Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 2, 429–451

Local Existence of the Solution to the Initial-Boundary Value Problem in Nonlinear Thermodiffusion in Micropolar Medium

J. Gawinecki

Abstract. We prove a theorem about local existence (in time) of the solution to the first initial-boundary value problem for a nonlinear hyperbolic-parabolic system of eight coupled partial differential equations of second order describing the process of thermodiffusion in a three-dimensional micropolar medium. At first, we prove existence, uniqueness and regularity of the solution to this problem for the associated linearized system by using the Faedo-Galerkin method and semi-group theory. Next, we prove (basing on this theorem) an energy estimate for the solution to the linearized system by applying the method of Sobolev spaces. At last, by using the Banach fixed point theorem we prove that the solution of our nonlinear problem exists and is unique.

Keywords: *Local existence of solutions, linear and nonlinear hyperbolic- parabolic systems, initial-boundary value problems, Sobolev spaces, semigroup theory, energy estimates, Banach fixed point theorem*

AMS subject classification: 35 G 25, 35 G 30, 35 M 50, 73 B 30, 73 C 15, 80 A 20

1. Introduction

In this paper, we consider the initial-boundary value problem (with Dirichlet boundary conditions) for a nonlinear hyperbolic-parabolic system of eight partial differential equations of second order describing the process of thermodiffusion in a three-dimensional micropolar medium ($i = 1, 2, 3$):

$$\partial_t^2 u_i - c_{i\alpha j\beta}(\nabla u, \nabla\varphi, \theta_1, \theta_2)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} - \alpha_{ij}(\nabla u, \nabla\varphi)\,\varepsilon_{jlk}\frac{\partial \varphi_k}{\partial x_l} + \tilde{c}^1_{i\alpha}(\nabla u, \theta_1, \theta_2)\frac{\partial \theta_1}{\partial x_\alpha} + \tilde{c}^2_{i\alpha}(\nabla u, \theta_1, \theta_2)\frac{\partial \theta_2}{\partial x_\alpha} = f_i \tag{1.1}$$

$$\partial_t^2 \varphi_i - d_{i\alpha j\beta}(\nabla u, \nabla\varphi, \theta_1, \theta_2)\frac{\partial^2 \varphi_j}{\partial x_\alpha x_\beta} + \tilde{\alpha}_{ij}(\nabla u, \nabla\varphi)\varphi_j - \alpha_{ij}(\nabla u, \nabla\varphi)\,\varepsilon_{jlk}\frac{\partial u_k}{\partial x_l} = Y_i \tag{1.2}$$

J. Gawinecki: Inst. Math. & Oper. Res., Military Univ. Techn., Warsaw, Poland

ISSN 0232-2064 / $ 2.50

$$c(\nabla u,\theta_1,\theta_2)\partial_t\theta_1 - a^1_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta}$$
$$+\bar{c}^1_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + d(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_2}{\partial t} = Q_1 \tag{1.3}$$
$$n(\nabla u,\theta_1,\theta_2)\partial_t\theta_2 - a^2_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta}$$
$$+\bar{c}^2_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + d(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_1}{\partial t} = Q_2. \tag{1.4}$$

Here we use the notations:

$u = u(t,x) = \big(u_1(t,x), u_2(t,x), u_3(t,x)\big)^*$ – displacement vector of the medium

$\varphi = \varphi(t,x) = \big(\varphi_1(t,x), \varphi_2(t,x), \varphi_3(t,x)\big)^*$ – microrotation vector

$\theta_1 = \theta(t,x)$ – temperature of the medium

$\theta_2 = \theta_2(t,x)$ $(t \in \mathbb{R}^+_0, x \in \Omega)$ – chemical potential

$\Omega \subset \mathbb{R}^3$ – bounded domain with smooth enough boundary $\partial\Omega$

$\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u), \nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$ – gradients of u and φ

$\nabla\theta_1 = (\partial_1\theta_1, \partial_2\theta_2, \partial_3\theta_3), \nabla\theta_2 = (\partial_1\theta_2, \partial_2\theta_2, \partial_1\theta_3)$ – gradients of θ_1 and θ_2

$\partial_t = \frac{\partial}{\partial t}$ and $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ $(\alpha = 1,2,3)$

$f = f(t,x) = \big(f_1(t,x) f_2(t,x) f_3(t,x)\big)^*$ – body force vector

$Y = Y(t,x) = \big(Y_1(t,x), Y_1(t,x), Y_2(t,x), Y_3(t,x)\big)^*$ – body couple vector

$Q_1 = Q_1(t,x)$ – intensity of heat source

$Q_2 = Q_2(t,x)$ – intensity of the source of diffusing mass.

