Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 1, 255–268

Asymptotic Expansions of Integral Functionals of Weakly Correlated Random Processes

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Abstract. In the paper asymptotic expansions for second-order moments of integral functionals of a family of random processes are considered. The random processes are assumed to be wide-sense stationary and ε-correlated, i.e. the values are not correlated excluding an ε-neighbourhood of each point. The asymptotic expansions are derived for $\varepsilon \to 0$. Using a special weak assumption there are found easier expansions as in the case of general weakly correlated random processes. Expansions are given for integral functionals of real-valued as well as of complex vector-valued processes.

Keywords: *Asymptotic expansion, second-order moment, random differential equation, weakly correlated process, stationary process, random vibration*

AMS subject classification: Primary 60 G 12, secondary 34 F 05, 41 A 60, 70 L 05

1. Introduction

In this paper asymptotic expansions for second-order moments of integral functionals of the type

$$^{\varepsilon}r := \int_{\mathcal{D}} Q(s)\, {}^{\varepsilon}f(s)\, ds \tag{1}$$

are considered, where Q is a deterministic function on an interval $\mathcal{D} \subset \mathbb{R}$ and $({}^{\varepsilon}f)_{\varepsilon>0}$ denotes a family of random functions, indexed by a parameter ε which describes the range of correlation. The random functions are defined on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, the expectation operator for random variables on this space is denoted by $\mathbf{E}\{\cdot\}$. Such integral functionals play an important role in many theoretical and practical mathematical problems. For example, differential equations with an inhomogeneous term containing ${}^{\varepsilon}f$ often possess solutions which can be represented in such a way (cf. Examples 2 and 3 in this paper). For an approximate description of those solutions and their characteristics asymptotic expansions with respect to $\varepsilon \to 0$ can be used if the values of ${}^{\varepsilon}f$ are correlated or stochastically dependent only in an ε-neighbourhood of each point.

We will suppose the validity of the following

Assumption 1.

1. ${}^{\varepsilon}f$ $(\varepsilon > 0)$ are wide-sense stationary processes with correlation functions

$$\mathbf{E}\{{}^{\varepsilon}f(s)\, {}^{\varepsilon}f(t)\} = {}^{\varepsilon}R_{ff}(t-s).$$

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ISSN 0232-2064 / \$ 2.50

2. ${}^{\varepsilon}f$ ($\varepsilon > 0$) are centered, i.e. $\mathbf{E}\{{}^{\varepsilon}f(s)\} = 0$ for $s \in \mathbb{R}$.
3. ${}^{\varepsilon}f$ ($\varepsilon > 0$) are ε-correlated, i.e. ${}^{\varepsilon}R_{ff}(s) = 0$ for $|s| \geq \varepsilon$.
4. The correlation functions ${}^{\varepsilon}R_{ff}$ ($\varepsilon > 0$) are generated by a correlation function R of a 1-correlated wide-sense stationary process, i.e. ${}^{\varepsilon}R_{ff}(s) = R(\frac{s}{\varepsilon})$ ($s \in \mathbb{R}, \varepsilon > 0$).
5. The correlation function R is continuous on $\mathbb{R}$, hence the processes ${}^{\varepsilon}f$ ($\varepsilon > 0$) are continuous in mean square on $\mathbb{R}$.

The integral in (1) is assumed to exist in mean square sense, under weak conditions it coincides a.s. with the pathwise integral. From condition 2 of Assumption 1 it follows that the random variables ${}^{\varepsilon}r$ ($\varepsilon > 0$) are centered, i.e. $\mathbf{E}\{{}^{\varepsilon}r\} = 0$.

In the first part of the paper we consider real-valued processes, after that complex vector-valued processes are investigated.

For example, $({}^{\varepsilon}f)_{\varepsilon>0}$ can be a family of so-called weakly correlated random processes. In the theory of these processes (cf. [8, 9]) asymptotic expansions with respect to $\varepsilon \to 0$ of the type

$$\mathbf{E}\{{}^{\varepsilon}r_1 \cdot {}^{\varepsilon}r_2 \cdots {}^{\varepsilon}r_m\} = \begin{cases} c_m \varepsilon^{\frac{m}{2}} + o(\varepsilon^{\frac{m}{2}}) & \text{for even } m \\ c_m \varepsilon^{\frac{m+1}{2}} + o(\varepsilon^{\frac{m+1}{2}}) & \text{for odd } m > 1 \end{cases}$$

with some real constants c_m are derived. The indices 1 to m refer to deterministic functions $Q_1, \ldots, Q_m$ and intervals $\mathcal{D}_1, \ldots, \mathcal{D}_m$ which are involved in the corresponding integral functionals. Here we will consider only second-order moments and propose a new method of obtaining such asymptotic expansions, which seems to be easier and clarifies in a certain sense the structure of asymptotic expansions in the case of correlation functions. The main difference to the general theory of weakly correlated random processes consists in the explicitely given generating condition 4 of Assumption 1 for the correlation functions ${}^{\varepsilon}R_{ff}$ ($\varepsilon > 0$).

