Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 1117–1122

On the Hilbert Inequality

Gao Mingzhe

Abstract. It is shown that the Hilbert inequality for double series can be improved by introducing the positive real number $\frac{1}{\pi^2}\big(\frac{s^2(a)}{\|a\|^2}+\frac{s^2(b)}{\|b\|^2}\big)$ where $s(x)=\sum_{n=1}^{\infty}\frac{x_n}{n}$ and $\|x\|^2=\sum_{n=1}^{\infty}x_n^2$ $(x=a,b)$. The coefficient π of the classical Hilbert inequality is proved not to be the best possible if $\|a\|$ or $\|b\|$ is finite. A similar result for the Hilbert integral inequality is also proved.

Keywords: *Hilbert inequality, binary quadratic form, exponential integral, inner product*
AMS subject classification: 26 D, 46 C

1. Introduction

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be arbitrary real sequences. Then the *Hilbert inequality* for double series can be written as

$$\Bigg(\sum_{m,n=1}^{\infty}\frac{a_m b_n}{m+n}\Bigg)^2 \leq \pi^2\Bigg(\sum_{n=1}^{\infty}a_n^2\Bigg)\Bigg(\sum_{n=1}^{\infty}b_n^2\Bigg). \tag{1}$$

Additionally,

$$\Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{a_m b_n}{m-n}\Bigg)^2 \leq \pi^2\Bigg(\sum_{n=1}^{\infty}a_n^2\Bigg)\Bigg(\sum_{n=1}^{\infty}b_n^2\Bigg) \tag{2}$$

is also called *Hilbert inequality*. Furthermore, if $f,g\in L^2(\mathbb{R}_+)$ where $\mathbb{R}_+=(0,\infty)$, then the inequality analogous to (1)

$$\Bigg(\iint_{\mathbb{R}_+^2}\frac{f(s)g(t)}{s+t}\,dsdt\Bigg)^2 \leq \pi^2\Bigg(\int_{\mathbb{R}_+}f^2(t)\,dt\Bigg)\Bigg(\int_{\mathbb{R}_+}g^2(t)\,dt\Bigg) \tag{3}$$

is called the *Hilbert integral inequality*. The constant π contained in these inequalities, especially in (1), was proved to be the best possible (see [3]). However, if $0<\sum_{n=1}^{\infty}a_n^2<\infty$ or $0<\sum_{n=1}^{\infty}b_n^2<\infty$, then we can select a number $r>0$ such that the right-hand side of (1) can be replaced by

$$\pi^2(1-r)\Bigg(\sum_{n=1}^{\infty}a_n^2\Bigg)\Bigg(\sum_{n=1}^{\infty}b_n^2\Bigg),$$

Gao Mingzhe: Xiangxi Educ. Coll., Dept. Math., Jishou, Hunan 416000, P.R. China

ISSN 0232-2064 / $ 2.50

i.e. an improvement of (1) will be obtained. Similarly, an improvement of (3) will be established. Namely, the right-hand side of (3) can be written as

$$\pi^2(1-R)\Big(\int_{\mathbb{R}_+} f^2(t)\,dt\Big)\Big(\int_{\mathbb{R}_+} g^2(t)\,dt\Big)$$

with a number $R > 0$. The main purpose of the present paper is to prove the existence of such numbers r and R and to find expressions for them.

We first introduce some notations and functions.

If α and β are elements of an inner product space E, then its inner product is denoted by (α,β) and the norm of α is given by $\|\alpha\| = \sqrt{(\alpha,\alpha)}$. Further, if $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ are two real sequences, then its inner product (a,b) and the norm $\|a\|$ of a are defined by

$$(a,b) = \sum_{n=1}^{\infty} a_n b_n \qquad \text{and} \qquad \|a\| = \sqrt{(a,a)}. \tag{4}$$

Analogously, for functions $f, g \in L^2(a,b)$ its inner product (f,g) and the norm $\|f\|$ of f are defined by

$$(f,g) = \int_a^b f(t)g(t)\,dt \qquad \text{and} \qquad \|f\| = \Big(\int_a^b f^2(t)\,dt\Big)^{\frac{1}{2}}. \tag{5}$$

