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Some New Conformal Covariants

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Abstract. By means of a certain conformal covariant differentiation process explicit formulae are derived for

(i) a conformally invariant generalized Bach tensor in dimension 6
(ii) conformally invariant differential operators acting on weighted functions, especially one with leading term $\square^4$
(iii) conformal covariants on symmetric, trace-free p-tensor bundles, especially one with leading term $\square^2$
(iv) conformal covariants on differential forms.

Furthermore, theorems for uniqueness, existence and non-existence of conformal covariants, in particular in dimension 4, are given.

Keywords: *Conformal transformation, conformally invariant tensor, conformally invariant differential operator, Bach tensor, conformal covariant derivative*

AMS subject classification: 53 A 30, 58 G 99

1. Introduction

The theory of conformal transformations of a pseudo-Riemannian manifold (M, g) plays a fundamental role in questions of geometry and physics. The investigation of conformally invariant differential operators and tensors on a conformal manifold is an active area of research (see, e.g., [2 - 7, 11, 12, 14, 16, 18 - 23, 25, 27, 29, 31, 34, 36, 37, 40, 41]). More generally, by a conformal covariant, we shall mean a universal polynomial expression in covariant derivatives with coefficients depending polynomially on the metric, its inverse, the curvature tensor, and its covariant derivatives which acts between conformally weighted tensor bundles and is unchanged when the metric is scaled. The Bach tensor in dimension 4 [1, 25, 37] and the conformally invariant Laplacian [43] are the most basic non- trivial examples. The polynomial conformal covariants have a great variety of applications, for example, in the representation theory of the conformal group (see, e.g., [9, 31]), in the "conformal extension" of the heat equation [10], in spectral theory [8, 9], for Lagrangian formulation of both general relativity and conformal field theories [1, 2, 16, 18, 40, 41], in describing massless fields and wave propagation in curved space-times [2, 13, 25, 30, 32, 39, 42]. It is an important problem to give a survey of all conformal covariants or, with less pretension, to give a method for generating special classes of conformal covariants. Some of these procedures and conformal covariants are well known (see, e.g., [2, 5, 16, 19, 22, 40]). The existence of a family of

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conformal covariants on weighted functions with the leading part being a power of the Laplacian is proved in [23].

In this paper we present a relatively simple derivation of explicit formulae for some conformal covariants using a method given in [38], in particular, the notation " *conformal covariant derivative*" is fundamental. Most but not all invariant operators in flat space admit a curved generalization.The basic reason for the non-existence in dimension 4 is an obstructing Bach tensor expression. For example, a curved version of $\Box^3$ in dimension $n \geq 5$ was found in [38], but there is no curved version of $\Box^3$ in dimension 4 (see [24]). By means of our method we simplify essentially the proof of this result and give further examples for the non-existence of curved analogues of flat operators.

The paper is organized as follows. In Section 2, the basic ideas and results of [27, 38] are presented. In Section 3 we give an extension of the conformally invariant Bach tensor to dimension 6, some remarks on Euler-Lagrange tensors and state two unsolved problems. In Section 4 we investigate the general structure of conformal covariants with leading terms $\Box^k$ acting on weighted functions by means of which one can derive explicit formulae for these operators without excessive calculation. In particular, a curved version of $\Box^4$, two second-order conformal covariants and a proof of the non-existence of a curved analogue of $\Box^4$ in dimension 6 are given. Furthermore, a generating system for the set of all conformal covariants with leading term $\Box^3$ is introduced. Conformal covariants acting on symmetric, trace-free p-tensors bundles are generated in Section 5. In particular, we derive a fourth-order operator $\overset{s}{D}_{(4,p)}$ with leading term $\Box^2$. For this operator the obstructing Bach tensor expression occuring in dimension 4 can be substituted by means of a suitable second-order operator. Finally, in Section 6 we consider conformal covariants acting on differential forms. A second-order and fourth-order conformal covariant with leading terms $\Box$ and $\Box^2$, respectively, was found by Branson [5] and the present author [40]. We present a new second-order conformal covariant and prove that there is no conformal covariant on 2-forms with leading term $\Box^2$ in dimension 4.

2. Conformal covariants

Let (M, g) be a pseudo-Riemannian C^∞ manifold of dimension n ($n \geq 4$) and

g_{ab}	the local components of the covariant metric tensor
g^{ab}	the local components of the contravariant metric tensor
∇_a	the Levi Civita connection
R_{abcd}	the curvature tensor
$R_{ab} := R_{a\cdot cb}^{\ c}$	the Ricci tensor
R	the scalar curvature tensor
C_{abcd}	the Weyl curvature tensor.

In the following we use the sign convention according to

$$\nabla_{[a}\nabla_{b]}T_c = -\frac{1}{2}R_{abc.}^{\ \ \ \ d}T_d \quad . \tag{2.1}$$

We consider *linear differential operators* acting on a space $\mathcal{T}$ of C^∞ tensor fields of a certain type with (possibly) some list of symmetry conditions. Suppose that the coefficients of these operators are polynomials in g_{ab} and the partial derivatives of g_{ab} up to a certain order.

Let $\mathcal{D}_k$ be the set of all polynomial operators of order less than k and $\mathcal{D} := \bigcup_k \mathcal{D}_k$. Then obviously $\mathcal{D}$ is an algebra [27, 40].

An operator $D(g) \in \mathcal{D}$ is said to be of *conformal weight* ω if under a uniform dilatation of the metric $\bar{g}_{ab} = e^{2\phi} g_{ab}$ with $\phi = \text{const}$, $D(g)$ transforms according to $D(g) = e^{2\omega\phi} D(g)$. Let $\mathcal{D}(\omega)$ be the set of all operators of conformal weight ω. Then $\mathcal{D} = \oplus_\omega \mathcal{D}(\omega)$.

Definition 2.1. An operator $D(g) \in \mathcal{D}(\omega)$ is said to be *conformally invariant* of conformal weight (ω, ω_0) on $\mathcal{T}$ (or shortly a *conformal covariant*) if a real number ω_o exists such that under a conformal change of the metric

$$\bar{g}_{ab} = e^{2\phi} g_{ab}, \qquad \phi \in C^\infty(M) \tag{2.2}$$

$D(g)$ transforms according to

$$D(\bar{g})[e^{2\omega_0\phi} u] = e^{2(\omega+\omega_0)\phi} D(g)[u] \tag{2.3}$$

for all $u \in \mathcal{T}$.

Remark 2.1. A zeroth-order conformal covariant is called a *conformally invariant tensor.*

Examples.

(i) The conformal Laplacian

$$D_{(2)} := \nabla^a \nabla_a - \frac{n-2}{4(n-1)} R \tag{2.4}$$

is a second-order conformal covariant on functions with $\omega = -1, \omega_0 = \frac{1}{4}(2-n)$ [43].

(ii) The Bach tensor

$$B_{i_1 i_2} := \nabla_a \nabla_b C^{a\ \ \ b}_{.i_1 i_2 .} + \frac{n-3}{n-2} C^{a\ \ \ b}_{.i_1 i_2 .} R_{ab} \tag{2.5}$$

is a conformally invariant tensor of weight $\omega = -1$ if $n = 4$ [1, 14, 25, 37, 39].

If $D(g) \in \mathcal{D}(\omega)$, then under the conformal change (2.2) $D(g)$ has a transformation law of the form

$$D(\bar{g})[e^{2\omega_0\phi} u] = e^{2(\omega+\omega_0)\phi} \left\{ D(g) + \sum_{k=1}^{m} P_k(\omega_0, g, \phi) \right\} [u] \qquad (u \in \mathcal{T}) \tag{2.6}$$

where the operators $P_k(\omega_0, g, \phi)$ are k-homogeneous in the derivatives of ϕ up to a certain order [27, 40]. $D(g)$ is conformally invariant if and only if the *conformal variation* $P_1(\omega_0, g, \phi)$ vanishes, i.e., for constructing conformal covariants it is sufficient to calculate only "to the first order in derivatives of ϕ" [27, 40].

