On the Hilbert Inequality

Gao Mingzhe

Abstract. It is shown that the Hilbert inequality for double series can be improved by in troducing the positive real number $\frac{1}{r^2}$ Gao Mingzhe

rt inequality for double series can be in

r($\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}$) where $s(x) = \sum_{n=1}^{\infty} \frac{z_1}{n}$

f the classical Hilbert inequality is proved

similar result for the Hilbert integral ine $\lVert \cdot \rVert$ and $\lVert x \rVert^2$ x_n^2 $(x = a, b)$. The coefficient π of the classical Hilbert inequality is proved not to be the best possible if $\|a\|$ or $\|b\|$ is finite. A similar result for the Hilbert integral inequality is also proved.

Keywords: *Hubert inequality, binary quadratic form, exponential integral, inner product* AMS subject classification: 26 D, 46 C

1. Introduction

double series can be written as

b|| is finite. A similar result for the Hilbert integral inequality is also
\nquality, binary quadratic form, exponential integral, inner product
\nratio: 26 D, 46 C

\ni
$$
\geq 1
$$
 be arbitrary real sequences. Then the Hilbert inequality for
\nritten as

\n
$$
\left(\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right) \left(\sum_{n=1}^{\infty} b_n^2\right).
$$
\n(1)\n
$$
\left(\sum_{\substack{m,n=1\\ \text{min}}}^{\infty} \frac{a_m b_n}{m-n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right) \left(\sum_{n=1}^{\infty} b_n^2\right).
$$

Additionally,

1. Introduction
\nLet
$$
(a_n)_{n\geq 1}
$$
 and $(b_n)_{n\geq 1}$ be arbitrary real sequences. Then the *Hilbert inequality* for
\ndouble series can be written as\n
$$
\left(\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right) \left(\sum_{n=1}^{\infty} b_n^2\right).
$$
\n(1)
\nAdditionally,\n
$$
\left(\sum_{\substack{m,n=1 \ m \neq n}}^{\infty} \frac{a_m b_n}{m-n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right) \left(\sum_{n=1}^{\infty} b_n^2\right).
$$
\n(2)
\nis also called *Hilbert inequality*. Furthermore, if $f, g \in L^2(\mathbb{R}_+)$ where $\mathbb{R}_+ = (0, \infty)$, then
\nthe inequality analogous to (1)
\n
$$
\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} dsdt\right)^2 \leq \pi^2 \left(\int_{\mathbb{R}_+} f^2(t) dt\right) \left(\int_{\mathbb{R}_+} g^2(t) dt\right).
$$
\n(3)
\nis called the *Hilbert integral inequality*. The constant π contained in these inequalities,

is also called *Hilbert inequality.* Furthermore, if $f, g \in L^2(\mathbb{R}_+)$ where $\mathbb{R}_+ = (0, \infty)$, then the inequality analogous to $\left(1\right)$

$$
\left(\sum_{\substack{m,n=1 \ n \neq n}} \frac{a_m b_n}{m-n}\right) \leq \pi^2 \left(\sum_{n=1} a_n^2\right) \left(\sum_{n=1} b_n^2\right) \tag{2}
$$
\n
$$
\text{Hilbert inequality. Furthermore, if } f, g \in L^2(\mathbb{R}_+) \text{ where } \mathbb{R}_+ = (0, \infty), \text{ then}
$$
\n
$$
\text{analogous to (1)}
$$
\n
$$
\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} \, dsdt\right)^2 \leq \pi^2 \left(\int_{\mathbb{R}_+} f^2(t) \, dt\right) \left(\int_{\mathbb{R}_+} g^2(t) \, dt\right) \tag{3}
$$

is called the *Hilbert integral inequality*. The constant π contained in these inequalities, especially in (1), was proved to be the best possible (see [3]). However, if $0 < \sum_{n=1}^{\infty} a_n^2 <$ ∞ or $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we can select a number $r > 0$ such that the right-hand side of (1) can be replaced by $\int_{\mathbb{R}_+}$
 inequality. The constant π coording to be the best possible (see [3])
 $\int_{\mathbb{R}_+}$ are can select a number $r >$
 $\pi^2(1-r)\left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right),$

$$
\pi^2(1-r)\bigg(\sum_{n=1}^\infty a_n^2\bigg)\bigg(\sum_{n=1}^\infty b_n^2\bigg),\,
$$

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i.e. an improvement of (1) will be obtained. Similarly, an improvement of (3) will be established. Namely, the right-hand side of (3) can be written as

$$
\pi^2(1-R)\bigg(\int_{\mathbb{R}_+}f^2(t)\,dt\bigg)\bigg(\int_{\mathbb{R}_+}g^2(t)\,dt\bigg)
$$

with a number $R > 0$. The main purpose of the present paper is to prove the existence of such numbers *r* and *R* and to find expressions for them.

We first introduce some notations and functions.

If α and β are elements of an inner product space E , then its inner product is denoted i.e. an improvement of (1) will be obtained. Similarly, an established. Namely, the right-hand side of (3) can be writ $\pi^2(1 - R) \left(\int_{\mathbb{R}_+} f^2(t) dt \right) \left(\int_{\mathbb{R}_+} g^2(t) dt \right)$ with a number $R > 0$. The main purpose of th Further, if $a = (a_n)_{n \geq 1}$ and $b = (b_n)_{n \geq 1}$ are two real sequences, then its inner product (a, b) and the norm $\|\bar{a}\|$ of *a* are defined by \overline{a} *and*
 anner prod
 given by \parallel
 as, then it
 $a_n b_n$ t) dt $\left(\int_{\mathbb{R}_+} g^2(t) dt\right)$
of the present paper is to prove the existence
essions for them.
functions.
uct space E, then its inner product is denoted
 $\alpha \parallel = \sqrt{(\alpha, \alpha)}$. Further, if $a = (a_n)_{n \ge 1}$ and
sinner product $(a$ inner product is denoted
ther, if $a = (a_n)_{n \ge 1}$ and
b) and the norm $||a||$ of a
 \overline{a} , \overline{a}). (4)
 g) and the norm $||f||$ of
 $f^2(t) dt$)^{$\frac{1}{2}$} (5).

$$
(a, b) = \sum_{n=1}^{\infty} a_n b_n
$$
 and $||a|| = \sqrt{(a, a)}.$ (4)

Analogously, for functions $f, g \in L^2(a, b)$ its inner product (f, g) and the norm $||f||$ of *f* are defined by

$$
(a,b) = \sum_{n=1}^{\infty} a_n b_n \quad \text{and} \quad ||a|| = \sqrt{(a,a)}.
$$
 (4)
for functions $f, g \in L^2(a,b)$ its inner product (f,g) and the norm ||f|| of
1 by

$$
(f,g) = \int_a^b f(t)g(t) dt \quad \text{and} \quad ||f|| = \left(\int_a^b f^2(t) dt\right)^{\frac{1}{2}}.
$$
 (5)
introduce a binary quadratic form $F(\cdot, \cdot)$ defined by

$$
F(x,y) = ||\alpha||^2 x^2 - 2(\alpha, \beta)xy + ||\beta||^2 y^2
$$
 (6)
3, γ) and $y = (\alpha, \gamma)$ for $\gamma \in E$. We further denote

$$
G(\alpha, \beta, \gamma) = F((\beta, \gamma), (\alpha, \gamma)).
$$
 (7)
nvolve $G(\alpha, \beta, \gamma)$ with α and β specified beforehand, and γ to be chosen
n felicity. It is obvious that if γ is orthogonal to both α and β then

