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Sobolev Inequalities on Sets with Irregular Boundaries

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Abstract. We derive (weighted) Sobolev-Poincaré inequalities for s-John domains and s-cusp domains, both with optimal exponents. These results are obtained as consequences of a more comprehensive criterion.

Keywords: *Sobolev inequality, Poincaré inequality, embeddings, weighted Sobolev spaces, John domains, cusp domains*

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1. Introduction

It is well known that the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ if Ω is a nice bounded domain in $\mathbb{R}^n$ and

$$1 \le p < \infty \qquad \text{and} \qquad q(n-p) \le np. \tag{1.1}$$

This fact, originally due to Sobolev, Gagliardo and Nirenberg, can nowadays be found in textbooks (cf. [12, 17]) and it is stated as the Sobolev-Poincaré inequality

$$\left(\int_\Omega |u-u_\Omega|^q dx\right)^{\frac{1}{q}} \le C\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{1.2}$$

The weighted case of Sobolev's imbedding has been developed by Nečas [14], Besov, Ilin and Nikolskii [3, 4], Kufner [7], Maz'ya [12] and others. It is not very difficult to give examples of domains having cusps for which the Sobolev-Poincaré inequality (1.2) fails to hold or the range for its validity differs from (1.1). The question of this embedding in non-smooth domains Ω is addressed by many authors. To mention but a few, we would like to refer to the books [12, 13], and point out that Besov [1, 2] obtained embeddings in domains satisfying "flexible cone conditions", Smith and Stegenga [15] proved Poincaré

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inequality with $q = p$ for s-John domains (that allow for twisted cusps of the type t^s with certain $s \geq 1$). Maz'ya [10] (see also Labutin [8]) established the optimal embedding for s-cusps. Hajłasz and Koskela [5] proved the optimal Sobolev-Poincaré inequality in s-John domains with $p = 1$ and the next to the optimal one for $p > 1$. Their result also involves weights. We refer to [5] also for further historical notes and references.

In this note we complete the picture for s-John domains and give a proof for the optimal Sobolev-Poincaré inequality in s-John domains, thus improving the results in [5] (see Theorem 2.3). We study also the weighted case where the weight is a power of the distance to the boundary. The result is obtained as a consequence of a slightly more general criterion, which may be used to illustrate why the optimal exponent for s-John domains is worse than the optimal exponent for domains with a single s-cusp. We use Hedberg's trick on the maximal operator [6], a truncation argument due to Maz'ya [11] and some ideas from Hajłasz and Koskela [5].The main new ingredient of our proof is a careful grouping of chains around a curve which we call a worm.

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$. If u is an integrable function defined at least on E, we let u_E stand for the average

$$u_E = \fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx.$$

The open n-dimensional ball with center at x and radius r is written as $B(x,r) = B_n(x,r)$. We use $\sharp F$ for the cardinality of a set F.

2. Main results

This section contains the results with proofs. We start with a general theorem and deduce the s-John domain result from it.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We consider exponents a, b, p, q satisfying

$$a \geq 0, \quad b \geq 1 - n \tag{2.1}$$

$$1 \leq p < q < \infty \tag{2.2}$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}. \tag{2.3}$$

Let μ the measure on $\mathbb{R}^n$ with

$$\frac{d\mu}{dx} = \begin{cases} \rho^a & \text{in } \Omega \\ 0 & \text{outside } \Omega \end{cases}.$$

Here and in what follows $\rho(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$.