In the following treatment the concept of correlation moments of wide-sense stationary processes is used.

Definition 1. Let R be a real continuous correlation function of a wide-sense stationary process and $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with

$$\int_{-\infty}^{\infty} |s|^j |R(s)|\, ds < \infty.$$

Then

$$\mu_j = \int_{-\infty}^{\infty} s^j R(s)\, ds = \begin{cases} 0 & \text{for odd } j \\ 2\int_0^{\infty} s^j R(s)\, ds & \text{for even } j \end{cases}$$

is called the *correlation moment* of j-th order of the correlation function or the random process and

$$\nu_j = \int_{-\infty}^{\infty} |s|^j R(s)\, ds = 2\int_0^{\infty} s^j R(s)\, ds$$

is called the *absolute correlation moment* of j-th order.

We remark some properties of correlation moments for real-valued wide-sense stationary processes:

1. From the positive definiteness of the correlation function, $\mu_0 = \nu_0 \geq 0$ follows.

2. Property 1 is not true for higher-order correlation moments, i.e. there exist correlation functions and numbers $j \in \mathbb{N}$ with $\nu_j < 0$.

3. For ε-correlated wide-sense stationary processes correlation moments of all orders exist and $\lim_{j\to\infty} \nu_j = \lim_{j\to\infty} \mu_j = 0$ holds.

We also note that for 1-correlated wide-sense stationary random processes the following version of the Shannon-Kotelnikov sampling theorem (see, e.g., [3]) is valid:

Proposition 1. *Let*

$$S(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(s)\exp(-i\alpha s)\,ds = \frac{1}{\pi}\int_0^1 R(s)\cos(\alpha s)\,ds$$

denote the spectral density of a 1*-correlated wide-sense stationary random process. Then the representation*

$$S(\alpha) = \lim_{m\to\infty}\sum_{n=-m}^{m}\frac{\sin(\alpha - n\pi)}{\alpha - n\pi}S(n\pi)$$

holds for all $\alpha \in \mathbb{R}$, *with* $\frac{\sin 0}{0} := 1$.

For a non-negative correlation function of a 1-correlated wide-sense stationary random process the corresponding spectral density is a positive definite function. In this case the correlation moments μ_j are closely related to the spectral moments and therefore to the variances of the mean-square derivatives of the adjoint stationary process (cf., e.g., [5: p. 368]).

In the following we will also suppose that the function Q satisfies

Assumption 2. The deterministic function Q is N times continuously differentiable on the interval $\mathcal{D}$ ($N \in \mathbb{N}_0$), $Q^{(N)}$ is absolutely continuous on $\mathcal{D}$ and the derivatives of Q up to the order $N+1$ belong to the space $L^2(\mathcal{D}) \cap L^1(\mathcal{D})$.

For such functions the Taylor expansion formula with exact integral representation of the remainder (cf., e.g., [4: Section 5.4]) is valid, for the integration by parts formula for absolutely continuous functions see, e.g., [6: Chapter IX/§7].

2. Expansions of variances

From (1) and Assumption 1 it follows that

$$\begin{aligned}
\mathbf{E}\{{}^{\varepsilon}r^2\} &= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\mathbf{E}\{{}^{\varepsilon}f(s)\,{}^{\varepsilon}f(t)\}dsdt \\
&= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\,{}^{\varepsilon}R_{ff}(t-s)\,dsdt \\
&= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\,R\Big(\frac{t-s}{\varepsilon}\Big)dsdt.
\end{aligned}$$

The substitution of the variables $t = t$ and $u = \frac{t-s}{\varepsilon}$ gives

$$\mathrm{E}\{{}^{\varepsilon}r^2\} = \varepsilon \iint_{{}^{\varepsilon}\mathcal{D}'} Q(t-\varepsilon u)Q(t)R(u)\,dtdu$$

with the transformed domain of integration ${}^{\varepsilon}\mathcal{D}' = \{(t,u) \in \mathbb{R}^2 : t \in \mathcal{D} \text{ and } t - \varepsilon u \in \mathcal{D}\}$.