We next introduce a binary quadratic form $F(\cdot, \cdot)$ defined by

$$
F(x,y) = ||\alpha||^2 x^2 - 2(\alpha,\beta)xy + ||\beta||^2 y^2
$$
 (6)

where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$ for $\gamma \in E$. We further denote

$$
G(\alpha, \beta, \gamma) = F((\beta, \gamma), (\alpha, \gamma)).
$$
\n(7)

The results involve $G(\alpha, \beta, \gamma)$ with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that if γ is orthogonal to both α and β , then $G(\alpha,\beta,\gamma) = 0$. It will turn out that if $(\alpha,\gamma)^2 + (\beta,\gamma)^2 > 0$ (see Lemma 1). Therefore, it is shrewd in every case to choose γ not orthogonal to both α and β . (a, γ) for $\gamma \in E$
 $G(\alpha, \beta, \gamma) =$

3, γ) with α and

is obvious that

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to choose γ not

itroduce yet the
 $\frac{a_m b_n}{n + n}$, $v(a, b)$ pecified beforehand, and γ is orthogonal to both $\alpha + (\beta, \gamma)^2 > 0$ (see Lemma
ogonal to both α and β .
ations
 $\sum_{m=1}^{\infty} \frac{a_m b_n}{m - n}$, $s(x) = \sum_{m=1}^{\infty} \frac{a_m}{n - n}$ $G(\alpha, \beta, \gamma) = F((\beta, \gamma))$
 u $G(\alpha, \beta, \gamma) = F((\beta, \gamma))$,
 u $G(\alpha, \beta, \gamma)$ with α and β specify
 u in the fiction of the solution of the solution
 u(*a, b*) = $\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}$, $v(a, b) = \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}$ her denote
 (α, γ)).

ied beforehand, and γ to

orthogonal to both α ar
 $(\gamma)^2 > 0$ (see Lemma 1).

al to both α and β .

s
 $\frac{a_m b_n}{n - n}$, $s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}$ *m*, α *m y* = (α , γ) for $\gamma \in E$. We further $G(\alpha, \beta, \gamma) = F((\beta, \gamma))$
G(α , β , γ) with α and β spectity. It is obvious that if γ is
will turn out that if $(\alpha, \gamma)^2 + (\gamma)$ asses to choose γ not

For convenience, we introduce yet the notations

num identity. It is obvious that if
$$
\gamma
$$
 is orthogonal to both α and $a = 0$. It will turn out that if $(\alpha, \gamma)^2 + (\beta, \gamma)^2 > 0$ (see Lemma 1). To find the very case to choose γ not orthogonal to both α and β .
\nOnvenience, we introduce yet the notations
\n
$$
u(a, b) = \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}, \quad v(a, b) = \sum_{\substack{m,n=1 \ m \neq n}}^{\infty} \frac{a_m b_n}{m-n}, \qquad s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.
$$

We shall frequently use these notations below.

 \boldsymbol{i}

2. Lemmas

To prove our theorems, we need the following results.

Lemma 1. Let $G(\alpha, \beta, \gamma)$ be defined as in (7). If $\alpha, \beta \in E$ are linearly independent *and* $(\alpha, \gamma)^2 + (\beta, \gamma)^2 > 0$, then $G(\alpha, \beta, \gamma) > 0$. *(a,* β *,* γ *) be defined as in (7). If* $\alpha, \beta \in E$ *are linearly independent*
hen $G(\alpha, \beta, \gamma) > 0$.
(a,fl,franceries)
(a,fl) be defined as defined in (7). If $\alpha, \beta \in E$ *are linearly*
*(a,fl)*² $\leq ||\alpha||^2 ||\beta$

Lemma 2. Let $G(\alpha,\beta,\gamma)$ be defined as defined in (7). If $\alpha,\beta \in E$ are linearly *dependent, then* $G(\alpha, \beta, \gamma) = 0$.

Lemma 3. Let $G(\alpha, \beta, \gamma)$ be defined as defined in (7). If $\alpha, \beta \in E$ are arbitrary *and* $\gamma \in E$ *with* $\|\gamma\| = 1$ *, then*

$$
(\alpha, \beta)^2 \le ||\alpha||^2 ||\beta||^2 - G(\alpha, \beta, \gamma), \tag{8}
$$

and equality holds in (8) if and only if α, β, γ are linearly dependent.

The proofs of Lemmas 1 and 2 have been given in our previous paper [1]. Lemma 3 is actually a sharpening of the Cauchy-Schwarz inequality. This result has been given also in the paper [1], and in (5]. Hence the proofs of all lemmas are omitted. **¹¹¹¹¹** *²* **¹¹** ⁹ **¹¹** *²* **-** *F(x, y) (9)*

Using the inner product defined by (5) and Lemma 3, we have the following result.

Corollary 1. *If* $f, g \in L^2(a, b)$, then

defined by (5) and Lemma 3, we have the following result.
\n
$$
f^{2}(a, b), \text{ then}
$$
\n
$$
(f, g)^{2} \leq ||f||^{2}||g||^{2} - F(x, y)
$$
\n(9)

where $F(x,y) = ||f||^2 x^2 - 2(f,g)xy + ||g||^2 y^2$ *with* $x = (g,\gamma)$ *and* $y = (f,\gamma), \gamma \in L^2(a,b)$ *with* $\|\gamma\| = 1$.

3. Main results

In this section we will combine the two forms (1) and (2) of the Hubert inequality into one similar form, and make inequalities (1) - (3) relaize significant improvements. The following theorems are the main results in this paper. publishing the two forms (1) and (2) of the Hilbert ake inequalities (1) - (3) relaize significant improblement main results in this paper.

(a_n) and $b = (b_n)$ are real sequences with non-
 $\langle |b|| < \infty$, then
 $u^2(a, b) + v$ **3. Main result**
In this section we wone similar form, and
following theorems:
Theorem 1. I_j
with $0 < ||a|| < \infty$ of
where $r = \frac{1}{\pi^2} (\frac{s^2(a)}{||a||^2})$
Proof. Let us

Theorem 1. If $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, *with* $0 < ||a|| < \infty$ or $0 < ||b|| < \infty$, then

$$
u^{2}(a,b) + v^{2}(a,b) < \pi^{2}(1-r)||a||^{2}||b||^{2} \tag{10}
$$

 $\|a\| < \infty \text{ or } 0$
 $\frac{1}{\pi^2} \left(\frac{s^2(a)}{\|a\|^2} + \frac{s}{\|a\|^2} \right)$ $+\frac{1}{2}^{2}(b)$

Proof. Let us define two real functions $f, g : (0, 2\pi) \to \mathbb{R}$ by

m 1. If
$$
a = (a_n)
$$
 and $b = (b_n)$ are real sequences with non-
\n $| \langle \infty \text{ or } 0 \rangle \langle |b| | \langle \infty, \text{ then}$
\n $u^2(a, b) + v^2(a, b) \langle \pi^2(1-r) ||a||^2 ||b||^2$
\n $\sigma \left(\frac{s^2(a)}{||a||^2} + \frac{s^2(b)}{||b||^2} \right).$
\nLet us define two real functions $f, g : (0, 2\pi) \to \mathbb{R}$ by
\n $f(t) = \sum_{n=1}^{\infty} a_n \sqrt{t} \sin(nt)$ and $g(t) = \sum_{n=1}^{\infty} b_n \sqrt{t} \cos(nt).$
\n $\log(a, b) + v(a, b) = \frac{1}{\pi} |(f, g)|.$

It is easily to deduce that, with the notations of the space $L^2(0, 2\pi)$,

$$
|u(a,b) + v(a,b)| = \frac{1}{\pi} |(f,g)|. \tag{11}
$$

According to (5) and (6) we have $(f,g)^2 \le ||f||^2 ||g||^2 - F(x,y)$ where $||f||^2 = \pi^2 ||a||^2$,
 $||g||^2 = \pi^2 ||b||^2$ and
 $F(x,y) = ||f||^2 x^2 - 2(f,g)xy + ||g||^2 y^2 \ge (||f||x - ||g||y)^2 = \pi^2 (||a||x - ||b||y)^2$.

