Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 2, 415–428

Weierstrass-Type Maximum Principle for Microstructure in Micromagnetics

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Abstract. We derive necessary and sufficient optimality conditions for a relaxed (in terms of Young measures) variational problem governing steady states of ferromagnetic materials. Such conditions here stated in the form of a generalized Weierstrass maximum principle enable us to establish uniqueness of a solution in some specific situations and can also be used in efficient numerical algorithms solving the relaxed problems, for instance.

Keywords: *Calculus of variations, convexification, ferromagnetism, micromagnetics, optimality conditions, relaxation, Young measures*

AMS subject classification: 49 K 20, 49 S 05

1. Introduction

Steady-state configurations of mechanical systems are usually governed by an energy-minimization type principle. In past centuries, this led to the development of variational calculus, which resulted in formulations of optimality conditions in terms of Euler-Lagrange equations or the Weierstrass maximum principle. Sometimes, the involved energy is not convex in highest derivatives, which causes "physically" the development of a microstructure and "mathematically" the failure of existence of a solution. To describe the microstructure in detail and to overcome the failure of existence, the original problem is to be extended suitably. In some situations, it may happen that the extended (relaxed) problem has a convex structure with respect to some geometry not necessarily compatible with the "natural" geometry of the original non-convex problem. Then one can formulate optimality conditions. For the case of scalar variational problems this results in one half of the Euler-Lagrange equation combined with the Weierstrass maximum principle (see [28: Section 5.3]). The identification of the linear structure that makes the relaxed problem convex and the formulation of corresponding optimality conditions is the basis for the construction of effective numerical algorithms for relaxed problems (cf. [7, 20, 28]). Let us still remark that other geometries applied to the relaxed

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ISSN 0232-2064 / $ 2.50

problem may lead to other optimality conditions (cf., e.g., Chipot and Kinderlehrer [8], DeSimone [12] or Pedregal [26]).

The goal of this paper is to adapt the above ideas to a steady-state micromagnetics. The variational problem, stated in Section 2, was already formulated in [3 - 5, 16] while its extension, stated here in Section 3, was formulated in [12, 24, 25, 27]. Our original results, i.e. the optimality conditions for the extended problem, are formulated in Sections 4 and 5 in terms of a Weierstrass-type maximum principle in integral form (Propositions 1 and 3) and also pointwise (Propositions 2 and 4). Some consequences are mentioned in Section 6.

2. Steady-state model of micromagnetics

In the classical theory of rigid ferromagnetic bodies, based mainly on works of Landau and Lifshitz [22], a magnetization $m : \Omega \to \mathbb{R}^n$, describing the state of the body $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ depends on a position $x \in \Omega$ and has a given temperature-dependent magnitude

$$|m(x)| = \mathrm{const}(T) \qquad \text{for a.a. } x \in \Omega$$

with $m(x) = 0$ for $T \geq T_c$, the so-called Curie point. We will treat the case when the temperature is fixed below the Curie point and thus we shall assume that $|m| = 1$ almost everywhere in Ω. In the so-called no-exchange formulation, the energy of a large rigid ferromagnetic body $\Omega \subset \mathbb{R}^n$ consists of three parts and the variational principle governing steady-state configurations can be stated as follows (see, e.g., Brown [3 - 5], Choksi and Kohn [9], James and Kinderlehrer [16], James and Müller [17], Kinderlehrer and Ma [18], Tartar [29], etc.):

$$\left.\begin{aligned} &\text{minimize} \\ &E(m,u) = \int_\Omega \big[\varphi(m(x)) - H_{\mathrm{e}}(x)\cdot m(x)\big]\,dx + \tfrac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx \\ &\text{subject to} \\ &|m| = 1 \text{ on } \Omega,\ \mathrm{div}(\nabla u - m\chi_\Omega) = 0 \text{ in } \mathbb{R}^n \quad \big(m \in L^\infty(\Omega;\mathbb{R}^n), u \in W^{1,2}(\mathbb{R}^n)\big) \end{aligned}\right\} \qquad (1)$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous, $m : \Omega \to \mathbb{R}^n$ is the magnetization, $H_{\mathrm{e}} : \mathbb{R}^n \to \mathbb{R}^n$ is a given external magnetic field, $u : \Omega \to \mathbb{R}$ is the potential of the induced magnetic field, and $\chi_\Omega : \mathbb{R}^n \to \{0, 1\}$ denotes the characteristic function of Ω. The first term in E is an anisotropy energy with density φ which is supposed to be an even non-negative function depending on material properties and exhibiting crystallographic symmetry. Two important cases are the uniaxial case, where φ attains its minimum along one axis, and the cubic case when it attains its minimum along three axes. The second term involving H_{e} is an interaction energy and the last term is a magnetostatic energy related with the magnetization field m through $\Delta u = \mathrm{div}(m\chi_\Omega)$. This equation stems from the Maxwell equations (omitting constants)

$$\left.\begin{aligned} \mathrm{div} B &= 0 \\ \mathrm{curl}\, H &= 0 \end{aligned}\right\} \qquad (2)$$

where B is the magnetic induction and H the intensity of the magnetic field. By definition, $B = H + m\chi_\Omega$ and $H = -\nabla u$. Then $\Delta u = \operatorname{div}(m\chi_\Omega)$ follows immediately. Let us notice that the weak formulation of this equation reads as

$$\int_{\mathbb{R}^n} \big[\nabla u(x) - m(x)\chi_\Omega(x)\big]\nabla v(x)\,dx = 0 \qquad \forall\ v \in W^{1,2}(\mathbb{R}^n). \tag{3}$$

