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Extension of the Bernstein Condition to Systems of Ordinary Differential Equations of General Form

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Abstract. The Bernstein condition of boundedness of the derivatives of an a priori bounded solution of a 2nd order ordinary differential equation is extended to systems in which each equation has its own order.

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1. Introduction

The Bernstein theorem for the equation

$$x''(t) = f(t, x(t), x'(t))$$

is well-known [1: Section 1.2]. According to it, the inequality

$$|f(t,x,x_1)| \le A\,x_1^2 + B \qquad (A, B \text{ constants})$$

guarantees the boundedness of x', if the solution x of the equation above is bounded. This theorem was extended in several directions. So, the vector equation

$$x^{(n)}(t) = f\big(t, x(t), x'(t), \dots, x^{(n-1)}(t)\big) \qquad \big(x(t) \in \mathbb{R}^m \ (m \ge 1), n \ge 2\big) \tag{1}$$

was considered in [2] with f continuous. There was proven that, if the function f satisfies the estimation

$$\big|f(t, x, x_1, \dots, x_{n-1})\big| \le A\left(|x_1|^n + |x_2|^{\frac{n}{2}} + \dots + |x_{n-1}|^{\frac{n}{n-1}}\right) + B \tag{2}$$

for $|x| \le a$ $(a > 0)$ and $A, B > 0$, then any solution $x : [t_0, T] \to \mathbb{R}^m$ of (1) which satisfies the a priori estimation $|x(t)| \le \alpha$ with sufficiently small α depending only

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on A, B and m, n can be continued onto the whole semiaxis $[t_0, \infty)$ and has bounded derivatives $x', \ldots, x^{(n-1)}$ on it. But if condition (2) is replaced by

$$\sup_{t\in[t_0,\infty)} \max_{|x|\leq a} \left|f(t, x, x_1, \ldots, x_{n-1})\right| = o\left(|x_1|^n + \ldots + |x_{n-1}|^{\frac{n}{n-1}}\right) \tag{3}$$

as

$$\left.\begin{matrix} |x_1| \\ \vdots \\ |x_{n-1}| \end{matrix}\right\} \to \infty,$$

for any fixed $a > 0$, then the condition of sufficient smallness of α is eliminated, i.e. any a priory bounded solution x has bounded derivatives $x', \ldots, x^{(n-1)}$ (this statement holds under estimation (2) only if $n = 2$ and $m = 1$, i.e. in the case covered by the Bernstein theorem).

The transition to a right-hand side of equation (1) which satisfies the Carathéodory conditions (see, e.g., [3: Section 18.4]), the replacement of boundedness of the solution x on its uniform L_p-boundedness on segments of fixed length, and some other generalizations are contained in [4, 5]. The results of [5] can be applied especially to the system of scalar equations

$$x_i^{(n_i)}(t) = f_i\left(t, \ldots, x_j^{(k)}(t), \ldots\right) \qquad \left(\begin{smallmatrix} i,j=1,\ldots,m \\ k=0,\ldots,n_j-1 \end{smallmatrix}\right). \tag{4}$$

The aim of the present paper is to give effective sufficiency conditions on the functions f_i for the possibility of a continuation onto the whole semi-axis of any a priory bounded solution of system (4) and the boundedness of all its derivatives $x_j^{(k)}(t)$ $(k \leq n_j - 1)$

2. General plan of the estimation of derivatives

2.1. We consider solutions of the system of scalar equations (4), whose right-hand sides are given for $t \in [0, \infty)$ and arbitrary values of other arguments and satisfy the Carathéodory condition. Uniqueness of the solution of any Cauchy problem is not supposed. Let the solution

$$t \mapsto x(t) = \left(x_1(t), \ldots, x_m(t)\right)$$

of system (4) be built starting from $t = 0$ in the direction of growth of t, and let be known that the values of this solution, being arbitrarily continued, cannot leave some domain

$$Q = [-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_m, \alpha_m] \qquad (\alpha_1, \ldots, \alpha_m \in (0, \infty)).$$

The problem is to find conditions on the functions f_i under which all derivatives of the solution of system (4) indicated in the right-hand sides of that system remain bounded. In particular, it follows from here that any such solution can be continued on the whole semi-axis $[0, \infty)$.