Further, $*$ means transposition, the nonlinear coefficients

$$\alpha_{ij},\ \tilde{c}^1_{i\alpha},\ \tilde{c}^2_{i\alpha},\ d_{i\alpha j\beta},\ \tilde{\alpha}_{ij},\ a^1_{\alpha\beta},\ c,\ \tilde{c}^1_{i\alpha},\ d,\ n,\ a^2_{\alpha\beta},\ c_{i\alpha j\beta}$$

depend from unknown functions and their gradients, and the symbol ε_{jlk} (j,l,k) is defined by

$$\varepsilon_{jlk} = \begin{cases} +1 & \text{when the permutation of the indexes } j,l,k \text{ is even} \\ -1 & \text{when the permutation of the indexes } j,l,k \text{ is odd.} \end{cases} \tag{1.5}$$

We will pose initial conditions

$$u(0,x) = u^0(x) \qquad (\partial_t u)(0,x) = u^1(x) \tag{1.6}$$
$$\varphi(0,x) = \varphi^0(x) \qquad (\partial_t\varphi)(0,x) = \varphi^1(x) \tag{1.7}$$
$$\theta_1(0,x) = \theta^0_1(x) \qquad \theta_2(0,x) = \theta^0_2(x) \tag{1.8}$$

with given data $u^0, \varphi^0, \theta^0_1$ and $u^1, \varphi^1, \theta^0_2,$ and boundary conditions (conditions of Dirichlet type)

$$\left.\begin{aligned} u(t,\cdot)|_{\partial\Omega} &= 0 \\ \varphi(t,\cdot)|_{\partial\Omega} &= 0 \\ \theta_1(t,\cdot)|_{\partial\Omega} &= 0 \\ \theta_2(t,\cdot)|_{\partial\Omega} &= 0 \end{aligned}\right\}. \tag{1.9}$$

Putting in system (1.1) - (1.4)

$$
\begin{aligned}
&c_{i\alpha j\beta} = \mu\delta_{\alpha\beta} + (\lambda+\mu)\delta_{ij} \\
&\alpha_{ij} = 2\alpha, \quad \bar{\alpha}_{ij} = 4\alpha, \quad \bar{c}^1_{i\alpha} = \gamma_1\delta_{i\alpha}, \quad \bar{c}^2_{i\alpha} = \gamma_2\delta_{i\alpha} \\
&d_{i\alpha j\beta} = (\gamma+\varepsilon)\delta_{\alpha\beta} + (\beta+\gamma-\varepsilon)\delta_{ij} \\
&a^1_{\alpha\beta} = \delta_{\alpha\beta}, \quad a^2_{\alpha\beta} = k\delta_{\alpha\beta}, \quad \bar{a}_{\alpha\beta} = \mathcal{D}\delta_{\alpha\beta} \\
&c = \text{const}, \quad d = \text{const}, \quad n = \text{const}
\end{aligned}
\tag{1.10}
$$

we obtain from (1.1) - (1.4) the linear hyperbolic-parabolic system of equation describing the process of thermodiffusion in micropolar medium (cf. [14, 17, 18]).

The initial-boundary value problem for the linear system of equations describing the process of thermodiffusion in micropolar medium was investigated by W. Nowacki [17] using the method of integral transformations. In the paper [5] a theorem about existence, uniqueness and regularity of the weak solution to the first initial-boundary value problem for the linear hyperbolic-parabolic system of thermodiffusion in micropolar medium was proved applying the Faedo-Galerkin method in suitably chosen Sobolev spaces.

The nonlinear hyperbolic-parabolic system (1.1) - (1.4) describing thermodiffusion in micropolar medium has not been investigated up till now. The aim of this paper is to prove a local existence theorem in the class of smooth functions with respect to the time variable and taking values in suitable Sobolev spaces with respect to the spatial variables.

For first order hyperbolic-parabolic systems local existence for the Cauchy problem has been studied in [9] and [24]. For the initial-boundary value problem in $\mathbb{R}^3$ there are results for systems with first order hyperbolic part with non-characteristic [22] or special characteristic boundary and admissible boundary conditions in the Friedrichs sense for bounded domains [12]. In [8] hyperbolic-parabolic systems in both bounded and unbounded domains with first order hyperbolic-parabolic part were studied. Some results devoted to existence (local in time) of solutions to initial-boundary value problems for quasi-linear hyperbolic-parabolic systems of first order were obtained in [20] using a method different from ours.

In our investigation, we use semigroup theory, methods of Sobolev spaces and energy estimates, and the Banach fixed point theorem. The paper is organized as follows: In Section 2 some notations and facts from the theory of Sobolev spaces are presented. Section 3 is devoted to the formulation of the main theorem (Theorem 3.1) of the paper. In Section 4 we prove an energy estimate to the linearized system of thermodiffusion in micropolar medium associated with the nonlinear one. Finally, in Section 5 the proof of the main theorem is presented.

