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# Nonlinear Hyperbolic Equations with Dissipative Temporal and Spatial Non-Local Memory

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**Abstract.** The equation governing the evolution of a displacement vector in an elastic body with dissipative temporal and spatial non-local memory is considered. The memory term is generated by a singular but integrable kernel. The existence of a global weak solution to the associated initial-boundary problem is established by constructing Galerkin approximations and deriving a suitable energy estimate.

**Keywords:** *Singular viscoelasticity, energy estimates, weak solutions*

**AMS subject classification:** 45 K 05, 73 F 15

## 1. Introduction

In this paper, the equation governing the evolution of a displacement vector in an elastic body is investigated. The body is assumed to occupy a reference domain $\Omega \subset \mathbb{R}^N$ at an initial time and to have unit density. The vector $u = (u_1, \ldots, u_N)$ represents the displacement and from Newton's laws of motion we obtain for it the wave equation

$$\ddot{u} = \operatorname{div} \sigma + f \tag{1}$$

where $\sigma_{ij}$ is the Cauchy stress tensor and $f = (f_1, \ldots, f_N)$ is the external body force per unit mass.

This equation holds for both the elastic and the plastic cases. The properties of a material are expressed by the constitutive law, which describes the relation between the stress $\sigma$ and the infinitesimal strain tensor $e_{ij}u = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The stress is a function of $eu$ for an elastic material, while it depends on the velocity of $eu$ in the case of the plastic one. In the linear cases, the relation in an elastic body is expressed by Hook's law (then (1) is a hyperbolic equation), while for a plastic material Newton's law holds (then (1) is a parabolic equation).

The stress tensor usually depends on the instantaneous strain

$$\sigma_{ij}(x,t) = \frac{\partial W}{\partial e_{ij}}(eu(x,t)) \tag{2}$$

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where $W = W(e_{ij})$ is a function of free energy. In the case of one space dimension the problem has been solved. The global existence of a weak solution of the mixed problem follows from a recent work of Di Perna [5] and is based on compensated compactness arguments. In spite of intensive efforts of many mathematicians the question of global existence of a solution to the general nonlinear elastic problem remains open. There are several results in some special cases for dimension $N \geq 2$.

Experience indicates that certain materials have memory. It means that the stress depends not only on the strain at the present time $t$, but also on the entire history of the strain from zero to time $t$. In this case, the instantaneous stress (2) in equation (1) is extended by the memory part, which usually has the form

$$\int_0^t h(t-\tau)\big(eu(x,t) - eu(x,\tau)\big)d\tau$$

($h$ denotes a suitable kernel). At first sight it is surprising that the existence of a solution to such an equation can be proved (see [2, 12]). However, there are other materials where the stress depends not only on the history of the strain at given $x$, but also on the history at all points located in a neigbourhood of $x$, more generally on the history at all points of $\Omega$. Our work generalizes the results from [2] for such a type of nonlinear elasticity memory choices, both time and spatial ones. (The singularity in the memory part of stress approaches the Dirac function.) It is the purpose of the present paper to prove global existence of a weak solution.

We will consider the equation

$$\ddot{u}_i(x,t) - \frac{\partial}{\partial x_j}\sigma_{ij}(x,t) = f_i(x,t) \quad \text{on } \Omega \times (0,\infty) \quad (i = 1,2,\ldots,N) \tag{3}$$

where the stress consists of both the instantaneous and the memory part $\sigma = \sigma^I + \sigma^M$,

$$\sigma_{ij}^I = \frac{\partial W}{\partial e_{ij}}(eu) \tag{4}$$

and

$$\sigma_{ij}^M = -\lambda \int_0^t \int_\Omega \big(e_{ij}u(\xi,\tau) - e_{ij}u(\xi,t)\big)\frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\xi d\tau, \tag{5}$$

with boundary conditions

$$u(x,\cdot) = 0 \qquad (x \in \partial\Omega) \tag{6}$$

and initial conditions

$$\left.\begin{aligned} u(\cdot,0) &= u^0 \\ \dot{u}(\cdot,0) &= u^1. \end{aligned}\right\} \tag{7}$$

Let the domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be bounded and let it possess a Lipschitz continuous boundary $\partial\Omega$. We assume that the function $W : \mathbb{R}^{2N} \to \mathbb{R}$ is continuous, has bounded second derivatives, $W(0) = \frac{\partial W}{\partial e_{ij}}(0) = 0$ and the condition of ellipticity is satisfied, i.e. there exists a real number $\kappa > 0$ such that

$$\frac{\partial^2 W}{\partial e_{ij}\,\partial e_{kl}}(q)\, a_{ij}a_{kl} \geq \kappa \|a\|^2 \tag{8}$$

holds for every $a, q \in \mathbb{R}^{2N}$. We also suppose $h(t) = e^{-t}t^{-\nu}$ where $0 < \nu < \frac{1}{2}, N-1 < \alpha < N$, $\lambda > 0$, and

$$\begin{aligned} f &\in W^{\frac{\nu}{2},2}((0,\infty); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \\ &\qquad \cap L^2((0,\infty); W^{-1,2}(\Omega;\mathbb{R}^N)) \cap L^\infty((0,\infty); L^2(\Omega;\mathbb{R}^N)) \\ u^0 &\in W_0^{1,2}(\Omega;\mathbb{R}^N) \\ u^1 &\in L^2(\Omega;\mathbb{R}^N). \end{aligned}$$

We use the Galerkin approximation. The operator $\frac{\partial}{\partial x_j}\sigma_{ij}$ is compact both in time and space. The memory part of the stress tensor allows us to establish the basic estimates.