We next introduce a binary quadratic form $F(\cdot,\cdot)$ defined by

$$F(x,y) = \|\alpha\|^2 x^2 - 2(\alpha,\beta)xy + \|\beta\|^2 y^2 \tag{6}$$

where $x = (\beta,\gamma)$ and $y = (\alpha,\gamma)$ for $\gamma \in E$. We further denote

$$G(\alpha,\beta,\gamma) = F((\beta,\gamma),(\alpha,\gamma)). \tag{7}$$

The results involve $G(\alpha,\beta,\gamma)$ with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that if γ is orthogonal to both α and β, then $G(\alpha,\beta,\gamma) = 0$. It will turn out that if $(\alpha,\gamma)^2 + (\beta,\gamma)^2 > 0$ (see Lemma 1). Therefore, it is shrewd in every case to choose γ not orthogonal to both α and β.

For convenience, we introduce yet the notations

$$u(a,b) = \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}, \quad v(a,b) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{a_m b_n}{m-n}, \qquad s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

We shall frequently use these notations below.

2. Lemmas

To prove our theorems, we need the following results.

Lemma 1. *Let $G(\alpha, \beta, \gamma)$ be defined as in (7). If $\alpha, \beta \in E$ are linearly independent and $(\alpha, \gamma)^2 + (\beta, \gamma)^2 > 0$, then $G(\alpha, \beta, \gamma) > 0$.*

Lemma 2. *Let $G(\alpha, \beta, \gamma)$ be defined as defined in (7). If $\alpha, \beta \in E$ are linearly dependent, then $G(\alpha, \beta, \gamma) = 0$.*

Lemma 3. *Let $G(\alpha, \beta, \gamma)$ be defined as defined in (7). If $\alpha, \beta \in E$ are arbitrary and $\gamma \in E$ with $\|\gamma\| = 1$, then*

$$(\alpha, \beta)^2 \leq \|\alpha\|^2 \|\beta\|^2 - G(\alpha, \beta, \gamma), \tag{8}$$

and equality holds in (8) if and only if α, β, γ are linearly dependent.

The proofs of Lemmas 1 and 2 have been given in our previous paper [1]. Lemma 3 is actually a sharpening of the Cauchy-Schwarz inequality. This result has been given also in the paper [1], and in [5]. Hence the proofs of all lemmas are omitted.

Using the inner product defined by (5) and Lemma 3, we have the following result.

Corollary 1. *If $f, g \in L^2(a, b)$, then*

$$(f, g)^2 \leq \|f\|^2 \|g\|^2 - F(x, y) \tag{9}$$

where $F(x, y) = \|f\|^2 x^2 - 2(f, g)xy + \|g\|^2 y^2$ with $x = (g, \gamma)$ and $y = (f, \gamma)$, $\gamma \in L^2(a, b)$ with $\|\gamma\| = 1$.

3. Main results

In this section we will combine the two forms (1) and (2) of the Hilbert inequality into one similar form, and make inequalities (1) - (3) relaize significant improvements. The following theorems are the main results in this paper.

Theorem 1. *If $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, with $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$, then*

$$u^2(a, b) + v^2(a, b) < \pi^2 (1 - r) \|a\|^2 \|b\|^2 \tag{10}$$

where $r = \frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}\right)$.

Proof. Let us define two real functions $f, g : (0, 2\pi) \to \mathbb{R}$ by

$$f(t) = \sum_{n=1}^{\infty} a_n \sqrt{t} \sin(nt) \qquad \text{and} \qquad g(t) = \sum_{n=1}^{\infty} b_n \sqrt{t} \cos(nt).$$

It is easily to deduce that, with the notations of the space $L^2(0, 2\pi)$,

$$|u(a, b) + v(a, b)| = \tfrac{1}{\pi} |(f, g)|. \tag{11}$$

According to (5) and (6) we have $(f,g)^2 \le \|f\|^2\|g\|^2 - F(x,y)$ where $\|f\|^2 = \pi^2\|a\|^2$, $\|g\|^2 = \pi^2\|b\|^2$ and

$$F(x,y) = \|f\|^2x^2 - 2(f,g)xy + \|g\|^2y^2 \ge (\|f\|x - \|g\|y)^2 = \pi^2(\|a\|x - \|b\|y)^2.$$