2. Basic notations

We shall use the notations

$$\partial_j = \frac{\partial}{\partial x_j} \qquad (j = 1, 2, 3)$$
$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \qquad (|\alpha| = \alpha_1 + \alpha_2 + \alpha_3).$$

For $0 < m < \infty$ we denote by $H^m(\Omega)$ and $H_0^m(\Omega)$ the usual Sobolev spaces with norms $\| \cdot \|_m$ [1]. For $1 \le p \le \infty$ we denote by $L^p(\Omega)$ the Lebesque function space on Ω with norm $\| \cdot \|_{L^p}$. However, the norm and the inner product in $L^2(\Omega)$ are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, respectively. For any integer $N \ge 0$ we denote

$$\mathcal{D}^N u = (\partial_t^j \partial_x^\alpha u;\ j + |\alpha| = N)$$
$$\bar{\mathcal{D}}^N u = (\partial_t^j \partial_x^\alpha u;\ j + |\alpha| \le N)$$
$$\mathcal{D}_x^N u = (\partial_x^\alpha u;\ |\alpha| = N)$$
$$\bar{\mathcal{D}}_x^N u = (\partial_x^\alpha u;\ |\alpha| \le N).$$

If $f \in X$ for a space with norm $\| \cdot \|_X$ means that each component $f_1, \ldots, f_n$ of f is in X, then

$$\|f\|_X = \|f_1\|_X + \ldots + \|f_n\|_X.$$

For any $0 \le m < \infty$ and $T > 0$ we also use the notation

$$|u|_{m,T} = \sup_{0 \le t \le T} \|u(t)\|_m$$

where $\| \cdot \|_0$ denotes $\| \cdot \|$.

3. Main theorem

In this section, we formulate the main theorem about local existence (in time) of the solution of the initial-boundary value problem to the nonlinear system (1.1) - (1.4). Before starting its formulation we notice that under the assumption $cn - d^2 > 0$ we can convert system (1.1) - (1.4) into the following form:

$$\partial_t^2 u_i - c_{i\alpha j\beta}(\nabla u, \nabla\varphi, \theta_1, \theta_2) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} - \alpha_{ij}(\nabla u, \nabla\varphi)\, \varepsilon_{jlk} \frac{\partial \varphi_k}{\partial x_l}$$
$$+ \bar{c}_{i\alpha}^1(\nabla u, \theta_1, \theta_2) \frac{\partial \theta_1}{\partial x_\alpha} + \bar{c}_{i\alpha}^2(\nabla u, \theta_1, \theta_2) \frac{\partial \theta_2}{\partial x_\alpha} = f_i \qquad (3.1)$$

$$\partial_t^2 \varphi_i - d_{i\alpha j\beta}(\nabla u, \nabla\varphi, \theta_1, \theta_2) \frac{\partial^2 \varphi_j}{\partial x_\alpha \partial x_\beta}$$
$$+ \bar{\alpha}_{ij}(\nabla u, \nabla\varphi)\varphi_j - \alpha_{ij}(\nabla u, \nabla\varphi)\, \varepsilon_{jlk} \frac{\partial u_k}{\partial x_l} = Y_i \qquad (3.2)$$

$$\partial_t\theta_1 - \bar{a}^{11}_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta}$$
$$-\bar{a}^{12}_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \tag{3.3}$$
$$\mathbf{C}^1_{i\alpha}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_1(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2,t,x)$$

$$\partial_t\theta_2 - \bar{a}^{21}_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta}$$
$$-\bar{a}^{22}_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \tag{3.4}$$
$$\mathbf{C}^2_{i\alpha}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_2(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2,x,t)$$

where, denoting $\delta = cn - d^2$,

$$\begin{aligned}
\bar{a}^{11}_{\alpha\beta} &= \frac{n}{\delta}a^1_{\alpha\beta} & \bar{a}^{12}_{\alpha\beta} &= -\frac{d}{\delta}a^2_{\alpha\beta}\\
\bar{a}^{21}_{\alpha\beta} &= -\frac{d}{\delta}a^2_{\alpha\beta} & \bar{a}^{22}_{\alpha\beta} &= \frac{c}{\delta}a^2_{\alpha\beta}\\
\mathbf{C}^1_{i\alpha} &= \frac{d\bar{c}^2_{i\alpha} - n\bar{c}^1_{i\alpha}}{\delta} & g_1 &= \frac{Q_1 n - dQ_2}{\delta}\\
\mathbf{C}^2_{i\alpha} &= \frac{d\bar{c}^1_{i\alpha} - c\bar{c}^2_{i\alpha}}{\delta} & g_2 &= \frac{Q_2 n - dQ_1}{\delta}.
\end{aligned} \tag{3.5}$$

Now we formulate our main theorem.

