_{pn}$ lying in $(0 <) q < |w| < 1$, and transformed into itself by the rotation $t = e^{i\frac{2\pi}{p}} w$. We shall write $B = B_0$ when all σ_j are circular arcs concentric with the circular ring. Let G be the family of all K-quasiconformal mappings $z = g(w)$ each of which maps B onto a circular ring $Q(g) < |z| < 1$ that has been slit along pn circular arcs $L_1(g), \ldots, L_{pn}(g)$ concentric with the

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circular ring such that $|z| = 1$, $|z| = Q$ and L_j correspond to $|w| = 1$, $|w| = q$ and σ_j $(j = 1, \ldots, pn)$, respectively, $g(1) = 1$ and satisfies the p times rotation symmetry

$$g(e^{i\frac{2\pi}{p}}w) = e^{i\frac{2\pi}{p}}g(w) \qquad (w \in B). \tag{1.1}$$

It is clear that the symmetry condition is trivial for $p = 1$. Therefore we see that the inverse mapping f of g, $g \in G$, belongs to the family F studied in [17], because from (1.1) for all $z \in A$, $A = g(B)$, and $f = g^{-1}$, $g \in G$, the relation

$$e^{i\frac{2\pi}{p}}f(z) = f(e^{i\frac{2\pi}{p}}z)$$

follows. Put

$R_j(g) = |z|$ with $z \in L_j(g)$ and $g \in G$

$c_j = \min|w|$ and $d_j = \max|w|$ with $w \in \sigma_j$ $(j = 1, \ldots, pn)$

$c = \min\{c_1, \ldots, c_{pn}\}$ and $d = \max\{d_1, \ldots, d_{pn}\}$

$\mu = \sup \prod_{j=1}^{J} \frac{r_j'}{r_j}$, where $r_j < |w| < r_j'$ are pairwise disjoint sets in B.

Furthermore, denote

by S the inner area of B

by s_j the external area of the compact set bounded by σ_j.

Clearly,

$$\frac{c}{qd} \leq \mu \leq \frac{1}{q} \tag{1.2}$$

and

$$S + s = \pi(1 - q^2) \qquad \text{with} \quad s = \sum_{j=1}^{pn} s_j. \tag{1.3}$$

Our task is to estimate the radii $Q(g)$ and $R_j(g)$ $(g \in G; j = 1, \ldots, pn)$ that are nothing but the *Riemann moduli* of the domain B in the case $K = 1$ (see [11: p. 334]); as well as $|g(w)|$ $(g \in G, w \in B)$, by at most the quantities $K, p, q, c_j, d_j, s_j, \mu, S$ and $|w|$. The bounds in the estimates will be explicitly calculated by simple functions or the auxiliary function $R(p, t, s)$ introduced in Section 4. Most of them are the best among the bounds that depend on the same quantities.

2. Estimates of Q

The estimate of Q plays an important role in establishing estimates for other quantities. Therefore we begin with this estimate.

Theorem 1. *Under the above hypothesis and notations we have, for every $g \in G$, the estimate*

$$\left(1 + \frac{S}{\pi q^2}\right)^{-\frac{K}{2}} \leq Q(g) \leq \mu^{-\frac{1}{K}}, \tag{2.1}$$

where the equality on the left-hand side holds if and only if $B = B_0$ and $g(w) = w|w|^{K-1}$ $(w \in B)$, and the equality on the right-hand side holds if and only if $B = B_0$ and $g(w) = w|w|^{\frac{1}{K}-1}$ $(w \in B)$.

Proof. Applying [17: Theorem 3.1] to the mapping $f = g^{-1}$, $g \in G$, we have

$$S \geq \pi q^2 (Q^{-\frac{2}{K}} - 1),$$

hence the lower bound of Q in (2.1) follows. Here the equality holds if and only if $f(z) = z|z|^{\frac{1}{K}-1}$ ($z \in A$), i.e. $B = B_0$ and $g(w) = w|w|^{K-1}$ ($w \in B$). On the other hand, applying [16: Theorem 1] to the mapping $g \in G$, we obtain

$$\pi \geq \pi Q^2 \mu^{\frac{2}{K}},$$

hence the upper bound of Q in (2.1) follows. Here with the help of [17: Formula (2.5)] we have the assertion on the occurence of the equality ∎

Corollary 1. *By (1.3), the lower bound of Q in (2.1) may be written in the form*

$$Q(g) \geq q^K \left(1 - \frac{s}{\pi} \right)^{-\frac{K}{2}} \qquad (g \in G) \tag{2.2}$$

hence

$$Q(g) \geq q^K \qquad (g \in G). \tag{2.3}$$

Equality in (2.2) and (2.3) can only occur if $B = B_0$ and $g(w) = w|w|^{K-1}$ ($w \in B$).