Hence

$$(f,g)^2 \le \pi^4\|a\|^2\|b\|^2 - \pi^2(\|a\|x - \|b\|y)^2 \tag{12}$$

where $x = (g,\gamma)$ and $y = (f,\gamma)$, $\gamma \in L^2(0,2\pi)$ with $\|\gamma\| = 1$. We can choose $\gamma = \frac{1}{2\pi}\sqrt{2t}$. Then $x = 0$ and $y = -\sqrt{2}\sum_{n=1}^{\infty}\frac{a_n}{n} = -\sqrt{2}\,s(a)$. Hence

$$(\|a\|x - \|b\|y)^2 = 2\|b\|^2s^2(a). \tag{13}$$

In virtue of (11) - (13) we obtain

$$|u(a,b) + v(a,b)|^2 \le \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{14}$$

Since the vectors f, g, γ are linearly independent, by Lemma 3, it is impossible to take equality in (14). Hence we have

$$|u(a,b) + v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{15}$$

Notice that $u(b,a) = u(a,b)$ and $v(b,a) = -v(a,b)$. Interchanging a and b in (11), similarly we obtain

$$|u(a,b) - v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|a\|^2s^2(b). \tag{16}$$

Adding (15) and (16), inequality (10) is yielded after some simplifications. Thus the proof of the theorem is completed ∎

Remark. Since $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, with $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$, it follows that $r > 0$. Hence inequality (10) is a significant refinement of the paper [4].

Corollary 2. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) + v^2(a,a) < \pi^2(1 - \tilde{r})\|a\|^4 \tag{17}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

If $v^2(a,b)$ in (10) is replaced by 0, then we have the following

Corollary 3. *With the assumptions of Theorem 1, then*

$$u^2(a,b) < \pi^2(1 - r)\|a\|^2\|b\|^2 \tag{18}$$

where $r = \frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}\right)$.

We see from the above Remark that inequality (18) is a significant improvement of (1). According to Corollary 2 we obtain at once the following

Corollary 4. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) < \pi^2(1-\tilde{r})\|a\|^4 \tag{19}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

Similarly, we can establish an improvement of the Hilbert integral inequality. For this we need the integral

$$e(t) = \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t}\,ds \qquad (t \in \mathbb{R}_+)$$

called *exponential integral with parameter t*.

Theorem 2. *Let $f, g \in L^2(\mathbb{R}_+)$ be positive. Then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\,dsdt\right)^2 < \pi^2(1-R)\|f\|^2\|g\|^2 \tag{20}$$

where $R = \frac{1}{\pi}\left(\frac{x}{\|g\|} - \frac{y}{\|f\|}\right)^2$ *with* $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(g,e)$ *and* $y = (2\pi)^{\frac{1}{2}}(f, e^{-s})$, *e being the exponential integral with parameter.*

Proof. Define functions F and G by

$$F(s,t) = \frac{f(s)}{(s+t)^{\frac{1}{2}}}\left(\frac{s}{t}\right)^{\frac{1}{4}} \qquad \text{and} \qquad G(s,t) = \frac{g(t)}{(s+t)^{\frac{1}{2}}}\left(\frac{t}{s}\right)^{\frac{1}{4}}.$$

Using inequality (9) we have in $L^2(\mathbb{R}_+^2)$

$$\begin{aligned}\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\,dsdt\right)^2 &= (F,G)^2 \\ &\le \|F\|^2\|G\|^2 - F(x,y) \\ &\le \|F\|^2\|G\|^2 - (\|F\|x - \|G\|y)^2\end{aligned} \tag{21}$$

where $x = (G,\gamma)$ and $y = (F,\gamma)$, $\gamma \in L^2(\mathbb{R}_+^2)$ with $\|\gamma\| = 1$. We can choose

$$\gamma(s,t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}}\left(\frac{s}{t}\right)^{\frac{1}{4}}.$$

Hence we get

$$x = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}(g,e) \qquad \text{and} \qquad y = (2\pi)^{\frac{1}{2}}(f,e^{-s}). \tag{22}$$

It is easy to deduce that

$$\|F\|^2 = \pi\|f\|^2 \qquad \text{and} \qquad \|G\|^2 = \pi\|g\|^2. \tag{23}$$

Substituting (22) and (23) into (21) we obtain

$$(F,G)^2 \le \pi^2\|f\|^2\|g\|^2 - \pi(\|f\|x - \|g\|y)^2. \tag{24}$$

Since F, G, γ are linearly independent, it is impossible to have equality in (24). Consequently, inequality (20) is obtained from (24) after some simplifications. Thus the theorem is proved ∎