Hence
 $(f,g)^2 \le \pi^4 ||a||^2 ||b||^2 - \pi^2 (||a||x - ||b||y)^2$ (12 $||g||^2 = \pi^2 ||b||^2$ and *F*(*x, y*) = $||f||^2 \le ||f||^2 ||g||^2 - F(x, y)$ where $||f||^2 \le ||f||^2 ||g||^2 - F(x, y)$ where $||f||^2$ and
 $F(x, y) = ||f||^2 x^2 - 2(f, g)xy + ||g||^2 y^2 \ge (||f||x - ||g||y)^2 = \pi^2 (||a||x - ||g||)^2$

re $(f, g)^2 \le \pi^4 ||g||^2 ||h||^2 - \pi^2 (||g||x - ||h||)^2$

$$
F(x,y) = ||f||^2 x^2 - 2(f,g)xy + ||g||^2 y^2 \ge (||f||x - ||g||y)^2 = \pi^2 (||a||x - ||b||y)^2.
$$

Hence

$$
(f,g)^2 \le \pi^4 ||a||^2 ||b||^2 - \pi^2 (||a||x - ||b||y)^2
$$
 (12)

 $\|f\|^2 \|g\|^2 - F(x, y)$ where $\|f\|$
 $\geq (\|f\|x - \|g\|y)^2 = \pi^2 (\|a\|x - \pi^2 (\|a\|x - \|b\|y)^2$
 π) with $\|\gamma\| = 1$. We can choose
 $s(a)$. Hence where $x = (g, \gamma)$ and $y = (f, \gamma), \, \gamma \in L^2(0, 2\pi)$ with $\|\gamma\| = 1$. We can choose $\gamma =$ Then $x = 0$ and $y = -\sqrt{2} \sum_{n=1}^{\infty} \frac{a_n}{n} = -\sqrt{2} s(a)$. Hence have $(f, g)^2 \le ||f||^2 ||g||^2 - F(x, y)$ where $||f||^2 = \pi^2 ||a||^2$,
 $g)xy + ||g||^2y^2 \ge (||f||x - ||g||y)^2 = \pi^2 (||a||x - ||b||y)^2$.
 $\frac{d}{dx} \le \pi^4 ||a||^2 ||b||^2 - \pi^2 (||a||x - ||b||y)^2$ (12)
 γ , $\gamma \in L^2(0, 2\pi)$ with $||\gamma|| = 1$. We can choose $\gamma = \frac{1}{2\pi}\$ $2^2 - 2(f, g)xy + ||g||^2y^2 \ge (||f||x - ||g||y)^2 = \pi^2(||a||x - ||g||y - ||g||y)^2$
 $[(f, g)^2 \le \pi^4 ||a||^2 ||b||^2 - \pi^2 (||a||x - ||b||y)^2$
 $y = (f, \gamma), \gamma \in L^2(0, 2\pi) \text{ with } ||\gamma|| = 1. \text{ We can choose } \gamma$
 $-\sqrt{2} \sum_{n=1}^{\infty} \frac{a_n}{n} = -\sqrt{2} s(a). \text{ Hence}$
 $(||a||x - ||b||y)^2 = 2||b||^2 s^2(a).$ $y = (f, \gamma), \gamma \in L^2(0, 2\pi)$ with $\|\gamma\| = 1$. We can choose γ
 $-\sqrt{2} \sum_{n=1}^{\infty} \frac{a_n}{n} = -\sqrt{2} s(a)$. Hence
 $(\|a\|x - \|b\|y)^2 = 2\|b\|^2 s^2(a)$.

3) we obtain
 $|u(a, b) + v(a, b)|^2 \le \pi^2 \|a\|^2 \|b\|^2 - 2\|b\|^2 s^2(a)$.
 y, γ are linea

$$
(\|a\|x - \|b\|y)^2 = 2\|b\|^2 s^2(a). \tag{13}
$$

In virtue of (11) - (13) we obtain

$$
|u(a,b) + v(a,b)|^2 \le \pi^2 ||a||^2 ||b||^2 - 2||b||^2 s^2(a).
$$
 (14)

Since the vectors f, g, γ are linearly independent, by Lemma 3, it is impossible to take equality in (14). Hence we have

$$
|u(a,b) + v(a,b)|^2 < \pi^2 \|a\|^2 \|b\|^2 - 2\|b\|^2 s^2(a). \tag{15}
$$

Notice that $u(b,a) = u(a,b)$ and $v(b,a) = -v(a,b)$. Interchanging *a* and *b* in (11), similarly we obtain 3) we obtain
 $|u(a, b) + v(a, b)|^2 \le \pi^2 ||a||^2 ||b||^2 - 2||b||^2 s^2(a)$.
 I, γ are linearly independent, by Lemma 3, it is impossit
 $|u(a, b) + v(a, b)|^2 < \pi^2 ||a||^2 ||b||^2 - 2||b||^2 s^2(a)$.
 $= u(a, b)$ and $v(b, a) = -v(a, b)$. Interchanging a and

$$
|u(a,b)-v(a,b)|^2<\pi^2||a||^2||b||^2-2||a||^2s^2(b).
$$
 (16)

Adding (15) and (16), inequality (10) is yielded after some simplifications. Thus the proof of the theorem is completed

Remark. Since $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, with $0 < ||a|| < \infty$ or $0 < ||b|| < \infty$, it follows that $r > 0$. Hence inequality (10) is a significant refinement of the paper [4]. $|b||^2 - 2||a||^2 s^2(b).$

after some simplificat

al sequences with non-

nat $r > 0$. Hence ined

ith non-negative term
 $(1 - \tilde{r})||a||^4$ $||| < \infty$, it follows that $r > 0$. Hence is

vaper [4].
 us a real sequence with non-negative te
 $a, a) + v^2(a, a) < \pi^2(1 - \tilde{r})||a||^4$

ced by 0, then we have the following
 usumptions of Theorem 1, *then*
 $u^2(a, b) < \pi^2($

Corollary 2. If $a = (a_n)$ *is a real sequence with non-negative terms and* $0 < ||a|| <$ ∞ , then Corollary 2.
 ∞ , then
 $where \space \tilde{r} = \frac{2}{\pi^2} \frac{s^2(a)}{\|a\|^2}$

$$
u^{2}(a,a) + v^{2}(a,a) < \pi^{2}(1-\tilde{r})||a||^{4} \tag{17}
$$

If $v^2(a, b)$ in (10) is replaced by 0, then we have the following

Corollary 3. *With the assumptions of Theorem 1, then*

$$
u^{2}(a,b) < \pi^{2}(1-r)\|a\|^{2}\|b\|^{2} \tag{18}
$$

where $r = \frac{1}{\pi^2} \left(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2} \right)$.