We will deal with spaces of functions with non-integer derivatives (see [1, 3, 10, 18]).

**Definition 1.** Let $0 \le s < 2$ and let $u : \Omega \to B$ be a function, where $B$ is a Banach space and $\Omega \subset \mathbb{R}^N$ is a domain with Lipschitz continuous boundary. We define

$$\begin{aligned} &\|u\|^2_{W^{s,2}(\Omega;B)} \\ &\quad = \begin{cases} \|u\|^2_{L^2(\Omega;B)} + \iint_{\Omega\times\Omega} \frac{\|u(x)-u(y)\|^2_B}{|x-y|^{N+2s}}\,dxdy & \text{if } 0 < s < 1 \\ \|u\|^2_{W^{1,2}(\Omega;B)} + \sum_{i=1}^N \iint_{\Omega\times\Omega} \frac{\|\frac{\partial u}{\partial x_i}(x)-\frac{\partial u}{\partial x_i}(y)\|^2_B}{|x-y|^{N+2(s-1)}}\,dxdy & \text{if } 1 < s < 2. \end{cases} \end{aligned}$$

The space $W^{s,2}(\Omega;B)$ contains the functions $u$ satisfying $\|u\|_{W^{s,2}(\Omega;B)} < \infty$, $W^{0,2}(\Omega;B) = L^2(\Omega;B)$, and $W^{1,2}(\Omega;B)$ is introduced as usual. (If $B = \mathbb{R}$, then we denote $W^{s,2}(\Omega) = W^{s,2}(\Omega;\mathbb{R})$.) The space $W_0^{s,2}(\Omega)$ can be introduced as the closure of $\mathcal{D}(\Omega)$ (test functions) in $W^{s,2}(\Omega)$ and we denote the dual space $W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))^*$. For $-\frac{1}{2} < s < +\frac{1}{2}$ we have $W^{s,2}(\Omega) = W_0^{s,2}(\Omega)$.

Let $\{w^n\}_{n\ge1}$ be a basis in $W_0^{1,2}(\Omega)$ which is orthonormal in $L^2(\Omega)$ consisting of the eigenfunctions of the equation

$$\Delta w + \theta w = 0 \qquad \text{on } \Omega$$

and let $\{\lambda_n\}_{n\ge1}$ denote the corresponding eigenvalues. Then in the space $W_0^{s,2}(\Omega)$ an equivalent norm can be given by

$$\|u\|^2_{W_0^{s,2}(\Omega)} \approx \sum_{i=1}^\infty \lambda_i^s c_i^2 \qquad \text{where } c_i = \int_\Omega u w^i dx$$

and

$$\|u\|^2_{W_0^{s,2}(\mathbb{R}^N)} \approx \int_{\mathbb{R}^N} (|\xi|^s |\widehat{u}(\xi)|)^2 d\xi$$

where $\widehat{u}$ means the Fourier transform of the function $u$:

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} u(x) e^{-i(\xi_1x_1+\ldots+\xi_Nx_N)}dx \qquad (\xi \in \mathbb{R}^n).$$

We shall use the Parseval equality

$$\int_{\mathbb{R}^N} u\,\bar{v}\,dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{u}\,\bar{\widehat{v}}\,d\xi \qquad (u, v \in L^2(\mathbb{R}^N)), \tag{9}$$

and rules for the Fourier transform of the convolution and derivatives:

$$\widehat{u * v} = \widehat{u}\,\widehat{v}, \tag{10}$$

$$\widehat{\frac{\partial u}{\partial x_j}}(\xi) = -i\,\xi_j\,\widehat{u}(\xi) \qquad \left(u \in \mathcal{S}^*(\mathbb{R}^N), v \in L^2(\mathbb{R}^N)\right). \tag{11}$$

Here the space of temperate distributions $\mathcal{S}^*(\mathbb{R}^N)$ means the dual space to

$$\mathcal{S}(\mathbb{R}^N) = \left\{\varphi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^N} \left|x^\beta D^\alpha \varphi(x)\right| < \infty \text{ for all multi-indexes } \alpha, \beta \in \mathbb{N}^N\right\}$$

and the convolution of $u$ and $v$ is introduced by the formula

$$(u * v)(x) = \int_{\mathbb{R}^N} u(\xi)\,v(x-\xi)\,d\xi. \tag{12}$$

We shall need also the Fourier transformation of the power $\frac{1}{|\cdot|^\alpha}$ for $\frac{N-1}{2} < \alpha < N$:

$$\widehat{\left(\frac{1}{|\cdot|^\alpha}\right)}(\xi) = (2\pi)^{\frac{N}{2}}\, 2^{\frac{N}{2}-\alpha}\,\frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\left(\sqrt{\xi_1^2 + \ldots + \xi_N^2}\right)^{\alpha-N} \tag{13}$$

(see [4, 6, 9]).

## 2. Galerkin approximation

After defining a weak solution of our problem we will construct its approximants by the Galerkin method.