Corollary 5. *If $f \in L^2(\mathbb{R}_+)$ is positive, then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)f(t)}{s+t}\, dsdt \right)^2 < \pi^2(1 - \tilde{R})\|f\|^4$$

where $\tilde{R} = \frac{1}{\pi}\frac{(x-y)^2}{\|f\|^2}$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(f,e)$ and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s})$, e being the exponential integral with parameter.

Obviously, this is an immediate consequence of Theorem 2.

4. Conclusions

Some classical reasults concerning the Hilbert inequality show that the constant π in (1) is the best possible (see, i.e., [1, 2, 5, 6]). We see from (18) that inequality in (1) can be obtained only if $r = 0$. However, to change r into 0, it is necessary to take both $\|a\|$ and $\|b\|$ infinite. Therefore, generally, the constant π in (1) is not the best possible because the constant r contained in (18) is not equal to 0 if $\|a\|$ or $\|b\|$ is finite. In other words, the factor π in (1) can be decreased if $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$.

Similarly, we see from (20) that strong inequality in (3) can be obtained only if $R = 0$. In other words, the factor π in (3) is also not the best possible if $\|f\|$ or $\|g\|$ is finite.

Acknowledgement. The author is indebted to the referees for many valuable suggestions in this subject.

References

[1] Gao Mingzhe, Tan Li and L. Debnath: *Some improvements on Hilbert's integral inequality.* J. Math. Anal. Appl. 229 (1999), 682 - 689.

[2] Gao Mingzhe and Yang Bichen: *On the extended Hilbert's inequality.* Proc. Amer. Math. Soc. 126 (1998), 751 - 759.

[3] Hardy, G. H., Littlewood, J. E. and G. Polya: *Inequalities.* Cambridge: Univ. Press 1952.

[4] Hu Ke: *On Hilbert's inequality.* Chin. Ann. Math. (Ser. B) 13 (1992), 35 - 39.

[5] Mitrinovic, D. S.: *Analytic Inequalities.* New York: Springer-Verlag 1970.

[6] Oleszkiewicz, K.: *An elementary proof on Hilbert's inequality.* Amer. Math. Monthly 100 (1993), 276 - 280.

Received 04.01.1999

Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 1123–1130

Recursion Formulae for $\sum_{m=1}^{n} m^k$

Sen-Lin Guo and Feng Qi

Abstract. Using elementary approach and mathematical induction, several recursion formulae for $S_k(n) = \sum_{m=1}^{n} m^k$ are presented which show that $S_{k+1}(n)$ could be obtained from $S_k(n)$. A method and a formula of calculating Bernoulli numbers are proposed.

Keywords: *Recursion formulas, sum of powers, mathematical induction, Bernoulli numbers*

AMS subject classification: Primary 11 B 37, secondary 11 B 68, 11 B 83

1. Introduction

By definition and geometric meanings of the definite integral, it is well-known that the area under the curve $y = x^k$ over the closed interval $[0, 1]$ equals

$$\lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{n}\left(\frac{m}{n}\right)^k = \lim_{n\to\infty} \frac{1}{n^{k+1}}\left(\sum_{m=1}^{n} m^k\right).$$

To complete the solution of this and many similar problems, it is then necessary to find the sums

$$S_k(n) = \sum_{m=1}^{n} m^k. \tag{1}$$

For small integer $k > 0$, the sums always appear in many calculus courses. For example,

$$S_7(n) = \frac{1}{24}n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)$$

and the like [6: p. 11]. Such sums are usually proved by induction or derived from simple geometric pictures. For arbitrary k, unfortunately, the standard closed forms involve Bernoulli numbers or Stirling numbers of the second kind [4: p. 119], which come from reasonably complicated recurrence relations.

H. J. Schultz [10] derived a procedure for finding $S_k(n)$, k a positive integer, that is easy to remember, arises naturally, and can be used with very little background.