We shall define a *worm*. This is a pair (γ, Δ), where $\gamma : [0,\ell] \to \Omega$ is a curve joining $y = \gamma(0)$ to $x_0 = \gamma(\ell)$, parametrized by its arc-length, and $\Delta = \{\xi_k\}$, $0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_m = \ell$, is a finite partition of $[0,\ell]$. With each worm we associate its *parameters*: the number m of the partition intervals $[\xi_{k-1},\xi_k]$, and for $k=1,\dots,m$ the quantities

$$\begin{aligned} \ell_k &= \xi_k - \xi_{k-1} \\ R_k &= \sup\big\{|\gamma(t)-y| : t\in[\xi_{k-1},\xi_k]\big\} \\ r_k &= \inf\big\{\rho(\gamma(t)) : t\in[\xi_{k-1},\xi_k]\big\}. \end{aligned}$$

Theorem 2.1. *Let a,b,p,q satisfy* (2.1) - (2.3). *Let*

$$\frac{1}{q}\geq\frac{n-p+b}{p(n+a)}. \tag{2.4}$$

Suppose that there is a constant $A>0$ and a point $x_0\in\Omega$ such that for each $y\in\Omega\setminus B(x_0,\frac{\rho(x_0)}{2})$ there is a worm (γ,Δ) joining y to x_0, with parameters m, $\{\ell_k\}$, $\{R_k\}$, $\{r_k\}$ and constants $\tau_1,\dots,\tau_m\in(0,1]$ (both parameters and τ_k's may depend on y), such that

$$\rho(y)\leq 3R_k \quad (k=1,\dots,m) \tag{2.5}$$

$$(1+A^{-1})\tau_{k-1}\leq\tau_k\leq A\tau_{k-1} \quad (k=2,\dots,m) \tag{2.6}$$

$$A^{-1}\big(\mu(B(y,3R_k))\big)^{\frac{1}{q}}\leq\tau_k\leq A\,r_k^{\frac{n+b-1}{p}}\,\ell_k^{\frac{1-p}{p}}. \tag{2.7}$$

Then there is a constant $C=C(n,p,a,b,A,\Omega)>0$ such that

$$\left(\int_\Omega|u-\bar u_a|^q\rho^a dx\right)^{\frac{1}{q}}\leq C\left(\int_\Omega|\nabla u|^p\rho^b dx\right)^{\frac{1}{p}}$$

for each $u\in C^1(\Omega)$ where $\bar u_a=⨍_\Omega u\,d\mu=\frac{1}{\mu(\Omega)}\int_\Omega u\,d\mu$.

We start the proof with the following lemma.

Lemma 2.2. *Suppose that Ω is a bounded open set. Let $z,z'\in\Omega$ and let $\gamma:[\xi,\xi']\to\mathbb{R}^n$ be a path of the length ℓ that joins z and z'. Suppose that $b\geq 1-n$ and that $\rho\geq r$ on γ. Let $u\in C^1(\Omega)$. Then*

$$\left|u_{B(z,\frac12\rho(z))}-u_{B(z',\frac12\rho(z'))}\right|\leq Cr^{\frac{1-b-n}{p}}\,\ell^{\frac{p-1}{p}}\left(\int_{D_\gamma}|\nabla u|^p\rho^b dx\right)^{\frac{1}{p}} \tag{2.8}$$

where $D_\gamma=\cup_{t\in[\xi,\xi']}B\big(\gamma(t),\frac12\rho(\gamma(t))\big)$.

Proof. Write $B=B(z,\frac12\rho(z))$ and $B'=B(z',\frac12\rho(z'))$. We construct a chain $\{B_i\}$, $B_i\equiv B(z_i,\frac12\rho(z_i))$ of balls and denote $\hat B_i=B(z_i,\frac14\rho(z_i))$. For the construction, it is enough to determine points t_i such that $z_i=\gamma(t_i)$. If $t_1,\dots,t_{j-1}$ are selected, we find the next as

$$t_j=\sup\Big\{t\in[t_{i-1},\xi'] : B\big(\gamma(t),\tfrac14\rho(\gamma(t))\big)\cap\hat B_{j-1}\neq\emptyset\Big\}.$$

If $t_j=\xi'$, we set $j_{\max}=j$, $t_j=\xi'$ and terminate the construction.