In particular, putting $v := u$ we have

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx = \int_\Omega m(x)\cdot\nabla u(x)\,dx \tag{4}$$

which gives

$$\|\nabla u\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le \|m\|_{L^2(\Omega;\mathbb{R}^n)}$$

by the Hölder inequality. It follows from the Lax-Milgram lemma that (3) has for any $m \in L^2(\Omega;\mathbb{R}^n)$ a unique solution $u \in W^{1,2}(\mathbb{R}^n)$ and that the mapping $m \mapsto \nabla u$ is linear and weakly continuous. Hence the magnetostatic energy $m \mapsto \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx$ is sequentially weakly lower semicontinuous.

As the set of admissible magnetizations $\{m \in L^\infty(\Omega;\mathbb{R}^n) : |m| = 1\}$ is not convex, we cannot rely on direct methods (see, e.g., [11]) in proving the existence of a solution to problem (1) (cf. [16] for failure of existence of a solution in a uniaxial case). More precisely, if the weak limit of some minimizing sequence of m's in problem (1) lives for almost all $x \in \Omega$ in the unit sphere, then this is the strong limit (cf. [21: p. 99]). Therefore, a so-called *fine structure* (or, in the "limit" we will speak about a *microstructure*) in m will typically develop, and we have to look for a notion of generalized solutions and to formulate a so-called relaxed problem. Let us emphasize that the fine structure in m is actually observed in real ferromagnetic materials (see [15]).

3. Relaxation in terms of Young measures

We need to describe suitably the oscillating character of sequences $\{(m^k, u^k)\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ minimizing sequence (1). It is well known (see [2, 10, 30]) that we can extract a subsequence (denoted, for simplicity, by the same indices) and find $u \in W^{1,2}(\mathbb{R}^n)$ and a family of probability measures $\nu \equiv \{\nu_x\}_{x\in\Omega}$ such that $\operatorname{supp}(\nu_x) \subset S^{n-1} := \{s \in \mathbb{R}^n : |s| = 1\}$ which is weakly measurable in the sense that $v \bullet \nu$ is Lebesgue measurable for any $v \in C(S^{n-1})$, and

$$\left.\begin{aligned} w\text{-}\lim_{k\to\infty} u^k &= u \\ w^*\text{-}\lim_{k\to\infty} v \circ m^k &= v \bullet \nu \end{aligned}\right\} \tag{5}$$

for any continuous function $v : \mathbb{R}^n \to \mathbb{R}$, where the limits refer respectively to the weak topology in $W^{1,2}(\mathbb{R}^n)$ and the weak* topology in $L^\infty(\Omega)$, and $[v \bullet \nu](x) := \int_{S^{n-1}} v(s)\nu_x(ds)$ for almost all $x \in \Omega$. Let us denote the set of all $\nu \equiv \{\nu_x\}_{x\in\Omega}$ with the above listed properties by $\mathcal{Y}(\Omega; S^{n-1})$; such ν's are called *Young measures*. Conversely, for any $\nu \in \mathcal{Y}(\Omega; S^{n-1})$ there is a sequence of measurable functions $m^k : \Omega \to S^{n-1}$ such that the later convergence in (5) is fulfilled.

The relaxation of problem (1) was done by DeSimone [12], Pedregal [24, 25], Rogers [27], etc. The continuously extended problem obtained by this way looks as follows:

$$\left.\begin{array}{l}\text{minimize}\\ \overline{E}(\nu,u) := \displaystyle\int_\Omega \int_{S^{n-1}} \big(\varphi(s) - H_{\rm e}(x)\cdot s\big)\nu_x(ds)dx + \tfrac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx\\ \text{subject to}\\ \displaystyle\int_{\mathbb{R}^n}\left[\nabla u(x) - \int_{S^{n-1}} \chi_\Omega(x)\, s\, \nu_x(ds)\right]\nabla v(x)\, dx = 0\\ \text{for all } v\in W^{1,2}(\mathbb{R}^n),\ \nu\in\mathcal{Y}(\Omega;S^{n-1}),\ u\in W^{1,2}(\mathbb{R}^n).\end{array}\right\} \quad (6)$$

The probability measure ν_x describes in a proper (we may say "mesoscopic") way the microstructure of the "limit" magnetization at a point x.

The extended problem (6) is a correct relaxation for the original problem (1). Indeed, by [12, 25] the infimum of $\overline{E}$ is attained and it is equal to the infimum of E. Moreover, having (ν, u) as solution to problem (6), there is a sequence $(m^k, u^k) \in L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ satisfying $\Delta u^k = \text{div}(m^k\chi_\Omega)$ in the weak sense, $|m^k| = 1$ a.e., minimizing E, and attaining (ν, u) in the sense (5). Conversely, every sequence $(m^k, u^k) \in L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ satisfying $\Delta u^k = \text{div}(m^k\chi_\Omega)$ weakly, $|m^k| = 1$ and minimizing E contains a subsequence attaining some $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ in the sense (5), and every (ν, u) obtained in such way solves the relaxed problem (6).