Sen-Lin Guo: Zhengzhou Textile Inst., Dept. Basic Sci., Zhengzhou City, Henan 450007, P. R. China

Feng Qi: Jiaozuo Inst. Techn., Dept. Math., Jiaozuo City, Henen 454000, P. R. China; e-mail: qifeng@jzit.edu.cn. Partially supported by NSF of Henan Province, P. R. China

ISSN 0232-2064 / \$ 2.50

However, he only illustrated the method by finding $S_6(n)$. According to [10], if one wants to compute, in general,

$$S_k(n) = A_{k+1}n^{k+1} + \ldots + A_1 n + A_0, \tag{2}$$

a system of $k+1$ equations

$$\sum_{i=j+1}^{k+1} (-1)^{i-j+1}\binom{i}{j} A_i = 0 \qquad (0 \le j \le k)$$

must be solved.

Let B_n be the n-th Bernoulli number defined in [6: p. 648] and [9: p. 632] by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| \le 2\pi). \tag{3}$$

Then A_1 obtained from the formula for $S_k(n)$ is the k-th Bernoulli number B_k (for details see [11: p. 320]). It is noted that the concept of Bernoulli polynomial is generalized in [8] by the second author.

There are many inequalities related to the sum $S_\alpha(n) = \sum_{m=1}^{n} m^\alpha$, where α is an arbitrary real number. For instance,

$$n^{\alpha+1} < (\alpha+1)S_\alpha(n) < (n+1)^{\alpha+1} - 1$$

$$(\alpha+1)[S_\alpha(n) - 1] < n^{\alpha+1} - 1 < (\alpha+1)S_\alpha(n-1)$$

$$(n+1)^{\alpha+1} - n^{\alpha+1} < (\alpha+1)\big[S_\alpha(n) - S_\alpha(n-1)\big] < n^{\alpha+1} - (n-1)^{\alpha+1}$$

for $\alpha > 0$, $\alpha < -1$ and $-1 < \alpha < 0$, respectively. The proofs of these inequalities could be found in [7: pp. 84 - 85].

In [5, 12, 13] the relationships between Bernoulli numbers and the sum (1) were also studied using the Euler-Maclaurin formula and other devices. It is worth noting that a fascinating account of the early history of the problem above and standard recursion formulas for $S_k(n)$ as originally stated by Pascal are given in [3].

In this article, we prove that $S_k(n)$ is a $(k+1)$-th degree polynomial for n with constant term 0 (that is, formula (2) is valid) and

$$S_{k+1}(n) = (k+1)\left(\frac{A_{k+1}}{k+2}n^{k+2} + \frac{A_k}{k+1}n^{k+1} + \ldots + \frac{A_2}{3}n^3 + \frac{A_1}{2}n^2\right) + b_1 n \tag{4}$$

where

$$b_1 = \begin{cases} 0 & \text{for even } k > 0 \\ 1 - (k+1)\sum_{i=1}^{k+1} \frac{A_i}{i+1} & \text{for odd } k > 0. \end{cases}$$

Formula (4) shows that we can use the coefficients A_i $(1 \le i \le k+1)$ in $S_k(n)$ to get the expression of $S_{k+1}(n)$. In fact, it also gives a method of computing Bernoulli numbers B_{k+1}. At last, other formulae for calculating Bernoulli numbers and $\sum_{m=1}^{n} m^k$ are given.

2. Lemmas

To obtain our main results, the following lemmas are necessary. Moreover, these lemmas also give some recursion formulae for $S_k(n)$.

Lemma 1. *For any integers $k \geq 0$ and $n > 0$, we have*

$$(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \binom{k+1}{i} S_i(n). \tag{5}$$

Proof. Recalling the binomial expansion $(1+m)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} m^i$ we obtain

$$\begin{aligned}(1+n)^{k+1} + S_{k+1}(n) - 1 &= \sum_{m=1}^{n} (1+m)^{k+1} \\ &= \sum_{m=1}^{n} \left(\sum_{i=0}^{k+1} \binom{k+1}{i} m^i \right) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\sum_{m=1}^{n} m^i \right) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} S_i(n).\end{aligned}$$

This is equivalent to

$$(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \binom{k+1}{i} S_i(n).$$

The proof of Lemma 1 is completed ∎

Lemma 1 shows that $S_k(n)$ could be deduced from $S_0(n), S_1(n), \ldots, S_{k-1}(n)$. Using Lemma 1 we can get