We observe that the balls $B(z_i, \frac{1}{4}\rho(z_i))$ $(i < j_{\max})$ are disjoint, and since their radii are bounded away from zero and Ω is bounded, the sequence really terminates after a finite number of steps. Fix $x \in \Omega$ and denote $I(x) = \{i < j_{\max} : x \in B_i\}$. Let $i \in I(x)$. Then

$$\left.\begin{aligned} \rho(z_i) \le \rho(x) + |x - z_i| \le \rho(x) + \tfrac{1}{2}\rho(z_i) \\ \rho(x) \le \rho(z_i) + |x - z_i| \le \rho(z_i) + \tfrac{1}{2}\rho(z_i) \end{aligned}\right\}$$

and thus

$$\rho(z_i) \le 2\rho(x) \qquad \text{and} \qquad \rho(x) \le 2\rho(z_i). \tag{2.9}$$

For any $y \in \hat{B}_i$ we have $|y - x| \le \rho(z_i) \le 2\rho(x)$ which means that $\cup_{i \in I(x)} \hat{B}_i \subset B(x, 2\rho(x))$. Since $\hat{B}_i$ $(i \in I(x))$ are disjoint, we have

$$|B(x, \tfrac{1}{8}\rho(x))|\, \sharp I(x) \le \sum_{i \in I(x)} |\hat{B}_i| \le |B(x, 2\rho(x))|$$

which implies $\sharp I(x) \le 16^n$. Thus we have proven that

$$\sum \chi_{B_i} \le 16^n + 1. \tag{2.10}$$

Next, consider $i \in \{1, \dots, j_{\max}\}$ and notice that there is a point $x \in \overline{\hat{B}_{i-1}} \cap \overline{\hat{B}_i}$. Then, as above, we infer that (2.9) holds and

$$\begin{aligned} B(x, \tfrac{1}{8}\rho(x)) \subset B(x, \tfrac{1}{4}\rho(z_{i-1})) \cap B(x, \tfrac{1}{4}\rho(z_i)) \subset B_{i-1} \cap B_i \\ B_{i-1} \cup B_i \subset B(x, \rho(z_{i-1})) \cup B(x, \rho(z_i)) \subset B(x, 2\rho(x)) \end{aligned}$$

so that

$$|B_{i-1} \cup B_i| \le 16^n |B_{i-1} \cap B_i|. \tag{2.11}$$

Also, it is clear that

$$\sum_{i=1}^{j_{\max}} \rho(z_i) \le C\ell. \tag{2.12}$$

Using (2.11) and the Poincaré inequality we have

$$\begin{aligned} |u_{B_i} - u_{B_{i-1}}| &\le |u_{B_i} - u_{B_i \cap B_{i-1}}| + |u_{B_i \cap B_{i-1}} - u_{B_{i-1}}| \\ &\le \fint_{B_i \cap B_{i-1}} |u - u_{B_i}|\, dx + \fint_{B_i \cap B_{i-1}} |u - u_{B_{i-1}}|\, dx \\ &\le \frac{|B_i|}{|B_i \cap B_{i-1}|} \fint_{B_i} |u - u_{B_i}|\, dx + \frac{|B_{i-1}|}{|B_i \cap B_{i-1}|} \fint_{B_{i-1}} |u - u_{B_{i-1}}|\, dx \\ &\le C\, \rho(z_i) \left(\fint_{B_i} |\nabla u|^p dx \right)^{\frac{1}{p}} + C\, \rho(z_{i-1}) \left(\fint_{B_{i-1}} |\nabla u|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Hence we can estimate by using (2.10) and (2.12) that