One can also think about a "coarser" relaxation in terms of the original "macroscopic" magnetization m. We denote by $\delta_{S^{n-1}}$ the indicator function of the unit sphere, i.e.

$$\delta_{S^{n-1}}(s) = \begin{cases} 0 & \text{if } |s| = 1\\ +\infty & \text{otherwise.}\end{cases}$$

Furthermore, by v^{**} we denote the second Fenchel conjugate (the convex envelope of v), i.e. $v^{**} = \sup\{w \text{ convex} :\ w \le v\}$. This can be used to pose the following relaxed problem:

$$\left.\begin{array}{l}\text{minimize}\\ \widetilde{E}(m,u) = \displaystyle\int_\Omega [\varphi + \delta_{S^{n-1}}]^{**}(m(x)) - H_{\rm e}(x)\cdot m(x)\, dx + \tfrac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx\\ \text{subject to}\\ \text{div}(\nabla u - m\chi_\Omega) = 0 \text{ in } \mathbb{R}^n \ \ \big(m\in L^\infty(\Omega;\mathbb{R}^n),\ u\in W^{1,2}(\mathbb{R}^n)\big);\end{array}\right\} \quad (7)$$

of course, the Poisson equation between u and m is again understood in the weak sense (3). Note that $[\varphi+\delta_{S^{n-1}}]^{**}$ equals $+\infty$ outside the unit ball in $\mathbb{R}^n$ so that any minimizer (m, u) of problem (7) must satisfy $|m(x)| \le 1$ for a.a. $x \in \Omega$. DeSimone [12] showed that $\widetilde{E}$ always attains its minimum on the considered admissible set, and this minimum is equal to the infimum of (1). For any $s \in \mathbb{R}^n$, one has

$$[\varphi + \delta_{S^{n-1}}]^{**}(s) = \inf_{\substack{\mu \text{ probability measure on } S^{n-1}\\ \int_{S^{n-1}} \sigma\mu(d\sigma) = s}} \int_{S^{n-1}} \varphi(\sigma)\mu(d\sigma). \quad (8)$$

Note that, for $|s| > 1$, the set of μ's considered in (8) is empty so that the infimum in (8) is $+\infty$. It is clear that if (ν, u) minimizes $\overline{E}$, then (m, u) with $m(x) = \int_{S^{n-1}} s\nu_x(ds)$ minimizes $\widetilde{E}$. Said differently, a unique minimizer of $\overline{E}$ implies a unique minimizer of $\widetilde{E}$. The opposite implication does not hold because, fixing some $m \in L^\infty(\Omega; \mathbb{R}^n)$ with values in the unit ball centered at the origin, we might still have many (even continuum of) minimizers of $\overline{E}$ with the first moment m (cf. Example 3 below). Clearly, the only term responsible for uniqueness/non-uniqueness is φ. First uniqueness results were obtained by DeSimone [12] who showed uniqueness of solutions to (7) and of Young measure solutions to (13) in the uniaxial case, i.e., if φ is non-negative and equals zero precisely at two points $\pm s \in S^{n-1}$ and has a given representation. Let us emphasize that the proof of the uniqueness of a solution to (7) is a very deep result. Recently, Carstensen and Prohl [6] found a new proof to show that, if φ corresponds to uniaxial ferromagnets, then $\widetilde{E}$ has a unique minimizer.

The Euler-Weierstrass-type optimality conditions for the corresponding $\overline{E}$ will enable us to establish sufficient conditions under which the uniqueness of a minimizer to (7) implies a unique Young measure-valued minimizer (see Proposition 5 and Examples 1 - 2 below). We note that a condition on support of a Young measure minimizer to (13) was also established in [12].

4. Optimality conditions in terms of ν and u

It is usual to identify a given Young measure $\nu \in \mathcal{Y}(\Omega; S^{n-1})$ with the linear functional in $L^1(\Omega; C(S^{n-1}))^*$ defined by

$$\langle \nu, h \rangle = \int_\Omega \int_{S^{n-1}} h(x, s)\nu_x(ds)\, dx. \tag{9}$$

Thus $\mathcal{Y}(\Omega; S^{n-1})$ can be considered as a convex weakly* compact subset of the space $L^1(\Omega; C(S^{n-1}))^*$ (see [28: Corollary 3.1.7]). Furthermore, let us define

$$\Pi : L^1(\Omega; C(S^{n-1}))^* \times W^{1,2}(\mathbb{R}^n) \to W^{1,2}(\mathbb{R}^n)^* \cong W^{1,2}(\mathbb{R}^n)$$

by the formula

$$\langle v, \Pi(\nu, u) \rangle = -\langle \nu, \chi_\Omega \nabla v \otimes \mathrm{id} \rangle + \int_\Omega \nabla v(x) \cdot \nabla u(x)\, dx \tag{10}$$

for $v \in W^{1,2}(\mathbb{R}^n)$, where naturally $[\nabla v \otimes \mathrm{id}](x, s) := \nabla v(x) \cdot s$. Let us note that $\Pi(\nu, u) = 0$ just means that u solves

$$\int_{\mathbb{R}^n} \left[\nabla u(x) - \int_{S^{n-1}} \chi_\Omega(x)\, s\, \nu_x(ds) \right] \nabla v(x)\, dx = 0 \qquad \forall\ v \in W^{1,2}(\mathbb{R}^n). \tag{11}$$