**Definition 2.** A *weak solution* to the mixed problem (3) - (7) is a function $u \in L^\infty\left((0,\infty); W_0^{1,2}(\Omega;\mathbb{R}^N)\right)$, for which

$$\left.\begin{array}{l} \dot{u} \in L^\infty\left((0,\infty); L^2(\Omega;\mathbb{R}^N)\right) \\ \ddot{u} \in L^2\left((0,T); W^{-1,2}(\Omega;\mathbb{R}^N)\right) \text{ for all } T > 0 \end{array}\right\}$$

and for all $v \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ and for almost all $T > 0$ the equality

$$\begin{aligned} &\int_0^T\int_\Omega \ddot{u}_i(x,t)v_i(x)\,dx dt + \int_0^T\int_\Omega \frac{\partial W}{\partial e_{ij}}(eu(x,t))\,e_{ij}v(x)\,dx dt \\ &\quad - \lambda\int_0^T\int_\Omega \left(\int_0^t\int_\Omega \left(e_{ij}u(\xi,\tau) - e_{ij}u(\xi,t)\right)\frac{h(t-\tau)}{|x-\xi|^\alpha}\,d\xi d\tau\right) e_{ij}v(x)\,dx dt \\ &= \int_0^T\int_\Omega f_i(x,t)v_i(x)\,dx dt \end{aligned} \tag{14}$$

holds (it is necessary to understand the integrals in the sense of distributions).

There exists a basis $\{w^n\}_{n\geq 1}$ in the space $W_0^{1,2}(\Omega)$ which is orthonormal in $L^2(\Omega)$. We construct Galerkin approximants $u^n$ of the form

$$u^n(x,t) = \sum_{k=1}^{n} c_k^{(n)}(t) w^k(x) \qquad (n \in \mathbb{N}).$$

Using successively $w^1, \dots, w^n$ as test functions in (14), we get the following conditions for the functions of time $c_1^{(n)}, c_2^{(n)}, \dots, c_n^{(n)}$:

$$\begin{aligned}
&\ddot{c}_m^{(n)}(t) + \int_\Omega \frac{\partial W}{\partial e_{ij}} \left( \sum_{k=1}^{n} c_k^{(n)}(t) e w^k(x) \right) e_{ij} w^m(x)\, dx \\
&\quad - \lambda \sum_{k=1}^{n} \iint_{\Omega\times\Omega} \frac{e_{ij} w^k(\xi) e_{ij} w^m(x)}{|x-\xi|^\alpha}\, d\xi dx \cdot \int_0^t \left(c_k^{(n)}(\tau) - c_k^{(n)}(t)\right) h(t-\tau)\, d\tau \\
&= \int_\Omega f_i(x,t) w_i^m(x)\, dx
\end{aligned}$$

with initial conditions

$$\left.\begin{aligned}
c_m^{(n)}(0) &= \int_\Omega u_i^0 w_i^m\, dx \\
\dot{c}_m^{(n)}(0) &= \int_\Omega u_i^1 w_i^m\, dx
\end{aligned}\right\} \qquad (m = 1, \dots, n).$$

This problem possesses a unique solution on an interval $[0,\delta)$ ($\delta$ is the maximal time of existence of the solution, see [8] or [11]). Thus there exist approximate solutions $u^n$ satisfying the equation

$$\begin{aligned}
&\int_\Omega \ddot{u}_i^n v_i dx + \int_\Omega \frac{\partial W}{\partial e_{ij}}(e u^n)\, e_{ij} v dx \\
&\quad - \lambda \int_\Omega \left( \int_0^t \int_\Omega \left(e_{ij} u^n(\xi,\tau) - e_{ij} u^n(\xi,t)\right) \frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\xi d\tau \right) e_{ij} v(x)\, dx \qquad (15)\\
&= \int_\Omega f_i v_i\, dx
\end{aligned}$$

for all $v \in \mathrm{sp}\,\{w^1, \dots, w^n\}$ (the subspace spanned by $w^1, \dots, w^n$).

## 3. Basic estimates

For the approximate solution introduced in the previous section we may establish the following estimates.

**Lemma 1.** *For any $T \in [0, \delta)$ the solution $u^n$ of (15) satisfies for some $C_1 > 0$ the inequality*

$$\frac{1}{2}\int_\Omega \|\dot{u}^n(T)\|^2 dx + \int_\Omega W(eu^n(T))\,dx$$
$$+ \lambda C_1 \int_0^T\int_0^T\int_{\mathbb{R}^N} \left(|\xi|^{1-\frac{N-\alpha}{2}}\left|\widehat{u}^n(\xi,t) - \widehat{u}^n(\xi,\tau)\right|\right)^2 d\xi \frac{d\tau\, dt}{(t-\tau)^{1+\nu}}$$
$$\leq \frac{1}{2}\int_\Omega \|u^1\|^2 dx + \int_\Omega W(eu^0)\,dx + \int_0^T\int_\Omega f_i \dot{u}_i^n dx dt.$$