$$\begin{aligned}
|u_{B'} - u_B| &\le \sum_{i=2}^{j_{\max}} |u_{B_i} - u_{B_{i-1}}| \\
&\le C \sum_{i=1}^{j_{\max}} \rho(z_i)^{1-\frac{n}{p}} \left(\int_{B_i} |\nabla u|^p dx \right)^{\frac{1}{p}} \\
&\le C \sum_{i=1}^{j_{\max}} \rho(z_i)^{1-\frac{1}{p}+\frac{1-n-b}{p}} \left(\int_{B_i} \rho(z_i)^b |\nabla u|^p dx \right)^{\frac{1}{p}} \\
&\le C \sum_{i=1}^{j_{\max}} r^{\frac{1-n-b}{p}} \rho(z_i)^{1-\frac{1}{p}} \left(\int_{B_i} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}} \\
&\le C r^{\frac{1-n-b}{p}} \left(\sum_{i=1}^{j_{\max}} \rho(z_i) \right)^{1-\frac{1}{p}} \left(\sum_{i=1}^{j_{\max}} \int_{B_i} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}} \\
&\le C r^{\frac{1-b-n}{p}} \ell^{\frac{p-1}{p}} \left(\int_{D_\gamma} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}}
\end{aligned} \tag{2.13}$$

since $b + n \ge 1$. The lemma is proven ∎

Proof of Theorem 2.1. Denote $B_0 = B(x_0, \frac{1}{2}\rho(x_0))$ and let $u \in C^1(\Omega)$. We may assume that

$$|\{u \ge 0\} \cap B_0| \ge \tfrac{1}{2}|B_0| \qquad \text{and} \qquad |\{u \le 0\} \cap B_0| \ge \tfrac{1}{2}|B_0|. \tag{2.14}$$

We will also assume as we may that

$$\int_\Omega |\nabla u|^p \rho^b dx = 1. \tag{2.15}$$

We shall first establish a weak type estimate

$$\mu(A_\lambda) \le C\lambda^{-q}, \tag{2.16}$$

where $A_\lambda = \{x \in \Omega : |u(x)| > \lambda\}$ and $\lambda > 0$. First observe that since the median of u is zero in B_0 by (2.14), we have

$$\int_{B_0} |u|^p dx \le c \int_{B_0} |\nabla u|^p dx \tag{2.17}$$

(see [17: Theorem 4.4.4]). Hence

$$|u_{B_0}| \le \left(\fint_{B_0} |u|^p dx \right)^{\frac{1}{p}} \le c \left(\fint_{B_0} |\nabla u|^p dx \right)^{\frac{1}{p}} \le c_0, \tag{2.18}$$

where c_0 is independent of u. Since $\mu(\Omega) < \infty$ it suffices to establish (2.16) for $\lambda > 3c_0$. To do so, we fix $\lambda > 3c_0$ and divide A_λ into three parts: write $B_y = B(y, \frac{1}{2}\rho(y))$ and let

$$E_\lambda = \{y \in A_\lambda \setminus B_0 : |u_{B_y}| > \tfrac{1}{2}\lambda\}$$
$$F_\lambda = A_\lambda \setminus (B_0 \cup E_\lambda).$$

The third part is

$$A_\lambda \cap B_0.$$

We treat E_λ first. Fix $y \in E_\lambda$ and let $(\gamma, \{\xi_k\})$ be a worm in Ω that connects y to x_0, with parameters $m, \{\ell_k\}, \{R_k\}, \{r_k\}$, and obeys the bounds of the theorem. We apply Lemma 2.2 to paths $\gamma_k = \gamma|_{[\xi_{k-1},\xi_k]}$ and points $z = z_k = \gamma(\xi_{k-1})$ and $z' = z'_k = \gamma(\xi_k)$. Let $x = \gamma(t)$ with $t \in [\xi_{k-1}, \xi_k]$. Then by (2.5)

$$\rho(x) \le \rho(y) + |y - x| \le 4R_k$$

and thus

$$B(x, \tfrac{1}{2}\rho(x)) \subset B(y, R_k + 2R_k)$$
$$D_{\gamma_k} \subset B(y, 3R_k).$$

Since $\lambda > 3c_0$, we have

$$\begin{aligned}
\lambda &\le 6\,|u_{B_y} - u_{B_0}| \\
&\le 6 \sum_{k=1}^{m} |u_{B_{z'_k}} - u_{B_{z_k}}| \\
&\le C \sum_k r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} \left(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p d\mu \right)^{\frac{1}{p}}.
\end{aligned}$$