Theorem 3.1 (Local existence in time). *Let the following conditions be satisfied:*

1° $s \geq [\frac{3}{2}] + 4 = 5$ *is an arbitrary but fixed integer.*

2°

$$\partial_t^k f_i, \partial_t^k Y_i, \partial_t^k Q_1, \partial_t^k Q_2 \in C^0\big([0,T], H^{s-2-k}(\Omega)\big) \quad (k = 0,1,\ldots,s-2)$$
$$\partial_t^{s-1} f_i, \partial_t^{s-1} Y_i, \partial_t^{s-1} Q_1, \partial_t^{s-1} Q_2 \in L^2\big([0,T], L^2(\Omega)\big).$$

3° *There are three constants* $\gamma_1,\gamma_2,\gamma_3$ *such that*

$$(C_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) \geq \gamma_1|\xi|^2|\eta|^2$$
$$(d_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) \geq \gamma_2|\xi|^2|\eta|^2$$
$$(\bar{a}_{\alpha\beta}\xi_\alpha\xi_\beta\bar{\eta},\bar{\eta}) \geq \gamma_3|\xi|^2|\bar{\eta}|^2$$

for $\xi = (\xi_1,\xi_2,\xi_3), \eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$ *and* $\bar{\eta} = (\bar{\eta}_1,\bar{\eta}_2) \in \mathbb{R}^2$, *where*

$$C_{\alpha\beta} = [c_{i\alpha j\beta}], \quad d_{\alpha\beta} = [d_{i\alpha j\beta}] \;\; (i,j = 1,2,3), \quad \bar{a}_{\alpha\beta} = [\bar{a}^{ij}_{\alpha\beta}] \;\; (i,j = 1,2)$$
$$d \in C^{s-1}(\mathbb{R}^{17}) \quad \text{and} \quad c,n \in C^{s-1}(\mathbb{R}^{11})$$
$$c_{i\alpha j\beta}, \alpha_{ij}, \bar{c}^1_{i\alpha}, \bar{c}^2_{i\alpha}, d_{i\alpha j\beta}, \bar{\alpha}_{ij} \in C^{s-1}(\mathbb{R}^{11})$$
$$c_{i\alpha j\beta} = c_{j\beta i\alpha}, \quad d_{i\alpha j\beta} = d_{j\beta i\alpha}, \quad \bar{a}^{ij}_{\alpha\beta} = \bar{a}^{ji}_{\beta\alpha}.$$

4° *The initial data* $u^0, \varphi^0, u^1, \varphi^1, \theta_1^0, \theta_2^0$ *shall satisfy*

$$\begin{aligned} u^0, \varphi^0, \theta_1^0, \theta_2^0 &\in H^s(\Omega) \cap H_0^1(\Omega) \quad , \\ u^1, \varphi^1 &\in H^{s-1}(\Omega) \cap H_0^1(\Omega) \end{aligned}$$

for $x \in \Omega$ *and the compatibility conditions:*

$$\begin{aligned} u^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (2 \le k \le s-1) \\ u^s &\in L^2(\Omega) \\ \varphi^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (2 \le k \le s-1) \\ \varphi^s &\in L^2(\Omega) \\ \theta_1^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (1 \le k \le s-2) \\ \theta_1^{s-1} &\in L^2(\Omega) \\ \theta_2^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (1 \le k \le s-2) \\ \theta_2^{s-1} &\in L^2(\Omega) \end{aligned}$$

where $u^k = \frac{\partial^k u(0,\cdot)}{\partial_t k}, \varphi^k = \frac{\partial^k \varphi(0,\cdot)}{\partial t^k}, \theta_1^k = \frac{\partial^k \theta_1(0,\cdot)}{\partial t^k}, \theta_2^k = \frac{\partial^k \theta_2(0,\cdot)}{\partial t^k}$ *and they are calculated formally (and recursively) in terms* $u^0, u^1, \varphi^0, \varphi^1, \theta_0^1, \theta_0^2$, *using system* $(3.1) - (3.4)$.

Then for sufficintly small $T > 0$ *there exists a unique solution* $(u, \varphi, \theta_1, \theta_2)$ *to the initial-boundary value problem* $(1.1) - (1.4), (1.6) - (1.9)$ *with the following properties:*

$$\begin{aligned} u &\in \cap_{k=1}^{s-1} C^k\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \\ \varphi &\in \cap_{k=1}^{s-1} C^k\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \\ \partial_t^s u &\in C^0\big([0,T], L^2(\Omega)\big) \\ \partial_t^s \varphi &\in C^0\big([0,T], L^2(\Omega)\big) \\ \theta_1 &\in \cap_{k=1}^{s-2} C^k\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \\ \partial_t^{s-1} \theta_1 &\in C^0\big([0,T], L^2(\Omega)\big) \\ \partial_t^{s-1} \nabla\theta_1 &\in L^2\big([0,T], L^2(\Omega)\big) \\ \theta_2 &\in \cap_{k=1}^{s-2} C^k\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \\ \partial_t^{s-1} \theta_2 &\in C^0\big([0,T], L^2(\Omega)\big) \\ \partial_t^{s-1} \nabla\theta_2 &\in L^2\big([0,T], L^2(\Omega)\big). \end{aligned}$$

The proof of Theorem 3.1 is devided into three steps:

1° Proof for the linear system of equations obtained by linearization of system (1.1) - (1.4) in the cases of of

a) two linear hyperbolic systems of equations
b) one linear parabolic system of equations.

2° Proof of an energy estimate for these systems of equations.

3° Proof of existence and uniqueness of the solution of the initial-boundary value problem to the nonlinear system of equations (1.1) - (1.4) by applying fixed point theory.