We shall use the Kolmogorov-Gorny inequality (see, e.g., [6: Supplement 37]) for any function $\psi \in C^s([a,b];\mathbb{R})$

$$\|\psi^{(k)}\| \le a_{s,k}\, \|\psi\|^{\frac{s-k}{s}} \left[\max\left\{\|\psi^{(s)}\|, \frac{s!}{(b-a)^s}\|\psi\|\right\}\right]^{\frac{k}{s}} \quad (k = 0,\dots,s-1) \tag{5}$$

where $\|\cdot\| = \max_{[a,b]}|\cdot|$ while $a_{s,k} > 0$ are absolute constants with $a_{s,0} = 1$.

The following simple lemma will be needed for us:

Lemma 1. *For any $s \in \mathbb{N}$ there exists $r_s > 0$ such that the implication*

$$a \in \mathbb{R}, b \in (a,\infty), \varphi \in C^s([a,b],\mathbb{R}) \quad \Longrightarrow \quad (b-a)^s \min|\varphi^{(s)}| \le r_s \max|\varphi|$$

holds.

2.2. Let $x : [0,t] \to Q$ $(0 < t < \infty)$ be a solution of system (4) and denote

$$M_i(t) = \max_{\tau\in[0,t]} |x_i^{(n_i-1)}(\tau)| \qquad (i = 1,\dots,m).$$

We find conditions under which all functions $M_i(t)$ remain bounded in the continuation process of any such solution of system (4). Then the boundedness of its derivatives of lower orders will follow from (5).

Consider the i-th equation of system (4). If

$$|x_i^{(n_i-1)}(\tau)| > \tfrac{1}{2}M_i(t) \qquad (\forall\,\tau \in [0,t]),$$

then from Lemma 1

$$\tfrac{1}{2}t^{n_i-1}M_i(t) \le r_{n_i-1}\alpha_i, \qquad \text{i.e. } M_i(t) \le 2\frac{r_{n_i-1}\alpha_i}{t^{n_i-1}} \tag{6}$$

follows. Let now be

$$\min_{\tau\in[0,t]} |x_i^{(n_i-1)}(\tau)| \le \tfrac{1}{2}M_i(t).$$

Then values $t_{i1}, t_{i2} \in [0,t]$ depending on t exist such that

$$\left.\begin{aligned} &|x_i^{(n_i-1)}(t_{i1})| = M_i(t)\\ &|x_i^{(n_i-1)}(t_{i2})| = \tfrac{1}{2}M_i(t)\\ &|x_i^{(n_i-1)}(\tau)| \in (\tfrac{1}{2}M_i(t), M_i(t))\ \forall\,\tau \text{ between } t_{i1} \text{ and } t_{i2}. \end{aligned}\right\}$$

Integrating both parts of equation (4) from t_{i1} up to t_{i2}, we obtain

$$M_i(t) = 2\left|\int_{t_{i1}}^{t_{i2}} f_i\big(\tau,\dots,x_j^{(k)}(\tau),\dots\big)\,d\tau\right|. \tag{7}$$

Moreover, from Lemma 1

$$\tfrac{1}{2}|t_{i2}-t_{i1}|^{n_i-1}M_i(t) \le r_{n_i-1}\alpha_i, \qquad \text{i.e. } |t_{i2}-t_{i1}| \le \left(2\frac{r_{n_i-1}\alpha_i}{M_i(t)}\right)^{\frac{1}{n_i-1}} \tag{8}$$

follows.

In order to estimate the right-hand side of (7), denote

$$\Phi_i(\ldots,b_{jk},\ldots;\delta,t) = \sup\left\{\left|\int_{t_1}^{t_1+h} f_i(\tau,\ldots,\varphi_{jk}(\tau),\ldots)\,d\tau\right| : \begin{array}{l} 0\le t_1\le t_1+h\le t, h\le\delta \\ \varphi_{jk}\in C\big([0,t],[-b_{jk},b_{jk}]\big)\end{array}\right\} \tag{9}$$

for $b_{jk} > 0$ $(j = 1,\ldots,m; k = 0,\ldots,n_i-1)$. Then we obtain from (5) - (7) (with $s = n_j - 1$) and (8) that

$$\begin{aligned} M_i(t) \le 2\max\Bigg\{ &\frac{r_{n_i-1}\alpha_i}{t^{n_i-1}}, \\ &\Phi_i\left(\ldots, a_{n_j-1,k}\alpha_j^{\frac{n_j-1-k}{n_j-1}}\left[\max\left\{M_j(t),(n_j-1)!\,\frac{\alpha_j}{t^{n_j-1}}\right\}\right]^{\frac{k}{n_j-1}},\ldots; \right. \\ &\left.\left[2\frac{r_{n_i-1}\alpha_i}{M_i(t)}\right]^{\frac{1}{n_i-1}},t\right)\Bigg\} \end{aligned} \tag{10}$$