Let $\vartheta(\omega)$ be the set of those elements of $\mathcal{D}(\omega)$ which contain only *first* order derivatives of ϕ in their transformation law. Furthermore, let $\vartheta := \oplus_\omega \vartheta(\omega)$ and ϑ_0 be the subset of zeroth-order operators (i.e. tensors) of ϑ. Then $D(g)$ is an element of ϑ if and only if $P_1(\omega_0, g, \Phi)$ has the form

$$P_1(\omega_0, g, \phi) := \nabla_\gamma \phi X^\gamma D. \tag{2.7}$$

Definition 2.2. The linear operator X^γ defined on ϑ by (2.7) is called the *infinitesimal generator* of D.

It is easy to show that X^γ is a derivation, i.e. X^γ obeys the Leibniz rule and commutes with contractions.

Corollary 2.1. *$D \in \mathcal{D}$ is a conformal covariant if and only if $D \in \vartheta$ and $X^\gamma D = 0$.*

If $D \in \vartheta$, then we have in general $\nabla_a D \notin \vartheta$.

Definition 2.3. For $D \in \vartheta$ the operator

$$\overset{c}{\nabla}_a D := \nabla_a D + P_{ak} X^k D, \tag{2.8}$$

where

$$P_{ab} := \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R g_{ab}\right) \tag{2.9}$$

is called the *conformal covariant derivative of D.*

The following theorem holds (see [27, 40]):

Therorem 2.1.

(i) *The conformal covariant derivative $\overset{c}{\nabla}_a$ a derivation.*

(ii) $\overset{c}{\nabla}_a : \vartheta \to \vartheta$.

(iii) *The "Ricci identity" for $\overset{c}{\nabla}_a$ has the form*

$$\overset{c}{\nabla}_{[a} \overset{c}{\nabla}_{b]} D(u) := (C, D[u])_{ab} + \frac{1}{2(n-3)} \nabla_u C_{ab\gamma .}{}^{u} X^\gamma D[u] \tag{2.10}$$

where $u \in \mathcal{T}$ and $(C, D[u])_{ab}$ is the term one obtains from the right-hand side of the usual Ricci identity $\nabla_{[a}\nabla_{b]} D(u) := (R, D[u])_{ab}$ by substitution of R by C.

(iv) *For $D \in \vartheta(\omega)$*

$$[X^\gamma, \overset{c}{\nabla}_a] D[u] := \{X^\gamma, \overset{c}{\nabla}_a - \overset{c}{\nabla}_a X^\gamma\} D[u] = 2(\omega + \omega_0)\delta^\gamma_a D[u] + \Delta^\gamma_a D[u] \tag{2.11}$$

is valid where

$$\begin{aligned}\Delta^\gamma_a (T_{ij\ldots}{}^{km\ldots}) = {} & \Delta^{\gamma k}_{ab} T_{ij\ldots}{}^{bm\ldots} + \Delta^{\gamma m}_{ab} T_{ij\ldots}{}^{kb\ldots} + \ldots \\ & - \Delta^{\gamma b}_{ai} T_{bj\ldots}{}^{km\ldots} - \Delta^{\gamma b}_{aj} T_{ib\ldots}{}^{km\ldots} - \ldots\end{aligned}$$

and

$$\Delta^{km}_{ai} := \delta^k_a \delta^m_i + \delta^k_i \delta^m_a - g_{ai} g^{km} \ .$$

(v) *The algebra ϑ_0 is generated by the tensors*

$$g^{ab}, \quad g_{ab}, \quad \overset{c}{\nabla}_{(i_1} \dots \overset{c}{\nabla}_{i_r} C^{a\phantom{.i_{r+1}i_{r+2})}b}_{.i_{r+1}i_{r+2}).} \ \ (r \in \mathbb{N}). \tag{2.12}$$

(vi) *Every operator $D \in \vartheta$ can be represented in the form*

$$D = \sum_{k=0}^{m} B^{(k)} \otimes \overset{c}{\nabla}^{(k)} \tag{2.13}$$

where $\overset{c}{\nabla}^{(k)}$ denotes $\{\overset{c}{\nabla}_{(i_1} \dots \overset{c}{\nabla}_{i_k)}\}$ and $B^{(k)} \in \vartheta_0$ $(k = 0, \dots, m)$.

Examples. We have (see [27, 40])

$$X^\gamma C_{abcd} = 0, \quad X^\gamma \overset{c}{\nabla}_a C^a_{.bcd} = (n-3) C^\gamma_{.bcd}, \quad \overset{c}{\nabla}_a C_{bdef} = \nabla_a C_{bdef} \tag{2.14}$$

$$\overset{c}{\nabla}_a u = \nabla_a u, \quad B_{i_1 i_2} = \overset{c}{\nabla}_a \overset{c}{\nabla}_b C^{ab}_{.i_1 i_2 .}, \quad X^\gamma B_{i_1 i_2} = 2(n-4) \nabla_a C^{a\gamma}_{.i_1 i_2 .} \tag{2.15}$$

$$X^\gamma \overset{c}{\nabla}_a u = 2\omega_0 \delta^\gamma_a u + \Delta^\gamma_a u, \quad \overset{c}{\nabla}_a \overset{c}{\nabla}_b u = \nabla_a \nabla_b u + P_{a\gamma} X^\gamma \overset{c}{\nabla}_b u \tag{2.16}$$

$$\overset{c}{\Box} := \overset{c}{\nabla}^a \overset{c}{\nabla}_a = \nabla^a \nabla_a + \frac{\omega_0}{n-1} R \quad \text{on } \omega_0\text{-weighted functions} \tag{2.17}$$

$$X^\gamma \overset{c}{\Box} = (4\omega_0 + n - 2) \overset{c}{\nabla}^\gamma = 0 \quad \Longleftrightarrow \quad 4\omega_0 = 2 - n. \tag{2.18}$$

3. On the generalization of the Bach tensor

When Latin indices with subindices (e.g. $i_1, \dots, i_r$) appear in the sequel, we assume that symmetrization has been carried out over the indices. We denote the trace-free symmetric part of a tensor T by $TS(T)$. Let $\mathcal{S}_r(\omega, n)$ be the set of all conformally invariant, symmetric, trace-free tensors with weight ω and covariant rank r. There is a natural generalization $B^{(k)}_{i_1 i_2} \in \mathcal{S}_2(\omega, n)$ of the Bach tensor $B^{(0)}_{i_1 i_2} := B_{i_1 i_2}$ in dimension four to any even dimension n with $\omega = \frac{2-n}{2} = -(k+1), k \in \mathbb{N}_0$ (see [3,19]). The tensor $B^{(k)}_{i_1 i_2}$ has only the *linear* leading term $\overset{c}{\Box}{}^{(k)} B_{i_1 i_2}$ (see [28]). Because of Theorem 2.1/(v) the Weyl curvature correction terms

$$T^{(k)}_{i_1 i_2} := B^{(k)}_{i_1 i_2} - \overset{c}{\Box}{}^{(k)} B_{i_1 i_2}$$

are generated by the tensors (2.12). Obviously, we cannot expext the uniqueness of the extension of $\overset{c}{\Box}{}^{(k)} B_{i_1 i_2}$ to a conformally invariant tensor for $k > 0$ (in the case of $k = 0$ see [37]).

Problem A. Find correction terms $T^{(k)}_{i_1i_2}$ for $k>0$.

Theorem 3.1. *The tensor*

$$\begin{aligned}B^{(1)}_{i_1i_2} :=& \overset{c}{\Box} B_{i_1i_2}\\ &+\left\{\frac{2(n-4)}{(n-3)}\nabla_a C^{a\ \ bc}_{.i_1..}\nabla_d C^{d}_{.i_2bc} - \nabla_a C^{abc}_{...i_1}\nabla_d C^{d}_{.bci_2}\right\}\\ &+2C_{i_1..i_2}^{\ \ ab}B_{ab}\end{aligned} \tag{3.1}$$

is an element of $\mathcal{S}_2(-2,6)$.