We see from the above Remark that inequality (18) is a significant improvement of (1). According to Corollary *2* we obtain at once the following

÷,

Corollary 4. *If* $a = (a_n)$ *is a real sequence with non-negative terms and* $0 < ||a|| <$ ∞ , then s a real sequence
 $u^2(a,a) < \pi^2(1$ $-\tilde{r}$)||a||⁴ (19)

$$
u^2(a,a) < \pi^2(1-\tilde{r})\|a\|^4 \tag{19}
$$

 $where \space \tilde{r} = \frac{2}{\pi^2} \frac{s^2(a)}{\|a\|^2}$

Similarly, we can establish an improvement of the Hilbert integral inequality. For this we need the integral

lish an improvement of the Hilb

$$
e(t) = \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t} ds \qquad (t \in \mathbb{R}_+)
$$

called exponential integral with parameter t.

Similarly, we can establish an improvement of the Hilbert integral inequality. For
\nwe need the integral
\n
$$
e(t) = \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t} ds \qquad (t \in \mathbb{R}_+)
$$
\n
$$
= \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t} ds \qquad (t \in \mathbb{R}_+)
$$
\n
$$
= \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t} ds \qquad (t \in \mathbb{R}_+)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} ds dt \qquad (20)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} ds dt \qquad (21)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} ds dt \qquad (22)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t} ds dt \qquad (23)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)}{s+t} ds dt \qquad (24)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)}{s+t} ds dt \qquad (25)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(s)}{s+t} ds dt \qquad (26)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{g(t)}{t} \Big|_{\mathbb{R}_+^2} \frac{f(t)}{t} ds dt \qquad (27)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{f(t)}{t} ds dt \qquad (28)
$$
\n
$$
= \int_{\mathbb{R}_+^2} \frac{g(t)}{t} \Big|_{\mathbb{R}_+^2} \frac{f(t)}{t} ds dt \qquad (29)
$$
\n
$$
= (F, G)^2
$$

where R onential interpresent the contract of $\left(\begin{array}{c} \left(\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right)^{-1} \frac{1}{\sqrt{2}} \end{array}\right)$
al integral windows $\left(\frac{y}{f\|}\right)^2$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(g,e)$ and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s}),$ *e being the exponential integral with parameter.* th $x = \left(\frac{4}{\pi}\right)$
eter.
and G by
 $\left(\frac{s}{t}\right)^{\frac{1}{4}}$ (g,e) and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s})$
and $G(s,t) = \frac{g(t)}{(s+t)^{\frac{1}{2}}} \left(\frac{t}{s}\right)^{\frac{1}{4}}$

Proof. Define functions *F* and G by

Integrals with parameter.

\nDefine functions
$$
F
$$
 and G by

\n
$$
F(s,t) = \frac{f(s)}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t}\right)^{\frac{1}{4}} \quad \text{and} \quad G(s,t) = \frac{g(t)}{(s+t)^{\frac{1}{2}}} \left(\frac{t}{s}\right)^{\frac{1}{4}}.
$$

Using inequality (9) we have in $L^2(\mathbb{R}^2_+)$

$$
(s+t)^{\frac{1}{2}} \setminus t
$$
\n
$$
(s+t)^{\frac{1}{2}} \setminus s
$$
\n
$$
y(9) \text{ we have in } L^{2}(\mathbb{R}_{+}^{2})
$$
\n
$$
\left(\iint_{\mathbb{R}_{+}^{2}} \frac{f(s)g(t)}{s+t} dsdt\right)^{2} = (F, G)^{2}
$$
\n
$$
\leq ||F||^{2}||G||^{2} - F(x, y)
$$
\n
$$
\leq ||F||^{2}||G||^{2} - (||F||x - ||G||y)^{2} \quad \text{where}
$$
\n
$$
\gamma) \text{ and } y = (F, \gamma), \gamma \in L^{2}(\mathbb{R}_{+}^{2}) \text{ with } ||\gamma|| = 1. \text{ We can choose}
$$
\n
$$
\gamma(s, t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t}\right)^{\frac{1}{4}}.
$$
\n
$$
(21)
$$

$$
\leq ||F||^2||G||^2 - (||F||x - ||G||y)^2 \qquad \Leftrightarrow
$$

where $x = (G, \gamma)$ and $y = (F, \gamma)$, $\gamma \in L^2(\mathbb{R}^2_+)$ with $\|\gamma\| = 1$. We can choose

$$
\gamma(s,t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t}\right)^{\frac{1}{4}}.
$$

Hence we get

$$
\gamma(s,t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t}\right)^{\frac{1}{4}}.
$$

\n
$$
\gamma(s,t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t}\right)^{\frac{1}{4}}.
$$

\n
$$
x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (g,e) \quad \text{and} \quad y = (2\pi)^{\frac{1}{2}} (f,e^{-s}).
$$

\nthat
\n
$$
||F||^2 = \pi ||f||^2 \quad \text{and} \quad ||G||^2 = \pi ||g||^2.
$$
 (23)
\nand (23) into (21) we obtain

It is easy to deduce that

that
\n
$$
||F||^2 = \pi ||f||^2 \quad \text{and} \quad ||G||^2 = \pi ||g||^2. \tag{23}
$$
\n
$$
|(23) \text{ into (21) we obtain}
$$
\n
$$
(F, G)^2 \le \pi^2 ||f||^2 ||g||^2 - \pi (||f||x - ||g||y)^2. \tag{24}
$$
\n
$$
|F||^2 = \pi^2 ||f||^2 ||g||^2 - \pi (||f||x - ||g||y)^2. \tag{25}
$$

Substituting (22) and **(23) into (21) we** obtain

$$
(F,G)^{2} \leq \pi^{2} \|f\|^{2} \|g\|^{2} - \pi (\|f\|x - \|g\|y)^{2}.
$$
 (24)

Since F, G, γ are linearly independent, it is impossible to have equality in (24). Consequently, inequality (20) is obtained from (24) after some simplifications. Thus the theorem is proved \blacksquare

Corollary 5. If $f \in L^2(\mathbb{R}_+)$ *is positive, then*

$$
\bigg(\iint_{\mathbb{R}^2_+} \frac{f(s)f(t)}{s+t} \, dsdt\bigg)^2 < \pi^2(1-\tilde{R})\|f\|^4
$$

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 corollary 5. If $f \in L^2(\mathbb{R}_+)$ is positive, then
 $\left(\iint_{\mathbb{R}^2_+} \frac{f(s)f(t)}{s+t} ds dt \right)^2 < \pi^2 (1 - \tilde{R}) ||f||^4$
 where $\tilde{R} = \frac{1}{\pi} \frac{(x-y)^2}{||f||^2}$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (f, e)$ and $y = (2\pi)^{\frac{$ *integral with parameter.*

Obviously, this is an immediate consequence of Theorem 2.

4. Conclusions

Some classical reasults concerning the Hilbert inequality show that the constant π in (1) is the best possible (see, i.e., $[1, 2, 5, 6]$). We see from (18) that inequality in (1) can be obtained only if $r = 0$. However, to change r into 0, it is necessary to take both $\|a\|$ and $\|b\|$ infinite. Therefore, generally, the constant π in (1) is not the best possible because the constant r contained in (18) is not equal to 0 if $\|a\|$ or $\|b\|$ is finite. In other words, the factor π in (1) can be decreased if $0 < ||a|| < \infty$ or $0 < ||b|| < \infty$.

Similarly, we see from (20) that strong inequality in (3) can be obtained only if $R = 0$. In other words, the factor π in (3) is also not the best possible if $||f||$ or $||g||$ is finite.

Acknowledgement. The author is indebted to the referees for many valuable suggestions in this subject.

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Recursion Formulae for $\sum_{m=1}^{n}$

Son Iⁱⁿ Gue and Forg Oi mk *m=i*

Sen-Lin Guo and Feng Qi

Abstract. Using elementary approach and mathematical induction, several recursion formulae for $S_k(n) = \sum_{m=1}^n m^k$ are presented which show that $S_{k+1}(n)$ could be obtained from $S_k(n)$. A method and a formula of calculating Bernoulli numbers are proposed.

