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CM-Selectors for Pairs of Oppositely Semicontinuous Multifunctions and Some Applications to Strongly Nonlinear Inclusions

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Abstract. We present a new approximate joint selection theorem which unifies Michael's theorem (1956) on continuous selections and Cellina's theorem (1969) on continuous ε-approximate selections. More precisely, we show that, given a convex-valued H-upper semicontinuous multifunction F and a convex-closed-valued lower semicontinuous multifunction G with $F(x) \cap G(x) \neq \emptyset$, one can find a continuous function f which is both a selection of G and an ε-approximate selection of F. We also prove a parametric version of this theorem for multifunctions F and G of two variables $(s,u) \in \Omega \times X$ where Ω is a measure space. Using this selection theorem, we obtain an existence result for elliptic systems involving a vector Laplacian and a strongly nonlinear multi-valued right-hand side, subject to Dirichlet boundary conditions.

Keywords: *Joint, continuous and ε-approximate selectors, H-upper and lower semicontinuous multifunctions, multifunctions satisfying one-side estimates, Dirichlet elliptic inclusions, multi-valued elliptic systems, problems with strong non-linearities, with lack of compactness and with critical exponents*

AMS subject classification: Primary 54 C 65, 54 C 60, 28 B 20, 35 R 70, secondary 47 H 04, 34 A 60

1. Introduction

The first purpose of this paper is to present a new continuous joint selection theorem (Theorem 2.1) which unifies two known theorems due to E. A. Michael [13] in 1956 and to A. Cellina [7] in 1969. More precisely, we prove that if F is an H-upper semicontinuous convex-valued multifunction from X to 2^Y, G is a lower semicontinuous convex-closed-valued multifunction from X to 2^Y, and $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$, then there exists a CM-selector for the pair (F, G), i.e. there exists a continuous function which is both a selector for G (as in Michael's theorem) and an ε-approximate selector for F (as in Cellina's theorem). In the case $G(x) \equiv Y$ Theorem 2.1 reduces to the Cellina

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theorem for F. In the case $F(x) \equiv Y$ it reduces to the Michael theorem for G. In the general case our theorem can be interpreted as an "intermediate" theorem between the Michael and Cellina theorems.

The notion of CM-selectors and the problem of their existence find motivation in our research on the existence of solutions of strongly nonlinear multi-valued problems: nonlinear Hammerstein multi-valued equations (inclusions) and elliptic boundary value problems with strongly nonlinear multi-valued right-hand sides F satisfying some one-sided estimates (the sign condition, the generalized sign condition, the Hammerstein one-sided estimate, etc.). We observed that in such a case each one-sided estimate generates in the multi-valued setting some pair (F, G), where the multifunction G is lower semicontinuous (see Theorems 2.2 and 3.1). The strong nonlinearity of $F(s, x)$ means that we have to consider problems involving F in the cases of the lack of compactness and of critical exponents in the exact non-compact Sobolev embedding theorems (Sobolev's and Pokhozaev-Trudinger's).

The second part of the present paper (Sections 4 and 5) is therefore devoted to some applications of ε-approximate CM-selectors as well as to the study of the simplest multi-valued strongly nonlinear problem (inclusion). To this end, we prove a simple parametric version of Theorem 2.1 (see Theorem 3.1) for multifunctions F and G of two variables $(s, x) \in \Omega \times X$ where Ω is a measure space. Next we apply the result to constructing a sequence of single-valued strongly nonlinear Dirichlet problems $-\Delta u(s) = f_n(s, u(s))$ approximating the original multi-valued strongly nonlinear Dirichlet problem $-\Delta u(s) \in F(s, u(s))$ in an "appropriate" sense such that the functions $f_n(s, x)$ satisfy the same one-sided estimate as the multifunction $F(s, x)$ (see our construction of CM-relaxations $f_n(s, x)$ in (5.1) - (5.2) of Step 1 in Section 5; cf. with usual truncated relaxations $f_n(s, x)$ in the proof of [4: Theorem 2] and [15: Formula (28)]).

Finally we formulate and prove an existence theorem (Theorem 4.1) for the above multi-valued strongly nonlinear problem (the simplest inclusion with lack of compactness), emphasizing seven main steps characteristic of our weak convergence analysis via the use of the above CM-relaxations (see Steps 1 - 7 in Section 5).

By the way, it is interesting to notice that in the proof of a recent result of Hu and Papageorgiou [10] on a generalization of Browder's degree for strongly nonlinear elliptic inclusions of $(S)_+$ type there is a gap in their construction of approximate single-valued scalar functions $g_\varepsilon(\cdot, \cdot)$ (see [10: p. 244[18]], where in fact it is impossible to use "line segments to make continuous connections" for defining their auxiliary function $\eta^*_\delta(r)$). This gap can be closed by using our "applied" ε-approximate CM-selection Theorems 2.2 and 3.1 together with Remark 2.1/(2) of Section 2.

2. CM-Selectors

For the convenience of the reader, we give the basic definitions and notations following [2, 5]. Let (X, ρ) be a metric space. For $x \in X$, $M \subset X$ and $\varepsilon > 0$ we denote by $d(x, M) = \inf\{\rho(x, y) : y \in M\}$ the distance from x to M, by $U_\varepsilon(M) = \{y \in X : d(y, M) < \varepsilon\}$ the ε-neighbourhood of M and by $B(x, r) = B_X(x, r)$ the open ball with center x and radius r. The distance in the product $X \times Y$ of metric spaces is defined by $d((x, y), (x_1, y_1)) =$

$\max\{\rho_X(x,x_1),\rho_Y(y,y_1)\}$. We assume that each multifunction considered has non-empty values, unless stated to the contrary. The graph of a multifunction $F : X \to 2^Y$ is the set $\operatorname{Gr} F = \{(x,y) \in X \times Y : y \in F(x)\}$. If $A \subset X$, then $F(A)$ denotes the set $\cup_{x\in A} F(x)$.