We split the last sum into two parts by $K = K(y)$ that is to be determined. First we notice that by (2.6) and (2.2)

$$\sum_{k>K} \tau_k^{-1} \le C\tau_{K+1}^{-1} \qquad \text{and} \qquad \sum_{k\le K} \tau_k^{\frac{q}{p}-1} \le C\tau_K^{\frac{q}{p}-1}. \tag{2.19}$$

If $K < m$, due to our normalization of u, (2.7) and (2.19) we have

$$\begin{aligned}
&\sum_{k>K} r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} \left(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p d\mu \right)^{\frac{1}{p}} \\
&\quad\le \left(\int_\Omega \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}} \sum_{k>K} r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} \\
&\quad= \sum_{k>K} r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} \\
&\quad\le C \sum_{k>K} \tau_k^{-1} \\
&\quad\le C\tau_{K+1}^{-1}.
\end{aligned} \tag{2.20}$$

Before treating the second part of the sum, we set

$$f = |\nabla u|^p \rho^{b-a} \qquad \text{and} \qquad g(x) = \left(\sup_{r>0} \frac{1}{\mu(B(x,r)} \int_{B(x,r)} f \, d\mu \right)^{\frac{1}{p}}.$$

Since the maximal operator with respect to μ is of weak type $(1,1)$ (see, e.g., [9: Theorem 2.19] or [16: p. 44/I.8.17]) and $\|f\|_{L^1(\mu)} = 1$, we have

$$\mu(\{g^p > t\}) \le C \, \frac{1}{t} \qquad (0 < t < \infty). \tag{2.21}$$

We estimate

$$\begin{aligned} \sum_{k \le K} r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} & \left(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p d\mu \right)^{\frac{1}{p}} \\ & \le \sum_{k \le K} r_k^{\frac{1-b-n}{p}} \ell_k^{\frac{p-1}{p}} \left(\mu(B(y,3R_k)) \right)^{\frac{1}{p}} g(y) \\ & \le C \sum_{k \le K} \tau_k^{-1} \tau_k^{\frac{q}{p}} \, g(y) \\ & \le C \tau_K^{-1+\frac{q}{p}} \, g(y). \end{aligned} \tag{2.22}$$

Now we specify the choice of K, distinguishing three cases. If $\tau_1^{-\frac{q}{p}} \le g(y)$, we choose $K = 0$. Then the sum over all $k = 1, \dots, m$ reduces to (2.20) and we have $\lambda \le C\tau_1^{-1} \le C \, g(y)^{\frac{p}{q}}$. If $\tau_m^{-\frac{q}{p}} \ge g(y)$, we choose $K = m$. Then the sum over $k = 1, \dots, m$ is treated in (2.22), and we have

$$\lambda \le C \tau_m^{-1+\frac{q}{p}} g(y) \le C g(y)^{\frac{p}{q}-1} g(y) = C g(y)^{\frac{p}{q}}.$$

The remaining case is that $\tau_m^{-\frac{q}{p}} < g(y) < \tau_1^{-\frac{q}{p}}$. Then we choose the integer $K < m$ so that $\tau_{K+1}^{-\frac{q}{p}} \le g(y) < \tau_K^{-\frac{q}{p}}$. Using (2.20) and (2.22) we obtain

$$\lambda \le C \tau_{K+1}^{-1} + C \tau_K^{-1+\frac{q}{p}} g(y) \le C g(y)^{\frac{p}{q}}.$$

Hence we always have $\lambda \le C g(y)^{\frac{p}{q}}$ for every $y \in E_\lambda$. Therefore by (2.21)

$$\mu(E_\lambda) \le \mu(\{g^p > (\lambda/C)^q\}) \le C \lambda^{-q}. \tag{2.23}$$