$(i = 1,\ldots,m)$. Here one must take $M_j(t)$ instead of the inner maximum if $k = n_j - 1$. If some $n_i = 1$, then the corresponding equation (10) is not considered and $M_i(t)$ is replaced by α_i in all other equations.

Thus we have obtained system (10) which contains m inequalities connecting m non-decreasing non-negative functions $t \mapsto M_i(t)$. According to that what has been said we obtain the following

Theorem 1. *If from the inequality system* (10) *the boundedness of all functions M_i for any $\{\alpha_j\}$ or any sufficiently small $\{\alpha_j\}$ follows, then by continuation of any a priori bounded solution or respectively any bounded with sufficiently small constants solution of system* (4), *all derivatives indicated in the right-hand sides of system* (4) *remain bounded and the continuation is possible for arbitrary large values of t.*

3. Examples

3.1. Consider equation (1) with $m = 1$, i.e. the scalar case, where condition (2) holds. Then we obtain from (9)

$$\Phi(b_0,\ldots,b_{n-1};\delta,t) \le A\,\delta\big(b_1^n + \ldots + b_{n-1}^{\frac{n}{n-1}}\big) + B\delta$$

for all $0 \le t < \infty$ and $|b_0| \le a$. We can assume that $t \ge t_0$ for some $t_0 > 0$ as it is possible to apply the existence theorem for the Cauchy problem on the interval $[0,t_0]$ for sufficiently small t_0. Then inequality (10) for $M(t) > 0$ takes the form

$$\begin{aligned} M(t) \le 2\max\Bigg\{ &\frac{r_{n-1}\alpha}{t_0^{n-1}}, \left[2\frac{r_{n-1}\alpha}{M(t)}\right]^{\frac{1}{n-1}} \\ &\times\left[B + A\sum_{k=1}^{n-1}\left(a_{n-1,k}\alpha^{\frac{n-1-k}{n-1}}\left[\max\left\{M(t),(n-1)!\,\frac{\alpha}{t_0^{n-1}}\right\}\right]^{\frac{k}{n-1}}\right)^{\frac{n}{k}}\right]\Bigg\} \end{aligned}$$

From here

$$M(t) \leq \max\left\{C_1\alpha, C_2\left[\frac{\alpha}{M(t)}\right]^{\frac{1}{n-1}} + C_3 \max\left\{\sum_{k=1}^{n-1} \alpha^{\frac{n-k}{k}} M(t), \alpha^{\frac{n^2-n-1}{k(n-1)}}\right\}\right\} \tag{11}$$

follows where the constants C_1, C_2, C_3 do not depend on α and $M(t)$. We see that the function $t \mapsto M(t)$ is bounded for all $t \geq t_0$ and sufficiently small α, and therefore Theorem 1 is applicable in the 2nd variant.

If condition (2) is repalced by (3), then the boundedness of $M(t)$ for any α follows from the fact that the value A and therefore C_3 in estimate (11) can be chosen arbitrarily small for sufficiently large $M(t)$. Therefore Theorem 1 is applicable in the 1st variant in this case.

3.2. Let be $m = 1$ and let the right-hand side of equation (1) admit the estimate

$$\left|f(t, x, x_1, \ldots, x_{n-1})\right| \leq \sum_{r=1}^{p} g_r(t)\, |x_1|^{\alpha_{r,1}} \cdots |x_{n-1}|^{\alpha_{r,n-1}}$$

$$\forall\, t \in [0, \infty),\ x \in [-\alpha, \alpha],\ x_1, \ldots, x_{n-1} \in \mathbb{R}$$

for some $\alpha_{i,j} \geq 0$ where $\int_{t_1}^{t_2} g_i(t)\, dt = o(|t_2 - t_1|^{\gamma_i})$ as $t_1, t_2 \in [0, \infty)$ with $0 < |t_2 - t_1| \to 0$ and $\gamma_i \in [0, 1]$ $(i = 1, \ldots, p)$. Then

$$\Phi(b_0, \ldots, b_{n-1}; \delta, t) = o\left(\sum_{r=1}^{p} \delta^{\gamma_r} b_1^{\alpha_{r,1}} \cdots b_{n-1}^{\alpha_{r,n-1}}\right) \qquad (\delta \to 0).$$