Let X, Y be metric spaces. A multifunction $F : X \to 2^Y$ is called

- *upper* or *lower semicontinuous at* x_0 if for any open set $V \subset Y$ with $F(x_0) \subset V$ or $F(x_0) \cap V \neq \emptyset$ one can find an open neighbourhood $U \subset X$ of x_0 such that $F(x) \subset V$ or $F(x) \cap V \neq \emptyset$, respectively, for all $x \in U$.
- *upper* or *lower semicontinuous*, if it is upper or lower semicontinuous, respectively, at every $x \in X$;
- *H-upper* or *H-lower semicontinuous at* x_0 if for any $\varepsilon > 0$ one can find $\delta > 0$ such that $F(B(x_0,\delta)) \subset U_\varepsilon(F(x_0))$ or $F(x_0) \subset U_\varepsilon(F(x))$, respectively, for all $x \in B(x_0,\delta)$.
- *H-upper* or *H-lower semicontinuous*, if it is H-upper or H-lower semicontinuous at every $x \in X$, respectively;

If F is upper semicontinuous, then it is H-upper semicontinuous; the converse is true if F takes compact values. If F is H-lower semicontinuous, then F is lower semicontinuous; the converse is true if F takes compact values.

If Y is a normed space, we denote by $\operatorname{conv} D$ and $\overline{\operatorname{conv}}\, D$ the convex hull and the closed convex hull of a subset D of Y, respectively.

The main purpose of this section is to prove a theorem, which is intermediate between two famous continuous selection theorems: the Michael theorem [13] and the Cellina theorem [7]. To formulate it, there is a need for a new notion which we immediately introduce.

Definition 2.1. Let $F, G : X \to 2^Y$ be two multifunctions, where X and Y are metric spaces, and let $\varepsilon > 0$ be an arbitrary positive number. By an *ε-approximate CM-selector* for the pair (F,G) we mean a continuous function $f : X \to Y$ which is both a selector for G (i.e., $f(x) \in G(x)$ for all $x \in X$) and an ε-approximate selector (ε-selector in short) for F (i.e. $\operatorname{Gr} f \subset U_\varepsilon(\operatorname{Gr} F)$).

Remark. If Y is a normed space, then $f : X \to Y$ is an ε-selector for F if and only if $f(x) \in F(B_X(x,\varepsilon)) + B_Y(0,\varepsilon)$ for all $x \in X$.

Theorem 2.1. *Let X be a metric space and Y a Banach space. Assume that $F, G : X \to 2^Y$ are multifunctions, F H-upper semicontinuous with convex values and G lower semicontinuous with closed convex values, and such that $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$. Then for every $\varepsilon > 0$ there exists an ε-approximate CM-selector for the pair (F,G).*

Proof. The proof of Theorem 2.1 will be carried out in two steps.

Step 1: Suppose first that Y is a normed space and $G : X \to 2^Y$ has convex values only. We claim that then for every $\varepsilon_1, \varepsilon_2 > 0$ there exists a continuous map $f : X \to Y$ such that $\operatorname{Gr} f \subset U_{\varepsilon_1}(\operatorname{Gr} F)$, $f(x) \in U_{\varepsilon_2}(G(x))$ for every $x \in X$, and $f(X) \subset \operatorname{conv} F(X)$.

For the proof fix $\varepsilon_1, \varepsilon_2 > 0$. Let y_x be an arbitrary element of $F(x) \cap G(x)$ with $x \in X$. F is H-upper semicontinuous, so for $x \in X$ there is $\delta_1(x) > 0$ such that

$\delta_1(x) < \varepsilon_1$ and $F(B(x, \delta_1(x))) \subset U_{\varepsilon_1}(F(x))$. The multifunction G is lower semicontinuous, therefore for $x \in X$ there exists $\delta_2(x) > 0$ such that $B(y_x, \varepsilon_2) \cap G(x') \neq \emptyset$, i.e. $y_x \in U_{\varepsilon_2}(G(x'))$ for $x' \in B(x, \delta_2(x))$. Denote $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ and $U_x = B(x, \frac{1}{2}\delta(x))$ for $x \in X$. Since $(U_x)_{x \in X}$ is an open covering of the metric space X and X is paracompact by the Stone theorem [12], we can find a locally finite refinement $(W_i)_{i \in I}$ of $(U_x)_{x \in X}$ and a continuous partition of unity $(\phi_i)_{i \in I}$ subordinate to $(W_i)_{i \in I}$. For each $i \in I$ choose $x_i \in X$ such that $\phi_i \equiv 0$ on $X \setminus U_{x_i}$. Denote $\delta(x_i) = \delta_i$, $U_{x_i} = U_i$ and $y_{x_i} = y_i$ for $i \in I$. Define the function $f : X \to Y$ by $f(x) = \sum_{i \in I} \phi_i(x)\, y_i$. Evidently, f is continuous, and as $f(x)$ is a convex combination of elements of $F(X)$, we have $f(x) \in \operatorname{conv} F(X)$ for every $x \in X$.