Corollary 2. *From (2.1) and (1.2) we obtain the estimate*

$$Q(g) \leq \left(\frac{dq}{c} \right)^{\frac{1}{K}} \ (< 1) \qquad (g \in G)$$

where the equality can only occur if $c = d$ and $g(w) = w|w|^{\frac{1}{K}-1}$ ($w \in B$).

3. Lower bounds of R_j

Since $R_j(g) > Q(g)$ ($g \in G$; $j = 1, \ldots, pn$), with the help of (2.2) or (2.3) we can get lower bounds of R_j. However, we want to establish other relations that, in certain situations, may give sharper estimates.

Theorem 2. *Under the hypothesis and notations given in Section 1, for every $g \in G$ with $Q(g) = Q > q^K$ we have the estimates*

$$R_j(g) > Q \left[\frac{ps_j}{\pi (Q^{\frac{2}{K}} - q^2)} \right]^{\frac{K}{2}} \qquad (j = 1, \ldots, pn) \tag{3.1}$$

and

$$\max_{1 \leq j \leq pn} R_j(g) > Q \left[\frac{s}{\pi (Q^{\frac{2}{K}} - q^2)} \right]^{\frac{K}{2}}. \tag{3.2}$$

Proof. First, we notice that by (1.1), each circle $|z| = R_j$ contains at least p slits belonging to the boundary of $A = g(B), g \in G$. By [17: Formula (3.1)], we therefore have

$$ps_j \leq \pi R_j^{\frac{2}{K}} (1 - q^2 Q^{-\frac{2}{K}}),$$

hence for every $g \in G$ with $q < Q^{\frac{1}{K}}$, i.e., $Q(g) > q^K$, estimate (3.1) follows. Similarly we obtain, with the help of [17: formula (3.2)], estimate (3.2) ∎

4. The auxiliary functions $R(p, t, s)$ and $T(p, r, s)$

In order to establish other estimates of R_j, Q and $|g(w)|$ we will introduce the following two real functions.

Definition. The real functions

$$\left.\begin{array}{l} r = R(p, t, s) \quad (0 \le s < t < 1) \\ t = T(p, r, s) \quad (0 \le s < r < 1) \end{array}\right\} \quad (p \in \mathbb{N})$$

are defined in such a way that the circular ring $s < |w| < 1$ with p radial slits

$$P_j = \left\{ w \,\middle|\, s \le |w| \le t \text{ and } \arg w = j\frac{2\pi}{p} \right\} \quad (j = 0, \ldots, p-1)$$

and the circular ring $r < |z| < 1$ can be schlicht conformal mapped onto each other.

Because of the monotony of the modulus of a doubly-connected domain (see [2: p. 176]) we have the following monotonies of the auxiliary function $R(p, t, s)$ with $p \in \mathbb{N}$:

$$s < R(p, t, s) < t \quad (0 \le s < t < 1) \tag{4.1}$$
$$R(p, t_1, s) < R(p, t_2, s) \quad (0 \le s < t_1 < t_2 < 1) \tag{4.2}$$
$$R(p, t, s_1) < R(p, t, s_2) \quad (0 \le s_1 < s_2 < t < 1) \tag{4.3}$$
$$R(p, t, s) > R(1, t, s) \quad (0 \le s < t < 1; p \ge 2)$$