Proof. The Ricci identity (see (2.10))

$$\nabla_{[a}\nabla_{b]}T_c := -\frac{1}{2}C_{abc.}^{\ \ \ \ d}T_d - \frac{1}{2(n-3)}\nabla_u C^u_{.\gamma ab}X^\gamma T_c \tag{3.2}$$

and the Bianchi identity

$$\nabla_{[a}C_{bc]ij} + \frac{1}{n-3}\left[g_{j[a}\nabla_{|u}C^u_{.i|bc]} - g_{i[a}\nabla_{|u}C^u_{.j|bc]}\right] = 0 \tag{3.3}$$

imply

$$\begin{aligned}(n-3)\left[\overset{c}{\Box} C_{baij} + 2\overset{c}{\nabla}_{[a|}\overset{c}{\nabla}_u C^u_{.|b]ij} - C_{udba}C^{ud}_{.\ .ij} - 4C_{ub[i|.}^{\ \ \ \ \ d}C^u_{.a|j]d}\right]&\\ +2\overset{c}{\nabla}_{[j|}\overset{c}{\nabla}_u C^u_{.|i]ba} + 4g_{[i[a}B_{j]b]} = 0&\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\overset{c}{\nabla}_u\overset{c}{\Box} C^{u\ \ \ b}_{.i_1i_2.} =& \overset{c}{\Box}\overset{c}{\nabla}_u C^{u\ \ \ b}_{.i_1i_2.} + 2\nabla^a C_{aui_1}^{\ \ \ \ k}C^{u\ \ \ \ \ b}_{.(ki_2).}\\ &+\nabla^a C_{au..}^{\ \ \ \ bk}C^u_{.i_1i_2k} + 4C_{aui_1.}^{\ \ \ \ \ k}\nabla^a C^{u\ \ \ \ \ b}_{.(ki_2).}\\ &+2C_{au..}^{\ \ \ \ bk}\nabla^a C^u_{.i_1i_2k} + \frac{1}{n-3}\nabla_l C^l_{.kau}X^k(\nabla^a C^{u\ \ \ b}_{.i_1i_2.}).\end{aligned} \tag{3.5}$$

Hence, on account of (2.11) and (2.15),

$$\begin{aligned}X^\gamma(\overset{c}{\Box} B_{i_1i_2}) &= [X^\gamma,\overset{c}{\nabla}_a]\overset{c}{\nabla}^a B_{i_1i_2} + \overset{c}{\nabla}^a[X^\gamma,\overset{c}{\nabla}_a]B_{i_1i_2}\\ &= (n-10)\overset{c}{\nabla}^\gamma B_{i_1i_2} + 4\overset{c}{\nabla}_{i_1}B_{i_2.}^{\ \ \gamma} + \overset{c}{\Box} X^\gamma B_{i_1i_2}\\ &= (n-6)[3\overset{c}{\nabla}^\gamma B_{i_1i_2} - 2\overset{c}{\nabla}_{i_1}B_{i_2}^{\ \ \gamma}] - X^\gamma(T^{(1)}_{i_1i_2}).\end{aligned}$$

Consequently,

$$X^\gamma(B^{(1)}_{i_1i_2}) = (n-6)[3\overset{c}{\nabla}^\gamma B_{i_1i_2} - 2\overset{c}{\nabla}_{i_1}B_{i_2}^{\ \ \gamma}]. \tag{3.6}$$

Remark 3.1. In dimension 6, the conformal tensor $B^{(1)}_{i_1i_2}$ has been already derived in [28]. A generating system for $\mathcal{S}_2(-2,n)$ is also given in [28, 39]. In [39] one can find applications of the tensor $B^{(1)}_{i_1 i_2}$ to the theory of Huygens' principle in dimension 6 (see also [36]).

Supposing a Lagrangian L to be an element of $\mathcal{S}_0(-\frac{n}{2}, n)$, by the action

$$\delta \int L\sqrt{|\det(g_{ab})|}dx =: \int E^{ab}(L)\delta g_{ab}\sqrt{|\det(g_{ab})|}\, dx = 0 \tag{3.7}$$

an Euler-Lagrange tensor $E_{ab}(L)$ is defined, which has the properties [15,39]

$$E_{ab}(L) \in \mathcal{S}_0\Big(-\frac{n}{2}+1, n\Big), \quad E_{[ab]}(L) = 0, \quad g^{ab}E_{ab}(L) = 0, \quad \nabla^a E_{ab}(L) = 0. \tag{3.8}$$

Example. $n = 4, E_{ab}(C_{ijkl}C^{ijkl}) = cB_{ab} \quad (c \in \mathbb{R} \setminus \{0\})$ (see, e.g., [39]).

A conformal tensor T is called *trivial* if it is generated by $\{g_{ab}, g^{ab}, C_{abcd}\}$.

In [28, 39] a generating system of $\mathcal{S}_0(-3, n)$ was found. The only *non-trivial* conformal tensors from $\mathcal{S}_0(-3, n)$ are multiples of

$$\begin{aligned} S_0(-3,n) := \frac{10-n}{2}&\left[\nabla_u C_{abcd}\nabla^u C^{abcd} - \frac{4(n-2)}{(n-3)^2}\nabla_u C^{uabc}_{\cdots\cdot}\nabla_v C^{v}_{.abc}\right] \\ &- 2\left(\Box - \frac{2}{n-1}R\right)C_{abcd}C^{abcd}. \end{aligned} \tag{3.9}$$

Independently, the conformal invariance of $S_0(-3, n)$ was verified also in [19]. The tensor $B^{(1)}_{i_1 i_2}$ given by (3.1) is not an Euler-Lagrange tensor, however in six dimensions an Euler-Lagrange tensor $E_{ab}(S_0(-3, n))$ is a linear combination of $B^{(1)}_{i_1 i_2}$ and further conformal tensors of $\mathcal{S}_0(-2, 6)$ (see [39]).

Problem B. Find non-trivial conformal tensors of $\mathcal{S}_0(\omega, n)$ if $\omega < -3$.

Conjecture A. *For every even n $(n \geq 4)$ there is an conformal Euler-Lagrange tensor $W^{(k)}_{i_1 i_2} \in \mathcal{S}_0(-\frac{n}{2}+1, n)$ $(k = \frac{n-4}{2} \in \mathbb{N}_0)$ with the linear leading term $\overset{c}{\Box}{}^k B_{i_1 i_2}$.*

4. Conformal covariants acting on weighted functions

Let $\mathcal{T}_0$ be the set of all C^∞ scalar ω_0-weighted functions and $\Box = \nabla^a\nabla_a, \overset{c}{\Box} = \overset{c}{\nabla}{}^a\overset{c}{\nabla}_a$. The following theorem was proved in [23].

Theorem 4.1. *If n is odd, then there is for each $k \geq 1$ a conformal covariant $D_{(2k)}$ of order $(2k)$ with $\omega = -k, 4\omega_0 = 2k - n$ and leading term $\Box^k$. If n is even, then the same result is true with the restriction $1 \leq k \leq \frac{n}{2}$.*

Using the notions and results of Section 2 here we analyze the general structure of $D_{(2k)}$ and present an algorithm for an explicit construction of $D_{(2k)}$, which simplifies those given in [23]. Relation (2.11) implies

$$\begin{aligned} X^\gamma \overset{c}{\Box}{}^k &= [X^\gamma, \overset{c}{\nabla}_a]\overset{c}{\nabla}{}^a\overset{c}{\Box}{}^{k-1} + \overset{c}{\nabla}{}^a[X^\gamma, \overset{c}{\nabla}_a]\overset{c}{\Box}{}^{k-1} \\ &= b_k\overset{c}{\nabla}{}^\gamma\overset{c}{\Box}{}^{k-1} + \overset{c}{\Box}\, X^\gamma\overset{c}{\Box}{}^{k-1} \end{aligned}$$

where $b_k := 4\omega_0 - 4k + n + 2$. Hence, by induction,

$$\begin{aligned} X^\gamma \overset{c}{\Box}{}^k &= b_k \overset{c}{\nabla}{}^\gamma \overset{c}{\Box}{}^{k-1} + b_{k-1} \overset{c}{\Box} \overset{c}{\nabla}{}^\gamma \overset{c}{\Box}{}^{k-2} + \ldots + b_1 \overset{c}{\Box}{}^{k-1} \overset{c}{\nabla}{}^\gamma \\ &= a_k \overset{c}{\nabla}{}^\gamma \overset{c}{\Box}{}^{k-1} + a_{k-1} [\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\Box}{}^{k-2} \\ &\quad + a_{k-2} \overset{c}{\Box} [\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\Box}{}^{k-3} + \ldots + a_1 \overset{c}{\Box}{}^{k-2} [\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \end{aligned} \tag{4.1}$$

where $[\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] := \overset{c}{\Box} \overset{c}{\nabla}{}^\gamma - \overset{c}{\nabla}{}^\gamma \overset{c}{\Box}$ and

$$a_m = \sum_{r=1}^{m} b_r = m(4\omega_0 + n - 2m) \qquad (m = 1, \ldots, k). \tag{4.2}$$