Observe that f is an ε_1-selector of F. Indeed, let $x \in X$ and denote $I(x) = \{i \in I : \phi_i(x) \neq 0\}$. The set $I(x)$ is finite and we have $f(x) = \sum_{i \in I(x)} \phi_i(x)\, y_i$. Define $j \in I(x)$ so that $\delta_j = \max_{i \in I(x)} \delta_i$. If $i \in I(x)$, then $\phi_i(x) > 0$ and hence $x \in U_i$. Now

$$\rho(x_i, x_j) \leq \rho(x_i, x) + \rho(x, x_j) < 2(\tfrac{1}{2}\delta_j) = \delta_j,$$

so $x_i \in B(x_j, \delta_j)$, and therefore $y_i \in U_{\varepsilon_1}(F(x_j))$ for $i \in I(x)$. Consequently, $f(x) \in U_{\varepsilon_1}(F(x_j))$ as the ε-neighbourhood $U_{\varepsilon_1}(F(x_j))$ of the convex set $F(x_j)$ in the normed space Y is convex. On the other hand, $x \in B(x_j, \varepsilon_1)$ because $\delta_j < \varepsilon_1$. Finally, $(x, f(x)) \in U_{\varepsilon_1}(\operatorname{Gr} F(x))$ for every $x \in X$, i.e. f is an ε_1-selector for F.

For the proof of the remaining part of our statement, let again $x \in X$. If $i \in I(x)$, then $x \in U_i$ and hence $y_i \in U_{\varepsilon_2}(G(x))$. Therefore $f(x) = \sum_{i \in I(x)} \phi_i(x)\, y_i \in U_{\varepsilon_2}(G(x))$ as the set $G(x)$ and hence also $U_{\varepsilon_2}(G(x))$ is convex.

Step 2: Assume now that Y is even a Banach space and that G takes closed convex values. We claim that for every $\varepsilon > 0$ there exists a CM-selector f for the pair (F, G). Indeed, fix $\varepsilon > 0$. By Step 1 there exists a continuous map $f_1 : X \to Y$ such that $\operatorname{Gr} f_1 \subset U_{\frac{\varepsilon}{2}}(\operatorname{Gr} F)$ and $f_1(x) \in U_{\frac{\varepsilon}{2}}(G(x))$ for $x \in X$. Consider the multifunction $G_1 : X \to 2^Y$ defined by $G_1(x) = \overline{G(x) \cap B(f_1(x), \frac{\varepsilon}{2})}$. Of course, G_1 has non-empty closed convex values. Moreover, G_1 is lower semicontinuous (see, e.g., [5: Proposition 1.1.5]). Thus, by the famous Michael theorem, G_1 has a continuous selector $f : X \to Y$. Note that f is also a selector for G as $G_1(x) \subset \overline{G(x)} = G(x)$ for $x \in X$.

It remains to show that $\operatorname{Gr} f \subset U_\varepsilon(\operatorname{Gr} F)$. Indeed, let $x \in X$. Since $(x, f_1(x)) \in U_{\frac{\varepsilon}{2}}(\operatorname{Gr} F)$, we have $\rho(x', x) < \frac{\varepsilon}{2}$ and $\rho(y, f_1(x)) < \frac{\varepsilon}{2}$ for some $x' \in X$ and $y \in F(x')$. Hence $\rho(y, f(x)) \leq \rho(y, f_1(x)) + \rho(f_1(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, because $f(x) \in G_1(x) \subset \overline{B}(f_1(x), \frac{\varepsilon}{2})$. Thus we have $d((x, f(x)), (x', y)) < \varepsilon$, and consequently f is an ε-selector for F ∎

Remarks 2.1. **1)** Theorem 2.1 is true also in the more general setting of Cellina's Theorem 1 from [8], i.e. when X is a paracompact, uniform space with countable base (in particular, metric space) and Y is a complete metric, locally convex space (i.e. Fréchet space). The L_p-decomposable nonconvex-valued version of the theorem is valid too (cf. [6]; the results were announced in H. T. Nguyễn [14] and are accepted for publication in [16]).

2) From Theorem 2.1 follows in addition the possibility to construct a CM-selector f such that $f(a) = f_0(a)$ $(a \in A)$, where $A \subset X$ is a fixed closed set (in particular, A is a fixed finite or closed countable set) and $f_0 : A \to Y$ is a fixed continuous function such that $f_0(a) \in F(a) \cap G(a)$ $(a \in A)$. For a proof put $G_0(x) = \{f_0(x)\}$ for $x \in A$ and $G_0(x) = G(x)$ for $x \notin A$. By [13] (see also [5]), G_0 is lower semicontinuous just as G. Applying the statement of Theorem 2.1 for the pair (F, G_0), we get its CM-selector f, which clearly is a CM-selector for the pair (F, G) with the additional property $f(x) = f_0(x)$ $(x \in A)$. The existence of ε-approximate selectors with this property, for an H-upper semicontinuous multifunction (as in Cellina's selection theorem [7]) seems to be unnoticed before (see recent references in the books [5, 11], and recent papers, for example [10]), although it is well-known that a lower semicontinuous multifunction of Michael's theorem has a continuous selector satisfying the additional property.

The following "applied" ε-approximate CM-selection theorem (and it together with the above Remark 2.1/(2)) is an example of how Theorem 2.1 can be applied to constructing ε-approximate continuous selectors satisfying some additional conditions.

Theorem 2.2. *Let X be a Banach space and X^* be its dual. Assume that $F : X \to 2^{X^*}$ is a H-upper semicontinuous multifunction with convex values and that $g : X \to \mathbb{R}$ is a continuous non-negative function. Define $G : X \to 2^{X^*}$ by*

$$G(x) = \begin{cases} \{y \in X^* : \langle x, y \rangle \le g(x)\} & \text{if } x \neq 0 \\ X^* & \text{if } x = 0 \text{ and } g(0) > 0 \\ \{0\} & \text{if } x = 0 \text{ and } g(0) = 0. \end{cases}$$

Assume that $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$. Then G is lower semicontinuous, and the pair (F, G) has an ε-approximate CM-selector for every $\varepsilon > 0$.