Keywords: *Recursion formulas, sum of powers, mathematical induction, Bernoulli numbers* AMS subject classification: Primary 11 B37, secondary 11 B68, 11 B83

1. Introduction

By definition and geometric meanings of the definite integral, it is well-known that the area under the curve $y = x^k$ over the closed interval [0, 1] equals

Intary approach and mathematical induction, see

\nare presented which show that
$$
S_{k+1}(n)
$$
 could be

\nla of calculating Bernoulli numbers are propose

\nformulas, sum of powers, mathematical inducti

\ncation: Primary 11 B 37, secondary 11 B 68, 11

\nmetric meanings of the definite integral, it is

\n $y = x^k$ over the closed interval $[0, 1]$ equals

\n
$$
\lim_{n \to \infty} \sum_{m=1}^n \frac{1}{n} \left(\frac{m}{n}\right)^k = \lim_{n \to \infty} \frac{1}{n^{k+1}} \left(\sum_{m=1}^n m^k\right).
$$

\nion of this and many similar problems, it is

To complete the solution of this and many similar problems, it is then necessary to find the sums

ngs of the definite integral, it is well-known that the closed interval [0, 1] equals\n
$$
\left(\frac{m}{n}\right)^k = \lim_{n \to \infty} \frac{1}{n^{k+1}} \left(\sum_{m=1}^n m^k\right).
$$
\nd many similar problems, it is then necessary to find\n
$$
S_k(n) = \sum_{m=1}^n m^k.
$$
\n(1)\n(2) We can use the following equations:\n(1)\n(2) We can use the following equations:\n(2) We can use the following equations:\

For small integer $k > 0$, the sums always appear in many calculus courses. For example,

$$
S_7(n) = \frac{1}{24}n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)
$$

and the like [6: p. 11]. Such sums are usually proved by induction or derived from simple geometric pictures. For arbitrary *k,* unfortunately, the standard closed forms involve Bernoulli numbers or Stirling numbers of the second kind [4: p. 119], which come from reasonably complicated recurrence relations.

H. J. Schultz [10] derived a procedure for finding $S_k(n)$, *k* a positive integer, that is easy to remember, arises naturally, and can be used with very little background.

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However, he only illustrated the method by finding $S_6(n)$. According to [10], if one wants to compute, in general, **Feng Qi**
 ated the method by finding $S_6(n)$ **. According to [10], if one

cording** $S_k(n) = A_{k+1}n^{k+1} + \ldots + A_1n + A_0$, (2)

ons

$$
S_k(n) = A_{k+1}n^{k+1} + \ldots + A_1n + A_0, \qquad (2)
$$

a system of $k + 1$ equations

General,

\n
$$
S_{k}(n) = A_{k+1}n^{k+1} + \ldots + A_{1}n + A_{0},
$$
\nitions

\n
$$
\sum_{i=j+1}^{k+1} (-1)^{i-j+1} {i \choose j} A_{i} = 0 \qquad (0 \leq j \leq k)
$$
\na Bernoulli number defined in [6: p. 648] a

\n
$$
\frac{x}{e^{x} - 1} = \sum_{n=0}^{\infty} B_{n} \frac{x^{n}}{n!} \qquad (|x| \leq 2\pi).
$$
\na the formula for $S_{k}(n)$ is the *k*-th Bernoulli noted that the concept of Bernoulli polyni.

must be solved.

Let B_n be the *n*-th Bernoulli number defined in [6: p. 648] and [9: p. 632] by

Let
$$
d
$$
 the method by $5e(n)$. According to [10], it one.

\n $S_k(n) = A_{k+1}n^{k+1} + \ldots + A_1n + A_0,$

\nand

\n $\sum_{i=1}^{n} (-1)^{i-j+1} \binom{i}{j} A_i = 0 \quad (0 \leq j \leq k)$

\nFrom $\sum_{i=1}^{n} (-1)^{i-j+1} \binom{i}{j} A_i = 0 \quad (0 \leq j \leq k)$

\nFrom $\sum_{i=1}^{n} B_n \frac{x^n}{n!}$ $(|x| \leq 2\pi).$

\n(3)

\nand B_k for details, if A_k is the function of $S_k(n)$ is the k -th Bernoulli number B_k (for details, if A_k is the correct of Bernoulli polynomial is generalized in the image.

Then A_1 obtained from the formula for $S_k(n)$ is the k-th Bernoulli number B_k (for details see [11: p. 320)). It is noted that the concept of Bernoulli polynomial is generalized in [8] by the second author.

There are many inequalities related to the sum $S_{\alpha}(n) = \sum_{m=1}^{n} m^{\alpha}$, where α is an trary real number. For instance,
 $n^{\alpha+1} < (\alpha+1)S_{\alpha}(n) < (n+1)^{\alpha+1} - 1$ arbitrary real number. For instance,

$$
n^{\alpha+1} < (\alpha+1)S_{\alpha}(n) < (n+1)^{\alpha+1} - 1
$$
\n
$$
(\alpha+1)[S_{\alpha}(n)-1] < n^{\alpha+1} - 1 < (\alpha+1)S_{\alpha}(n-1)
$$
\n
$$
(n+1)^{\alpha+1} - n^{\alpha+1} < (\alpha+1)[S_{\alpha}(n)-S_{\alpha}(n-1)] < n^{\alpha+1} - (n-1)^{\alpha+1}
$$
\n
$$
0, \alpha < -1, \text{ and } -1 < \alpha < 0, \text{ respectively. The proofs of these inequality}
$$

for $\alpha > 0$, $\alpha < -1$ and $-1 < \alpha < 0$, respectively. The proofs of these inequalities could be found in [7: pp. $84 - 85$].

In [5, 12, 13] the relationships between Bernoulli numbers and the sum (1) were also studied using the Euler-Maclaurin formula and other devices. It is worth noting that a fascinating account of the early history of the problem above and standard recursion formulas for $S_k(n)$ as originally stated by Pascal are given in [3].

In this article, we prove that $S_k(n)$ is a $(k + 1)$ -th degree polynomial for *n* with constant term 0 (that is, formula (2) is valid) and

values for
$$
S_k(n)
$$
 as originally stated by Pascal are given in [3].

\nIn this article, we prove that $S_k(n)$ is a $(k + 1)$ -th degree polynomial for n with tant term 0 (that is, formula (2) is valid) and

\n $S_{k+1}(n) = (k + 1) \left(\frac{A_{k+1}}{k+2} n^{k+2} + \frac{A_k}{k+1} n^{k+1} + \ldots + \frac{A_2}{3} n^3 + \frac{A_1}{2} n^2 \right) + b_1 n$ (4)

\nre

\n $b_1 = \begin{cases} 0 & \text{for even } k > 0 \\ 1 - (k+1) \sum_{i=1}^{k+1} \frac{A_i}{i+1} & \text{for odd } k > 0. \end{cases}$

\nwhile (4) shows that we can use the coefficients A_i (1 < i < k + 1) in $S_k(n)$ to

where

$$
b_1 = \begin{cases} 0 & \text{for even } k > 0\\ 1 - (k+1) \sum_{i=1}^{k+1} \frac{A_i}{i+1} & \text{for odd } k > 0. \end{cases}
$$

Formula (4) shows that we can use the coefficients A_i ($1 \leq i \leq k+1$) in $S_k(n)$ to get the expression of $S_{k+1}(n)$. In fact, it also gives a method of computing Bernoulli numbers B_{k+1} . At last, other formulae for calculating Bernoulli numbers and $\sum_{m=1}^{n} m^k$ are given.