In order to show how the method works the case $\mathcal{D} = \mathbb{R}$ is considered explicitely. We deal with the random variables

$${}^{\varepsilon}r = \int_{-\infty}^{\infty} Q(s)\,{}^{\varepsilon}f(s)\,ds \qquad (\varepsilon > 0)$$

where

$$\begin{aligned}\mathrm{E}\{{}^{\varepsilon}r^2\} &= \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(t-\varepsilon u)Q(t)R(u)\,dtdu \\ &= \varepsilon \int_{-1}^{1} R(u) \int_{-\infty}^{\infty} Q(t-\varepsilon u)Q(t)\,dtdu\end{aligned}$$

can be obtained. Using the notation

$$\phi(u) = \int_{-\infty}^{\infty} Q(t-u)Q(t)\,dt$$

we write

$$\mathrm{E}\{{}^{\varepsilon}r^2\} = \varepsilon \int_{-1}^{1} \phi(\varepsilon u)R(u)\,du$$

and it can be seen that the value of the variances depends on the correlation function R and the behavior of the function ϕ in a neighbourhood of zero. Now the Taylor expansion of the function ϕ is applied. From Assumption 2 we find for all $t \in \mathbb{R}$ and $u \in [-1,1]$

$$Q(t-\varepsilon u) = \sum_{j=0}^{N} Q^{(j)}(t)\frac{(-\varepsilon u)^j}{j!} + \tilde{\rho}_{N+1}(t,u,\varepsilon)$$

with

$$\tilde{\rho}_{N+1}(t,u,\varepsilon) = \frac{1}{N!}\int_{t}^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N\,dv,$$

and

$$\begin{aligned}\mathrm{E}\{{}^{\varepsilon}r^2\} = &\sum_{j=0}^{N} \frac{(-1)^j\varepsilon^{j+1}}{j!} \cdot \int_{-\infty}^{\infty} Q(t)Q^{(j)}(t)\,dt \cdot \int_{-1}^{1} u^j R(u)\,du \\ &+ \frac{\varepsilon}{N!}\int_{-1}^{1}\int_{-\infty}^{\infty}\int_{t}^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N Q(t)R(u)\,dvdtdu\end{aligned} \tag{2}$$

follows. Integration by parts with respect to the quantities

$$q_j = \int_{-\infty}^{\infty} Q(t)Q^{(j)}(t)\,dt \qquad (j = 0,\ldots,N) \tag{3}$$

which do not depend on the random processes leads to

$$\begin{aligned}\int_{-\infty}^{\infty} Q(t)Q^{(j)}(t)\,dt &= \left[Q(t)Q^{(j-1)}(t)\right]_{t\to-\infty}^{t\to\infty} - \int_{-\infty}^{\infty} Q'(t)Q^{(j-1)}(t)\,dt \\ &= -\int_{-\infty}^{\infty} Q'(t)Q^{(j-1)}(t)\,dt \\ &\;\vdots \\ &= (-1)^j \int_{-\infty}^{\infty} Q^{(j)}(t)Q(t)\,dt.\end{aligned}$$

Then

$$\left.\begin{aligned} q_{2k} &= (-1)^k \int_{-\infty}^{\infty} [Q^{(k)}(t)]^2 dt && (k = 0,\dots,[\tfrac{N}{2}]) \\ q_{2k+1} &= (-1)^{2k+1} q_{2k+1} = 0 && (k = 0,\dots,[\tfrac{N-1}{2}]) \end{aligned}\right\}$$

follows and the following asymptotic expansion for $\mathbf{E}\{{}^{\varepsilon}r^2\}$ can be given:

Theorem 1. *Let $({}^{\varepsilon}f)_{\varepsilon>0}$ be a family of random processes satisfying Assumption* 1 *and Q a function satisfying Assumption* 2 *with $\mathcal{D} = \mathbb{R}$ and $N \in \mathbb{N}_0$. Then*

$$\mathbf{E}\{{}^{\varepsilon}r^2\} = \sum_{\substack{j=0\\ j\text{ even}}}^{N} \frac{\varepsilon^{j+1}}{j!} q_j\mu_j + \rho_{N+1}(\varepsilon),$$

where μ_j denotes the correlation moment of j-th order of the correlation function R, q_j is given in (3) *and $\rho_{N+1}(\varepsilon)$ is the last term in* (2).

Example 1. For the function $Q(t) = \exp(-\frac{t^2}{2})$,

$$\mathbf{E}\{{}^{\varepsilon}r^2\} = \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!}\mu_{2k}\varepsilon^{2k+1}$$

holds for all $\varepsilon > 0$. This equation can be proved using the formula

$$Q^{(l)}(t) = (-1)^l H_l(t)\exp\left(-\frac{t^2}{2}\right)$$

where H_l denotes the Hermite polynomial of l-th order with the representation

$$H_{2k}(t) = \sum_{j=0}^{k}(-1)^j \frac{(2k)!}{2^j j!(2k-2j)!}\, t^{2(k-j)}$$

for even values $l = 2k$ ($k \in \mathbb{N}$). The convergence to zero of the remainder can be shown using, for example, [1: Estimation 22.14.17].