Proof. It suffices to show that G satisfies the assumptions of Theorem 2.1. It is clear that G has non-empty closed convex values. It remains to show that it is lower semicontinuous. Indeed, assume that G is not lower semicontinuous at some $x_0 \in X$. Then there exist an open set $V \subset X^*$ such that $G(x_0) \cap V \neq \emptyset$ and a sequence $(x_n) \subset X \setminus \{0\}$,which converges to x_0 and such that $G(x_n) \cap V = \emptyset$ for $n \in \mathbb{N}$. Therefore, for every $y \in V$ we have $\langle x_n, y \rangle > g(x_n)$ for $n \in \mathbb{N}$ and hence $\langle x_0, y \rangle \ge g(x_0)$ by the continuity of $\langle \cdot, \cdot \rangle$ and g. This is a contradiction if $g(0) > 0$ and $x_0 = 0$.

Assume now that $x_0 \neq 0$ and take $y_0 \in G(x_0) \cap V$. Then $B(y_0, r) \subset V$ for some $r > 0$. From the above it follows that $\langle x_0, y_0 \rangle = g(x_0)$. On the other hand, by the classical Hahn-Banach theorem [9], there exists $z \in X^*$ with $\|z\| = 1$ and $\langle x_0, z \rangle = \|x_0\| \neq 0$. Then for $y = y_0 - \frac{r}{2} z$ we have $y \in B(y_0, r) \subset V$ and $\langle x_0, y \rangle = g(x_0) - \frac{r}{2} \|x_0\| < g(x_0)$ which is a contradiction.

The lower semicontinuity of G at $x_0 = 0$ when $g(0) = 0$ is obvious, since in this case by definition $G(0) = \{0\}$. So if $G(0) \cap V \neq \emptyset$ where V is open, then $0 \in V$. But of course $0 \in G(x)$ for every $x \in X$, hence $G(x) \cap V \neq \emptyset$ for every $x \in X$ ∎

Remark that the inequality $\langle x, y \rangle \le g(x)$ is called in the literature *one-sided estimate.*

3. CM-selectors for multifunctions of two variables

Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space with a complete, σ-finite measure μ on a σ-algebra $\mathfrak{A}$. Let X and Y be separable complete metric spaces. A multifunction $F : \Omega \to 2^X$ with closed values is called *measurable* if the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is measurable for every open subset U of X. For other equivalent notions of measurability of multifunctions see, e.g., [2, 5].

We recall that $f : \Omega \times X \to Y$ is called a *Carathéodory function* if $f(s, \cdot)$ is continuous for almost all $s \in \Omega$ and $f(\cdot, x)$ is measurable for all $x \in X$. Following e.g. [2], a multifunction $F : \Omega \times X \to 2^Y$ is called *H-upper Carathéodory* if $F(s, \cdot)$ is H-upper semicontinuous for almost all $s \in \Omega$ and $F(\cdot, x)$ is measurable for all $x \in X$.

Further, a multifunction $F : \Omega \times X \to 2^Y$ is called *$(mod\,0)$-measurable* if $F(\cdot, \cdot)$ is measurable on $(\Omega \backslash D_0) \times X$ with respect to the algebra $\mathfrak{A} \times \mathcal{B}(X)$ where D_0 is some measurable set with $\mu(D_0) = 0$ and $\mathcal{B}(X)$ is the algebra of all Borel subsets of X. More information concerning multifunctions of two variables can be found e.g. in [2].

The following "applied" ε-approximate CM-selection theorem is a parametric version of Theorem 2.2. Note that Remark 2.1/(2) is valid also for Theorem 3.1, and is useful in applications.

Theorem 3.1. *Let $F : \Omega \times \mathbb{R}^m \to 2^{R^m}$ be an H-upper Carathéodory and $(mod\,0)$-measurable multifunction, taking convex compact values. Further, let $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a non-negative Carathéodory function, define $G : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}$ by*

$$G(s,x) = \begin{cases} \{y \in \mathbb{R}^m : \langle x, y \rangle \leq g(s,x)\} & \text{if } x \neq 0 \\ \mathbb{R}^m & \text{if } x = 0 \text{ and } g(s,0) > 0 \\ \{0\} & \text{if } x = 0 \text{ and } g(s,0) = 0. \end{cases}$$

for any $s \in \Omega$, and assume that $F(s,x) \cap G(s,x) \neq \emptyset$ for almost all $s \in \Omega$ and all $x \in \mathbb{R}^m$. Then for every positive measurable function $\varepsilon : \Omega \to \mathbb{R}_+$ there exists a Carathéodory function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that $f(s, \cdot)$ is a CM-selector for the pair $(F(s, \cdot), G(s, \cdot))$ with respect to $\varepsilon(s) > 0$ for almost all $s \in \Omega$.

Proof. Let $\varepsilon : \Omega \to \mathbb{R}_+$ be an arbitrary measurable positive function. Define $\hat{F}(s,x) = F(s, \overline{B}(x, \frac{1}{2}\varepsilon(s)) + \overline{B}(0, \frac{1}{2}\varepsilon(s))$ for $s \in \Omega$ and $x \in \mathbb{R}^m$. It is easy to show that $\hat{F}(s, \cdot)$ is H-upper semicontinuous and has closed values for almost all $s \in \Omega$. We contend also that $\hat{F}(\cdot, x)$ is measurable for all $x \in X$. In fact, it is known that the properties of F ensure that the multifunction $s \mapsto F(s, Z(s))$ is measurable for every measurable multifunction $Z : \Omega \to 2^{\mathbb{R}^m}$ (see, e.g., [2]). Taking $Z(s) \equiv \overline{B}(x, \frac{1}{2}\varepsilon(s))$ for $s \in \Omega$ and fixed $x \in \mathbb{R}^m$ we obtain that the multifunction $F(\cdot, \overline{B}(x, \frac{1}{2}\varepsilon(\cdot)))$, and hence also $\hat{F}(\cdot, x)$ is measurable.