2. Lemmas

To obtain our main results, the following lemmas are necessary. Moreover, these lemmas also give some recursion formulae for $S_k(n)$.

Lemma 1. For any integers $k \geq 0$ and $n > 0$, we have

Lemmas

\nRecursion formulae for
$$
\sum_{m=1}^{n} m^{k}
$$
 1125

\nLemma our main results, the following lemmas are necessary. Moreover, these lemmas give some recursion formulae for $S_k(n)$.

\nLemma 1. For any integers $k \geq 0$ and $n > 0$, we have

\n
$$
(1+n)^{k+1} = 1 + \sum_{i=0}^{k} {k+1 \choose i} S_i(n).
$$
\nProof. Recalling the binomial expansion $(1+m)^{k+1} = \sum_{i=0}^{k+1} \left(\frac{k+1}{i}\right) m^i$ we obtain

$$
\lim_{i=0} \left(1 + n\right)^{k+1} = \sum_{i=0}^{k+1} \left(\frac{k+1}{i}\right)^{k+1}
$$
\n
$$
(1+n)^{k+1} + S_{k+1}(n) - 1 = \sum_{m=1}^{n} (1+m)^{k+1}
$$
\n
$$
= \sum_{m=1}^{n} \left(\sum_{i=0}^{k+1} {k+1 \choose i} m^{i}\right)
$$
\n
$$
= \sum_{i=0}^{k+1} \left(\frac{k+1}{i}\right) \left(\sum_{m=1}^{n} m^{i}\right)
$$
\n
$$
= \sum_{i=0}^{k+1} \left(\frac{k+1}{i}\right) S_{i}(n).
$$

This is equivalent to

$$
= \sum_{i=0}^{k} \left(\frac{k+1}{i}\right) S_i(
$$

$$
(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \left(\frac{k+1}{i}\right) S_i(n).
$$

The proof of Lemma 1 is completed \blacksquare

Lemma 1 shows that $S_k(n)$ could be deduced from $S_0(n), S_1(n), \ldots, S_{k-1}(n)$. Using Lemma 1 we can get

Lemma 2. For arbitrary integer $k > 0$,

$$
(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \left(\frac{k+1}{i}\right) S_i(n).
$$

s completed \blacksquare

$$
S_k(n)
$$
 could be deduced from $S_0(n), S_1(n), \ldots, S_{k-1}(n)$. Using
itrary integer $k > 0$,

$$
S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_i n^i.
$$
 (6)
atical induction on k , the result that $S_k(n)$ is a $(k+1)$ -th degree
at term 0 follows straightforwardly. Equation the coefficients on

Proof. By mathematical induction on k , the result that $S_k(n)$ is a $(k+1)$ -th degree polynomial with constant term 0 follows straightforwardly. Equating the coefficients on the two sides of (5), it is deduced easily that the coefficients of n^{k+1} and n^k in $S_k(n)$ are $\frac{1}{k+1}$ and $\frac{1}{2}$, respectively. This completes the proof of Lemma 2 $\frac{k}{1}n^{k+1} + \frac{1}{2}n^k + \sum_{i=1}^{k-1} A_i$

on on k, the result that

was straightforwardly.

sily that the coefficie:

mpletes the proof of L

lies
 $A_i = 12 - \frac{1}{k+1}$.

Since $S_k(1) = 1$, formula (6) implies

$$
\sum_{i=1}^{k-1} A_i = 12 - \frac{1}{k+1}.
$$
 (7)

For any integer $k > 0$, let $\langle k \rangle$ stand for the largest odd number less than k . Then

$$
\log Qi
$$
\n
$$
k \rightarrow \langle k \rangle = \begin{cases}\n1 & \text{for any even } k \\
2 & \text{for any odd } k\n\end{cases}
$$
\n3, and so forth.

For example, $\langle 2 \rangle = 1$, $\langle 5 \rangle = 3$, and so forth.

Let $A_p^{(q)}$ denote the coefficient of n^p in $S_q(n)$. Then

Lemma 3. For any integer $k > 1$,

(k)1 k (2i) Sk(fl) = 1 + flk +! E *(2i flk2, (9) k+1 2 2 (k)1 k —i--)*

that is,

$$
A_{k-2i+1}^{(k)} = \frac{1}{2i} {k \choose 2i-1} A_1^{(2i)} \qquad (1 \le i \le \frac{(k)+1}{2})
$$
 (10)

where $A_1^{(2i)}$ is the coefficient of the term n in $S_{2i}(n)$.

Proof. We will use mathematical induction on *k*. It is clear that formula (9) is for $k = 2$. Suppose the result is true for 3,..., $k - 1$. From Lemma 2, we have $S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k$ true for $k = 2$. Suppose the result is true for $3, \ldots, k - 1$. From Lemma 2, we have **Proof.** We will use mathematical induction on *k*. It is clear
true for $k = 2$. Suppose the result is true for 3,..., $k - 1$. From Ler
 $S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k-i}$.
Equating the coefficien

$$
\frac{1}{k+1}n^{k+1} + \frac{1}{2}n^{k} + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i} {n \choose 2i-1} A_{1}^{(2i)}
$$
\n(k)
\n(k)
\n(k)
\n(k)
\n(k)
\n $k-2i+1 = \frac{1}{2i} {n \choose 2i-1} A_{1}^{(2i)}$ $(1 \le i \le \frac{(k)+1}{2})$
\nfficient of the term n in $S_{2i}(n)$.
\nuse mathematical induction on k. It is clear
\nuse the result is true for 3,..., k - 1. From I
\n
$$
S_k(n) = \frac{1}{k+1}n^{k+1} + \frac{1}{2}n^{k} + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k-i}.
$$
\nents of n^{k-j} for $j = 1, 3, ..., \langle k \rangle$ in (5) gives
\n
$$
A_{k-j}^{(k)} = \frac{1}{k+1} \left[\frac{1}{2} {k+1 \choose k-j} - \frac{1}{k-j} {k+1 \choose k-j} - 1 \right]
$$

$$
S_{k}(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^{k} + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k-i}.
$$

Equating the coefficients of n^{k-j} for $j = 1, 3, ..., (k)$ in (5) gives us

$$
A_{k-j}^{(k)} = \frac{1}{k+1} \left[\frac{1}{2} {k+1 \choose k-j} - \frac{1}{k-j} {k+1 \choose k-j-1} - \sum_{i=0}^{\frac{i-3}{2}} A_{k-j}^{(k-j+2i+1)} {k-j+2i+1} \right].
$$

By the inductive assumption, we have

$$
A_{k-j}^{(k-j+2i+1)} = A_1^{(2(i+1))} \frac{1}{k-j+2(i+1)} {k-j+2(i+1) \choose 2(i+1)}
$$
 (12)
for $0 \le i \le \frac{i-3}{2}$. Combining (11) and (12) yields

$$
A_1^{(k)} = \frac{1}{k-j} \left[\frac{1}{2} {k+1 \choose k-j} - \frac{1}{2} {k+1 \choose k} \right]
$$