4. Energy estimate

4.1 Linearized system of thermodiffusion in micropolar medium. In this subsection, we shall investigate three initial-boundary value problems for two linear hyperbolic systems and one linear parabolic system. These systems arise from the linearized system of equations (1.1) - (1.4). So we shall investigate the solvability of the following problems:

1° The initial-boundary value problem for the linear hyperbolic system of equations

$$\partial_t^2 u_i - \bar{c}_{i\alpha j\beta}(t,x)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = \bar{h}_i(t,x) \qquad \big((t,x) \in [0,T] \times \Omega;\; i = 1,2,3) \tag{4.1}$$

with initial conditions

$$\left.\begin{aligned} u_i(0,x) &= u_i^0(x) \\ (\partial_t u_i)(0,x) &= u_i^1(x) \end{aligned}\right\} \tag{4.2}$$

and boundary condition

$$u_i(t,\cdot)|_{\partial\Omega} = 0 \qquad (t \in [0,T]). \tag{4.3}$$

2° The initial-boundary value problem for the linear hyperbolic systems of equations

$$\partial_t^2 \varphi_i - \bar{d}_{i\alpha j\beta}(t,x)\frac{\partial^2 \varphi_j}{\partial x_\alpha \partial x_\beta} = \bar{k}_i(x,t) \qquad \big((t,x) \in [0,T] \times \Omega; i = 1,2,3) \tag{4.4}$$

with initial conditions

$$\left.\begin{aligned} \varphi_i(0,x) &= \varphi_i^0(x) \\ (\partial_t \varphi_i)(0,x) &= \varphi_i^1(x) \end{aligned}\right\} \tag{4.5}$$

and boundary condition

$$\varphi_i(t,\cdot)|_{\partial\Omega} = 0 \qquad (t \in [0,T]). \tag{4.6}$$

3° The initial-boundary value problem for the linear parabolic system of equations

$$\partial_t \theta_1 - a^{11}_{\alpha\beta}(t,x)\frac{\partial^2 \theta_1}{\partial x_\alpha \partial x_\beta} - a^{12}_{\alpha\beta}(t,x)\frac{\partial^2 \theta_2}{\partial x_\alpha \partial x_\beta} = \bar{g}_1(t,x) \tag{4.7}$$

$$\partial_t \theta_2 - a^{21}_{\alpha\beta}(t,x)\frac{\partial^2 \theta_1}{\partial x_\alpha \partial x_\beta} - a^{22}_{\alpha\beta}(t,x)\frac{\partial^2 \theta_2}{\partial x_\alpha \partial x_\beta} = \bar{g}_2(t,x) \tag{4.8}$$

with initial conditions

$$\left.\begin{aligned} \theta_1(0,x) &= \theta_1^0(x) \\ \theta_2(0,x) &= \theta_2^0(x) \end{aligned}\right\} \tag{4.9}$$

and boundary conditions

$$\left.\begin{aligned} \theta_1(t,\cdot)|_{\partial\Omega} &= 0 \\ \theta_2(t,\cdot)|_{\partial\Omega} &= 0 \end{aligned}\right\} \qquad (t \in [0,T]). \tag{4.10}$$

4.2 Energy estimate for the solution of the initial-boundary value problem for linear hyperbolic systems. At the first step, we start with a results on the existence of the solution to the initial-boundary value problem (4.1) - (4.3).

Theorem 4.1 (Existence, uniqueness and regularity to problem (4.1) - (4.9)). *Let the following assumptions be satisfied:*

1° $s > \left[\frac{3}{2}\right] + 4 = 5$ *is an arbitrary but fixed integer.*

2°

$$\begin{aligned} &\bar{c}_{i\alpha j\beta} \in C^0([0,T] \times \bar{\Omega}) \cap L^\infty\big([0,T], L^\infty(\Omega)\big) \\ &\mathcal{D}_x \bar{c}_{i\alpha j\beta} \in L^\infty\big([0,T], H^{s-2}(\Omega)\big) \\ &\partial_t^k \bar{c}_{i\alpha j\beta} \in L^\infty\big([0,T], H^{s-1-k}(\Omega)\big) \quad (k = 1, \dots, s-1). \end{aligned}$$

3° $\bar{c}_{i\alpha j\beta} = \bar{c}_{j\beta i\alpha}$ *for* $(t,x) \in [0,T] \times \bar{\Omega}$*, and if* $u \in H_0^1(\Omega)$*, then*

$$\|u\|_1^2 \le \gamma_0 \left\{ \left(\bar{c}_{i\alpha j\beta}(t) \frac{\partial u_j}{\partial x_\beta}, \frac{\partial u_i}{\partial x_\alpha} \right) + \|u\|^2 \right\}$$

for $t \in [0,T]$ *where* $\gamma_0 > 0$ *is some constant.*

4° $\bar{c}_{i\alpha j\beta} \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega)$ *for a.e.* $t \in [0,T]$*, and if* $u \in H_0^1(\Omega)$*, then* $u \in H^{k+2}(\Omega)$ *and*

$$\|u\|_{k+2} \le \gamma_1 \left(\left\| -\bar{c}_{i\alpha j\beta}(t) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} \right\|_k + \|u\| \right)$$

$(0 \le k \le s-2)$ *for* $t \in [0,T]$ *where* $\gamma_1 > 0$ *is some constant.*

5° $\partial_t^k \bar{h} \in C^0\big([0,T], H^{s-2-k}(\Omega)\big)$ $(0 \le k \le s-2)$ *and* $\partial_t^{s-1} \bar{h} \in L^2\big([0,T], L^2(\Omega)\big)$.