With the help of Hersch's [9: p. 316] and Nehari's [11: p. 295] formulae, I myself found in [14: pp. 101 - 104] the following expression for $R(p, t, s)$:

$$R(p, t, 0) = \exp\left\{ \frac{-\pi K'(t^p)}{2pK(t^p)} \right\} \quad (0 < t < 1; p \in \mathbb{N}) \tag{4.4}$$

with

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$
$$K'(k) = K(\sqrt{1 - k^2})$$

and for $0 < s < t < 1$

$$R(p, t, s) = \exp\left\{ \frac{-\pi K'(u)}{2pK(u)} \right\}$$

with

$$u = 1 + h - \sqrt{h(2 + h)}$$

where

$$h = \frac{(1 - k)(1 - ak)}{k(1 + a)}, \quad k = 4s^p \prod_{n=1}^{\infty} \left[\frac{1 + s^{4pn}}{1 + s^{2p(2n-1)}} \right]^4$$

$$a = sn\left(b + \frac{i2pb}{\pi} \log\frac{t}{s}, k \right), \quad b = K(k).$$

Here $sn(z, k)$ means the Jacobian elliptic sinus with the parameter k. Another expression for $R(p, t, s)$ was shown by Graeser in [3: pp. 77 - 78].

In view of [14: Formula (1.21)] we have the estimate

$$4^{-\frac{1}{p}}t < R(p, t, s) < t \qquad (0 \le s < t < 1; \, p \in \mathbb{N}), \tag{4.5}$$

hence $R(p, t, s) \approx t$ when $p \to \infty$.

The evaluation of $K(t^p)$ and $K'(t^p)$ (see [18: p. 177]) yields the asymptotic behaviour of $R(p, t, 0)$:

$$R(p, t, 0) \approx 4^{-\frac{1}{p}}t \qquad \text{when } t \to 0 \tag{4.6}$$

and

$$1 - R(p, t, 0) \approx \frac{\pi^2}{2p \log \frac{8}{p(1-t)}} \qquad \text{when } t \to 1.$$

Successive approximations for $R(1, t, 0)$ are given by Lehto [10: p. 64]. The expression for $T(p, r, s)$, that is not needed here, was shown by Thao ([14: pp. 102 - 105] or [17: p. 61]).

5. Other estimates of R_j, Q and $|g(w)|$

Using the auxiliary functions studied in Section 4, other estimates for R_j, Q and $|g(w)|$ will be given. In particular, when $s_j = 0$ or $s = 0$, they may be sharper than ones of (3.1) and (2.2).

Theorem 3. *Under the hypothesis and notations given in Sections 1 and 4, for every $g \in G$, $w \in B$ and $j = 1, \ldots, pn$, we have the estimates*

$$R^K(p, d_j, q) < R_j(g) < Q(g)R^{-K}\left(p, \frac{q}{c_j}, q\right) \tag{5.1}$$

$$Q(g) > R^K(p, d_j, q) R^K\left(p, \frac{q}{c_j}, q\right) \tag{5.2}$$

$$R^K(p, |w|, q) < |g(w)| < Q(g)R^{-K}\left(p, \frac{q}{|w|}, q\right). \tag{5.3}$$

Proof. Considering the mapping $f = g^{-1}$, $g \in G$, with the help of [17: Theorem 6.1] we obtain

$$d_j < T(p, R_j^{\frac{1}{k}}, q) = t \qquad \text{and} \qquad \frac{q}{c_j} < T\left[p, \left(\frac{Q}{R_j}\right)^{\frac{1}{k}}, q\right] = t'.$$

Hence the definition of the auxiliary functions and the monotony (4.2) yield the relations

$$R_j^{\frac{1}{k}} = R(p, t, q) > R(p, d_j, q) \qquad \text{and} \qquad \left(\frac{Q}{R_j}\right)^{\frac{1}{k}} = R(p, t', q) > R\left(p, \frac{q}{c_j}, q\right).$$

Thus we have estimate (5.1). Estimate (5.2) is just a consequence of (5.1). Using [17: Formula (6.10)], we obtain similar estimate (5.3) ∎

Corollary 3. *From Theorem 3 and (4.5), for every $g \in G$, $w \in B$ and $j = 1, \ldots, pn$ we obtain the simple estimates*

$$4^{-\frac{K}{p}} d_j^K < R_j(g) < 4^{\frac{K}{p}} Q(g) \left(\frac{c_j}{q}\right)^K \tag{5.4}$$

$$Q(g) > 4^{-2\frac{K}{p}} \left(\frac{q d_j}{c_j}\right)^K \tag{5.5}$$

$$4^{-\frac{K}{p}} |w|^K < |g(w)| < 4^{\frac{K}{p}} Q(g) \left(\frac{|w|}{q}\right)^K. \tag{5.6}$$

In view of (4.3) and (4.6) we see that the coefficients in (5.4) - (5.6) are the best possible.

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