Lemma 2. *For arbitrary integer $k > 0$,*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_i n^i. \tag{6}$$

Proof. By mathematical induction on k, the result that $S_k(n)$ is a $(k+1)$-th degree polynomial with constant term 0 follows straightforwardly. Equating the coefficients on the two sides of (5), it is deduced easily that the coefficients of n^{k+1} and n^k in $S_k(n)$ are $\frac{1}{k+1}$ and $\frac{1}{2}$, respectively. This completes the proof of Lemma 2 ∎

Since $S_k(1) = 1$, formula (6) implies

$$\sum_{i=1}^{k-1} A_i = 12 - \frac{1}{k+1}. \tag{7}$$

For any integer $k > 0$, let $\langle k \rangle$ stand for the largest odd number less than k. Then

$$k - \langle k \rangle = \begin{cases} 1 & \text{for any even } k \\ 2 & \text{for any odd } k. \end{cases}$$

For example, $\langle 2 \rangle = 1$, $\langle 5 \rangle = 3$, and so forth.

Let $A_p^{(q)}$ denote the coefficient of n^p in $S_q(n)$. Then

Lemma 3. *For any integer $k > 1$,*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \frac{1}{2} \sum_{i=1}^{\frac{\langle k \rangle + 1}{2}} \frac{1}{i} \binom{k}{2i-1} A_1^{(2i)} n^{k-2i+1}, \tag{9}$$

that is,

$$A_{k-2i+1}^{(k)} = \frac{1}{2i} \binom{k}{2i-1} A_1^{(2i)} \qquad \left(1 \le i \le \tfrac{\langle k \rangle + 1}{2}\right) \tag{10}$$

where $A_1^{(2i)}$ is the coefficient of the term n in $S_{2i}(n)$.

Proof. We will use mathematical induction on k. It is clear that formula (9) is true for $k = 2$. Suppose the result is true for $3, \ldots, k-1$. From Lemma 2, we have

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k-i}.$$

Equating the coefficients of n^{k-j} for $j = 1, 3, \ldots, \langle k \rangle$ in (5) gives us

$$\begin{aligned} A_{k-j}^{(k)} = \frac{1}{k+1} \Bigg[\frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} \\ - \sum_{i=0}^{\frac{j-3}{2}} A_{k-j}^{(k-j+2i+1)} \binom{k+1}{k-j+2i+1} \Bigg]. \end{aligned} \tag{11}$$

By the inductive assumption, we have

$$A_{k-j}^{(k-j+2i+1)} = A_1^{(2(i+1))} \frac{1}{k-j+2(i+1)} \binom{k-j+2(i+1)}{2(i+1)} \tag{12}$$

for $0 \le i \le \frac{j-3}{2}$. Combining (11) and (12) yields

$$\begin{aligned} A_{k-j}^{(k)} = \frac{1}{k+1} \Bigg[\frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} \\ - \sum_{i=1}^{\frac{j-1}{2}} A_1^{(2i)} \frac{1}{k-j+2i} \binom{k-j+2i}{2i} \binom{k+1}{k-j+2i-1} \Bigg]. \end{aligned} \tag{13}$$

From (7) and the inductive assumption, it follows that

$$A_1^{(j+1)} = \frac{1}{2} - \frac{1}{j+2} - \sum_{i=1}^{\frac{j-1}{2}} A_{2i+1}^{(j+1)} \tag{14}$$

and

$$A_{j-2i}^{(j+1)} = A_1^{(2(i+1))} \frac{1}{j+2} \binom{j+2}{2(i+1)} \qquad (0 \le i \le \tfrac{j-3}{2}). \tag{15}$$

Substituting (15) into (14) produces

$$\begin{aligned} A_1^{(j+1)} \frac{1}{k+1} \binom{k+1}{j+1} &= \frac{1}{k+1} \left[\frac{1}{2} \binom{k+1}{j+1} - \frac{1}{j+2} \binom{k+1}{j+1} \right. \\ &\quad \left. - \sum_{i=1}^{\frac{j-1}{2}} A_1^{2i} \frac{1}{j+2} \binom{j+2}{2i} \binom{k+1}{j+1} \right] \\ &= \frac{1}{k+1} \left[\frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} \right. \\ &\quad \left. - \sum_{i=1}^{\frac{j-1}{2}} A_1^{2i} \frac{1}{k-j+2i} \binom{k-j+2i}{2i} \binom{k+1}{k-j+2i-1} \right]. \end{aligned} \tag{16}$$