Then there exists a unique solution $u = (u_1, u_2, u_3)^*$ *of problem* $(4.1) - (4.3)$ *with properties*

$$\begin{aligned} &\partial_t^s u \in C^0\big([0,T], L^2(\Omega)\big) \\ &\partial_t^k u \in C^0\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \quad (0 \le k \le s-1). \end{aligned} \tag{4.11}$$

Sketch of proof. The proof follows from theorems of semigroup theory (cf. [7, 16]). We can convert problem (1.1) - (1.4) into an equivalent (evolution) problem of the form

$$\partial_t V + AV = F \tag{4.12}$$

$$V(0,x) = V(x) \tag{4.13}$$

where

$$V = \big(u_1, u_2, u_3, \partial_t u_1, \partial_t u_2, \partial_t u_3\big)^* \tag{4.14}$$

$$V(0) = V^0 = \big(u_1^0, u_2^0, u_3^0, u_1^1, u_2^1, u_3^1\big)^* \tag{4.15}$$

$$F = (0, \bar{h}) \tag{4.16}$$

$$A(t) = \begin{pmatrix} 0 & -I \\ -\bar{c}_{i\alpha j\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} & 0 \end{pmatrix}, \tag{4.17}$$

the operator

$$A : \mathcal{D}(A) \to X_0 \tag{4.18}$$

being defined by (4.11) with the domains

$$\begin{aligned} \mathcal{D}(A) &= H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \\ X_0 &= H_0^2(\Omega) \times L^2(\Omega). \end{aligned} \tag{4.19}$$

Next we show that in terms of definition $(A, X_0, \mathcal{D}(A))$ is a $C\mathcal{D}$-system. To show that $A(t)$ satisfies also the other conditions of [7: Theorem 2], we introduce a double scale of real Hilbert spaces X_j and Y_j defined by

$$\left.\begin{aligned} X_j &= H^{j+1}(\Omega) \cap H_0^1(\Omega) \times H^j(\Omega) \\ Y_j &= H^{j+1}(\Omega) \cap H_0^1(\Omega) \times H^j(\Omega) \cap H_0(\Omega) \end{aligned}\right. \qquad (j \geq 1)$$

with $X_0 = Y_0$ and equipped with usual norms and inner products. At last, the desired regularity follows from [7: Theorems 1.2 and 3.3] ∎

Now, we formulate an energy estimate to problem (4.1) - (4.4).

Theorem 4.2 (Energy estimate to problem (4.1) - (4.4)). *If the assumptions of Theorem* 4.1 *are satisfied, then the solution of problem* (4.1)$-$(4.3) *guaranted by Theorem* 4.1 *satisfies the inequality*

$$|\bar{\mathcal{D}}^s u|_{0,T}^2 \leq K_0 K_1 e^{K_2 \zeta(T)} \tag{4.20}$$

with positive constants K_0, K_1, K_2 *where*

$$K_0 = \sum_{k=0}^{s} \|u^k\|_{s-k}^2 + (1+T)|\bar{\mathcal{D}}^{s-2}\bar{h}|_{0,T}^2 + T^{\frac{1}{2}} \int_0^T \|\partial_t^{s-1}\bar{h}(t)\|^2 dt,$$

$K_1 = K_1(L_0, \gamma_0, \gamma_1)$ *and* $K_2 = K_2(L, \gamma_0, \gamma_1)$ *depend continuously on their arguments where*

$$L_0 = \|\bar{c}_{i\alpha j\beta}(0)\|_{L^\infty} + \|\mathcal{D}_x \bar{c}_{i\alpha j\beta}(0)\|_{s-3}$$

$$L = \sup_{0 \leq t \leq T} \|\bar{c}_{i\alpha j\beta}(t)\|_\infty + |\mathcal{D}_x \bar{c}_{i\alpha j\beta}|_{s-2,T} + \sum_{k=1}^{s-1} |\partial_t^k \bar{c}_{i\alpha j\beta}|_{s-1-k,T}$$

and

$$\xi(T) = T^{\frac{1}{2}}\big(1 + T^{\frac{1}{2}} + T + T^{\frac{3}{2}}\big). \tag{4.21}$$