We obtain by means of (2.10)

$$[\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\Box}{}^m = \frac{1}{n-3} \big[B^\gamma_{.a} - 2(\overset{c}{\nabla}_u C^{u.b\gamma}_{.a}) \overset{c}{\nabla}_b \big] X^a \overset{c}{\Box}{}^m =: \frac{1}{n-3} F^\gamma_{.a} X^a \overset{c}{\Box}{}^m \tag{4.3}$$

if $m \geq 1$ and $[\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\Box}{}^m = 0$ if $m = 0$.

Choosing a flat metric as a test metric we get from (4.1)

Lemma 4.1. *Necessary for $D_{(2k)}$ to be a conformal covariant is*

$$a_k = k(4\omega_0 + n - 2k) = 0. \tag{4.4}$$

In the case $4\omega_0 = 2k - n$ (see Lemma 4.1) one has

$$a_m = 2m(k - m) \qquad (m = 1, \ldots, k). \tag{4.5}$$

Thus we have proved

Lemma 4.2. *For every $k \geq 1$ and $4\omega_0 = 2k - n$ there holds*

$$X^\gamma \overset{c}{\Box}{}^k = \begin{cases} 2 \sum_{m=2}^{k-1} m(k-m) \overset{c}{\Box}{}^{k-m-1} [\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\Box}{}^{m-1} & \text{if } k \geq 3 \\ 0 & \text{if } k \leq 2. \end{cases} \tag{4.6}$$

Now from (2.8), Theorem 2.1 and $\overset{c}{\nabla}_a B^a_{.b} = 0$ we imply immediately

Lemma 4.3. *The coefficients of all Weyl curvature correction terms of $D_{(2k)} - \Box^k$ which are linear with respect to the conformal covariant derivatives of the Weyl tensor are up to real numbers conformal covariant derivatives of the Bach tensor of order p with $0 \leq p \leq k - 2$.*

Conjecture B. *For every n and k with $2k \leq n$ and $n \geq 4$ there is a conformal covariant $D_{(2k)}$, which reduces to $\Box^k$ for an Einstein metric.*

Examples.

(i) The operator

$$D_{(2)} \equiv \overset{c}{\Box} = \Box - \frac{n-2}{4(n-1)} R \tag{4.7}$$

is the only second order conformal covariant on $\mathcal{T}_0$ with $\omega = -1, \omega_0 = 2-n$ and leading term $\Box$ (see, e.g., [40, 43]).

(ii) If $k = 2$, then we get $X^\gamma \overset{c}{\Box}{}^2 = 0$ from Lemma 4.2. Consequently, a fourth-order conformal covariant on $\mathcal{T}_0$ with $\omega = -2, 4\omega_0 = 4-n$ and leading term $\Box^2$ has the form

$$D_{(4)} := \overset{c}{\Box}{}^2 + c_0 C_{abcd}C^{abcd} \qquad (c_0 = \text{const}). \tag{4.8}$$

The operator $\overset{c}{\Box}{}^2$ is exactly that conformal covariant which S. Paneitz introduced in 1983 [33]. The proof of the uniqueness is very easy (see, e.g., [40]).

(iii) If $k = 3$, then because of

$$\left.\begin{aligned} X^\gamma \overset{c}{\Box} &= (4\omega_0 + n - 2)\overset{c}{\nabla}{}^\gamma = 4\overset{c}{\nabla}{}^\gamma \\ X^\gamma TS(\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2}) &= (4-n)TS(\delta^\gamma_{i_1}\overset{c}{\nabla}_{i_2}) \end{aligned}\right\} \tag{4.9}$$

we have

$$\begin{aligned} X^\gamma(\overset{c}{\Box}{}^3) &= 4[\overset{c}{\Box}, \overset{c}{\nabla}{}^\gamma]\overset{c}{\Box} \\ &= \frac{16}{n-3}\big[B^{\gamma a} - 2\overset{c}{\nabla}_u C^{uab\gamma}\overset{c}{\nabla}_b\big]\overset{c}{\nabla}_a \\ &= \frac{16}{n-3}\Big[B^{\gamma a} - \frac{1}{n-4}X^\gamma(B^{ab})\overset{c}{\nabla}_b\Big]\overset{c}{\nabla}_a \\ &= -\frac{16}{(n-3)(n-4)}X^\gamma(B^{ab}\overset{c}{\nabla}_b\overset{c}{\nabla}_a). \end{aligned} \tag{4.10}$$

Hence, the sixth-order operator found by the present author in [40]

$$D^0_{(6)} := \overset{c}{\Box}{}^3 + \frac{16}{(n-3)(n-4)}B^{ab}\overset{c}{\nabla}_a\overset{c}{\nabla}_b \tag{4.11}$$

is for $n > 4$ a conformal covariant on $\mathcal{T}_0$ with $\omega = -3, 4\omega_0 = 6-n$ and leading term $\Box^3$. If $n = 4$, then $B^{ab}\nabla_a\nabla_b$ is a second-order conformal covariant on $\mathcal{T}_0$. On account of Lemma 4.3 it is easy to see that $cB^{ab}\nabla_a\nabla_b$ with $c = \text{const}$ is the only possible curvature correction term of $D^0_{(6)}$ with a *linear* coefficient. As a conclusion, there is no operator $D_{(6)}$ such that

$$X^\gamma(D_{(6)} - \overset{c}{\Box}{}^3) = -16\big[B^{\gamma a} - 2\overset{c}{\nabla}_u C^{uab\gamma}\overset{c}{\nabla}_b\big]\overset{c}{\nabla}_a$$

in dimension 4. Thus, by means of the method given in Section 2, we obtain a simple proof of the following theorem due to Graham [24].

Theorem 4.1. *If $n = 4$, then there is no conformal covariant on $\mathcal{T}_0$ with the leading term $\Box^3$.*

Theorem 4.2. *The following operators are second-order conformal covariants on $\mathcal{T}_0$ with $\omega = -3$ and $4\omega_0 = 6-n$ $(n \geq 4)$:*

$$\begin{aligned} D^1_{(2)} = \overset{c}{\nabla}_a &\left[\left\{C^a_{.def}C^{bdef} - \frac{1}{4}g^{ab}C^{gdef}C_{gdef}\right\}\overset{c}{\nabla}_b\right] \\ &+ \frac{(n-4)(n-6)}{4(n-3)^2}\nabla_a C^{adef}\nabla_b C^b_{.def} \end{aligned} \tag{4.12}$$

$$D^2_{(2)} = 2(n-10)\overset{c}{\nabla}_a[C^{bdef}C_{bdef}\overset{c}{\nabla}{}^a] + (n-6)\overset{c}{\Box}[C_{abde}C^{abde}]. \tag{4.13}$$

Proof. From (2.11) and (4.9), it follows after a straight computation

$$\begin{aligned}
X^\gamma \left\{ \overset{c}{\nabla}_a \left[\left\{ C^a_{.def} C^{bdef} - \frac{1}{4} g^{ab} C^{gdef} C_{gdef} \right\} \overset{c}{\nabla}_b \right] \right\} \\
&= -\frac{(n-4)(n-6)}{2(n-3)} \nabla_a C^{adef} C^\gamma_{.def} \\
&= -\frac{(n-4)(n-6)}{2(n-3)^2} X^\gamma \big[\nabla_a C^a_{.def} \nabla_b C^{bdef} \big],
\end{aligned}$$

hence $X^\gamma D^1_{(2)} = 0$. The proof of $X^\gamma D^2_{(2)} = 0$ is analogous ∎

Corollary 4.1. *Every sixth-order conformal covariant on $\mathcal{T}_0$ with $\omega = -3, 4\omega_0 = 6 - n$ $(n > 4)$ and leading term $\square^3$ has the form*

$$D_{(6)} = D^0_{(6)} + c_1 D^1_{(2)} + c_2 D^2_{(2)} + S \tag{4.14}$$

with

$$S = d_1 S_0(-3, n) + d_2 C^{aebf} C_{acdf} C_{e..b}^{\ cd} + d_3 C^{ab}_{..cd} C^{cd}_{..ef} C^{ef}_{..ab} \tag{4.15}$$

were $c_1, c_2 \in \mathbb{R}$ and $d_1, d_2, d_3 \in \mathbb{R}$.