**Proof.** Let us extend $u^n$ by zero outside $\Omega$. We put the time derivatives $\dot{u}^n(\cdot,t)$ as test functions into expression (15) and integrate over $(0,T)$:

$$\frac{1}{2}\int_\Omega \|\dot{u}^n(T)\|^2 dx - \frac{1}{2}\int_\Omega \|\dot{u}^n(0)\|^2 dx + \int_\Omega W(eu^n(T))\,dx - \int_\Omega W(eu^n(0))\,dx$$
$$+ \lambda \int_0^T\int_\Omega \left(\int_0^t\int_\Omega \left(e_{ij}u^n(\xi,t) - e_{ij}u^n(\xi,\tau)\right)\frac{h(t-\tau)}{|x-\xi|^\alpha}\,d\xi d\tau\right) e_{ij}\dot{u}^n(x,t)\,dx dt \tag{16}$$
$$= \int_0^T\int_\Omega f_i \dot{u}_i^n \,dx dt.$$

We can write the last integral on the left-hand side of (16) as a convolution (12), then use the Parseval equality (9) and the properties of the Fourier transform of a convolution, derivatives and powers (10), (11) and (13) to get

$$\iint_{\Omega\times\Omega} \frac{e_{ij}u^n(\xi,t) - e_{ij}u^n(\xi,\tau)}{|x-\xi|^\alpha} e_{ij}\dot{u}^n(x,t)\,d\xi dx$$
$$= \int_{\mathbb{R}^N} \left[(e_{ij}u^n(t) - e_{ij}u^n(\tau)) * \frac{1}{|\cdot|^\alpha}\right] \overline{e_{ij}\dot{u}^n(t)}\,dx$$
$$= (2\pi)^{\frac{N}{2}} 2^{\frac{N}{2}-\alpha} \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{2}\left[\int_{\mathbb{R}^N} |\xi|^{2+\alpha-N}\left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau)\right)\overline{\widehat{\dot{u}_i^n}(\xi,t)}\,d\xi\right.$$
$$\left. + \int_{\mathbb{R}^N} |\xi|^{\alpha-N}\xi_i\xi_j\left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau)\right)\overline{\widehat{\dot{u}_j^n}(\xi,t)}\,d\xi\right].$$

Denoting by $\mathbf{n} = (n_t, n_\tau)$ the outer normal to $\partial M_T$, where

$$M_T = \left\{(t,\tau) : 0 < t < \tau \text{ and } 0 < \tau < T\right\},$$

we compute that for some $d_1 > 0$

$$2\int_0^T\int_0^t\int_{\mathbb{R}^N} |\xi|^{2+\alpha-N}\left(\widehat{u_i^n}(\xi,t)-\widehat{u_i^n}(\xi,\tau)\right)\overline{\widehat{u_i^n}(\xi,t)}\,d\xi\; h(t-\tau)\,d\tau dt$$
$$= \int_{\mathbb{R}^N}|\xi|^{2+\alpha-N}\int_{M_T}\frac{d}{dt}\left|\widehat{u^n}(\xi,t)-\widehat{u^n}(\xi,\tau)\right|^2 h(t-\tau)\,d\tau dt d\xi$$
$$= \int_{\mathbb{R}^N}|\xi|^{2+\alpha-N}\Big(\int_{\partial M_T}\left|\widehat{u^n}(\xi,t)-\widehat{u^n}(\xi,\tau)\right|^2 h(t-\tau)n_t\,dS$$
$$-\int_{M_T}\frac{d}{dt}\left|\widehat{u^n}(\xi,t)-\widehat{u^n}(\xi,\tau)\right|^2 h'(t-\tau)\,d\tau dt\Big)d\xi$$
$$\geq d_1\int_0^T\int_0^T\int_{\mathbb{R}^N}\left(|\xi|^{1-\frac{N-\alpha}{2}}\left|\widehat{u^n}(\xi,t)-\widehat{u^n}(\xi,\tau)\right|\right)^2 d\xi\,\frac{d\tau dt}{(t-\tau)^{1+\nu}}$$

holds (we use integration by parts and take into account that $h'$ is negative on $(0,\infty)$). Similarly

$$\int_0^T\int_0^t\int_{\mathbb{R}^N}|\xi|^{\alpha-N}\xi_i\xi_j\left(\widehat{u_i^n}(\xi,t)-\widehat{u_i^n}(\xi,\tau)\right)\overline{\widehat{u_j^n}(\xi,t)}h(t-\tau)\,d\tau dt$$
$$= \int_{\mathbb{R}^N}|\xi|^{\alpha-N}\int_{M_T}\frac{d}{dt}\left[\sum_{i=1}^N\xi_i\left(\widehat{u_i^n}(\xi,t)-\widehat{u_i^n}(\xi,\tau)\right)\right]^2 h(t-\tau)\,d\tau dt d\xi$$
$$\geq 0$$

and the lemma is proved ∎

It follows from Lemma 1 that the approximants $u^n$ are defined on the whole interval $[0,\infty)$.

Using the Gronwall lemma we obtain the following corollary.