3. Expansion of correlation functions

Now the mean square continuous wide-sense stationary processes

$$^{\varepsilon}g(t) := \int_{-\infty}^{t} Q(t-s)\,^{\varepsilon}f(s)\,ds = \int_{0}^{\infty} Q(u)\,^{\varepsilon}f(t-u)\,du \qquad (\varepsilon > 0)$$

are examined where the deterministic function Q satisfies Assumption 2 on $\mathbb{R}_+$. In this case the correlation functions can be written as

$$\begin{aligned}
^{\varepsilon}R_{gg}(\tau) &:= \mathbf{E}\{\,^{\varepsilon}g(t)\,^{\varepsilon}g(t+\tau)\} \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} Q(t-s_1)Q(t+\tau-s_2)\mathbf{E}\{\,^{\varepsilon}f(s_1)\,^{\varepsilon}f(s_2)\}\,ds_2 ds_1 \\
&= \int_{0}^{\infty}\int_{0}^{\infty} Q(u_1)Q(u_2)\,^{\varepsilon}R_{ff}(\tau+u_1-u_2)\,du_2 du_1.
\end{aligned}$$

Substituting the variables we get

$$\begin{aligned}
^{\varepsilon}R_{gg}(\tau) &= \int_{-\infty}^{0} \,^{\varepsilon}R_{ff}(\tau+v_1)\int_{-v_1}^{\infty} Q(v_1+v_2)Q(v_2)\,dv_2 dv_1 \\
&\quad + \int_{0}^{\infty} \,^{\varepsilon}R_{ff}(\tau+v_1)\int_{0}^{\infty} Q(v_1+v_2)Q(v_2)\,dv_2 dv_1 \\
&= \int_{-\infty}^{0} \,^{\varepsilon}R_{ff}(\tau+v_1)\int_{0}^{\infty} Q(v_2)Q(v_2-v_1)\,dv_2 dv_1 \\
&\quad + \int_{0}^{\infty} \,^{\varepsilon}R_{ff}(\tau+v_1)\int_{0}^{\infty} Q(v_1+v_2)Q(v_2)\,dv_2 dv_1 \\
&= \int_{-\infty}^{\infty} \,^{\varepsilon}R_{ff}(\tau+v_1)\int_{0}^{\infty} Q(v_2+|v_1|)Q(v_2)\,dv_2 dv_1 \\
&= \int_{-\infty}^{\infty} \,^{\varepsilon}R_{ff}(w)\int_{0}^{\infty} Q(u+|w-\tau|)Q(u)\,du dw \\
&= \int_{-\infty}^{\infty} R\left(\frac{w}{\varepsilon}\right)\int_{0}^{\infty} Q(u+|w-\tau|)Q(u)\,du dw \\
&= \varepsilon\int_{-1}^{1} R(v)\int_{0}^{\infty} Q(u+|\varepsilon v-\tau|)Q(u)\,du dv.
\end{aligned}$$

Applying the Taylor expansion of the function $Q(u+|\varepsilon v-\tau|)$ in neighbourhoods of the points $u+|\tau|$,

$$Q(u+|\varepsilon v-\tau|) = \sum_{j=0}^{N}\frac{1}{j!}Q^{(j)}(u+|\tau|)(|\varepsilon v-\tau|-|\tau|)^j + \tilde{\rho}_{N+1}(u,v,\varepsilon,\tau)$$

follows with a remainder term $\tilde{\rho}_{N+1}(u,v,\varepsilon,\tau)$. Hence

$$^{\varepsilon}R_{gg}(\tau) = \sum_{j=0}^{N}\frac{\varepsilon}{j!}\int_{0}^{\infty} Q^{(j)}(u+|\tau|)Q(u)\,du\int_{-1}^{1} R(v)(|\varepsilon v-\tau|-|\tau|)^j dv + \rho_{N+1}(\varepsilon,\tau)$$

with

$$\rho_{N+1}(\varepsilon,\tau) = \varepsilon \int_{-1}^{1} R(v) \int_{0}^{\infty} \tilde{\rho}_{N+1}(u,v,\varepsilon,\tau) Q(u)\, du dv$$

holds. Evaluation of the integral terms containing the correlation function R for $\tau \geq 0$ leads to