By the inductive assumption, we have

$$
A_{k-j}^{(k-j+2i+1)} = A_1^{(2(i+1))} \frac{1}{k-j+2(i+1)} {k-j+2(i+1) \choose 2(i+1)}
$$
(12)

$$
-\sum_{i=0}^{\frac{r-5}{2}} A_{k-j}^{(k-j+2i+1)} \binom{k+1}{k-j+2i+1}.
$$
\n
$$
A_{k-j}^{(k-j+2i+1)} = A_1^{(2(i+1))} \frac{1}{k-j+2(i+1)} \binom{k-j+2(i+1)}{2(i+1)}
$$
\n
$$
A_{k-j}^{(k)} = \frac{1}{k+1} \left[\frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} - \frac{\frac{i-1}{2}}{k-j+2i} \binom{k+1}{k-j+2i} - \frac{\frac{i-1}{2}}{k-j+2i} \binom{k-j+2i}{2i} \binom{k+1}{k-j+2i-1} \right].
$$
\n(13)

From (7) and the inductive assumption, it follows that

Recursion Formulae for
$$
\sum_{m=1}^{n} m^{k}
$$
 1127
assumption, it follows that

$$
A_{1}^{(j+1)} = \frac{1}{2} - \frac{1}{j+2} - \sum_{i=1}^{j-1} A_{2i+1}^{(j+1)}
$$
(14)

$$
\frac{A_{1}^{(2(i+1))} - 1}{j+2} \left(\frac{j+2}{2(i+1)}\right) \qquad (0 \le i \le \frac{j-3}{2}).
$$
(15)
produces

and

Recursion Formulae for
$$
\sum_{m=1}^{n} m^{k}
$$
 1127
\ne inductive assumption, it follows that
\n
$$
A_{1}^{(j+1)} = \frac{1}{2} - \frac{1}{j+2} - \sum_{i=1}^{j-1} A_{2i+1}^{(j+1)}
$$
\n
$$
A_{j-2i}^{(j+1)} = A_{1}^{(2(i+1))} \frac{1}{j+2} {j+2 \choose 2(i+1)} \qquad (0 \le i \le \frac{j-3}{2}).
$$
\n(15)
\ninto (14) produces

Substituting (15) into (14) produces

$$
A_1^{(j+1)} = \frac{1}{2} - \frac{1}{j+2} - \sum_{i=1}^{j} A_{2i+1}^{(j+1)}
$$
(14)
\n1
\n1
\n
$$
A_{j-2i}^{(j+1)} = A_1^{(2(i+1))} \frac{1}{j+2} \left(\frac{j+2}{2(i+1)} \right)
$$
(0 \le i \le \frac{j-3}{2}). (15)
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\n
$$
A_1^{(j+1)} \frac{1}{k+1} {k+1 \choose j+1} = \frac{1}{k+1} \left[\frac{1}{2} {k+1 \choose j+1} - \frac{1}{j+2} {k+1 \choose j+1} \right]
$$

\n
$$
- \sum_{i=1}^{\frac{j-1}{2}} A_i^{2i} \frac{1}{j+2} {j+2 \choose 2i} {k+1 \choose j+1}
$$

\n
$$
= \frac{1}{k+1} \left[\frac{1}{2} {k+1 \choose k-j} - \frac{1}{k-j} {k+1 \choose k-j-1} - \sum_{i=1}^{\frac{j-1}{2}} A_i^{2i} \frac{1}{k-j+2i} {k-j \choose 2i} {k+1 \choose k-j+2i-1} \right].
$$

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From (13) and (16),

3),
\n
$$
A_{k-j}^{(k)} = \frac{1}{k+1} {k+1 \choose j+1} A_1^{(j+1)} \qquad (j = 1, 3, ..., (k))
$$
\narly, by mathematical induction, we can prove that

\n
$$
A_{k-i}^{(k)} = 0 \qquad (i = 2, 4, 6, ..., (k) + 1).
$$

is obtained. Similarly, by mathematical induction, we can prove that

$$
A_{k-i}^{(k)} = 0 \qquad (i = 2, 4, 6, \ldots, (k) + 1).
$$

The proof of Lemma 3 is completed \blacksquare

Note Lemma 3 shows that the coefficients of the term *n* in $S_2(n), \ldots, S_{2i-2}(n)$ can be used to calculate $S_{2i-1}(n)$ and $S_{2i}(n)$.

3. Main results

l,

Now we use Lemma 3 to prove

Main Theorem. For any integer $k > 1$, let

ma 3 to prove
\nrem. For any integer
$$
k > 1
$$
, let
\n
$$
S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{\frac{k+1}{2}} A_{k-2i+1} n^{k-2i+1}.
$$

Then

$$
S_{k+1}(n) = \frac{1}{k+2} n^{k+2} + \frac{1}{2} n^{k+1} + (k+1) \sum_{i=1}^{\frac{(k)+1}{2}} \frac{A_{k-2i+1}}{k-2(i-1)} n^{k-2(i-1)} + b_1 n
$$

re

$$
b_1 = \begin{cases} 0 & \text{for even } k \\ \frac{1}{2} - \left[\frac{1}{k+2} + (k+1) \sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2} \right] & \text{for odd } k. \end{cases}
$$
(17)
Proof. From (10) we know that the coefficients of n^{k-j} $(j = 1, 3, ..., \langle k \rangle)$ in $S_k(n)$

where

$$
b_1 = \begin{cases} 0 & \text{for even } k \\ \frac{1}{2} - \left[\frac{1}{k+2} + (k+1) \sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2} \right] & \text{for odd } k. \end{cases} \tag{17}
$$

m (10) we know that the coefficients of n^{k-j} $(j = 1, 3, ..., (k))$ in $S_k(n)$

$$
A_{k-j}^{(k)} = \frac{1}{k+1} {k+1 \choose j+1} A_1^{(j+1)}.
$$

are $\begin{aligned} \text{at the coefficients of } n^k \ = \frac{1}{n-1} \binom{k+1}{A} A_1^{(j+1)} \end{aligned}$

$$
A_{k-j}^{(k)} = \frac{1}{k+1} {k+1 \choose j+1} A_1^{(j+1)}
$$

Therefore

$$
\frac{1}{k+2}n^{k+2} + \frac{1}{2}n^{k+1} + (k+1) \sum_{i=1}^{\frac{k+1}{2}} \frac{A_{k-2i+1}}{k-2(i-1)}n^{k+1}
$$
\n
$$
= \begin{cases}\n0 & \text{for even} \\
\frac{1}{2} - \left[\frac{1}{k+2} + (k+1)\sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2}\right] & \text{for odd } k\n\end{cases}
$$
\n
$$
= \begin{cases}\n0 & \text{for even} \\
\frac{1}{2} - \left[\frac{1}{k+2} + (k+1)\sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2}\right] & \text{for odd } k\n\end{cases}
$$
\n
$$
A_{k-j}^{(k)} = \frac{1}{k+1} \begin{pmatrix} k+1 \\ j+1 \end{pmatrix} A_1^{(j+1)}.
$$
\n
$$
A_{k-j}^{(k)} = A_1^{(j+1)} \frac{1}{k+1} \begin{pmatrix} k+1 \\ j+1 \end{pmatrix} \frac{k+1}{k-1} = A_1^{(j+1)} \frac{1}{k+2} \begin{pmatrix} k+2 \\ j+1 \end{pmatrix}
$$
\n
$$
= A_{k-j+1}^{(k+1)}
$$

is the coefficient of n^{k+1-j} $(j = 1,3,..., (k))$ in $S_{k+1}(n)$. If *k* is even, since $k - (k) + 1 =$ $(k + 1) - (k + 1)$, then $b_1 = 0$ follows from (9). If *k* is odd, formula (17) follows from (7). This completes the proof \blacksquare