**Corollary.** *There exists a constant $C_2 > 0$ such that*

$$\left.\begin{aligned}\|u^n\|_{L^\infty((0;\infty);W_0^{1,2}(\Omega;\mathbb{R}^N))} &\leq C_2\\ \|\dot u^n\|_{L^\infty((0;\infty);L^2(\Omega;\mathbb{R}^N))} &\leq C_2\end{aligned}\right\}.$$

*For any $T > 0$ there exists a constant $C_3(T) > 0$ such that*

$$\|u^n\|_{W^{\frac{\nu}{2},2}((0;T);W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))} \leq C_3(T).$$

**Lemma 2.** *For any $T > 0$ there exists a constant $C_4(T) > 0$ such that the solution $u^n$ of (15) satisfies*

$$\|\ddot u^n\|_{W^{\frac{\nu}{2},2}((0;T);W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))} \leq C_4(T).$$

**Proof.** Let $\varepsilon = \frac{N-\alpha}{2}$ and denote by $R^n$ the projection operator mapping the space $W_0^{1+\varepsilon,2}(\Omega;\mathbb{R}^N)$ to $\mathrm{sp}(w^1,\dots,w^n)$. The starting point of our consideration will be the definition

$$\|\ddot u^n\|^2_{W^{\frac{\nu}{2},2}((0;T);W^{-1-\varepsilon,2}(\Omega;\mathbb{R}^N))} \approx \int_0^T\int_0^T\frac{\|\ddot u^n(t_1)-\ddot u^n(t_2)\|^2_{W^{-1-\varepsilon,2}}}{|t_1-t_2|^{1+\nu}}\,dt_1dt_2$$

and

$$\begin{aligned}
&\left\|\ddot{u}^n(\cdot,t_1)-\ddot{u}^n(\cdot,t_2)\right\|_{W^{-1-\epsilon,2}(\Omega;\mathbb{R}^N)}\\
&\quad=\sup_{\|\psi\|_{W_0^{1+\epsilon,2}}\le 1}\int_\Omega \left(\ddot{u}_i^n(t_1)-\ddot{u}_i^n(t_2)\right)\psi_i\,dx\\
&\quad=\sup_{\|\psi\|_{W_0^{1+\epsilon,2}}\le 1}\int_\Omega \left(\ddot{u}_i^n(t_1)-\ddot{u}_i^n(t_2)\right)(R^n\psi)_i\,dx.
\end{aligned}$$

If we extend any function $\varphi\in W_0^{1+\epsilon,2}(\Omega;\mathbb{R}^N)$ by zero outside $\Omega$, then

$$\|\varphi\|^2_{W_0^{s,2}(\mathbb{R}^N)}\approx\int_{\mathbb{R}^N}(|\xi|^s|\widehat{\varphi}(\xi)|)^2d\xi\qquad(-\tfrac{3}{2}<s<+\tfrac{3}{2}).$$

We choose any function $\psi\in W_0^{1+\epsilon,2}(\Omega;\mathbb{R}^N)$ with $\|\psi\|_{W_0^{1+\epsilon,2}(\Omega;\mathbb{R}^N)}\le 1$, denote $\varphi=R^n\psi\in\operatorname{sp}(w^1,\dots,w^n)$ and use equality (15) for $u^n(t_1)$ and $u^n(t_2)$. First we estimate

$$\begin{aligned}
&\left|\int_\Omega\left(\frac{\partial W}{\partial e_{ij}}(eu^n(x,t_1))-\frac{\partial W}{\partial e_{ij}}(eu^n(x,t_2))\right)\frac{\partial\varphi_i}{\partial x_j}(x)\,dx\right|\\
&\quad\le d_2\sum_{i,j,k,l}\left|\int_{\mathbb{R}^N}\frac{\partial}{\partial x_l}(u_k^n(x,t_1)-u_k^n(x,t_2))\frac{\partial\varphi_i}{\partial x_j}(x)\,dx\right|\\
&\quad\le d_3\sum_{i,k}\int_{\mathbb{R}^N}\left(|\xi|^{1-\epsilon}\left|\widehat{u_k^n}(\xi,t_1)-\widehat{u_k^n}(\xi,t_2)\right|\right)^2(|\xi|^{1+\epsilon}|\widehat{\varphi}_i(\xi)|)^2d\xi\\
&\quad\le d_4\left\|u^n(t_1)-u^n(t_2)\right\|^2_{W^{1-\epsilon,2}}\|\varphi\|^2_{W^{1+\epsilon,2}}.
\end{aligned}\tag{17}$$

Let us remark that $\|u^n\|_{W^{\frac{\epsilon}{2}}((0,T);W^{1-\epsilon,2}(\Omega;\mathbb{R}^N))}\le C_3(T)$ by Corollary. We can proceed similarly in the case of the difference of the right-hand sides of equation (15). It is sufficient to look at the term in (15) generated by the memory portion of the stress $\sigma_{ij}^M$. For $0\le t_2<t_1<T$ we have

$$\begin{aligned}
&\int_\Omega\left[\int_0^{t_1}\!\!\int_\Omega\frac{e_{ij}u^n(\xi,t_1)-e_{ij}u^n(\xi,\tau)}{|x-\xi|^\alpha}h(t_1-\tau)\,d\xi d\tau\right.\\
&\qquad\left.-\int_0^{t_2}\!\!\int_\Omega\frac{e_{ij}u^n(\xi,t_2)-e_{ij}u^n(\xi,\sigma)}{|x-\xi|^\alpha}h(t_2-\sigma)\,d\xi d\sigma\right]e_{ij}\varphi(x)\,dx\\
&=\iint_{\Omega\times\Omega}\left[\int_0^{t_1}\left(e_{ij}u^n(\xi,t_1)-e_{ij}u^n(\xi,t_1-s)\right)h(s)\,ds\right.\\
&\qquad\left.-\int_0^{t_2}\left(e_{ij}u^n(\xi,t_2)-e_{ij}u^n(\xi,t_2-s)\right)h(s)\,ds\right]\frac{e_{ij}\varphi(x)}{|x-\xi|^\alpha}\,d\xi dx\\
&=\int_{\mathbb{R}^N}\left\{\int_0^{t_2}\left(e_{ij}u^n(\xi,t_1)-e_{ij}u^n(\xi,t_2)\right)h(s)\,ds\right.\\
&\qquad-\int_0^{t_2}\left(e_{ij}u^n(\xi,t_1-s)-e_{ij}u^n(\xi,t_2-s)\right)h(s)\,ds\\
&\qquad\left.+\int_{t_2}^{t_1}\left(e_{ij}u^n(\xi,t_1)-e_{ij}u^n(\xi,t_1-s)\right)h(s)\,ds\right\}\left(e_{ij}\varphi*\frac{1}{|\cdot|^\alpha}\right)(\xi)\,d\xi.
\end{aligned}\tag{18}$$