Denote $H(s,x) = \hat{F}(s,x) \cap G(s,x)$ $(s \in \Omega, x \in \mathbb{R}^m)$ and let $C(\mathbb{R}^m, \mathbb{R}^m)$ be the separable metric space of all continuous functions from $\mathbb{R}^m$ into $\mathbb{R}^m$, with the topology of uniform convergence on compact sets. From Theorem 2.2 it follows that $H(s, \cdot)$ has a continuous selector for almost all $s \in \Omega$, so the multifunction $\Phi : \Omega \to 2^{C(\mathbb{R}^m, \mathbb{R}^m)}$ defined by $\Phi(s) = \{f \in C(\mathbb{R}^m, \mathbb{R}^m) : f(x) \in H(s,x) \text{ for all } x \in \mathbb{R}^m\}$ has a.e. non-empty values. We noted earlier that $\hat{F}(s, \cdot)$ is H-upper semicontinuous for almost all

$s \in \Omega$, and therefore for almost all $s \in \Omega$ and for any continuous function $f : \mathbb{R}^m \to \mathbb{R}^m$ the implication

$$f(r) \in \hat{F}(s,r)\ (\forall\ r \in \mathbb{Q}^m) \quad \Longrightarrow \quad f(x) \in \hat{F}(s,x)\ (\forall\ x \in \mathbb{R}^m)$$

is valid. In view of this fact, and because H takes closed values, we can write

$$\begin{aligned} \operatorname{Gr}\Phi &= \Big\{(s,f) \in \Omega \times C(\mathbb{R}^m,\mathbb{R}^m) : d(f(x),H(s,x)) = 0\ (\forall\ x \in \mathbb{R}^m)\Big\} \\ &= \Big\{(s,f) \in \Omega \times C(\mathbb{R}^m,\mathbb{R}^m) : d(f(r),H(s,r)) = 0\ (\forall\ r \in \mathbb{Q}^m)\Big\} \\ &= \bigcap_{r\in\mathbb{Q}^m} \Big\{(s,f) \in \Omega \times C(\mathbb{R}^m,\mathbb{R}^m) : d(f(r),H(s,r)) = 0\Big\}. \end{aligned}$$

Clearly, $H(\cdot,x)$ is measurable for all $x \in \mathbb{R}^m$, so for each r of the countable set $\mathbb{Q}^m$ the function $(s,f) \mapsto d(f(r),H(s,r))$ is a Carathéodory function from $\Omega \times C(\mathbb{R}^m,\mathbb{R}^m)$ into $\mathbb{R}$. Hence as is well-known (see, e.g., [2, 5]) we have

$$\Big\{(s,f) \in \Omega \times C(\mathbb{R}^m,\mathbb{R}^m) : d(f(r),H(s,r)) = 0\Big\} \in \mathfrak{A} \otimes \mathcal{B}(C(\mathbb{R}^m,\mathbb{R}^m))$$

where $\mathcal{B}(M)$ is the algebra of all Borel subsets of a metric space M. It follows that $\operatorname{Gr}\Phi \in \mathfrak{A} \otimes \mathcal{B}(C(\mathbb{R}^m,\mathbb{R}^m))$. By the von Neumann-Aumann selection theorem (see [5]) Φ has a measurable selector $h : \Omega \to C(\mathbb{R}^m,\mathbb{R}^m)$. Set $f(s,x) = (h(s))(x)$. Then f is a Carathéodory function with the desired properties ∎

Theorem 3.1 (and a more general parametric version of Theorem 2.1) can also be proved in the framework of Fréchet or Banach spaces (by a different but rather complicated technique). Its various modifications cover multi-valued versions of many generalized Hammerstein one-sided estimates and all generalized sign conditions.

4. The Dirichlet problem for multi-valued elliptic differential systems with strong non-linearities

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and $F : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}$ some multifunction of two variables $(s,u) \in \Omega \times \mathbb{R}^m$. We shall consider the problem

$$\left.\begin{aligned} -\Delta_m u(s) \in F(s,u(s)) \quad &\text{for a.a. } s \in \Omega \\ u|_{\partial\Omega} = 0 \end{aligned}\right\} \tag{4.1}$$

where $\Delta_m = (\Delta, ..., \Delta)$ is an m-vector Laplacian.

In what follows we shall denote the scalar product and norm in the Euclidean space $\mathbb{R}^m$ by $(\cdot,\cdot)$ and $|\cdot|$, respectively, and the scalar product and norm in the Lebesgue space $L_2 = L_2(\Omega,\mathbb{R}^m)$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. As usual, $H^1 = H^1(\Omega,\mathbb{R}^m)$ is the Sobolev space defined by the norm $\|u\|_1 = \|u\| + \|\nabla_m u\|$, while $H_0^1 = H_0^1(\Omega,\mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^m)$ with respect to this norm. Denote by H^{-1} the dual space to H_0^1 with respect to the L_2-pairing $\langle\cdot,\cdot\rangle$. Given a Young function $M : \Omega \times \mathbb{R} \to [0,+\infty)$, the term *Orlicz space* (see, e.g., [17]) will refer to the space $L_M = L_M(\Omega, \mathbb{R}^m)$ (of all equivalence classes) of all measurable functions u on Ω taking values in $\mathbb{R}^m$, which is equipped with the Luxemburg norm $\|u\|_M = \inf\{k > 0 : \int_\Omega M(s, \|u(s)\|/k)\, ds \le 1\}$. In particular, we shall be interested in Orlicz spaces X with the property that $X \subset L_2 \subset X'$ where X' denotes the *Köthe associate space* of X (see, e.g., [4]). Remember that if $M(s,\alpha) = |\alpha|^p \ \ (1 \le p < +\infty)$, we get $L_M = L_p$.