$$\begin{aligned}
&\int_{-1}^{1} R(v)(|\varepsilon v - \tau| - |\tau|)^j dv \\
&= \int_{-1}^{\frac{\tau}{\varepsilon}\wedge 1} R(v)(\tau - \varepsilon v - \tau)^j dv + \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1} R(v)(\varepsilon v - \tau - \tau)^j dv \\
&= \int_{-1}^{\frac{\tau}{\varepsilon}\wedge 1} R(v)(-\varepsilon v)^j dv + \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1} R(v)(\varepsilon v - 2\tau)^j dv \\
&= \int_{-1}^{1} R(v)(-\varepsilon v)^j dv + \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1} R(v)\left[(\varepsilon v - 2\tau)^j - (-\varepsilon v)^j\right] dv \\
&= (-\varepsilon)^j \mu_j + \mathbf{1}_{[0,\varepsilon)}(\tau) \int_{\frac{\tau}{\varepsilon}}^{1} R(v)\left[(\varepsilon v - 2\tau)^j - (-\varepsilon v)^j\right] dv
\end{aligned}$$

where $a \wedge b := \min(a,b)$, and analogously for $\tau \leq 0$ leads to

$$\begin{aligned}
&\int_{-1}^{1} R(v)(|\varepsilon v - \tau| - |\tau|)^j dv \\
&\qquad = \varepsilon^j \mu_j + \mathbf{1}_{(-\varepsilon,0]}(\tau) \int_{\frac{|\tau|}{\varepsilon}}^{1} R(v)\left[(\varepsilon v - 2|\tau|)^j - (-\varepsilon v)^j\right] dv.
\end{aligned}$$

Using that $\mu_j = 0$ for odd j we have finally:

Theorem 2. *Let $({}^{\varepsilon}f)_{\varepsilon>0}$ be a family of random processes satisfying Assumption 1 and Q a function satisfying Assumption 2 with $\mathcal{D} = \mathbb{R}_+$ and $N \in \mathbb{N}_0$. Then*

$${}^{\varepsilon}R_{gg}(\tau) = \sum_{\substack{j=0\\ j \text{ even}}}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau)\mu_j + \mathbf{1}_{(-\varepsilon,\varepsilon)}(\tau) \sum_{j=1}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau) c_j(\tau) + \rho_{N+1}(\varepsilon,\tau) \tag{4}$$

with the quantities

$$q_j(\tau) = \int_0^{\infty} Q^{(j)}(u + |\tau|) Q(u)\, du$$

and

$$c_j(\tau) = \int_{\frac{|\tau|}{\varepsilon}}^{1} R(v) \left[\left(v - 2\frac{|\tau|}{\varepsilon} \right)^j - (-v)^j \right] dv.$$

For fixed values of τ and $\varepsilon \to 0$ the expansions

$$\left.\begin{aligned}
{}^{\varepsilon}R_{gg}(0) &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(0)\nu_j + o(\varepsilon^{N+1}) \\
{}^{\varepsilon}R_{gg}(\tau) &= \sum_{\substack{j=0\\ j \text{ even}}}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau)\mu_j + o(\varepsilon^{N+1}) \quad (\tau \neq 0)
\end{aligned}\right\}$$

are valid. We can see that a discontinuity in the expansions of the correlation function at the point $\tau = 0$ arises if $q_j(0)$ or ν_j for odd values j do not vanish.

Examining asymptotic expansions of ${}^{\varepsilon}R_{gg}(\tau)$ as a function of τ it is necessary to consider not only the first terms in (4) but also the correction terms in the second sum of (4) for $|\tau| < \varepsilon$.

Example 2. The stationary solution of the second-order linear differential equation with constant coefficients and a random weakly correlated wide-sense stationary inhomogeneous term

$$\ddot{x} + 2\delta\dot{x} + w_0^2 x = {}^{\varepsilon}f(t)$$

is given by

$$x(t) = \frac{1}{w}\int_{-\infty}^{t} e^{-\delta(t-s)} \sin(w(t-s))\, {}^{\varepsilon}f(s)\, ds$$

with $w = \sqrt{w_0^2 - \delta^2}, 0 < \delta < w_0$. In this case the kernel function of the integral functional reads as

$$Q(u) = \frac{1}{w} e^{-\delta u} \sin(wu)$$

and straightforward calculations lead to

$$q_j(\tau) = a_j e^{-\delta|\tau|} \cos(w|\tau|) + b_j e^{-\delta|\tau|} \sin(w|\tau|) \qquad (j \in \mathbb{N},\ \tau \in \mathbb{R})$$

with

$$\left.\begin{aligned} a_j &= \sum_{l=0}^{j} \binom{j}{l} (-\delta)^l w^{j-l-2} \int_0^{\infty} e^{-2\delta u} \sin\left(wu + (j-l)\frac{\pi}{2}\right) \sin(wu)\, du \\ b_j &= \sum_{l=0}^{j} \binom{j}{l} (-\delta)^l w^{j-l-2} \int_0^{\infty} e^{-2\delta u} \cos\left(wu + (j-l)\frac{\pi}{2}\right) \sin(wu)\, du. \end{aligned}\right\}$$