Also note that Π is (weak* $\times$ weak,weak)-continuous and surjective in the sense that

$$\forall\ f \in W^{1,2}(\mathbb{R}^n)^*\ \exists\ u \in W^{1,2}(\mathbb{R}^n)\ \exists\ \nu \in \mathcal{Y}(\Omega; S^{n-1}) : \qquad \Pi(\nu, u) = f \tag{12}$$

which follows immediately from the Lax-Milgram lemma. The relaxed problem (6) now takes the following abstract form:

$$\left.\begin{array}{l}\text{minimize}\\ \overline{E}(\nu,u)\\ \text{subject to}\\ \Pi(\nu,u)=0\quad \big((\nu,u)\in\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)\big).\end{array}\right\} \tag{13}$$

Note that $\overline{E}$ is convex, Π is linear and $\mathcal{Y}(\Omega;S^{n-1})$ is convex, so that problem (13) has a convex structure. As $\overline{E}$ is Gateâux differentiable and $0\in\operatorname{int}(\Pi(\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)))$ due to (12), it is known (see, e.g., Aubin and Ekeland [1: p. 175]) that the first-order optimality conditions looks as follows:

$$\begin{aligned}\overline{E}'(\nu,u) &\in -N_{\operatorname{Ker}\Pi\,\cap\,(\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n))}(\nu,u)\\ &= \operatorname{Range}\Pi^* - N_{\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)}(\nu,u)\\ &= \operatorname{Range}\Pi^* - N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)\times\{0\}\end{aligned}$$

where

$$\begin{aligned}\overline{E}'=(\overline{E}'_\nu,\overline{E}'_u):\ L^1(\Omega;C(S^{n-1}))^*\times W^{1,2}(\mathbb{R}^n) &\to \big(L^1(\Omega;C(S^{n-1}))^{**}\times W^{-1,2}(\mathbb{R}^n)\big)\\ &\cong L^1(\Omega;C(S^{n-1}))^{**}\times W^{1,2}(\mathbb{R}^n)\end{aligned}$$

denotes the Gâteaux differential of E and

$$\Pi^*=(\Pi^*_\nu,\Pi^*_u):\ W^{1,2}(\mathbb{R}^n)\to L^1(\Omega;C(S^{n-1}))^{**}\times W^{1,2}(\mathbb{R}^n)$$

is the adjoint operator to Π. Moreover, $N_{\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)}(\nu,u)$ denotes the normal cone to the convex set $\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)$ at the point (ν,u), and analogously $N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)$ is the normal cone to $\mathcal{Y}(\Omega;S^{n-1})$ at ν, i.e.

$$N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)=\Big\{\xi\in L^1(\Omega;C(S^{n-1}))^{**}\,\Big|\,\langle\xi,\tilde\nu-\nu\rangle\le 0\ \forall\,\tilde\nu\in\mathcal{Y}(\Omega;S^{n-1})\Big\}.$$

Therefore, we can deduce that, if $(\nu,u)\in L^1(\Omega;C(S^{n-1}))^*\times W^{1,2}(\mathbb{R}^n)$ solves problem (13), then there is a Lagrange multiplier $\lambda\in W^{1,2}(\mathbb{R}^n)\cong W^{1,2}(\mathbb{R}^n)^*$ such that

$$\Pi^*_u\lambda-\overline{E}'_u(\nu,u)=0 \tag{14}$$

$$\Pi^*_\nu\lambda-\overline{E}'_\nu(\nu,u)\in N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \tag{15}$$

(see [28: Subsection 5.3]). As problem (13) is convex, conditions (14) - (15) are also sufficient in the sense that, if $(\nu,u)\in\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)$ satisfies $\Pi(\nu,u)=0$ and (14) - (15) for some multiplier $\lambda\in W^{1,2}(\mathbb{R}^n)$, then (ν,u) solves problem (13).

The abstract conditions (14) - (15) turns for the concrete data $\overline{E}$ from (6), Π from (10) and $\mathcal{Y}(\Omega;S^{n-1})$ into the following integral maximum principle:

Proposition 1. *Let* $H_\mathrm{e} \in L^2(\Omega;\mathbb{R}^n)$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be continuous, let* $(\nu, u) \in L^1(\Omega; C(S^{n-1}))^* \times W^{1,2}(\mathbb{R}^n)$ *solve problem* (13) *with the data from* (6) *and* (10). *Then*

$$\langle \nu, \mathcal{H}_u \rangle = \sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n)\\ |m|=1 \ a.e.}} \int_\Omega \mathcal{H}_u(x, m(x))\,dx \tag{16}$$

where the Hamiltonian $\mathcal{H}_u : \Omega \times \mathbb{R}^n \times \mathbb{R}$ *is defined by*

$$\mathcal{H}_u(x,s) := -\nabla u(x)\cdot s + H_\mathrm{e}(x)\cdot s - \varphi(s). \tag{17}$$

Conversely, if $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ *satisfies* $\Pi(\nu, u) = 0$ *and if the maximum principle* (16) *holds, then* (ν, u) *solves problem* (13).