Skech of proof. Differentiating (4.1) $(n-1)$-times $(1 \leq n \leq s-1)$ formally with respect to t, multiplying by $\partial_t^n u_i$ and then integrating over $(0,t) \times \Omega$, using integration by parts with respect to x, the Schwartz inequality, Friedrichs mollifier (in order to estimate $\partial_t^s u(t,x)$), and the assumptions of Theorem 4.1, we get

$$\begin{aligned} \|\bar{\mathcal{D}}^s u(t)\|^2 &= C(L, \gamma_0, \gamma_1) K_0 \\ &\quad + C(L, \gamma_0, \gamma_1)\Big(1 + T^{\frac{1}{2}} + T + \frac{1}{T^{\frac{1}{2}}}\Big) \int_0^t \|\bar{\mathcal{D}}^s u\|^2 d\tau. \end{aligned} \tag{4.22}$$

Applying the Gronwall inequality to (4.16) we get immediately the energy estimate stated ∎

At a second step, we start with the formulation of a theorem about the existence of the solution of the initial-boundary value problem (4.4) - (4.6). Namely, we have

Theorem 4.3 (Existence, uniqueness and regularity to problem (4.4) - (4.6)). *Let the following assumptions be satisfied:*

1° $s \geq \left[\frac{3}{2}\right] + 4 > 5$ *is an arbitrary but fixed integer.*

2°

$$\begin{aligned} \bar{d}_{i\alpha j\beta} &\in C^0([0,T]\times\bar{\Omega}) \cap L^\infty\big([0,T], L^\infty(\Omega)\big) \\ \mathcal{D}_x \bar{d}_{i\alpha j\beta} &\in L^\infty\big([0,T], H^{s-2}(\Omega)\big) \quad (k=1,\ldots,s-1) \\ \partial_t^k \bar{d}_{i\alpha j\beta} &\in L^\infty\big([0,T], H^{s-1-k}(\Omega)\big). \end{aligned}$$

3° $\bar{d}_{i\alpha j\beta} = \bar{d}_{j\beta i\alpha}$ *for* $(t,x) \in [0,T]\times\bar{\Omega}$, *and if* $\varphi \in H_0^1(\Omega)$, *then*

$$\|\varphi\|_1^2 \leq \gamma_0' \left[\left(\bar{d}_{i\alpha j\beta}(t)\frac{\partial\varphi_j}{\partial x_\beta}, \frac{\partial\varphi_i}{\partial x_\alpha}\right) + \|\varphi\|^2\right]$$

for $t \in [0,T]$ *where* $\gamma_0' > 0$ *is some constant.*

4° $\bar{d}_{i\alpha j\beta}\frac{\partial^2\varphi_j}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega)$ *for a.e.* $t \in [0,T]$, *and if* $\varphi \in H_0^1(\Omega)$, *then* $\varphi \in H^{k+2}(\Omega)$ *and*

$$\|\varphi\|_{k+2} \leq \gamma_1' \left(\left\|-\bar{d}_{i\alpha j\beta}(t)\frac{\partial^2\varphi_j}{\partial x_\alpha \partial x_\beta}\right\|_k + \|\varphi\|\right)$$

$(0 \leq k \leq s-2)$ *for* $t \in [0,T]$ *where* $\gamma_1' > 0$ *is some constant.*

5°

$$\begin{aligned} \partial_t^{s-1}\bar{k} &\in L^2\big([0,T], L^2(\Omega)\big) \quad (s>5) \\ \partial_t^k \bar{k} &\in C^0\big([0,T], H^{s-2-k}(\Omega)\big) \quad (0 \leq k \leq s-2). \end{aligned}$$

Then there exists a unique solution $\varphi = (\varphi_1, \varphi_2, \varphi_3)^*$ *of problem* $(4.4) - (4.6)$ *with the properties*

$$\left.\begin{aligned} &\partial_t^s \varphi \in C^0\big([0,T], L^2(\Omega)\big) \\ &\partial_t^k \varphi \in C^0\big([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)\big) \quad (0 \leq k \leq s-1). \end{aligned}\right\} \tag{4.23}$$

Skech of proof. Introducing the vector $U = \big(\varphi_1, \varphi_2, \varphi_3, \partial_t\varphi_1, \partial_t\varphi_2, \partial_t\varphi_3\big)^*$ we can convert problem (4.4) - (4.6) into an equivalent (evolution) problem of the form

$$\partial_t U + AU = G \tag{4.24}$$

$$U(0,x) = U^0(x) \tag{4.25}$$

where

$$\begin{aligned} U(0) = U^0 &= \big(\varphi_1^0, \varphi_2^0, \varphi_3^0, \varphi_1^1, \varphi_2^1, \varphi_3^1\big)^* \\ G &= (0, \bar{k})^* \\ \mathcal{A}(t) &= \begin{pmatrix} 0 & -I \\ -\bar{d}_{i\alpha j\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta} & 0 \end{pmatrix} \end{aligned}$$

and from here the proof runs similary as that for Theorem 4.1 ∎

Using the same approach as in Theorem 4.2 we can obtain also the following energy estimate to the solution of problem (4.4) - (4.6).