Throughout this section, we denote by Z the special Lebesgue or Orlicz space

$$Z = \begin{cases} L_{\frac{2n}{n-2}} & \text{if } n > 2 \\ L_N & \text{if } n = 2 \end{cases} \tag{4.2}$$

where $N(s,\alpha) = \exp(|\alpha|^2) - 1$. By the Sobolev exact embedding theorem (the case $n > 2$) and the Pokhozaev-Trudinger exact embedding theorem (the case $n = 2$), the Sobolev space H_0^1 is always continuously non-compactly embedded into Z (see, e.g., [18]). By e.g. [4: Lemma 1], H_0^1 is compactly embedded into X, if X is an Orlicz space such that the space Z is absolutely continuously embedded into the space X, i.e. the elements of the unit ball of Z have uniformly absolutely continuous norms in X:

$$\lim_{\text{mes}\,(D)\to 0} \sup_{\|u\|_Z \le 1} \|P_D u\|_X = 0.$$

Here P_D denotes the multiplication operator by the characteristic function of a measurable set D. Following e.g. [2], we define the *multi-valued superposition operator* N_F by

$$N_F(u) = \{v : v \text{ is measurable and } v(s) \in F(s, u(s)) \text{ a.e.}\}. \tag{4.3}$$

We shall use one of the following acting conditions:

(AC1) $m = 1$ (i.e. the case of scalar equations), $X = Z$, and the multi-valued superposition operator N_F acts from Z into $2^{Z'}$ where $Z' = L_{\frac{2n}{n+2}}$ if $n > 2$ and $Z' = L_{N^*}$ with N^* the dual to the Young function N if $n = 2$.

(AC2) $m > 1$ (i.e. the case of a system of equations), and either

(a) $n > 2$, $Z \subset X$ strictly, the multi-valued superposition operator N_F acts from X into $2^{Z'}$, Z is absolutely continuously embedded into X

or

(b) $n = 2$, $Z \subset X$, the multi-valued superposition operator N_F acts from X into $2^{Z'}$, and the equality $\lim_{\text{mes}\,(D)\to 0} \sup_{y \in N(x), \|x\|_X \le r} \langle y, P_D z\rangle = 0$

holds for each $z \in Z$ and $r > 0$.

Later on, we denote by μ_Δ the *first Dirichlet eigenvalue* of the Laplacian $-\Delta$ on Ω.

The main result for this section is the following

Theorem 4.1. *Let Z be the space in* (4.2) *and X be an Orlicz space such that*

$$H_0^1 \subset Z \subset X \subset L_2 \subset X' \subset Z' \subset H^{-1} \tag{4.4}$$

continuously. Suppose condition (AC1) *for the case $m = 1$ and condition* (AC2) *for the case $m > 1$. Suppose in addition that the following conditions are satisfied:*

1) *$F(\cdot,\cdot)$ has non-empty compact convex values and is an H-upper Carathéodory as well as an $(mod\,0)$-measurable multifunction.*

2) *For almost all $s \in \Omega$ there exists $w \in F(s,u)$ such that the one-sided inequality*

$$(u,w) \leq \gamma(u,u) + \delta(s) \tag{4.5}$$

holds where $0 < \gamma < \mu_\Delta$ and $\delta \in L_1(\Omega,\mathbb{R})$ is positive.

Then problem (4.1) has at least one solution $u_* \in H_0^1$.

The proof of Theorem 4.1 will be given in Section 5.

Remarks 4.1.

1) Sufficient (and necessary) conditions guaranteeing that the multi-valued superposition (Nemytskij) operator N_F acts as desired in conditions (AC1) - (AC2) of Theorem 4.1 are completely analogous to those for the single-valued superposition operator (for the latter case see, e.g., [4]). For example, when $m = 1$ and $n > 2$ we may assume the polynomial growth condition

$$\sup_{w\in F(s,u)} |w| \leq a(s) + b\,|u|^{\frac{n+2}{n-2}} \tag{4.6}$$

for some $a \in L_{\frac{2n}{n+2}}(\Omega,\mathbb{R})$ and $b \in [0,+\infty)$; when $m = 1$ and $n = 2$ we may assume the analogous non-polynomial exponential growth condition, using the Young function N and its dual Young function N^*. It is well-known that all the exponents in (4.6) as well as in the above non-polynomial exponential growth condition are critical (and the inclusion under consideration is non-compact-type strongly nonlinear) since they all correspond to the exact continuous non-compact embeddings in the above-mentioned Sobolev/Pokhozaev-Trudinger theorems (see the discussions for the single-valued case, e.g., in [4, 15]).

2) The compact-type nonlinear inclusions were treated, e.g., in [3] and the references cited therein.

3) Analogous existence results are valid for more complicated strongly nonlinear inclusions such as multi-valued versions of strongly nonlinear problems, which were studied, e.g., in [4, 15] and the references cited therein.

Example 4.1. Let $n > 2$ and $D \subset \mathbb{R}$ be a fixed closed non-empty set (finite or countable, or uncountable such as a Cantor "middle thirds" set). Put $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1(u) = |u|^{\frac{n+2}{n-2}}$ for $u \leq 0$ and $\varphi_1(u) = 0$ for $u > 0$, and $\varphi_2(u) = 0$ for $u \in D$ and $\varphi_2(u) = 1$ for $u \notin D$ (then the set of discontinuity points of φ coincides with D). Define [2] the so-called (Krasovskij) convexification of the discontinuous φ by $\varphi^*(u) = \cap_{\eta>0}\overline{\mathrm{co}}\,(\varphi([u-\eta,u+\eta]))$. Given any $h \in L_2(\Omega)$ with $h \notin L_p(\Omega)$ for all $p > n$, Theorem 4.1 allows us to state the solvability result for (4.1) with $F(s,u) = \varphi^*(u)+h(s)$, while the example cannot be treated by [3], and papers cited therein.