Proof. It is easy to see that the operator $D^0_{(6)} - D_{(6)}$ can only contain the (linear independent) curvature monomials from $D^1_{(2)}, D^2_{(2)}$ and S. Now, from Lemma 4.3, Theorem 4.2 and the fact that $\mathcal{S}_0(-3, n)$ is generated by the monomials of (4.15) (see [18, 28, 39]) the assertion follows ∎

Theorem 4.3. *The following operator is for $n \geq 5, n \neq 6$ a conformal covariant on $\mathcal{T}_0$ with $\omega = -4, 4\omega_0 = 8 - n$ and leading term $\square^4$:*

$$\begin{aligned}
D_{(8)} :=\ & \overset{c}{\square}{}^4 \\
&+ \frac{96}{(n-3)(n-4)} \big[B^{ab} \overset{c}{\nabla}_a \overset{c}{\nabla}_b \overset{c}{\square} + (\overset{c}{\nabla}{}^a B^{bd}) \overset{c}{\nabla}_a \overset{c}{\nabla}_b \overset{c}{\nabla}_d \big] \\
&+ \frac{48}{(n-3)(n-6)} (\overset{c}{\square} B^{ab}) \overset{c}{\nabla}_a \overset{c}{\nabla}_b \\
&- \frac{192}{(n-3)^2(n-6)} \big[\overset{c}{\nabla}_u C^{ueda}_{....} \overset{c}{\nabla}_\nu C^{\nu\ \ \ b}_{.ed.} - \overset{c}{\nabla}_u C^{uaed}_{....} \overset{c}{\nabla}_\nu C^{\nu b}_{..ed} \big] \overset{c}{\nabla}_a \overset{c}{\nabla}_b \\
&- \frac{96(n-8)}{(n-3)(n-4)(n-6)} C^{aedb}_{....} B_{ed} \overset{c}{\nabla}_a \overset{c}{\nabla}_b - \frac{96}{(n-3)^2} B^{bd} (\overset{c}{\nabla}_u C^{u\ \ \ a}_{.bd.}) \overset{c}{\nabla}_a \\
&- \frac{24(n-2)(n-8)}{(n-3)^2(n-4)^2} B^{ab} B_{ab}\ .
\end{aligned}$$

Proof. Supposing (4.4), i.e. $4\omega_0 = 8 - n$, from Lemma 4.2 and (4.3), (4.5) there follows

$$\begin{aligned}
X^\gamma \overset{c}{\square}{}^4 &= 8 \overset{c}{\square} [\overset{c}{\square}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\square} + 6 [\overset{c}{\square}, \overset{c}{\nabla}{}^\gamma] \overset{c}{\square}{}^2 \\
&= \frac{2}{n-3} \big[4 \overset{c}{\square} (F^\gamma_{.a} X^a \overset{c}{\square}) + 3 F^\gamma_{.a} X^a \overset{c}{\square}{}^2 \big] \\
&= \frac{48}{n-3} \big[\overset{c}{\square} (F^{\gamma a} \overset{c}{\nabla}_a) + F^{\gamma a} \overset{c}{\nabla}_a \overset{c}{\square} \big].
\end{aligned}$$

Now, we have to compute the action of X^γ on all Weyl curvature correction terms of $D_{(8)}$, using relation (2.11). For example, we obtain

$$\begin{aligned} &X^\gamma(B^{ab}\overset{c}{\nabla}_a\overset{c}{\nabla}_b\overset{c}{\Box}\,) \\ &\quad = X^\gamma(B^{ab})\overset{c}{\nabla}_a\overset{c}{\nabla}_b\overset{c}{\Box}\, + B^{ab}\Big[[X^\gamma,\overset{c}{\nabla}_a]\overset{c}{\nabla}_b\overset{c}{\Box}\, + \overset{c}{\nabla}_a[X^\gamma,\overset{c}{\nabla}_b]\overset{c}{\Box}\, + \overset{c}{\nabla}_a\overset{c}{\nabla}_b X^\gamma\overset{c}{\Box}\,\Big] \\ &\quad = 2(n-4)\nabla_u C^{u(ab)\gamma}\overset{c}{\nabla}_a\overset{c}{\nabla}_b\overset{c}{\Box}\, + (2-n)B^{\gamma a}\overset{c}{\nabla}_a\overset{c}{\Box}\, + 6B^{ab}\overset{c}{\nabla}_a\overset{c}{\nabla}_b\overset{c}{\nabla}^\gamma. \end{aligned}$$

Substituting $\overset{c}{\Box}\,B_{ab}$ by $B^{(1)}_{ab}$ (see (3.1)) and using (3.6), one finds

$$X^\gamma(B^{(1)}_{ab}\overset{c}{\nabla}^a\overset{c}{\nabla}^b) = (n-6)\big[3\overset{c}{\nabla}^\gamma B^{ab} - 2\nabla^{(a}B^{b)\gamma}\big]\overset{c}{\nabla}_a\overset{c}{\nabla}_b + (6-n)B^{(1)\gamma}_a\overset{c}{\nabla}^a.$$

In an analogous manner one computes the other actions. Using successively identity (2.10), for example

$$\begin{aligned} &TS\big[\overset{c}{\Box}\,\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2} - \overset{c}{\nabla}_{i_1}\overset{c}{\Box}\,\overset{c}{\nabla}_{i_2}\big] \\ &\qquad = \frac{2(5-n)}{n-3}\nabla^a C_{ai_1i_2.}{}^{k}\overset{c}{\nabla}_k - 2C^{a}_{.i_1i_2.}{}^{k}\overset{c}{\nabla}_a\overset{c}{\nabla}_k - \frac{n-8}{2(n-3)}B_{i_1i_2}\ , \end{aligned}$$

one verifies after length straight computations the assertion $X^\gamma D_{(8)} = 0$ ∎

Corollary 4.2.

(i) *If $n=6$, then*

$$B^{(1)}_{ab}\overset{c}{\nabla}^a\overset{c}{\nabla}^b \tag{4.15}$$

is a second-order conformal covariant on $\mathcal{T}_0$ with $\omega=-4$ and $\omega_0=\frac{1}{2}$ (see (3.1)).

(ii) *If $n=6$, then there is no conformal covariant $D_{(8)}$ on $\mathcal{T}_0$ with leading term $\Box^4$.*

Proof. Assertion (i) follows multiplying $D_{(8)}$ by $(n-6)$, using (3.1). In order to prove assertion (ii) we remember that by Lemma 4.3 the linear coefficients of the correction terms of any operator $D^*_{(8)}$ are conformal covariant derivatives up to second order of the Bach tensor. It is easy to see that the operator $D_{(8)}$ of Theorem 4.3 contains all possible conformal covariant derivatives of B_{ab} (including the zeroth order derivatives). The real coefficients of all monomials of $D_{(8)}$ are determined uniquely. In particular, the term

$$48[(n-3)(n-6)]^{-1}(\overset{c}{\Box}\,B^{ab})\overset{c}{\nabla}_a\overset{c}{\nabla}_b$$

cannot be substituted by other "linear" terms. Consequently, we obtain assertion (ii) ∎

Conjecture C. *There is no conformal covariant on $\mathcal{T}_0$ with leading term $\Box^k$ in even dimension n with $n<2k$ and $k\geq 2$.*

5. Conformal covariants acting on symmetric, trace-free tensor fields

Let $\overset{s}{\mathcal{T}}_p$ be the space of all symmetric, trace-free C^∞-tensor fields of covariant rank p. In [40] the following theorem was proved.