Proof. Let us evaluate the differential of $\overline{E}$. As for $\overline{E}'_u(\nu,u) \in \mathcal{L}(W^{1,2}(\mathbb{R}^n),\mathbb{R}) = W^{1,2}(\mathbb{R}^n)$, we have

$$\langle E'_u(\nu,u), v\rangle = \int_\Omega \nabla u \cdot \nabla v\,dx \tag{18}$$

while for $\overline{E}'_\nu(\nu,u) \in L^1(\Omega; C(S^{n-1}))^{**} = \mathcal{L}(L^1(\Omega;C(S^{n-1}))^*,\mathbb{R})$ we have

$$[E'_\nu(\nu,u)](\tilde\nu) = \langle \tilde\nu, 1\otimes\varphi - H_\mathrm{e}\otimes \mathrm{id}\rangle \tag{19}$$

where naturally $[1\otimes\varphi](x,s) = \varphi(s)$ and $[H_\mathrm{e}\otimes\mathrm{id}](x,s) = H_\mathrm{e}(x)\cdot s$. Equation (14) now gives

$$\int_\Omega \nabla v\cdot\nabla\lambda\,dx = \langle\lambda, \Pi(0,v)\rangle = \langle \Pi_u^*\lambda, v\rangle = \langle \overline{E}'_u(\nu,u), v\rangle = \int_\Omega \nabla u\cdot\nabla v\,dx \tag{20}$$

for any $v \in W^{1,2}(\mathbb{R}^n)$, from which we get simply $\lambda = u + \text{constant}$. As λ should live in $W^{1,2}(\mathbb{R}^n)$, this constant must vanish so that we eventually have $\lambda = u$.

Inclusion (15) results in the inequality

$$\begin{aligned} 0 &\geq \langle \Pi_\nu^*\lambda - \overline{E}'_\nu(\nu,u), \tilde\nu - \nu\rangle \\ &= \langle \lambda, \Pi(\tilde\nu-\nu, 0)\rangle - \langle\tilde\nu - \nu, 1\otimes\varphi - H_\mathrm{e}\otimes\mathrm{id}\rangle \\ &= \langle \tilde\nu - \nu, -\chi_\Omega\nabla\lambda\otimes\mathrm{id} - 1\otimes\varphi + H_\mathrm{e}\otimes\mathrm{id}\rangle \end{aligned}$$

for all $\tilde\nu \in \mathcal{Y}(\Omega; S^{n-1})$. This gives $\langle\tilde\nu - \nu, \mathcal{H}_\lambda\rangle \leq 0$ with the Hamiltonian $\mathcal{H}_\lambda = \mathcal{H}_u$ given by (17). By (20), $\mathcal{H}_\lambda = \mathcal{H}_u$. In other words, we got

$$\langle \nu, \mathcal{H}_u\rangle = \max_{\tilde\nu\in\mathcal{Y}(\Omega;S^{n-1})} \langle\tilde\nu, \mathcal{H}_u\rangle. \tag{21}$$

As E'_ν as well as Π_ν^* take their values in the space $L^1(\Omega;C(S^{n-1}))$ rather than in the space $L^1(\Omega;C(S^{n-1}))^{**}$, we can take into considerations only the intersection of the normal cone $N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \subset L^1(\Omega;C(S^{n-1}))^{**}$ with $L^1(\Omega;C(S^{n-1}))$ as was already done in [28]. Hence,

$$\begin{aligned} &N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \cap L^1(\Omega;C(S^{n-1})) \\ &\quad = \left\{ h\in L^1(\Omega;C(S^{n-1})) \,\middle|\, \langle\tilde\nu, h\rangle \leq \langle\nu, h\rangle \ \text{ for all } \ \tilde\nu\in\mathcal{Y}(\Omega;S^{n-1}) \right\} \\ &\quad = \left\{ h\in L^1(\Omega;C(S^{n-1})) \,\middle|\, \langle\nu, h\rangle = \sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n)\\ |m|=1 \ a.e.}} \int_\Omega h(x,m(x))\,dx \right\} \end{aligned}$$

which eventually gives us (16). As problem (13) is convex, the maximum principle (16) is also sufficient in the above specified sense ∎

Thanks to the special form of the set of admissible magnetizations in (1) admitting arbitrary oscillations of m, the integral maximum principle (16) can be localized into the following pointwise maximum principle, which gives a very explicit restriction on possible steady-state microstructures.