5. Proof of Theorem 4.1

We shall divide the proof of Theorem 4.1 into 7 steps.

Step 1: By Theorem 3.1 (if $F(s,\cdot)$ is independent of s, it is sufficient to apply Theorem 2.2) there exists for each $\varepsilon > 0$ a Carathéodory function $g_\varepsilon(\cdot,\cdot) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\operatorname{Gr} g_\varepsilon(s,\cdot) \subset \Big\{(u,v) \in \mathbb{R}^m \times \mathbb{R}^m : d((u,v), \operatorname{Gr} F(s,\cdot)) < \varepsilon\Big\} \text{ a.e.} \tag{5.1}$$

and, moreover, g_ε satisfies the one-sided estimate

$$(u, g_\varepsilon(s,u)) \le \gamma(u,u) + \delta(s) \tag{5.2}$$

where γ and $\delta(\cdot)$ are the same as in (4.5) and do not depend on ε. Choosing $\varepsilon_n = \frac{1}{n}$ we define so-called *CM-relaxations* f_n by

$$f_n(s,u) = \begin{cases} g_{\frac{1}{n}}(s,u) & \text{if } |g_{\frac{1}{n}}(s,u)| \le n \\ n\frac{g_{\frac{1}{n}}(s,u)}{|g_{\frac{1}{n}}(s,u)|} & \text{if } |g_{\frac{1}{n}}(s,u)| > n. \end{cases} \tag{5.3}$$

By (5.2) and (5.3), f_n are Carathéodory functions and

$$f_n(s,u) = \theta_n(s,u) g_{\frac{1}{n}}(s,u) \tag{5.4}$$

where $0 < \theta_n(s,u) < 1$ and

$$(u, f_n(s,u)) \le \gamma(u,u) + \delta(s). \tag{5.5}$$

Step 2: As is well-known (see, e.g., [18]), the operator L generated by the Laplacian $-\Delta_m$ is continuous and invertible from H_0^1 into H^{-1}, and

$$\langle Lu, u\rangle \ge \alpha \|u\|_1^2 \qquad (u \in H_0^1) \tag{5.6}$$

for some $\alpha > 0$ (see, e.g., [18]). Remember that the solvability of (4.1) in H_0^1 means the existence of $u \in H_0^1$ and $v \in N_F(u)$ such that $v \in H^{-1}$ and $Lu = v$. Now we consider the approximate single-valued problem in H_0^1

$$\left.\begin{aligned} -\Delta_m u(s) &= f_n(s,u(s)) \quad \text{a.e.} \\ u|_{\partial\Omega} &= 0 \end{aligned}\right\} \tag{5.7}$$

where f_n is defined in (5.4) and satisfies (5.5). Clearly, the single-valued superposition operator F_n, where $F_n(x) = f_n(\cdot, x(\cdot))$, maps the space L_2 into itself. In the presence of (5.5) via *CM*-selections, by [4: Lemma 5], the continuous compact operator $L^{-1}F_n$ has a fixed point $u_n \in L_2$ (i.e. $u_n = L^{-1}F_n u_n$) such that

$$\|u_n\|_{L_2}^2 \le \|u_n\|_1^2 \le \frac{d}{c} \tag{5.8}$$

where $d = \|\delta(\cdot)\|_{L_1}$ and $c = \alpha(\mu_\Delta - \gamma)\mu_\Delta^{-1}$ do not depend on $n \in \mathbb{N}$.

Step 3: In the presence of (5.8) and (5.5) via CM-selections, by the same argument [4], we deduce that the inequality

$$\int_\Omega \big|(F_n u_n(s), u_n(s))\big|\, ds \le 2\sigma \tag{5.9}$$

holds for u_n from Step 2, where $\sigma = d(1 + \frac{\gamma}{c})$ does not depend on $n \in \mathbb{N}$.

Step 4: We claim additionally that for u_n from Step 2 we have the equality

$$\lim_{\mathrm{mes}\,(D)\to 0} \sup_n \langle F_n u_n, P_D z\rangle = 0 \tag{5.10}$$

for each $z \in Z$. In the presence of (5.9) and (5.5) via CM-selections, this can be verified directly as in the single-valued case [4].

Step 5: From (5.10) via the Dunford-Pettis type $\sigma(Z', Z)$-weak precompactness and $\sigma(Z', Z)$-weak completeness theorems in Z' (see, e.g., [9, 16]) it follows that there exist some subsequence n_k and some $v_* \in Z'$ such that $\langle F_{n_k} u_{n_k}, z\rangle \to \langle v_*, z\rangle$ for each $z \in Z$. Further, since L^{-1} acts continuously from H^{-1} into H_0^1 and $H_0^1 \subset Z \subset Z' \subset H^{-1}$ continuously (see (4.2)), L^{-1} acts continuously from Z' into Z. So the dual operator $(L^{-1})^*$ acts continuously from Z^* into $(Z')^*$. Remark that $Z' \subset Z^*$ continuously and $(Z')^* = (Z')' = Z$, since Z is a perfect space and $Z' = (Z')^0$ (see [17]) by our choice of Z in (4.2). Therefore, $(L^{-1})^*$ acts continuously from Z' into Z, and so $\langle L^{-1}v, z\rangle = \langle v, (L^{-1})^* z\rangle$ for all $v, z \in Z'$. Consequently,

$$\langle L^{-1}(F_{n_k} u_{n_k}), z\rangle = \langle F_{n_k} u_{n_k}, (L^{-1})^* z\rangle \to \langle v_*, (L^{-1})^* z\rangle = \langle L^{-1} v_*, z\rangle$$

for each $z \in Z'$, as $k \to +\infty$. So $L^{-1}(F_{n_k} u_{n_k})$ converges in the weak topology $\sigma(Z, Z')$ to $L^{-1} v_*$.