The last line of (18) contains three parts. We can write the symmetric parts of the gradient and estimate the corresponding integrals (similarly as in (17)) as

$$\begin{aligned}&\left|\int_0^{t_2}\int_{\mathbb{R}^N}\left(\frac{\partial\varphi_i}{\partial x_j}*\frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u_k^n}{\partial x_l}(t_1)-\frac{\partial u_k^n}{\partial x_l}(t_2)\right)d\xi\, h(s)\,ds\right|\\ &\qquad\le d_5(T)\,\|\varphi\|_{W^{1+\epsilon,2}}\|u^n(t_1)-u^n(t_2)\|_{W^{1-\epsilon,2}}\\ &\qquad\le d_6(T)\,\|u^n(t_1)-u^n(t_2)\|_{W^{1-\epsilon,2}}.\end{aligned} \tag{19}$$

The second integral can be estimated as

$$\begin{aligned}&\left|\int_0^{t_2}\int_{\mathbb{R}^N}\left(\frac{\partial\varphi_i}{\partial x_j}*\frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u_k^n}{\partial x_l}(t_1-s)-\frac{\partial u_k^n}{\partial x_l}(t_2-s)\right)d\xi\, h(s)\,ds\right|\\ &\qquad\le d_7(T)\,\|\varphi\|_{W^{1+\epsilon,2}}\int_0^{t_2}\left\|u^n(t_1-s)-u^n(t_2-s)\right\|_{W^{1-\epsilon,2}}h(s)\,ds\\ &\qquad\le d_8(T)\left\{\int_0^{t_2}\|u^n(t_1)-u^n(t_2)\|^2_{W^{1-\epsilon,2}}ds\right\}^{\frac12}\end{aligned} \tag{20}$$

and

$$\int_0^T\int_0^{t_1}\int_0^{t_2}\frac{\|u^n(t_1-s)-u^n(t_2-s)\|^2_{W^{1-\epsilon,2}}}{|(t_1-s)-(t_2-s)|^{\nu+1}}\le d_9(T)\,\|u^n\|_{W^{\frac{\nu}{2}}((0,T);W^{1-\epsilon,2}(\Omega;\mathbb{R}^N))}.$$

Analogously we get

$$\begin{aligned}&\int_{t_2}^{t_1}\int_{\mathbb{R}^N}\left(\frac{\partial\varphi_i}{\partial x_j}*\frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u_k^n}{\partial x_l}(t_1)-\frac{\partial u_k^n}{\partial x_l}(t_1-s)\right)d\xi\, h(s)\,ds\\ &\qquad\le d_{10}(T)\int_{t_2}^{t_1}\|u^n(t_1)-u^n(t_1-s)\|_{W^{1-\epsilon,2}}h(s)\,ds\\ &\qquad\le d_{10}(T)\int_{t_2}^{t_1}\|u^n(t_1)-u^n(t_1-s)\|_{W^{1,2}}h(s)\,ds.\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^T\int_0^{t_1}\left[\int_{t_2}^{t_1}\left\|u^n(t_1)-u^n(t_1-s)\right\|_{W^{1,2}}h(s)\,ds\right]^2\frac{dt_2dt_1}{(t_1-t_2)^{\nu+1}}\\ &\qquad\le\sup_{\tau\in[0,T]}\|u^n(\tau)\|^2_{W^{1,2}}\int_0^T\int_0^{t_1}\left(\int_{t_2}^{t_1}h(s)\,ds\right)^2\frac{dt_2dt_1}{(t_1-t_2)^{\nu+1}}\\ &\qquad\le d_{11}(T)\,\|u^n\|^2_{L^\infty((0,T);W^{1,2}(\Omega;\mathbb{R}^N))}.\end{aligned} \tag{21}$$

Lemma 2 follows now from definitions and estimates (19) - (21) and Corollary ∎

## 4. Interpolation

Let $1 < \mu < \frac{3}{2}$ and $-\frac{3}{2} < \beta < +\frac{3}{2}$. We can introduce spaces $W^{\mu,2}((0,T);W_0^{\beta,2}(\Omega))$ by Definition 1. Then $v \in W^{\mu,2}((0,T);W_0^{\beta,2}(\Omega))$ may be expanded into the double Fourier series

$$v = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\, h_i(t) w^j(x)$$

where $h_0(t) = \frac{1}{\sqrt{T}}$ and $h_i(t) = \sqrt{\frac{2}{T}} \cos \frac{i\pi}{T} t$ $(i \in \mathbb{N})$. We use the equivalent norm

$$\|v\|^2_{W^{\mu,2}((0,T);W_0^{\beta,2}(\Omega))} \approx \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{i,j}^2 (1+i^2)^{\mu} \lambda_j^{\beta}.$$