From (13) and (16),

$$A_{k-j}^{(k)} = \frac{1}{k+1} \binom{k+1}{j+1} A_1^{(j+1)} \qquad (j = 1, 3, \ldots, \langle k \rangle)$$

is obtained. Similarly, by mathematical induction, we can prove that

$$A_{k-i}^{(k)} = 0 \qquad (i = 2, 4, 6, \ldots, \langle k \rangle + 1).$$

The proof of Lemma 3 is completed ∎

Note Lemma 3 shows that the coefficients of the term n in $S_2(n), \ldots, S_{2i-2}(n)$ can be used to calculate $S_{2i-1}(n)$ and $S_{2i}(n)$.

3. Main results

Now we use Lemma 3 to prove

Main Theorem. *For any integer $k > 1$, let*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{\frac{\langle k \rangle + 1}{2}} A_{k-2i+1} n^{k-2i+1}.$$

Then

$$S_{k+1}(n) = \frac{1}{k+2} n^{k+2} + \frac{1}{2} n^{k+1} + (k+1) \sum_{i=1}^{\frac{\langle k\rangle+1}{2}} \frac{A_{k-2i+1}}{k-2(i-1)} n^{k-2(i-1)} + b_1 n$$

where

$$b_1 = \begin{cases} 0 & \text{for even } k \\ \frac{1}{2} - \left[\frac{1}{k+2} + (k+1)\sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2}\right] & \text{for odd } k. \end{cases} \tag{17}$$

Proof. From (10) we know that the coefficients of n^{k-j} $(j = 1, 3, \ldots, \langle k\rangle)$ in $S_k(n)$ are

$$A_{k-j}^{(k)} = \frac{1}{k+1}\binom{k+1}{j+1} A_1^{(j+1)}.$$

Therefore

$$\begin{aligned} A_{k-j}^{(k)} \frac{k+1}{k-j+1} &= A_1^{(j+1)} \frac{1}{k+1}\binom{k+1}{j+1}\frac{k+1}{k-j+1} \\ &= A_1^{(j+1)} \frac{1}{k+2}\binom{k+2}{j+1} \\ &= A_{k-j+1}^{(k+1)} \end{aligned}$$

is the coefficient of n^{k+1-j} $(j = 1, 3, \ldots, \langle k\rangle)$ in $S_{k+1}(n)$. If k is even, since $k - \langle k\rangle + 1 = (k+1) - \langle k+1\rangle$, then $b_1 = 0$ follows from (9). If k is odd, formula (17) follows from (7). This completes the proof ∎

Corollary. *Let A_i be the coefficients of the terms n^i $(1 \le i \le k+1)$ in $S_k(n)$ and let B_i $(i > 1)$ be the i-th Bernoulli numbers. Then*

$$B_{2j+1} = 0$$

$$B_{2j} = \frac{1}{2} - \left[\frac{1}{2j+1} + 2j \sum_{i=1}^{j-1} \frac{A_{2(j-i)}}{2(j-i)+1}\right]$$

for every integer $j \ge 1$,

Remark. By Lemmas 1 - 3 and Main Theorem, calculating directly we obtain

$$S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

$$S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

$$S_{12}(n) = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n$$

$$\begin{aligned} S_{20}(n) = &\frac{1}{21}n^{21} + \frac{1}{2}n^{20} + \frac{5}{3}n^{19} - \frac{19}{2}n^{17} + \frac{1292}{21}n^{15} - 323n^{13} + \frac{41990}{33}n^{11} \\ &- \frac{223193}{63}n^9 + 6460n^7 - \frac{68723}{10}n^5 + \frac{219335}{63}n^3 - \frac{174611}{330}n \end{aligned}$$

$$\begin{aligned} S_{21}(n) = &\frac{1}{22}n^{22} + \frac{1}{2}n^{21} + \frac{7}{4}n^{20} - \frac{133}{12}n^{18} + \frac{323}{4}n^{16} - \frac{969}{2}n^{14} + \frac{146965}{66}n^{12} \\ &- \frac{223193}{30}n^{10} + \frac{33915}{2}n^8 - \frac{481061}{20}n^6 + \frac{219335}{12}n^4 - \frac{1222277}{220}n^2. \end{aligned}$$

From here the Bernoulli numbers

$$B_{10} = 566, \qquad B_{12} = -\frac{691}{2730}, \qquad B_{20} = -\frac{174611}{330}$$

are obtained.