Step 6: From (5.8) and the Rellich-Kondrashov theorem (see [18]) we get that $\{u_n\}_n$ is precompact in L_2 and in measure. Therefore we may choose a subsequence of $\{n_k\}_k$, which we shall for simplicity denote again by $\{n_k\}$, such that u_{n_k} converges in L_2 to u_* and $u_{n_k}(s) \to u_*(s)$ for almost all $s \in \Omega$, for some measurable function $u_* \in L_2$. From (5.8) we get that $u_{n_k}, u_* \in Z$. From our choice of n_k, we get also that (see Step 5) $\langle F_{n_k} u_{n_k}, z\rangle \to \langle v_*, z\rangle$ $(k \to +\infty)$ for each $z \in Z = (Z')^*$, i.e. $F_{n_k} u_{n_k} \to v_*$ in the weak topology $\sigma(Z', (Z')^*)$. Invoking, e.g., [1: Theorem 8] about sequential strong-weak continuous dependence, we get $v_* \in N_F(u_*)$. We draw attention of the reader to the fact that for to apply the theorem it is crucial that in definition (5.3) of the functions f_n there is involved property (5.1) of CM-selectors. Let us remark here that in the single-valued case of $F(\cdot,\cdot)$ one can get by the Nemytskij theorem (see [4, 15, 17]) also the convergence in measure of $F_{n_k} u_{n_k}$ to $N_F(u_*)$ (and so one can get immediately $v_* = N_F(u_*)$); in the multi-valued case this is not true.

Step 7: From Step 2 we get $u_{n_k} = L^{-1}(F_{n_k} u_{n_k})$. From the results of Steps 5 and 6 we then obtain $u_* = L^{-1} v_*$ (since by the well-known Hahn-Saks-Vitali theorem[9] the limit in measure and the $\sigma(Z, Z')$-weak limit coincide in Z) and $v_* \in N_F(u_*)$. Therefore $u_* \in L^{-1} N_F(u_*)$, and so $L u_* \in N_F(u_*)$ ∎

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References

[1] Amrani, A., Castaing, C. and M. Valadier: *Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1.* Arch. Rat. Mech. Anal. 117 (1992), 167 – 191.

[2] Appell, J., De Pascale, E., Nguyêñ, H. T. and P. P. Zabrejko: *Multivalued Superpositions* (Diss. Math.: Vol. 345). Warsaw: Polish Acad. Sci. 1995, pp. 1 – 97.

[3] Appell, J., De Pascale, E., Nguyêñ, H. T. and P. P. Zabrejko: *Nonlinear Integral Inclusions of Hammerstein type.* Top. Methods Nonlin. Anal. 5 (1995), 109 –122.

[4] Appell, J., Nguyêñ, H. T. and P. P. Zabrejko: *General existence theorems for quasilinear elliptic systems without monotonicity.* J. Math. Anal. Appl. 145 (1990), 26 – 38.

[5] Aubin, J.-P. and A. Cellina: *Differential Inclusions. Set Valued Maps and Viability Theory.* Berlin et al.: Springer-Verlag 1984.

[6] Bressan, A. and G. Colombo: *Extensions and selections of maps with decomposable values.* Studia Math. 90 (1988), 69 – 86.

[7] Cellina, A.: *Approximation of set-valued functions and fixed point theorems.* Ann. Mat. Pura Appl. 82 (1969), 17 – 24.

[8] Cellina, A. and A. Lasota: *A new approach to the definition of topological degree for multi-valued mappings.* Atti Accad. Naz. Lincei 47 (1969), 434 – 440.

[9] Dunford, N. and J. Schwartz: *Linear Operators.* Vol. I: *General Theory.* New York: Intersci. Publ. 1958.

[10] Hu, Sch. Ch. and N. S. Papageorgiou: *Generalizations of Browder's degree theory.* Trans. Amer. Math. Soc. 347 (1995), 233 – 259.

[11] Hu, Sch. Ch. and N. S. Papageorgiou: *Handbook of Multivalued Analysis.* Vol. I: *Theory.* Dordrecht: Kluwer Acad. Publ. 1997.

[12] Kuratowski, K.: *Topology.* Vol. I and II. New York: Acad. Press 1966 and 1968.

[13] Michael, E. A.: *Continuous Selections.* Part I. Ann. Math. 63 (1956), 361 – 381.

[14] Nguyêñ, H. T.: *Equations and inclusions with strong nonlinearities.* In: Abstracts of Invited Lectures on V-th Vietnamese Math. Conf. (Hanoi, 17 - 20 September, 1997). Hanoi: Viet. Acad. Sci. 1997, p. 88.

[15] Nguyêñ, H. T.: *Existence theorems for boundary value problems for strongly nonlinear elliptic systems.* Z. Anal. Anw. 18 (1999), 585 – 610.

[16] Nguyen, H. T., Juniewicz, M. and J. Zieminska: *CM-selectors for pairs of oppositely semicontinuous multi-valued maps with L_p-decomposable values.* Studia Math. (accepted).

[17] Rao, M. M. and Z. D. Ren: *Theory of Orlicz Spaces.* New York: Marcel Dekker 1991.

[18] Zeidler, E.: *Nonlinear Functional Analysis and its Applications.* Parts 1 - 4. Berlin et al.: Springer-Verlag 1986 - 1998.

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