Theorem 5.1. *The operator $\overset{s}{D}_{(2,p)}$ defined on $\overset{s}{\mathcal{T}}_p$ by*

$$\begin{aligned}(\overset{s}{D}_{(2,p)}[u])_{i_1\ldots i_p} := \overset{c}{\Box}\, u_{i_1\ldots i_p} &- \frac{4p}{n+2p-2}\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^k u_{ki_2\ldots i_p}\\ &+ \frac{4p(p-1)}{(n+2p-2)(n+2p-4)} g_{i_1i_2}\overset{c}{\nabla}{}^k\overset{c}{\nabla}{}^m u_{kmi_3\ldots i_p}\end{aligned} \tag{5.1}$$

is a second-order conformal covariant with $\omega = -1$ and $4\omega_0 = 2+2p-n$ $(n \geq 4)$.

Remark 5.1. Explicitly, we have

$$\begin{aligned}\overset{c}{\Box}\, u_{i_1\ldots i_p} &= \left(\Box - \frac{n-2}{4(n-1)}R\right) u_{i_1\ldots i_p}\\ \overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^k u_{ki_1\ldots i_p} &= \nabla_{i_1}\nabla^k u_{ki_2\ldots i_p} + \frac{n+2p-p}{2}P^k_{i_1}u_{ki_2\ldots i_p} \quad .\end{aligned}$$

Theorem 5.2. *If $n+2p > 6$, the operator $\overset{s}{D}_{(4,p)}$ defined on $\overset{s}{\mathcal{T}}_p$ by*

$$\begin{aligned}&(\overset{s}{D}_{(4,p)}[u])_{i_1\ldots i_p}\\ &\quad := TS\Big[\overset{c}{\Box}{}^2 u_{i_1\ldots i_p} + a_1\overset{c}{\Box}\,\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^a u_{ai_2\ldots i_p} + a_2\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}{}^a\overset{c}{\nabla}{}^b u_{abi_3\ldots i_p}\\ &\qquad + C^{a}_{.i_1i_2.}{}^{b}\Big\{a_3\overset{c}{\Box}\, u_{abi_3\ldots i_p} + a_4\overset{c}{\nabla}_a\overset{c}{\nabla}{}^k u_{kbi_3\ldots i_p}\Big\} + a_5 C_{i_1\ldots}{}^{abk}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_k u_{abi_3\ldots i_p}\\ &\qquad + \overset{c}{\nabla}_u C^{uab}_{\ldots}\Big\{a_6\overset{c}{\nabla}_a u_{bi_2\ldots i_p} + a_7\overset{c}{\nabla}_b u_{ai_3\ldots i_p} + a_8\overset{c}{\nabla}_{i_2}u_{abi_3\ldots i_p}\Big\}\\ &\qquad + a_9\overset{c}{\nabla}_u C^{u}_{.i_1i_2.}{}^{a}\overset{c}{\nabla}{}^k u_{kai_3\ldots i_p} + a_{10}B^a_{i_1}u_{ai_2\ldots i_p}\Big]\end{aligned}$$

with

$$\left.\begin{aligned}a_1 &= \frac{-8p}{n+2p}\\ a_2 &= \frac{-16p(p-1)}{(n+2p)(n+2p-2)}\\ a_3 &= -\frac{1}{2}a_5 = \frac{-4p(p-1)}{n+2p-6}\end{aligned}\right\}$$

$$\left.\begin{aligned}
a_4 &= \frac{-8p(p-1)(n+2p-12)}{(n+2p)(n+2p-6)} \\
a_6 &= \frac{4p(n-2p-2)}{(n-3)(n+2p-6)} \\
a_7 &= \frac{a_8}{(1-p)} = \frac{-4p(n-4)}{(n-3)(n+2p-6)} \\
a_9 &= \frac{-4p(p-1)(n^2+2np-12n+24)}{(n-3)(n+2p)(n+2p-6)} \\
a_{10} &= \frac{2p(n-6)}{(n-3)(n+2p-6)}
\end{aligned}\right\}$$

is a fourth-order conformal covariant with $\omega = -2$ and $4\omega_0 = 4 + 2p - n$.

Proof. In the following the relation $T_1 \overset{*}{=} T_2$ means $TS(T_1 - T_2) = 0$. From (2.11) there follows (see also [35] and [40: p. 277])

$$\begin{aligned}
&X^\gamma\big(\overset{c}{\Box}{}^2 u_{i_1\dots i_p}\big)\\
&\quad\overset{*}{=}\big([X^\gamma,\overset{c}{\nabla}_a]\overset{c}{\nabla}{}^a + \overset{c}{\nabla}{}^a[X^\gamma,\overset{c}{\nabla}_a]\big)\overset{c}{\Box} + \overset{c}{\Box}\,\big([X^\gamma,\overset{c}{\nabla}_a]\overset{c}{\nabla}{}^a + \overset{c}{\nabla}{}^a[X^\gamma,\overset{c}{\nabla}_a]\big)\Big\}u_{i_1\dots i_p}\\
&\quad= 2(4\omega_0 - 2p + n - 4)\overset{c}{\nabla}{}^\gamma\overset{c}{\Box}\,u_{i_1\dots i_p} - 4p\overset{c}{\Box}\,\big(\delta^\gamma_{i_1}\overset{c}{\nabla}{}^d u_{di_2\dots i_p} - \overset{c}{\nabla}_{i_1}u^{\gamma}_{.i_2\dots i_p}\big)\\
&\qquad+ (4\omega_0 - 2p + n - 2)\big\{\overset{c}{\Box}\,\overset{c}{\nabla}{}^\gamma - \overset{c}{\nabla}{}^\gamma\overset{c}{\Box}\,\big\}u_{i_1\dots i_p} - 2p\big\{\overset{c}{\Box}\,\overset{c}{\nabla}_{i_1} - \overset{c}{\nabla}_{i_1}\overset{c}{\Box}\,\big\}u^{\gamma}_{.i_2\dots i_p}\\
&\qquad+ 2p\delta^\gamma_{i_1}\big\{\overset{c}{\Box}\,\overset{c}{\nabla}{}^d - \overset{c}{\nabla}{}^d\overset{c}{\Box}\,\big\}u_{di_2\dots i_p},
\end{aligned}$$

$$\begin{aligned}
&X^\gamma\big(\overset{c}{\Box}\,\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_a u^{a}_{.i_2\dots i_p}\big)\\
&\quad\overset{*}{=}\big([X^\gamma,\overset{c}{\nabla}_a]\overset{c}{\nabla}{}^a + \overset{c}{\nabla}{}^a[X^\gamma,\overset{c}{\nabla}_a]\big)\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^d u_{di_2\dots i_p} + \overset{c}{\Box}\,X^\gamma\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^k u_{ki_2\dots i_p}\\
&\quad\overset{*}{=}(2\omega_0 - p - 1)\delta^\gamma_{i_1}\overset{c}{\Box}\,\overset{c}{\nabla}{}^k u_{ki_2\dots i_p}\\
&\qquad+ (2\omega_0 + n - 2)\overset{c}{\Box}\,\overset{c}{\nabla}_{i_1}u^{\gamma}_{.i_2\dots i_p} - 2(p-1)\delta^\gamma_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_a\overset{c}{\nabla}_b u^{ab}_{..i_3\dots i_p}\\
&\qquad+ 2(p-1)\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}{}^k u^{\gamma}_{k.i_3\dots i_p} + (4\omega_0 - 2p + n - 4)\overset{c}{\nabla}{}^\gamma\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_k u^{k}_{.i_2\dots i_p}\\
&\qquad- 4(p-1)\delta^\gamma_{i_1}\overset{c}{\nabla}_{[l}\overset{c}{\nabla}_{i_2]}\overset{c}{\nabla}_k u^{kl}_{..i_3\dots i_p} - 2(\overset{c}{\nabla}{}^\gamma\overset{c}{\nabla}_{i_1} - \overset{c}{\nabla}_{i_1}\overset{c}{\nabla}{}^\gamma)\overset{c}{\nabla}_k u^{k}_{.i_2\dots i_p}.
\end{aligned}$$

and

$$\begin{aligned}
&X^\gamma(\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_a\overset{c}{\nabla}_b u^{ab}_{..i_3\dots i_p})\\
&\quad\overset{*}{=}(4\omega_0 - 4p + n - 2)\delta^\gamma_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_a\overset{c}{\nabla}_b u^{ab}_{..i_3\dots i_p} + (4\omega_0 + 2p + n - 6)\overset{c}{\nabla}_{i_1}\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_k u^{k\gamma}_{..i_3\dots i_p}.
\end{aligned}$$