4. Another formulae for $\sum_{m=1}^{n} m^k$ and Bernoulli numbers

In this section, another formulae for computing Bernoulli numbers and $\sum_{m=1}^{n} m^k$ will be given, from which we can get the Bernoulli numbers more easily (see [1] and [2: pp. 246 - 265]).

Define functions B_n by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (|z| < 2\pi)$$

and write $B_n = B_n(0)$ for the Bernoulli numbers. Then formula (3) follows by putting $x = 0$. We can equate coefficients of z^n in

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z}{e^z - 1} \cdot e^{xz} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n\right)$$

to get

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{18}$$

Also, since

$$\frac{z\, e^{(x+1)z}}{e^z - 1} - \frac{z\, e^{xz}}{e^z - 1} = z\, e^{xz},$$

we have

$$\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+1},$$

and by equating coefficients of z^n we get

$$B_n(x+1) - B_n(x) = n\, x^{n-1}. \tag{19}$$

So putting $x = 0$ we have

$$B_n = B_n(0) = B_n(1) \qquad (n \neq 1). \tag{20}$$

Thus for $n \geq 2$ we can put $x = 1$ in (18) and use (20) to obtain

$$B_n = B_n(1) = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

This is a much simpler recursion formula for computing Bernoulli numbers.

Result (19) can be used, taking $x = 1, 2, \ldots, k-1, k$ and adding, to give

$$\begin{aligned} B_n(k+1) - B_n(1) &= \sum_{i=0}^{k-1} \left[B_n(k+1-i) - B_n(k-i)\right] \\ &= n \cdot k^{n-1} + n(k-1)^{n-1} + \ldots + n \cdot 2^{n-1} + n \cdot 1^{n-1} \\ &= n \sum_{m=1}^{k} m^{k-1}, \end{aligned}$$

that is,

$$\sum_{m=1}^{k} m^{k-1} = \frac{B_n(k+1) - B_n}{n}.$$

References

[1] Apostol, T.: *Mathematical Analysis*, 2nd edition. New York: Addison-Wesley 1974.

[2] Apostol, T.: *Introduction to Analytic Number Theory*. Berlin: Springer-Verlag 1976.

[3] Boyer, C. B.: *Pascal's formula for the sums of powers of the integers*. Scripta Math. 9 (1943), 237 - 244.

[4] Brualdi, R.: *Introductory Combinatories*. Amsterdam: North-Holland 1977.

[5] Jing-Tian Cao: *A method of summing series and some corollaries* (in Chinese). Mathematics in Practice and Theory 20 (1990)2, 77 - 84.

[6] Group of Compilation: *Handbook of Mathematics* (in Chinese). Beijing: Higher Education Press 1979.

[7] Ji-Chang Kuang: *Applied Inequalities*, 2nd ed. (in Chinese). Changsha: Hunan Education Press 1993.

[8] Feng Qi: *Generalized Bernoulli polynomial*. Mathematics and Informatics Quarterly (to appear).

[9] Mitrinović, D. S., Pečarić, J. E. and A. M. Fink: *Classical and New Inequalities in Analysis*. Dordrecht - Boston - London: Kluwer Acad. Publ. 1993.

[10] Schultz, H. J.: *The sums of the k-th powers of the first n integers*. Amer. Math. Monthly 87 (1980), 478 - 481.

[11] Struik, D. J.: *A Source Book in Mathematics 1200 - 1800*. Cambridge (Mass.): Harvad Univ. Press, Cambridge 1969.

[12] Bi-Cheng Yang: *Formulae related to Bernoulli number and for sums of the same power of natural numbers* (in Chinese). Mathematics in Practice and Theory 24 (1994)4, 52 - 56 and 74.

[13] Nan-Yue Zhang: *Euler's number and some sums related to Zeta function* (in Chinese). Mathematics in Practice and Theory 20 (1990)4, 62 - 70.

Received 10.12.1998