Proposition 2. *Let $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ solve the relaxed problem* (13)*. Then*

$$\int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(ds) = \max_{s\in S^{n-1}} \mathcal{H}_u(x,s) \quad \text{for a.a.} \quad x \in \Omega \tag{22}$$

with the Hamiltonian again from (17)*. In other words,*

$$\operatorname{supp}(\nu_x) \subset \operatorname{Argmax} \mathcal{H}_u(x,\cdot) \tag{23}$$

where

$$\operatorname{Argmax} \mathcal{H}_u(x,\cdot) := \Big\{ s \in S^{n-1} \Big| \, \mathcal{H}_u(x,s) = \max \mathcal{H}_u(x, S^{n-1}) \Big\}.$$

Conversely, if $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ satisfies $\Pi(\nu, u) = 0$ and (23) *holds for a.a. $x \in \Omega$, then (ν, u) solves the relaxed problem* (13)*.*

Proof. We will show that (16) and (22) are equivalent to each other. Due to [13: Theorem 1.2/Chapter VIII], there exists $\tilde{m} : \Omega \to S^{n-1}$ measurable such that $\mathcal{H}_u(x, \tilde{m}(x)) = \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)$ for a.a. $x \in \Omega$. First, suppose that (16) is fulfilled. Therefore,

$$\begin{aligned}
\langle \nu, \mathcal{H}_u \rangle &= \int_\Omega \int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(ds)dx \\
&= \sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n) \\ |m|=1 \ a.e.}} \int_\Omega \mathcal{H}_u(x, m(x))\, dx \\
&\geq \int_\Omega \mathcal{H}_u(x, \tilde{m}(x))\, dx \\
&= \int_\Omega \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\, dx.
\end{aligned}$$

In other words,

$$\int_\Omega \left(\int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(ds) - \max_{s\in S^{n-1}} \mathcal{H}_u(x,s) \right) dx \geq 0.$$

At the same time,

$$\int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(ds) \leq \max_{s\in S^{n-1}} \mathcal{H}_u(x,s) \qquad \text{for a.a. } x \in \Omega$$

which shows that (22) holds.

Let now (22) be satisfied. Integrating it over Ω one gets

$$\begin{aligned}
\langle \nu, \mathcal{H}_u \rangle &= \int_\Omega \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\, dx \\
&\geq \sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n) \\ |m|=1 \ a.e.}} \int_\Omega \mathcal{H}_u(x, m(x))\, dx \\
&\geq \int_\Omega \mathcal{H}_u(x, \tilde{m}(x))\, dx.
\end{aligned}$$

On the other hand,

$$\int_\Omega \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\,dx = \int_\Omega \mathcal{H}_u(x,\widetilde{m}(x))\,dx$$

and thus (16) holds ∎

5. Optimality conditions in terms of ν and m

One can also alternatively consider optimality conditions when the energy functional is supposed to depend on the "mesoscopic" Young-measure magnetization ν and the "macroscopic" magnetization m. Interestingly, it turns out that such optimality conditions are the same as those derived in the previous section. In order to show this we will define a new functional

$$e:\ L^1(\Omega;C(S^{n-1}))^*\times L^2(\Omega;\mathbb{R}^n)\to\mathbb{R}$$

by

$$e(\nu,m)=\langle\nu,1\otimes\varphi\rangle-\int_\Omega H_{\rm e}(x)\cdot m(x)\,dx+\tfrac12\int_{\mathbb{R}^n}|\nabla u(x)|^2dx$$

with ∇u determined via $\Delta u=\operatorname{div}(m\chi_\Omega)$. Eventually, we define

$$\pi:\ L^1(\Omega;C(S^{n-1}))^*\times L^2(\Omega;\mathbb{R}^n)^*\to L^2(\Omega;\mathbb{R}^n)$$

by

$$\pi(\nu,m)=\mathrm{id}\bullet\nu-m$$

and we see that $\pi(\mathcal{Y}(\Omega;S^{n-1})\times L^2(\Omega;\mathbb{R}^n))=L^2(\Omega;\mathbb{R}^n)$. Thus we are concerned with the problem

$$\left.\begin{array}{l}\text{minimize}\\ e(\nu,m)\\ \text{subject to}\\ \pi(\nu,m)=0\quad\big((\nu,m)\in\mathcal{Y}(\Omega;S^{n-1})\times L^2(\Omega;\mathbb{R}^n)\big).\end{array}\right\}\tag{24}$$

Note that π is continuous and linear and e is convex. We will also show that e is Gateâux differentiable. The first order optimality conditions read in this case that, if $(\nu,m)\in\mathcal{Y}(\Omega;S^{n-1})\times L^2(\Omega;\mathbb{R}^n)$ solve problem (24), then there is a Lagrange multiplier $\ell\in L^2(\Omega;\mathbb{R}^n)$ such that

$$\pi_m^*\ell-e_m'(\nu,m)=0\tag{25}$$

$$\pi_\nu^*\ell-e_\nu'(\nu,m)\in N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)\tag{26}$$

where

$$e'=(e_\nu',e_m'):\ L^1(\Omega;C(S^{n-1}))^*\times L^2(\Omega;\mathbb{R}^n)\to L^1(\Omega;C(S^{n-1}))^{**}\times L^2(\Omega;\mathbb{R}^n)$$

$$\pi^*=(\pi_\nu^*,\pi_m^*):\ L^2(\Omega;\mathbb{R}^n)\to L^1(\Omega;C(S^{n-1}))^{**}\times L^2(\Omega;\mathbb{R}^n).$$

Proposition 3. *Let* $H_\mathrm{e} \in L^2(\Omega;\mathbb{R}^n)$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ *continuous, let* $(\nu, m) \in \mathcal{Y}(\Omega; S^{n-1}) \times L^2(\Omega;\mathbb{R}^n)$ *solve problem* (24) *and let* u *solve equation* (3). *Then*

$$\langle \nu, h_\ell \rangle = \sup_{\substack{\widetilde{m}\in L^2(\Omega;\mathbb{R}^n)\\ |\widetilde{m}|=1 \ a.e.}} \int_\Omega h_\ell(x, \widetilde{m}(x))\, dx \tag{27}$$

where the Hamiltonian in now defined as

$$h_\ell = \ell \otimes \mathrm{id} - \varphi \tag{28}$$

with $\ell = H_\mathrm{e} - \nabla u$.