**Lemma 3.** *Let $0 < \delta < \frac{1}{2}$, $0 < \varepsilon < \frac{1}{2}$ and $0 \le \gamma \le 1$. Then there exists a constant $C_5 > 0$ such that*

$$\|v\|_{W^{(1+\delta)\gamma,2}((0,T);W^{-(1+\epsilon)\gamma,2}(\Omega))} \le C_5 \|v\|^{1-\gamma}_{L^2((0,T);L^2(\Omega))} \|v\|^{\gamma}_{W^{1+\delta,2}((0,T);W^{-1-\epsilon,2}(\Omega))}.$$

**Proof.** We compute directly that

$$\begin{aligned}
&\|v\|_{W^{(1+\delta)\gamma,2}((0,T);W^{-(1+\epsilon)\gamma,2}(\Omega))} \\
&\quad = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2 (1+i^2)^{(1+\delta)\gamma} \lambda_j^{-(1+\varepsilon)\gamma} \\
&\quad = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{ij}^{2\gamma} (1+i^2)^{\gamma(1+\delta)} \cdot \lambda_j^{-\gamma(1+\varepsilon)} c_{ij}^{2(1-\gamma)} \\
&\quad \le C_5 \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \left(c_{ij}^2 (1+i^2)^{(1+\delta)} \lambda_j^{-(1+\varepsilon)}\right)^{\gamma} \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \left(c_{ij}^2 (1+i^2)^0 \lambda_j^0\right)^{1-\gamma} \\
&\quad = C_5 \|v\|^{1-\gamma}_{L^2((0,T);L^2(\Omega))} \|v\|^{\gamma}_{W^{1+\delta,2}((0,T);W^{-1-\epsilon,2}(\Omega))}
\end{aligned}$$

and the lemma is proved ∎

## 5. Existence of a weak solution

The following theorem establishes the Lipschitz continuity of the operator $\sigma^M$.

**Theorem 1.** *There exists $p > 0$, independent of $T$, such that*

$$\|\sigma^M u - \sigma^M v\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^{N^2}))} \le \lambda p \, \|eu - ev\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^{N^2}))}.$$

**Proof.** Using twice the Schwarz inequality we get

$$\begin{aligned}
&\frac{1}{\lambda^2}\left(\sigma_{ij}^M u - \sigma_{ij}^M v\right)^2 \\
&= \left(\int_0^t \int_\Omega \left[\left(e_{ij}u(\xi,t) - e_{ij}v(\xi,t)\right) - \left(e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau)\right)\right] \frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\xi d\tau\right)^2 \\
&\le \int_\Omega \left(e_{ij}u(\xi,t) - e_{ij}v(\xi,t)\right)^2 \frac{d\xi}{|x-\xi|^\alpha} \int_\Omega \frac{d\xi}{|x-\xi|^\alpha} \left(\int_0^t h(t-\tau)\, d\tau\right)^2 \\
&\quad + \int_\Omega \int_0^t \left(e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau)\right)^2 \frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\tau d\xi \int_\Omega \frac{d\xi}{|x-\xi|^\alpha} \int_0^t h(t-\tau)\, d\tau.
\end{aligned}$$

We denote $p_1 = \int_0^\infty h(\tau)\, d\tau$ and $p_2 = \int_{B(0,\operatorname{diam}\Omega)} \frac{1}{|x|^\alpha}\, dx$. Changing the order of integration we go on to compute

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\|\sigma^M u - \sigma^M v\right\|_{L^2(L^2)}^2 \\
&\le p_1^2 p_2 \int_\Omega \left(\int_0^T \left(e_{ij}u(\xi,t) - e_{ij}v(\xi,t)\right)^2 dt\right)\left(\int_\Omega \frac{dx}{|x-\xi|^\alpha}\right) d\xi \\
&\quad + p_1 p_2 \int_\Omega \int_0^T \left(e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau)\right)^2 \left(\int_\tau^T h(t-\tau)\, dt\right)\left(\int_\Omega \frac{d(\xi)}{|x-\xi|^\alpha}\right) d\tau d\xi \\
&\le 2p_1^2 p_2^2 \left\|eu - ev\right\|_{L^2(L^2)}^2
\end{aligned}$$

and the theorem is proved ∎

**Theorem 2** (Existence of weak solutions). *Let us consider equation* $(3)-(5)$ *with boundary and initial conditions* $(6)$ *and* $(7)$. *Let the just introduced assumptions be satisfied. Moreover, let* $\nu > N - \alpha$ *and* $c_0\kappa > p\lambda$ *where* $c_0$ *is the constant in Korn's inequality. Then problem* $(3)-(7)$ *possesses weak solutions* $u$ *on* $(0,\infty)$. *These solutions satisfy*

$$\left.\begin{aligned}
&u \in L^\infty\left((0,\infty); W_0^{1,2}(\Omega;\mathbb{R}^N)\right) \\
&\dot u \in L^\infty\left((0,\infty); L^2(\Omega;\mathbb{R}^N)\right) \\
&\ddot u \in L^2\left((0,T); W^{-1,2}(\Omega;\mathbb{R}^N)\right) \\
&u \in W^{\frac{\nu}{2},2}\left((0,T); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)\right) \\
&\ddot u \in W^{\frac{\nu}{2},2}\left((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)\right)
\end{aligned}\right\}$$

*for all* $T > 0$.