Corollary. Let A_i be the coefficients of the terms n^i $(1 \le i \le k+1)$ in $S_k(n)$ and let B_i $(i > 1)$ be the *i*-th Bernoulli numbers. Then

$$
= A_{k-j+1}^{(k+1)}
$$

\n
$$
+1-j \quad (j = 1, 3, ..., (k)) \text{ in } S_{k+1}(n). \text{ If } k \text{ is even}
$$

\n
$$
h_1 = 0 \text{ follows from (9). If } k \text{ is odd, form}
$$

\n
$$
h_i \text{ be the coefficients of the terms } n^i \quad (1 \le i \le i-k. \text{ Bernoulli numbers. Then}
$$

\n
$$
B_{2j+1} = 0
$$

\n
$$
B_{2j} = \frac{1}{2} - \left[\frac{1}{2j+1} + 2j \sum_{i=1}^{j-1} \frac{A_{2(j-i)}}{2(j-i)+1} \right]
$$

\n
$$
1,
$$

\n
$$
n \text{ mass } 1 - 3 \text{ and Main Theorem, calculating } c
$$

\n
$$
+ \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n
$$

\n
$$
+ \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{11}{12}n^4 + \frac{11}{12}n^8 + \frac{11}{12}n^8 - \frac{11}{8}n^4 + \frac{11}{12}n^8 +
$$

for every integer $j \geq 1$,

Corollary. Let
$$
A_i
$$
 be the coefficients of the terms n^i $(1 \le i \le k+1)$ in $S_k(n)$
\n B_i $(i > 1)$ be the *i*-th Bernoulli numbers. Then
\n $B_{2j+1} = 0$
\n $B_{2j} = \frac{1}{2} - \left[\frac{1}{2j+1} + 2j\sum_{i=1}^{j-1} \frac{A_{2(j-i)}}{2(j-i)+1}\right]$
\nevery integer $j \ge 1$,
\nRemark. By Lemmas 1 - 3 and Main Theorem, calculating directly we obtain
\n $S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$
\n $S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$
\n $S_{12}(n) = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n$
\n $S_{20}(n) = \frac{1}{21}n^{21} + \frac{1}{2}n^{20} + \frac{5}{3}n^{19} - \frac{19}{2}n^{17} + \frac{1292}{21}n^{15} - 323n^{13} + \frac{41990}{33}n^{11}$
\n $- \frac{223193}{63}n^9 + 6460n^7 - \frac{68723}{10}n^5 + \frac{219335}{63}n^3 - \frac{174611}{330}n$
\n $S_{21}(n) = \frac{1}{22}n^{22} + \frac{1}{2}n^{21} + \frac{7}{4}n^{20} - \frac{133}{12}n$

From here the Bernoulli numbers

Recursion Formulae for
$$
\sum
$$

rnoulli numbers
 $B_{10} = 566$, $B_{12} = -\frac{691}{2730}$, $B_{20} = -\frac{174611}{330}$

are obtained.

4. Another formulae for $\sum_{m=1}^{n} m^k$ and Bernoulli numbers

In this section, another formulae for computing Bernoulli numbers and $\sum_{m=1}^{n} m^k$ will
be given, from which we can get the Bernoulli numbers more easily (see [1] and [2: pp.
246 – 265]).
Define functions B_n by
 $\frac{ze$ be given, from which we can get the Bernoulli numbers more easily (see [1] and [2: pp. $246 - 265$). m here the Bernoulli numbers
 $B_{10} = 566$, $B_{12} = -\frac{6}{2}$

obtained.
 Another formulae for $\sum_{m=1}^{n} n$

his section, another formulae for compu

iven, from which we can get the Bernou

- 265]).

Define functions B

$$
\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (|z| < 2\pi)
$$

and write $B_n = B_n(0)$ for the Bernoulli numbers. Then formula (3) follows by putting $x = 0$. We can equate coefficients of z^n in

1. another formulae for computing Bernoulli numbers and
$$
\Delta
$$
.
\n2. which we can get the Bernoulli numbers more easily (see $|z| < 2\pi$)

\n2. The graph of the formula is given by:

\n
$$
\frac{ze^{xz}}{e^{z}-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (|z| < 2\pi)
$$
\n
$$
= B_n(0) \text{ for the Bernoulli numbers. Then formula (3) follow a equation of the formula, we get:
$$
\n
$$
B_n(x) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n\right)
$$
\n
$$
B_n(x) = \sum_{k=0}^n {n \choose k} B_k x^{n-k}.
$$
\n
$$
\frac{ze^{(x+1)z}}{e^z-1} - \frac{ze^{xz}}{e^z-1} = ze^{xz},
$$
\n
$$
= \sum_{n=0}^{\infty} \frac{ze^{nx}}{e^x-1}.
$$

to get

$$
\frac{z}{1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (|z| < 2\pi)
$$

\nthe Bernoulli numbers. Then formula (3) follows by putting
\nents of z^n in
\n
$$
\frac{z}{e^z - 1} \cdot e^{zz} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n\right)
$$

\n
$$
B_n(x) = \sum_{k=0}^n {n \choose k} B_k x^{n-k}.
$$

\n
$$
\frac{z e^{(z+1)z}}{e^z - 1} - \frac{z e^{zz}}{e^z - 1} = z e^{zz},
$$

\n
$$
\frac{z e^{(z+1)z}}{n!} - \frac{z e^{zz}}{e^z - 1} = z e^{zz},
$$

\n
$$
\frac{z e^{(z+1)z}}{n!} z^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+1},
$$

\n
$$
f z^n \text{ we get}
$$

\n
$$
B_n(x+1) - B_n(x) = n x^{n-1}.
$$

\n(19)

Also, since

$$
\frac{z\,e^{(x+1)z}}{e^z-1}-\frac{z\,e^{xz}}{e^z-1}=z\,e^{xz},
$$

we have

$$
-z^* = \frac{1}{e^z - 1} \cdot e^{iz} = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n\right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n\right)
$$

\n
$$
B_n(x) = \sum_{k=0}^n {n \choose k} x^{n-k}.
$$

\n
$$
\frac{z e^{(z+1)z}}{e^z - 1} - \frac{z e^{zz}}{e^z - 1} = z e^{zz},
$$

\n
$$
\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+1},
$$

\n
$$
\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} = n x^{n-1}.
$$

\n(19)
\n
$$
B_n = B_n(0) = B_n(1) \quad (n \neq 1).
$$

\n(20)
\n
$$
p_n(x+1) = 1 \text{ in (18) and use (20) to obtain}
$$

and by equating coefficients of z^n we get

$$
B_n(x+1) - B_n(x) = n x^{n-1}.
$$
 (19)

So putting $x = 0$ we have

$$
B_n = B_n(0) = B_n(1) \qquad (n \neq 1). \tag{20}
$$

Thus for $n \geq 2$ we can put $x = 1$ in (18) and use (20) to obtain

$$
B_n=B_n(1)=\sum_{k=0}^n\binom{n}{k}B_k.
$$

This is a much simpler recursion formula for computing Bernoulli numbers.

Result (19) can be used, taking $x = 1, 2, \ldots, k - 1, k$ and adding, to give

$$
B_n(k+1) - B_n(1) = \sum_{i=0}^{k-1} \left[B_n(k+1-i) - B_n(k-i) \right]
$$

= $n \cdot k^{n-1} + n(k-1)^{n-1} + \dots + n \cdot 2^{n-1} + n \cdot 1^{n-1}$
= $n \sum_{m=1}^{k} m^{k-1}$,

$$
\sum_{m=1}^{k} m^{k-1} = \frac{B_n(k+1) - B_n}{n}.
$$

that is,

$$
\sum_{m=1}^{k} m^{k-1} = \frac{B_n(k+1) - B_n}{n}
$$

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