Now we have to compute the action of X^γ on all Weyl curvature correction terms of

$\overset{s}{D}_{(4,p)}$, using relation (2.11). For instance, we obtain

$$
\begin{aligned}
&X^\gamma(C^{a\ \ \ b}_{.i_1i_2.}\overset{c}{\Box}\, u_{abi_3\ldots i_p}) \\
&\quad \overset{*}{=} C^{a\ \ \ b}_{.i_1i_2.}\Big[(4\omega_0 - 2p + n - 2)\overset{c}{\nabla}{}^\gamma u_{abi_3\ldots i_p} - 4\delta^\gamma_a \overset{c}{\nabla}_k u^k_{.bi_3\ldots i_p} \\
&\qquad + 4\overset{c}{\nabla}_a u_{b.i_3\ldots i_p}^{\ \gamma} - 2(p-2)\delta^\gamma_{i_3}\overset{c}{\nabla}_k u^k_{.abi_4\ldots i_p} + 2(p-2)\overset{c}{\nabla}_{i_3} u^\gamma_{.abi_4\ldots i_p}\Big].
\end{aligned}
$$

In an analogous manner one computes the other actions. Using $4\omega_0 + n - 2p - 4 = 0$ and successively identity (2.10), for instance

$$
\begin{aligned}
\big(\overset{c}{\Box}\,\overset{c}{\nabla}{}^\gamma - \overset{c}{\nabla}{}^\gamma\overset{c}{\Box}\,\big)u_{i_1\ldots i_p} &= 2\big(\overset{c}{\nabla}_{[a}\overset{c}{\nabla}_{\gamma]}\overset{c}{\nabla}{}^a + \overset{c}{\nabla}{}^a\overset{c}{\nabla}_{[a}\overset{c}{\nabla}_{\gamma]}\big)u_{i_1\ldots i_p} \\
&= -2pC_{\gamma.\,.i_i}^{\ \ ab}\overset{c}{\nabla}_a u_{bi_2\ldots i_p} + \frac{p(n-2)}{n-3}\overset{c}{\nabla}_u C^u_{.\gamma a i_1} u^a_{.i_2\ldots i_p} \quad ,
\end{aligned}
$$

one verifies after lengthy straight computations the assertion $X^\gamma \overset{s}{D}_{(4,p)} = 0$ ∎

Theorem 5.3. *If $n + 2p > 6$, the operator $\overset{s}{D}{}^*_{(2,p)}$ defined on $\overset{s}{\mathcal{T}}$ by*

$$
\begin{aligned}
(\overset{s}{D}{}^*_{(2,p)}[u])_{i_1\ldots i_p} := TS\Big[& C_{i_1\ldots}^{\ \ abd}\big(\overset{c}{\nabla}_a\overset{c}{\nabla}_b u_{di_2\ldots i_p} - 2b_1\overset{c}{\nabla}_{i_2}\overset{c}{\nabla}_d u_{abi_3\ldots i_p}\big) \\
&+ b_1 C^{a\ \ \ b}_{.i_1i_2}\big(\overset{c}{\Box}\, u_{abi_3\ldots i_p} - 2\overset{c}{\nabla}_a\overset{c}{\nabla}{}^k u_{kbi_3\ldots i_p}\big) \\
&+ \overset{c}{\nabla}_u C^{uab}_{\ldots i_1}\big(b_2\overset{c}{\nabla}_{i_2}u_{abi_3\ldots i_p} + b_3\overset{c}{\nabla}_a u_{bi_2\ldots i_p} + b_4\overset{c}{\nabla}_b u_{ai_2\ldots i_p}\big) \\
&+ b_2\overset{c}{\nabla}_u C^{u\ \ \ a}_{.i_1i_2.}\overset{c}{\nabla}{}^k u_{kai_3\ldots i_p} + b_5 B^a_{i_1} u_{ai_2\ldots i_p}\Big]
\end{aligned}
$$

with

$$
\left.\begin{aligned}
\beta_0 &= (n+2p-6)^{-1} \\
\beta_1 &= [4(n-3)]^{-1}\beta_0
\end{aligned}\right\}
$$

and

$$
\left.\begin{aligned}
b_1 &= (1-p)\beta_0 \\
b_2 &= 4(p-1)(n-4)\beta_1 \\
b_3 &= 2(n-4)(n+2p-2)\beta_1 \\
b_4 &= -4(n-4)(n+2p-5)\beta_1 \\
b_5 &= (n-6)(n+2p-4)\beta_1
\end{aligned}\right\}
$$

is a second-order conformal covariant with $\omega = -2$ and $4\omega_0 = 4 + 2p - n$.

Proof. The operator $\overset{s}{D}{}^*_{(2,p)}$ is a linear combination of the Weyl curvature terms of $\overset{s}{D}_{(4,p)}$ and the term $TS[C_{i_1\ldots}^{\ \ abd}\overset{c}{\nabla}_a\overset{c}{\nabla}_b u_{di_2\ldots i_p}]$. Using $4\omega_0 + n - 2p - 4 = 0$ we obtain

$$
\begin{aligned}
X^\gamma\big(C_{i_1\ldots}^{\ \ abd}\overset{c}{\nabla}_a\overset{c}{\nabla}_b u_{di_2\ldots i_p}\big) \overset{*}{=} (p-1)\Big[&\delta^\gamma_{i_1} C_{i_2\ldots}^{\ \ abd}\overset{c}{\nabla}_d u_{abi_3\ldots i_p} - C^{a\ \ \ b}_{.i_1i_2.}\overset{c}{\nabla}_a u^\gamma_{.bi_3\ldots i_p}\Big] \\
&+ (n-3)C_{i_1\ldots}^{\ \ ab\gamma}\overset{c}{\nabla}_a u_{bi_2\ldots i_p} - \frac{n}{2}C_{i_1\ldots}^{\ \ ab\gamma}\overset{c}{\nabla}_b u_{.ai_2\ldots i_p}
\end{aligned}
$$

and after lengthy straight computations one verifies the result $X^\gamma \overset{s}{D}{}^*_{(2,p)} = 0$ ∎

Remark 5.2. The operator $\overset{s}{D}{}^{*}_{(2,1)}$ has been found already by Graham [24]. Only if $n = 4$ and $p = 1$, Bach tensor expressions obstruct the existence of $\overset{s}{D}_{(4,1)}$ and $\overset{s}{D}{}^{*}_{(2,1)}$. However, by means of $\overset{s}{D}{}^{*}_{(2,1)}$ one can substitute the Bach tensor expression of $\overset{s}{D}_{(4,1)}$ such that one obtains a conformal covariant also in the critical case $n = 4$ and $p = 1$ by a suitable linear combination of $\overset{s}{D}_{(4,1)}$ and $\overset{s}{D}{}^{*}_{(2,1)}$ (see also [24]).