Next, we estimate the measure of F_λ. Using the Besicovitch covering theorem (cf. [9: Theorem 2.7]) we can cover F_λ with balls $B_{x_i} = B(x_i, \frac{1}{2}\rho(x_i))$ so that $x_i \in F_\lambda$ and

$\sum_i \chi_{B_{x_i}} \le N$. Then $|u - u_{B_{x_i}}| \ge \frac{1}{2}\lambda$ on F_λ whence we have by using the Sobolev-Poincaré inequality that

$$\begin{aligned} \mu(F_\lambda) &\le \sum_i \mu(B_{x_i} \cap F_\lambda) \\ &\le \sum_i \int_{B_{x_i}\cap F_\lambda} \rho^a dx \\ &\le C \sum_i \rho(x_i)^a \int_{B_{x_i}\cap F_\lambda} dx \\ &\le C\lambda^{-q} \sum_i \rho(x_i)^a \int_{B_{x_i}\cap F_\lambda} |u - u_{B_{x_i}}|^q dx \\ &\le C\lambda^{-q} \sum_i \rho(x_i)^{a+q+n(1-\frac{q}{p})} \left(\int_{B_{x_i}} |\nabla u|^p dx \right)^{\frac{q}{p}} \\ &\le C\lambda^{-q} \sum_i \left(\int_{B_{x_i}} |\nabla u|^p \rho^{p(\frac{a+n}{q}+1-\frac{n}{p})} dx \right)^{\frac{q}{p}} \\ &\le C\lambda^{-q} \left(\int_\Omega |\nabla u|^p \rho^b dx \right)^{\frac{q}{p}} \\ &\le C\lambda^{-q} \end{aligned} \tag{2.24}$$

since $p\big(\frac{a+n}{q} + 1 - \frac{n}{p}\big) \ge b$ by (2.4).

Finally, combining (2.17) and the usual Sobolev inequality in the ball B_0, we obtain the weak type estimate $\mu(A_\lambda \cap B_0) \le C\lambda^{-q}$. Hence by estimates (2.23) and (2.24)

$$\mu(A_\lambda) \le \mu(E_\lambda) + \mu(F_\lambda) + \mu(A_\lambda \cap B_0) \le C\lambda^{-q}.$$

In conclusion, (2.16) holds for all $\lambda > 0$ or, without normalization (2.15),

$$\sup_{\lambda>0} \lambda\, \mu(\{|u| > \lambda\})^{\frac{1}{q}} \le C \left(\int_\Omega |\nabla u|^p \rho^b dx \right)^{\frac{1}{p}}. \tag{2.25}$$

A truncation argument shows that the weak type estimate (2.25) implies the desired embedding. Indeed, for each $t > 0$ the truncated functions

$$u_t(x) = \begin{cases} \frac{1}{2}t & \text{if } |u(x)| > t \\ |u(x)| - \frac{1}{2}t & \text{if } \frac{1}{2}t < |u(x)| < t \\ 0 & \text{if } |u(x)| < \frac{1}{2}t \end{cases}$$

satisfy (2.14). Thus we may use (2.25) to conclude

$$\begin{aligned} \left(\int_{\{t<u\le 2t\}} |u|^q d\mu \right)^{\frac{1}{q}} &\le Ct\, \mu(\{|u| > t\})^{\frac{1}{q}} \\ &\le Ct\, \mu(\{u_t \ge \tfrac{1}{2}t\})^{\frac{1}{q}} \end{aligned}$$

$$\leq C\left(\int_\Omega |\nabla u_t|^p \rho^b dx\right)^{\frac{1}{p}}$$

$$= C\left(\int_{\{\frac{1}{2}t<|u|\leq t\}} |\nabla u|^p \rho^b dx\right)^{\frac{1}{p}}.$$