Proof. First we prove that $e'(\nu, m) = (\varphi, -H_e + \nabla u)$. The first component of e', namely e'_ν, is obvious because $e(\cdot, m)$ is affine. As to the second component, we denote by w the solution to equation (3) with arbitrary $v \in L^2(\Omega;\mathbb{R}^n)$ instead of m. Then we have

$$\begin{aligned}
[e'_m(\nu, m)](v) &= -H_e \cdot v + \tfrac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u(x) + t\nabla w(x)|^2 dx\Big|_{t=0} \\
&= -H_e \cdot v + \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla w(x)\, dx \\
&= -H_e \cdot v + \int_\Omega \nabla u(x) \cdot v(x)\, dx \\
&= (-H_e + \nabla u) \cdot v
\end{aligned}$$

where we used, beside (3) with v instead of m, also the linearity of the mapping $m \mapsto \nabla u$. Furthermore, $\pi^*\ell = (\ell \otimes \mathrm{id}, -\ell)$ holds for any $\ell \in L^2(\Omega;\mathbb{R}^n)$ because

$$\langle \pi^*\ell, (\nu, m)\rangle = \langle \ell, \pi(\nu, m)\rangle = \langle \ell, \mathrm{id} \bullet \nu\rangle - \langle \ell, m\rangle = \langle \nu, \ell \otimes \mathrm{id}\rangle - \langle \ell, m\rangle.$$

Relations (25), (26) now turn into

$$\ell = H_e - \nabla u \tag{29}$$

$$-\varphi + \ell \otimes \mathrm{id} \in N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu), \tag{30}$$

respectively. Again, since e'_ν as well as π^*_ν take their values in $L^1(\Omega; C(S^{n-1}))$ rather than in $L^1(\Omega; C(S^{n-1}))^{**}$, we obtain the claimed maximum principle ∎

Proposition 4. *Under the assumptions of the previous proposition*

$$[h_\ell \bullet \nu](x) = \max_{s\in S^{n-1}} h_\ell(x, s) \qquad \text{for a.a. } x \in \Omega. \tag{31}$$

The proof of the above point-wise version is analogous to that of Proposition 2.

We point out that $h_\ell = \mathcal{H}_u$ provided $\ell = H_\mathrm{e} - \nabla u$ so that, in fact, Propositions 1 and 2 are equivalent with Propositions 3 and 4, respectively.

6. Some consequences

The following proposition gives a sufficient condition under which the relaxed problem (13) has a unique minimizer. This condition is indeed satisfied in some physically relevant situations (see Examples 1 and 2 below) while in other situations admitting many minimizers is not satisfied (see Example 3).

Proposition 5. *Let problem* (7) *possess a unique minimizer and let, for any* $r \in \mathbb{R}^n$*, the function* $S^{n-1} \to \mathbb{R} : s \mapsto r \cdot s - \varphi(s)$ *attain its maximum at a finite number* $\kappa(r) \leq n+1$ *of points* σ_r^l, $l = 1, \ldots, \kappa(r)$*. Then problem* (13) *has a unique solution.*

Proof. The proof paraphrases that of [28: Corollary 5.3.4]. Let $(m, u) \in L^2(\Omega; \mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ be the unique minimizer of problem (7) and let (ν^1, u^1) and (ν^2, u^2) be two different solutions to problem (13). Let us denote $m^1 = \mathrm{id} \bullet \nu^1$ and $m^2 = \mathrm{id} \bullet \nu^2$. As (m^1, u^1) and (m^2, u^2) must solve problem (7), we get by our uniqueness assumptions

$$\left.\begin{aligned} m^1 &= m = m^2 \\ u^1 &= u = u^2. \end{aligned}\right\} \tag{32}$$

Then the Hamiltonian is determined uniquely, i.e. $\mathcal{H}_{u^1} = \mathcal{H}_{u^2}$. By (23) and the assumption, the probability measure ν_x^i must be supported at a finite number $k(x) = \kappa(H_\mathrm{e}(x) - \nabla u(x))$ of points $s^l(x) = \sigma^l_{H_\mathrm{e}(x)-\nabla u(x)}$, i.e. $\nu_x^i = \sum_{l=1}^{k(x)} a_l^i(x)\delta_{s^l(x)}$ with $a_l^i \geq 0$ and $\sum_{l=1}^{k(x)} a_l^i = 1$ a.e. in Ω. By (32), we have

$$\sum_{l=1}^{k(x)} (a_l^1(x) - a_l^2(x)) s^l(x) = m^1(x) - m^2(x) = 0.$$

The assumed restriction on $\kappa(r)$ and therefore also on $k(x)$ yields the linear independency of $\{s^l(x)\}_{l=1}^{k(x)}$ for almost all $x \in \Omega$ and thus $a^1 = a^2$ a.e. in Ω ∎

Proposition 6. *Let the assumptions of Proposition* 5 *be fulfilled. Then the original problem* (1) *possesses a solution* (*being equal just to* (u, m)*, the assumed unique solution to problem* (7)) *if and only if, for a.a.* $x \in \Omega$,

$$m(x) \in \{\sigma_r^l\}_{l=1}^{\kappa(r)} \qquad \text{with } \ r = H_\mathrm{e}(x) - \nabla u(x) \tag{33}$$

holds.