**Proof.** Let us choose any $T > 0$. As $\dot u_i^n$ is a bounded sequence in $L^2\left((0,T); L^2(\Omega)\right)$ and $W^{1+\frac{\nu}{2},2}\left((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega)\right)$, we get from Lemma 3 (putting $v = \dot u_i^n$) that $\dot u_i^n$ is bounded also in the space $W^{\gamma(1+\frac{\nu}{2}),2}\left((0,T); W^{-\gamma(1+\frac{N-\alpha}{2}),2}(\Omega)\right)$ where $\frac{1}{1+\frac{\nu}{2}} < \gamma < \frac{1}{1+\frac{N-\alpha}{2}}$. It is possible to choose such $\gamma$, because $\nu > N - \alpha$. The space $W^{\gamma(1+\frac{\nu}{2}),2}\left((0,T); W^{-\gamma(1+\frac{N-\alpha}{2}),2}(\Omega)\right)$ is compactly embedded into $W^{1,2}\left((0,T); W^{-1,2}(\Omega)\right)$. Thus we can choose a subsequence $u^{n_k}$ which converges to a certain function $u$ in the following sense:

$$\left.\begin{array}{lll} u^{n_k} \rightharpoonup u & \text{in} & L^2\big((0,T); W_0^{1,2}(\Omega;\mathbb{R}^N)\big) \\ \dot u^{n_k} \rightharpoonup \dot u & \text{in} & L^2\big((0,T); L^2(\Omega;\mathbb{R}^N)\big) \\ u^{n_k} \rightharpoonup u & \text{in} & W^{\frac{\nu}{2},2}\big((0,T); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)\big) \\ \ddot u^{n_k} \rightharpoonup \ddot u & \text{in} & W^{\frac{\nu}{2},2}\big((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)\big) \\ \ddot u^{n_k} \to \ddot u & \text{in} & L^2\big((0,T); W^{-1,2}(\Omega;\mathbb{R}^N)\big) \end{array}\right\}.$$

Now, let $P_n$ be the projection operator from $L^2\big((0,T); W_0^{1,2}(\Omega;\mathbb{R}^N)\big)$ to the space spanned by the vectors $c_j(t)\,w^j(x)$ where $c_j \in L^2(0,T)$ $(j = 1,\dots,n)$. We have $P_n u \to u$ in $L^2\big((0,T); W_0^{1,2}(\Omega;\mathbb{R}^N)\big)$. We put $v = u^n - P_n u$ as a test function into equality (15) to obtain

$$\begin{aligned} &\int_0^T\int_\Omega \ddot u_i^{n_k}\big(u_i^{n_k} - (P_{n_k}u)_i\big)\,dx dt + \int_0^T\int_\Omega \frac{\partial W}{\partial e_{ij}}(eu^{n_k})\,e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \\ &\quad - \lambda\int_0^T\int_\Omega\bigg[\int_0^t\int_\Omega \big(e_{ij}u^{n_k}(\xi,\tau) - e_{ij}u^{n_k}(\xi,t)\big)\,\frac{d\xi}{|x-\xi|^\alpha}h(t-\tau)\,d\tau\bigg] \\ &\quad \times e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \\ &\quad = \int_0^T\int_\Omega f_i\,\big(u_i^{n_k} - (P_{n_k}u)_i\big)\,dxdt. \end{aligned}$$

The first and the last integrals tend to 0. We obtain a lower estimate from the condition of ellipticity (8) and Korn's inequality:

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial W}{\partial e_{ij}}(e(u^{n_k} - P_{n_k}u))e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \\ &\qquad \geq \kappa c_0\,\|u^{n_k} - P_{n_k}u\|^2_{L^2((0,T);W_o^{1,2}(\Omega;\mathbb{R}^n))}. \end{aligned}$$

As $\sigma^M$ is Lipschitz continuous we have

$$\begin{aligned} &\int_0^T\int_\Omega \sigma_{ij}^M(u^{n_k} - P_{n_k}u)\,e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \\ &\qquad \leq \lambda p\,\|u^{n_k} - P_{n_k}u\|^2_{L^2((0,T);W_0^{1,2}(\Omega;\mathbb{R}^N))}. \end{aligned}$$

As

$$\left.\begin{array}{l} \displaystyle\int_0^T\int_\Omega \frac{\partial W}{\partial e_{ij}}(eP_{n_k}u)\,e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \to 0 \\ \displaystyle\int_0^T\int_\Omega \sigma_{ij}^M(P_{n_k}u)\,e_{ij}(u^{n_k} - P_{n_k}u)\,dxdt \to 0 \end{array}\right\}$$

and $\kappa c_0 - \lambda p > 0$, then $u^{n_k} - P_{n_k}u \to 0$ and also $u^{n_k} \to u$ in $L^2\big((0,T); W_0^{1,2}(\Omega;\mathbb{R}^N)\big)$. The existence of the required weak solutions follows as a direct consequence of (15) ∎

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