Lemma 5.1. *The operators $TS\big[\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_k} u_{i_{k+1}\dots i_{k+p}}\big]$ are conformal covariants on $\overset{s}{\mathcal{T}}_p$ with $\omega = 0$ and $2\omega_0 = 2p + k - 1$.*

The proof is a direct consequence of (see [42])

$$X^{\gamma} TS\big[\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_k} u_{i_{k+1}\dots i_{k+p}}\big] = k[2\omega_0 - 2p - k + 1] TS\big[\delta^{\gamma}_{i_1} \overset{c}{\nabla}_{i_2} \dots \overset{c}{\nabla}_{i_k} u_{i_{k+1}\dots i_{k+p}}\big].$$

Corollary 5.1. *A symmetric, trace-free tensor $E_{i_1\dots i_{k+p}}$ from ϑ_0 is conformally invariant if and only if the operator*

$$E^{i_1\dots i_{k+p}} \overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_k} u_{i_{k+1}\dots i_{k+p}} \tag{5.2}$$

is a conformal covariant with $2\omega_0 = 2p + k - 1$.

Proof. The first part is a consequence of Lemma 5.1. Conversely, since the derivatives of $u_{i_1\dots i_p}$ at any fixed point can be chosen arbitrarily, the assertion $X^{\gamma} E^{i_1\dots i_{k+p}} = 0$ follows from the conformal invariance of (5.2) ∎

Corollary 5.2. *The nonlinear operators $T_{(k,p)}$ defined on $\mathcal{T}_p$ by*

$$T_{(k,p)}[u] := \big(TS\big[\overset{c}{\nabla}{}^{i_1} \dots \overset{c}{\nabla}{}^{i_k} u^{i_{k+1}\dots i_{k+p}}\big]\big)\big[\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_k} u_{i_{k+1}\dots i_{k+p}}\big]$$

are conformal covariants with $2\omega_0 = 2p + k - 1$.

6. Conformal covariants acting on differential forms

Let $\Lambda^p (p \geq 1)$ denote the space of all p-forms of class C^{∞}. When $\alpha_1, \dots, \alpha_p$ appear in the sequel, we assume that alternation has been carried out over these indices. Branson found in [5] a second-order conformal covariant $D_{2,p}$ on Λ^p with $\omega = -1, 4\omega_0 = 2+2p-n$ and a fourth-order conformal covariant $D_{4,p}$ on Λ^p with $\omega = -2, 4\omega_0 = 4 + 2p - n$ $(n > 4)$. The operator $D_{4,p}$ differs only by a real factor to the conformal covariant $D_{(4,p)}$ constructed by the present author in [40] by means of the method given in Section 2. We have $D_{(4,1)} = -\frac{n+2}{4} \overset{s}{D}_{(4,1)}$ (see Theorem 5.2).

As for $D_{(6)}$ and $\overset{s}{D}_{(4,1)}$, a Bach tensor expression of the form

$$\frac{c(n,p)}{n-4} B_{\alpha_1 \cdot}{}^{k} u_{k\alpha_2\dots\alpha_p} \qquad \text{with } c(n,p) \in \mathbb{R} \setminus \{0\} \tag{6.1}$$

obstructs the existence of $D_{(4,p)}$ in dimension 4 (if $p = 1$, see Remark 5.2).

Problem C. Is there a fourth-order conformal covariant on Λ^2 with the leading term $\Box^2$ in dimension 4?

The proof of the following theorem is analogous to the proofs of Theorems 5.2 and 5.3.

Theorem 6.1. *Suppose that $n > 4, 2p \neq n+6$ or $p = 2, n = 4$. Let $D^*_{(2,p)}$ be the operator defined on Λ^p by*

$$
\begin{aligned}
(D^*_{(2,p)}[u])_{\alpha_1\ldots\alpha_p} := {} & C_{\alpha_1\ldots}{}^{abd}\big(\overset{c}{\nabla}_a\overset{c}{\nabla}_b u_{d\alpha_2\ldots\alpha_p} + 2c_1\overset{c}{\nabla}_{\alpha_2}\overset{c}{\nabla}_a u_{bd\alpha_3\ldots\alpha_p}\big)\\
& + c_1 C_{\alpha_1\alpha_2..}{}^{ab}\big(\overset{c}{\Box}\, u_{ab\alpha_3\ldots\alpha_p} + 2\overset{c}{\nabla}_a\overset{c}{\nabla}{}^k u_{kb\alpha_3\ldots\alpha_p}\big)\\
& - 4c_1\overset{c}{\nabla}_{\alpha_1} C_{\alpha_2\ldots}{}^{abd}\overset{c}{\nabla}_a u_{bd\alpha_3\ldots\alpha_p} + c_2\overset{c}{\nabla}_{\alpha_1}\big(\overset{c}{\nabla}_u C^{u}{}_{.\alpha_2..}{}^{ab} u_{ab\alpha_3\ldots\alpha_p}\big)\\
& + c_3\big(\overset{c}{\nabla}_u C^{u}{}_{.\alpha_2..}{}^{ab}\overset{c}{\nabla}_{\alpha_2} u_{ab\alpha_3\ldots\alpha_p} + \overset{c}{\nabla}_u C^{ua}{}_{..\alpha_1\alpha_2}\overset{c}{\nabla}{}^k u_{ka\alpha_2\ldots\alpha_p}\big)\\
& + \overset{c}{\nabla}_u C^{uab}{}_{\ldots\alpha_1}\big(c_4\overset{c}{\nabla}_b u_{a\alpha_2\ldots\alpha_p} + c_5\overset{c}{\nabla}_a u_{b\alpha_2\ldots\alpha_p}\big)\\
& + c_6 B_{\alpha_1.}{}^{a} u_{a\alpha_2\ldots\alpha_p} + c_7 C_{\alpha_1\alpha_2}{}^{ab}\overset{c}{\nabla}_{\alpha_3}\overset{c}{\nabla}{}^k u_{kab\alpha_4\ldots\alpha_p}
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
c_1 &= \frac{3(p-1)}{2(n-2p+6)}\\
c_2 &= \frac{2(n-2)}{n-3}c_1\\
c_3 &= \frac{n}{n-3}c_1\\
c_4 &= \frac{6(1-p)+(3-n)(n-2p+6)}{(n-3)(n-2p+6)}\\
c_5 &= \frac{n}{2(n-3)}\\
c_6 &= \frac{(n-2)[12(p-1)+(n-6)(n-2p+6)]}{4(n-3)(n-4)(n-2p+6)}\\
c_7 &= 0.
\end{aligned}
\right\}
$$

Then $D^*_{(2,p)}$ is a second-order conformal covariant with $\omega = -2$ and $4\omega_0 = 4 + 2p - n$.

Remark 6.1. As expected there holds $D^*_{(2,1)} = \overset{s}{D}{}^*_{(2,1)}$. As already noted, one can annihilate the obstructing Bach tensor expression of $D_{(4,1)}$ in dimension 4 by means of $\overset{s}{D}{}^*_{(2,1)}$. If $p = 3$, then we have an analogous situation. If $p = 2$, we have $c_6 = \frac{n(n-2)}{4(n-3)(n+2)}$. Hence, the operator $D^*_{(2,2)}$ is a *regular* operator in the important case $p = 2$ and $n = 4$. Consequently, the obstructing Bach tensor expression (6.1) of $D_{(4,2)}$ cannot be substituted by $D^*_{(2,2)}$.

Theorem 6.2. *If $n = 4$, there is no conformal covariant on Λ^2 with leading term $\Box^2$.*

Proof. Under consideration of (3.2) and (3.3), it is easy to see that the operator $D^*_{(2,p)}$ contains all possible curvature monomials if $\omega = -2$ and $4\omega_0 = 4 + 2p - n$ (modulo trivial order zero actions of the Weyl tensor). Furthermore, the real coefficients of $D^*_{(2,1)}$ and $D_{(4,p)}$ (see [40]: p. 279]) are determined uniquely. Now the fact that the obstructing Bach tensor expression (6.1) of $D_{(4,2)}$ cannot be substituted by $D^*_{(2,2)}$ implies the assertion ∎

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