Hence

$$\int_\Omega |u|^q \rho^a dx \leq \sum_{j=-\infty}^{\infty} \int_{\{2^j<|u|\leq 2^{j+1}\}} |u|^q \rho^a dx$$

$$\leq C \sum_{j=-\infty}^{\infty} \left(\int_{\{2^{j-1}<|u|\leq 2^j\}} |\nabla u|^p \rho^b dx\right)^{\frac{q}{p}}$$

$$\leq C\left(\int_\Omega |\nabla u|^p \rho^b dx\right)^{\frac{q}{p}},$$

and the theorem is proved, since $\int_\Omega |u-\bar{u}_a|^q \rho^a dx \leq C\int_\Omega |u|^q \rho^a dx$ ∎

Following Smith and Stegenga [15] we call a bounded domain Ω an *s-John domain* ($s \geq 1$), if there is a point $x_0 \in \Omega$ and a constant $c_0 \geq 1$ such that each point $x \in \Omega$ can be joined to x_0 in Ω by a rectifiable curve (called an *s-John core*) $\gamma : [0,\ell] \to \Omega$ such that γ is parametrized by the arc length, $\gamma(0) = x$, $\gamma(\ell) = x_0$, and $\operatorname{dist}(\gamma(t), \partial\Omega) \geq c_0^{-1} t^s$ for all $t \in [0,\ell]$.

The next theorem improves the main result of [5].

Theorem 2.3. *Suppose that Ω is an s-John domain. Let a, b, p, q satisfy* (2.1) - (2.3) *and*

$$\frac{1}{q} \geq \frac{s(n+b-1)-p+1}{p(n+a)}.$$

Then there is a constant $C = C(n,p,q,a,b,\Omega) > 0$ such that

$$\left(\int_\Omega |u-\bar{u}_a|^q \rho^a dx\right)^{\frac{1}{q}} \leq C\left(\int_\Omega |\nabla u|^p \rho^b dx\right)^{\frac{1}{p}}$$

for each $u \in C^1(\Omega)$.

Proof. We will verify the assumptions of Theorem 2.1. First we notice that $s \geq 1$ implies

$$\frac{1}{q} \geq \frac{s(n+b-1)-p+1}{p(n+a)} \geq \frac{n+b-p}{p(n+a)}$$

so that (2.4) is true. For fixed $y \in \Omega \setminus B(x_0, \frac{1}{2}\rho(x_0))$, the s-John core γ on $[0,\ell]$ gives us the desired worm: Let $d = \sup\{|\gamma(t)-y| : t \in [0,\ell]\}$. Find the integer m with $3d < 2^m \rho(y) \leq 6d$. Since

$$\rho(y) \leq \rho(x_0) + |y-x_0| \leq 3|y-x_0| \leq 3d,$$

we have $m \geq 1$. Set

$$\xi_k = \sup\Big\{t \in [0,\ell] : |\gamma(s)-y| \leq 2^{k-m} d \ \text{ for all } s \in [0,t]\Big\}.$$

Then $(\gamma, \{\xi_k\})$ is a worm with parameters m, $\{\ell_k\}$, $\{R_k\}$, $\{r_k\}$, and

$$\left.\begin{aligned} &\ell_k \le \xi_k \\ &\xi_k \ge R_k = 2^{k-m} d \\ &r_k \ge c_0 \xi_k^s. \end{aligned}\right\}$$

The inequality $\rho(y) \le 6 \cdot 2^{-m} d \le 3R_k$ verifies (2.5). Since

$$\frac{n+a}{q} \ge \frac{s(n+b-1)+1-p}{p}$$

we have by choosing $\tau_k = 2^{(k-m)\frac{n+a}{q}}$ that

$$\mu(B(y,R_k))^{\frac{1}{q}} \le R_k^{\frac{n+a}{q}} \le C\tau_k$$

and

$$r_k^{-\frac{n+b-1}{p}} \ell_k^{\frac{p-1}{p}} \le (c_0\xi_k)^{-s\frac{n+b-1}{p}} \xi_k^{\frac{p-1}{p}} \le C\xi_k^{-\frac{n+a}{q}} \le C\tau_k^{-1}.$$

Hence the claim follows from Theorem 2.1 ∎

Remark. The exponent q of Theorem 2.3 is the best possible in the class of s-John domains (see [5]).