Proof. The unique solution (ν, u) to problem (13) is 1-atomic at a given $x \in \Omega$, i.e. of the form $\nu_x = \delta_{m(x)}$, if and only if (33) holds. If (33) holds a.e., (m, u) then solves problem (1). Conversely, if (m, u) solves problem (1), then (ν, u) with $\nu_x = \delta_{m(x)}$ solves problem (13). As this solution is unique, failure of (33) for x from a positive Lebesgue measure set implies failure of existence of any solution to problem (1) ∎

Example 1 (Uniaxial magnets I). Let us take $n = 2$ and $\varphi : \mathcal{S}^1 \to \mathbb{R}$ as

$$\varphi(s) = s_1^2 + \min\big\{(s_2 - 1)^2, (s_2 + 1)^2\big\}.$$

Such potential is related with the so-called uniaxial ferromagnet. For this type of ferromagnets there is a unique solution to problem (7) if φ is suitable; (cf. DeSimone [12] or also Carstensen and Prohl [6]). Suppose that for our φ (7) has a unique solution. As $n = 2$, we have $S^1 = \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}$ and, for $\phi(\theta) := \varphi(\cos\theta, \sin\theta)$, it is easy to see that $\phi(\theta) = 2 - 2|\sin\theta|$. For $r = (r_1, r_2) \in \mathbb{R}^2$, let us further denote

$$f(\theta, r) = \phi(\theta) - r_1 \cos\theta - r_2 \sin\theta.$$

Differentiating f with respect to θ we have

$$\frac{\partial f}{\partial\theta}(\theta, r) = \begin{cases} r_1 \sin\theta - (r_2 + 2)\cos\theta & \text{if } \theta \in (0, \pi) \\ r_1 \sin\theta - (r_2 - 2)\cos\theta & \text{if } \theta \in (\pi, 2\pi). \end{cases}$$

We see that $f(\cdot, r)$ has at most four local extrema at $\theta = 0$, $\theta = \pi$, $\theta = \theta_1 \in [0, \pi]$ and $\theta = \theta_2 \in [\pi, 2\pi]$ where

$$\theta_1 := \begin{cases} \arctan \frac{r_2+2}{r_1} & \text{if } r_1 \neq 0 \\ \frac{\pi}{2} & \text{if } r_1 = 0 \end{cases} \qquad \text{and} \qquad \theta_2 := \begin{cases} \arctan \frac{r_2-2}{r_1} & \text{if } r_1 \neq 0 \\ \frac{3\pi}{2} & \text{if } r_1 = 0. \end{cases}$$

On the other hand, there is no $r \in \mathbb{R}^2$ for which $f(0, r) = f(\pi, r) = f(\theta_1, r) = f(\theta_2, r)$. This shows together with Proposition 3 that $\kappa(r) \leq 3 \ \ (r \in \mathbb{R}^2)$, and that for φ as above problem (13) has a unique solution. In truth, one can even show that $\kappa(r) \leq 2$, so that problem (13) has a solution (ν, u) with $\nu_x = \lambda(x)\delta_{s_1(x)} + (1 - \lambda(x))\delta_{s_2(x)}$ for almost all $x \in \Omega$.

Example 2 (Uniaxial magnets II). By similar arguments one obtains uniqueness also for $n = 2$ and $\varphi : \mathcal{S}^1 \to \mathbb{R}$ as

$$\varphi(s) = \varphi(s_1, s_2) = s_1^2 \ .$$

The uniqueness of the solution to (7) as well as of (13) has been already observed in [12].

Example 3 (Cubic magnets). Let us take $n = 3$ and $\varphi : \mathcal{S}^2 \to \mathbb{R}$ as

$$\varphi(s) = \varphi(s_1, s_2, s_3) = s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2.$$

Then one can see that for $H_{\mathrm{e}} = 0$ there are many solutions to problem (13), for example, $\nu_x = \frac{1}{2}\delta_{(0,0,1)} + \frac{1}{2}\delta_{(0,0,-1)} \ \ (x \in \Omega)$, $u = 0$ or $\nu_x = \frac{1}{2}\delta_{(1,0,0)} + \frac{1}{2}\delta_{(-1,0,0)} \ \ (x \in \Omega)$, $u = 0$. Note that the assumptions of Proposition 5 are indeed not satisfied because $\kappa(0) = 6 > n + 1 = 4$.

Acknowledgment. The research was conducted during the stay of M. K. in the Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany whose hospitality is gratefully acknowledged. The work was supported by the grant A 107 5707 (Grant Agency of the Academy of Sciences of the Czech Republic).

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Received 28.05.1999