Hence we obtain, arguing as in Example 3.1, that if

$$\sum_{k=1}^{n-1} k\alpha_{r,k} - \gamma_r \leq n - 1 \qquad (r = 1, \ldots, p),$$

then the function $t \mapsto M(t)$ is bounded for all $t \geq 0$ and any $\alpha > 0$, i.e. Theorem 1 is applicable in its 1-st variant.

3.3. Consider the system of scalar equations with bounded functions g_i

$$\left.\begin{aligned} x_1''(t) &= g_1\big(t, x(t), x'(t)\big)\, |x_1'(t)|^{\beta_{11}} |x_2'(t)|^{\beta_{12}} \\ x_2''(t) &= g_2\big(t, x(t), x'(t)\big)\, |x_1'(t)|^{\beta_{21}} |x_2'(t)|^{\beta_{22}} \end{aligned}\right\} \qquad (0 \leq t < \infty) \tag{12}$$

for some $\beta_{ij} > 0$ $(1 \leq i, j \leq 2)$ and $x = (x_1, x_2)$. Here

$$\Phi_i(b_{11}, b_{21}; \delta, t) \leq G\, b_{11}^{\beta_{i1}} b_{21}^{\beta_{i2}} \delta \qquad (i = 1, 2;\ G > 0),$$

therefore the system of inequalities (10) has the form

$$M_i(t) \leq \max\left\{C_1\alpha_i, C_2 M_1(t)^{\beta_{i1}} M_2(t)^{\beta_{i2}} \alpha_i M_i(t)^{-1}\right\} \qquad (i = 1, 2) \tag{13}$$

for $t \geq t_0 > 0$ where $C_1 > 0$ and $C_2 > 0$ are certain constants. After taking the logarithm of both sides of (13) and denoting $y_i = \ln M_i(t)$ we obtain the inequality system

$$\left.\begin{array}{l} y_1 \leq \max\left\{\ln(C_1\alpha_1), \ln(C_2\alpha_1) + (\beta_{11} - 1)y_1 + \beta_{12}y_2\right\} \\ y_2 \leq \max\left\{\ln(C_1\alpha_2), \ln(C_2\alpha_2) + \beta_{21}y_1 + (\beta_{22} - 1)y_2\right\} \end{array}\right\}. \tag{14}$$

Inequality $(14)_1$ defines an angle in the (y_1, y_2)-plane which is larger than π and bounded by two rays given as

$$\left.\begin{array}{l} y_1 = \ln(C_1\alpha_1) \\ y_2 = \dfrac{2-\beta_{11}}{\beta_{12}}y_1 - \dfrac{\ln(C_2\alpha_1)}{\beta_{12}} \end{array}\right\}$$

where the first goes downwards and the second one to the right. Further, inequality $(14)_2$ defines an angle which is larger than π and bounded by two rays given as

$$\left.\begin{array}{l} y_2 = \ln(C_1\alpha_2) \\ y_1 = \dfrac{2-\beta_{22}}{\beta_{21}}y_2 - \dfrac{\ln(C_2\alpha_2)}{\beta_{21}} \end{array}\right\}$$

where the first goes to the left and the second one upwards. The direct consideration of the intersection of these angles shows that the conditions

$$\beta_{11} < 2, \ \frac{\beta_{12}}{2-\beta_{11}} < \frac{2-\beta_{22}}{\beta_{21}} \qquad \text{or} \qquad \beta_{11} < 2, \ \frac{\beta_{12}}{2-\beta_{11}} \leq \frac{2-\beta_{22}}{\beta_{21}} \tag{15}$$

are necessary and sufficient for the boundedness from above both coordinates of its points for any, or respectively any sufficiently small values of α_1 and α_2.

Thus Theorem 1 is applicable to system (12) in the 1-st or 2-nd variant, if inequalities $(15)_1$ or equality in $(15)_2$ hold.

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