Example 2.4. An example of an s-John domain is an s-cusp domain. Surprisingly, the optimal embedding exponent for the s-cusp obtained in [8, 10, 13] is better than that for general s-John domains. The reason is that complicated s-John domains may contain "rooms and corridors" so that the upper estimate for $\mu(B(y,R)\cap\Omega)$ must be more carefully examined. We show that the optimal embedding for s-cusp domains can be deduced from Theorem 2.1. Let us write $x \in \mathbb{R}^n$ as $x = (\hat{x}, x^*)$, where $\hat{x} \in \mathbb{R}^{n-1}$ and x^* is the last coordinate of x. We will consider the s-cusp domain

$$\Omega = \Big\{ x \in \mathbb{R}^n : |\hat{x}| \le (x^*)^s \ \text{ and } \ 0 < x^* < 2 \Big\}$$

and show that if (2.1) - (2.3) are verified, Theorem 2.1 yields embedding of $W^{1,p}(\Omega,\rho^b)$ into $L^q(\Omega,\rho^a)$, where

$$\frac{1}{q} \ge \frac{s(n+b-1)-p+1}{p(s(n+a-1)+1)}.$$

We choose $x_0 = e_n = (0,1)$. If $y \in \Omega \setminus B(x_0, \frac{1}{2}\rho(x_0))$, we set $\ell = \ell(y) = |\hat{y}| + |y^*-1|$ and define the worm curve $\gamma : [0,\ell] \to \Omega$ as

$$\gamma(t) = \begin{cases} \big((1-\frac{t}{|\hat{y}|})\hat{y}, y^*\big) & \text{if } 0 \le t \le |\hat{y}| \\ \big(1+\frac{\ell-t}{\ell-|\hat{y}|}(y^*-1)\big)e_n & \text{if } |\hat{y}| \le t \le \ell. \end{cases}$$

In other words, worm curve starts at y, goes first on line segment connecting y with y^*e_n and then turns to the line segment connecting y^*e_n with e_n. We find a partition $\{\xi_0,\dots,\xi_m\}$ of $[0,\ell]$ in such a way that $\xi_0 = 0$,

$$\xi_0 = 0,\ \xi_k = 2^{k-m}\ell \ (k = 1,\dots,m)$$
$$\rho(y) < \xi_1 < 2\rho(y),$$

where the last is what determines m and guarantees (2.5).

From now we treat only the interesting case that $y^* < 1$. Then

$$\left.\begin{aligned} &\ell_1 = \xi_1, \ell_k = \frac{1}{2}\xi_k \quad (k = 2, \dots, m) \\ &\ell_k^s \le r_k \\ &\xi_k \le R_k \le 2\xi_k \\ &B(y, R_k) \cap \Omega \subset B_{n-1}(\hat{y}, Cr_k) \times (y^* - R_k, y^* + R_k) \\ &\rho \le Cr_k \quad \text{on } B(y, R_k). \end{aligned}\right\} \tag{2.26}$$

Set $\tau_k = (\xi_k^{n+a-1}\ell_k)^{\frac{1}{q}}$. It is easy to observe that τ_k satisfy (2.6). From $(2.26)_2$ we obtain

$$r_k^{\frac{n+b-1}{p}} \ell_k^{\frac{1-p}{p}} \ge r_k^{\frac{n+a-1}{q}} \ell_k^{\frac{1}{q}} \ge C^{-1}\tau_k.$$

The additional information provided by $(2.26)_{4-5}$ has no counterpart in the case of a general s-John domain. We use it to estimate $\mu(B(y, 3R_k))$:

$$C\mu(B(y, R_k))^{\frac{1}{q}} \le C(R_k r_k^{n-1+a})^{\frac{1}{q}} \le C(\xi_k\, r_k^{n-1+a})^{\frac{1}{q}} \le C\tau_k.$$

Hence (2.7) is verified and Theorem 2.1 yields the result.

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