For example, from the relations

$$\left.\begin{aligned} \int_0^{\infty} e^{-2\delta u} \sin^2(wu)\, du &= \frac{1}{4} \frac{w^2}{\delta(\delta^2 + w^2)} \\ \int_0^{\infty} e^{-2\delta u} \cos(wu) \sin(wu)\, du &= \frac{1}{4} \frac{w}{\delta^2 + w^2}, \end{aligned}\right\}$$

for $j = 0, 1, 2$ and $\tau \in \mathbb{R}$

$$\left.\begin{aligned} q_0(\tau) &= \frac{e^{-\delta|\tau|}}{4(\delta^2 + w^2)} \Big(\frac{1}{\delta} \cos(\omega|\tau|) + \frac{1}{w} \sin(\omega|\tau|)\Big) \\ q_1(\tau) &= -\frac{e^{-\delta|\tau|}}{4\delta w} \sin(\omega|\tau|) \\ q_2(\tau) &= \frac{e^{-\delta|\tau|}}{4} \Big(-\frac{1}{\delta} \cos(\omega|\tau|) + \frac{1}{w} \sin(\omega|\tau|)\Big) \end{aligned}\right\}$$

follows. Choosing the hat-like correlation function

$$R(v) = \begin{cases} 1 - |v| & \text{for } |v| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

the correlation moments can be found by

$$\mu_j = \frac{2}{(j+1)(j+2)}$$

for even values of j, and for the correction terms we get by setting $\alpha := \frac{|\tau|}{\varepsilon}$

$$\begin{aligned} c_j(\tau) &= \int_\alpha^1 R(v)\big[(v-2\alpha)^j - (-v)^j\big]\,dv \\ &= \frac{2(-1)^{j+1}}{j+1}(\alpha-1)\alpha^{j+1} + \frac{(1-2\alpha)^{j+2} - (-1)^j}{(j+1)(j+2)}. \end{aligned}$$

Especially,

$$\left.\begin{aligned} c_1(\tau) &= \frac{1}{3} - \alpha + \alpha^2 - \frac{1}{3}\alpha^3 \\ c_2(\tau) &= -\frac{2}{3}\alpha + 2\alpha^2 - 2\alpha^3 + \frac{2}{3}\alpha^4. \end{aligned}\right\}$$

4. Expansion of covariance matrices

Now complex vector-valued processes are investigated. In this case the non-commutativity of matrix multiplication has to be taken into account.

We assume the validity ot the corresponding versions of Assumption 1 for $\mathbb{C}^n$-valued wide-sense stationary processes ${}^\varepsilon f$ ($\varepsilon > 0$), and of Assumption 2 for $m \times n$-matrix-valued deterministic functions Q ($m, n \in \mathbb{N}$). So for the matrix correlation functions of the $\mathbb{C}^n$-valued wide-sense stationary processes ${}^\varepsilon f$ ($\varepsilon > 0$)

$${}^\varepsilon R_{ff}(s) := \mathbf{E}\{{}^\varepsilon f(t)\,{}^\varepsilon f^*(t+s)\} = R\left(\frac{s}{\varepsilon}\right)$$

holds according to condition 4 of Assumption 1. Here * denotes the conjugate-complex transposed of matrices and vectors where vectors are assumed to be columns. From the relation $R(s) = R^*(-s)$ we find in this case for the correlation moments

$$\left.\begin{aligned} \mu_j &= \int_{-\infty}^{\infty} s^j R(s)\,ds = \nu_j^+ + (-1)^j[\nu_j^+]^* \\ \nu_j &= \int_{-\infty}^{\infty} |s|^j R(s)\,ds = \nu_j^+ + [\nu_j^+]^* \end{aligned}\right\}$$

with $\nu_j^+ = \int_0^\infty s^j R(s)\,ds$, hence $\mu_j^* = (-1)^j \mu_j$ and $\nu_j^* = \nu_j$ hold. For functionals of the type

$${}^\varepsilon r = \int_{-\infty}^{\infty} Q(s)\,{}^\varepsilon f(s)\,ds \qquad (\varepsilon > 0)$$

we get

$$\mathbf{E}\{{}^\varepsilon r\,{}^\varepsilon r^*\} = \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(t - \varepsilon u) R(u) Q^*(t)\,dt du$$

and we find in the same way as for real-valued processes:

Theorem 3. *Let $({}^{\varepsilon}f)_{\varepsilon>0}$ be a family of ε-correlated $\mathbb{C}^n$-valued wide-sense stationary random processes satisfying Assumption 1 and Q a matrix-valued function satisfying Assumption 2 on $\mathcal{D} = \mathbb{R}$ and $N \in \mathbb{N}_0$. Then*

$$\mathbf{E}\{{}^{\varepsilon}r\,{}^{\varepsilon}r^*\} = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j + \rho_{N+1}(\varepsilon)$$

with

$$q_j = \int_{-\infty}^{\infty} Q^{(j)}(t)\mu_j^* Q^*(t)\,dt$$

and the remainder term

$$\rho_{N+1}(\varepsilon) = \frac{\varepsilon}{N!}\int_{-1}^{1}\int_{-\infty}^{\infty}\int_{t}^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N R(u)Q^*(t)\,dvdtdu.$$

5. Expansion of matrix correlation functions

Considering the complex vector-valued wide-sense stationary processes

$${}^{\varepsilon}g(t) := \int_{-\infty}^{t} Q(t-s)\,{}^{\varepsilon}f(s)\,ds = \int_{0}^{\infty} Q(u)\,{}^{\varepsilon}f(t-u)\,du \qquad (\varepsilon > 0)$$

the matrix correlation functions can be written as

$${}^{\varepsilon}R_{gg}(\tau) := \mathbf{E}\{{}^{\varepsilon}g(t)\,{}^{\varepsilon}g^*(t+\tau)\} = \int_0^{\infty}\int_0^{\infty} Q(u_1)\,{}^{\varepsilon}R_{ff}(\tau + u_1 - u_2)Q^*(u_2)\,du_2du_1$$

and it can be obtained

$$\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau) &= \int_{-\infty}^{0}\int_{-v_1}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2dv_1 \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2dv_1 \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} Q(v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2-v_1)\,dv_2dv_1 \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2dv_1 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} Q(v_2+v_1^+)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2+v_1^-)\,dv_2dv_1 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} Q(u+(w-\tau)^+)\,{}^{\varepsilon}R_{ff}(w)Q^*(u+(w-\tau)^-)\,dudw \\
&= \varepsilon\int_{-1}^{1}\int_{0}^{\infty} Q(u+(\varepsilon v-\tau)^+)R(v)Q^*(u+(\varepsilon v-\tau)^-)\,dudv.
\end{aligned}$$

Now, Taylor expansion of the matrix function Q in neighbourhoods of the points $u+|\tau|$ is considered. In the case of $\tau = 0$,

$$\begin{aligned}
{}^{\varepsilon}R_{gg}(0) &= \varepsilon \int_0^1 \int_0^\infty Q(u+|\varepsilon v|)R(v)Q^*(u)\,du dv \\
&\quad + \varepsilon \int_{-1}^0 \int_0^\infty Q(u)R(v)Q^*(u+|\varepsilon v|)\,du dv \\
&= \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!} \left\{ \int_0^1 \int_0^\infty Q^{(j)}(u)v^j R(v)Q^*(u)\,du dv \right. \\
&\quad \left. + \left[\int_0^1 \int_0^\infty Q^{(j)}(u)v^j R(v)Q^*(u)\,du dv \right]^* \right\} + \rho_{N+1}(\varepsilon, 0) \\
&= \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!} \{q_j + q_j^*\} + \rho_{N+1}(\varepsilon, 0)
\end{aligned}$$

with

$$q_j = \int_0^\infty Q^{(j)}(u)\nu_j^+ Q^*(u)\,du.$$

In an analogous manner to Section 3 we obtain for $\tau > 0$

$$\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau) &= \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!} \left\{ q_j(\tau) \right. \\
&\quad + \mathbf{1}_{(0,\varepsilon)}(\tau) \left\{ \int_0^\infty Q^{(j)}(u+\tau) \int_{\frac{\tau}{\varepsilon}}^1 \left(v - \frac{2\tau}{\varepsilon}\right)^j R(v)\,dv\, Q^*(u)\,du \right. \\
&\quad \left. \left. + (-1)^{j+1} \int_0^\infty Q(u) \int_{\frac{\tau}{\varepsilon}}^1 v^j R(v)\,dv\, [Q^{(j)}(u+\tau)]^* du \right\} \right\} + \rho_{N+1}(\varepsilon, \tau)
\end{aligned}$$

with

$$q_j(\tau) = \int_0^\infty Q(u)\mu_j^* [Q^{(j)}(u+\tau)]^* du$$

and the following theorem holds:

Theorem 4. *Let $({}^{\varepsilon}f)_{\varepsilon>0}$ be a family of $\mathbb{C}^n$-valued random processes satisfying Assumption 1 and Q a matrix-valued function satisfying Assumption 2 on $\mathcal{D} = \mathbb{R}_+$ and $N \in \mathbb{N}_0$. Then for $\tau > 0$*

$${}^{\varepsilon}R_{gg}(\tau) = \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!} \left\{ q_j(\tau) + \mathbf{1}_{(0,\varepsilon)}(\tau) c_j(\tau) \right\} + \rho_{N+1}(\varepsilon, \tau)$$

with

$$q_j(\tau) = \int_0^\infty Q(u)\mu_j^* [Q^{(j)}(u+\tau)]^* du$$

and

$$c_j(\tau) = \int_0^\infty Q^{(j)}(u+\tau) \int_{\frac{\tau}{\varepsilon}}^1 \left(v - \frac{2\tau}{\varepsilon}\right)^j R(v)\, dv\, Q^*(u)\, du$$
$$+ (-1)^{j+1} \int_0^\infty Q(u) \int_{\frac{\tau}{\varepsilon}}^1 v^j R(v) dv\, [Q^{(j)}(u+\tau)]^* du.$$

For $\tau = 0$, the relation

$${}^{\varepsilon}R_{gg}(0) = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} \{q_j + q_j^*\} + \rho_{N+1}(\varepsilon, 0)$$

is valid with

$$q_j = \int_0^\infty Q^{(j)}(u) \nu_j^+ Q^*(u)\, du$$

and the values for $\tau < 0$ can be calculated from the relation ${}^{\varepsilon}R_{gg}(\tau) = {}^{\varepsilon}R_{gg}^*(-\tau)$.

Example 3. Let us consider a system of n ordinary differential equations of first order with constant coefficients and an ε-correlated wide-sense stationary inhomogeneous term

$${}^{\varepsilon}\dot{x} = A\, {}^{\varepsilon}x + {}^{\varepsilon}f.$$

The matrix A is assumed to be stable (i.e. all eigenvalues $\lambda_1, \ldots, \lambda_n$ have negative real parts) and diagonalizable (i.e. $A = V\Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$). Then a stationary solution

$${}^{\varepsilon}x(t) = \int_{-\infty}^{t} e^{A(t-s)}\, {}^{\varepsilon}f(s)\, ds$$

of the system exists (see, e.g., [2, 7, 9]).

Then for $\tau = 0$

$${}^{\varepsilon}R_{xx}(0) = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} \{q_j + q_j^*\} + o(\varepsilon^{N+1})$$

as $\varepsilon \to 0$ with

$$q_j = V\Lambda^j \int_0^\infty e^{\Lambda u} B_j e^{\Lambda^* u} du\, V^* = -V \left(\frac{b_{jkl}\lambda_k^j}{\lambda_k + \overline{\lambda_l}} \right)_{k,l=1}^{n} V^*$$
$$B_j = (b_{jkl})_{k,l=1}^n = V^{-1} \nu_j^+ [V^{-1}]^*$$

and for fixed $\tau > 0$

$${}^{\varepsilon}R_{xx}(\tau) = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau) + o(\varepsilon^{N+1})$$

as $\varepsilon \to 0$ with

$$q_j(\tau) = V \int_0^\infty e^{\Lambda u} C_j e^{\Lambda^* u} du\, [\Lambda^j]^* e^{\Lambda^* \tau} V^* = -V \left(\frac{c_{jkl} e^{\overline{\lambda_l}\tau} \overline{\lambda_l}^j}{\lambda_k + \overline{\lambda_l}} \right)_{k,l=1}^{n} V^*$$
$$C_j = (c_{jkl})_{k,l=1}^n = V^{-1} \mu_j^* [V^{-1}]^*$$

holds.

6. Conclusion

Asymptotic expansions as $\varepsilon \to 0$ of variances or correlation functions of integral functionals involving ε-correlated wide-sense stationary random functions have been derived. In these expansions the influence of the deterministic kernel function and of the random function can be separated using the concept of correlation moments and certain characteristics of the kernel function. In the case of random variables the expansions have the form of a power series in ε, in the case of a correlation function additional correction terms for $|\tau| < \varepsilon$ arise. For given kernel functions and a generating correlation function it is easy to compute (at least numerically) the terms of the expansion. An estimation of the remainder term is also possible.

With respect to applications it is worth to note that the asymptotic expansions can over- or underestimate the true value. The statement of the overestimation of the true value in [9: p. 49] results from the special type of the correlation functions considered there.

Further expansions of second-order characteristics of integral functionals of random processes can be found in [10, 11]. An extension of the results to scalar- or vector-valued random fields and certain classes of non-stationary processes is also possible and will be considered in a subsequent paper.

Acknowledgment. The support by the Deutsche Forschungsgemeinschaft is greatfully appreciated.

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Received 20.04.1999