Theorem 4.4 (Energy estimate to problem (4.4) - (4.6)). *If the assumptions of Theorem 4.3 are satisfied, then the solution of problem* (4.4) – (4.6) *guaranteed by Theorem 4.2 satisfies the inequality*

$$|\bar{\mathcal{D}}^s\varphi|^2_{0,T} \leq M_0 M_1 e^{M_2\eta(T)} \tag{4.26}$$

with positive constants M_0, M_1, M_2 *where*

$$M_0 = \sum_{k=0}^{s} \|\varphi^k\|^2_{s-k} + (1+T)|\bar{\mathcal{D}}^{s-2}\bar{k}|^2_0 + T^{\frac{1}{2}} \int_0^T \|\partial_t^{s-1}\bar{k}(t)\|^2 dt,$$

$M_1 = M_1(P_0, \gamma_0', \gamma_1')$ *and* $M_2 = M_2(P, \gamma_0', \gamma_1')$ *depend continuously on their arguments where*

$$P_0 = \|\bar{d}_{i\alpha j\beta}(0)\|_{L^\infty} + \|\mathcal{D}_x\bar{d}_{i\alpha j\beta}(0)\|_{s-3}$$

$$P = \sup_{0\leq t\leq T} \|\bar{d}_{i\alpha j\beta}(t)\|_{L^\infty} + |\mathcal{D}_x\bar{d}_{i\alpha j\beta}|_{s-2,T} + \sum_{k=1}^{s-1} |\partial_t^k\bar{d}_{i\alpha j\beta}|_{s-1-k,T}$$

and

$$\eta(T) = T^{\frac{1}{2}}\big(1 + T^{\frac{1}{2}} + T + T^{\frac{3}{2}}\big). \tag{4.27}$$

Proof. It runs in the same way as that of Theorem 4.1 and we leave details to the reader ∎

At the third step, we consider the solvability of the initial-boundary value problem to the linear parabolic system (4.7) - (4.8) with initial condition (4.8) and boundary conditions (4.10). Before starting the next consideration we introducing the vector $V = (\theta_1, \theta_2)^*$ and can convert this way the initial-boundary value problem (4.7) - (4.9) into the form

$$\partial_t V - a_{\alpha\beta}(t,x)\frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} = G(t,x) \tag{4.28}$$

with

$$\left.\begin{aligned} V(0,x) &= V^0(x) \\ V(t,\cdot)|_{\partial\Omega} &= 0 \end{aligned}\right\} \tag{4.29}$$

where

$$a_{\alpha\beta}(t,x) = \begin{pmatrix} a^{11}_{\alpha\beta}(t,x) & a^{12}_{\alpha\beta}(t,x) \\ a^{21}_{\alpha\beta}(t,x) & a^{22}_{\alpha\beta}(t,x) \end{pmatrix} \tag{4.30}$$

$$G(t,x) = \big(\bar{g}_1(t,x), \bar{g}_2(t,x)\big)^*. \tag{4.31}$$

In order to prove an energy estimate to problem (4.22) - (4.25), we present two theorems.

Theorem 4.5 (Energy estimate to problem (4.22) - (4.25)). *Let the following conditions be satisfied* $(i,j=1,2)$*:*

$$\begin{aligned}
\bar{\mathcal{D}}^1 a^{ij}_{\alpha\beta}(t,x) &\in C^0([0,T]\times\bar{\Omega})\cap L^\infty\big([0,T],L^\infty(\Omega)\big)\\
\partial_t\nabla a^{ij}_{\alpha\beta}(t,x) &\in L^\infty\big([0,T],L^\infty(\Omega)\big)\\
G &\in C^0\big([0,T],L^2(\Omega)\big)\\
\partial_t G &\in L^2\big([0,T],H^{-1}(\Omega)\big)\\
V^0 &\in H^1_0(\Omega)\\
V^1 = a_{\alpha\beta}(0)\frac{\partial^2 V^0}{\partial x_\alpha \partial x_\beta} + G(0) &\in L^2(\Omega)
\end{aligned}$$

and

$$a^{ij}_{\alpha\beta}(t,x) = a^{ji}_{\beta\alpha}(t,x) \qquad \big((t,x)\in[0,T]\times\bar{\Omega}\big) \tag{4.32}$$

$$(a_{\alpha\beta}\zeta_\alpha\zeta_\beta\eta,\eta) \geq \gamma_3|\zeta|^2|\eta|^2 \qquad \big((\zeta,\eta)\in\mathbb{R}^3\times\mathbb{R}^2\big) \tag{4.33}$$

for some constant $\gamma_3>0$*. Then there exists a unique solution* $V=(\theta_1,\theta_2)^*$ *to problem* $(4.24)-(4.25)$ *with the properities*

$$\left.\begin{aligned}
\theta_1 &\in C^0\big([0,T],H^2(\Omega)\cap H^1_0(\Omega)\big)\\
\partial_t\theta_1 &\in C^0\big([0,T],L^2(\Omega)\big)\\
\partial_t\nabla\theta_1 &\in L^2\big([0,T],L^2(\Omega)\big)
\end{aligned}\right\}$$

and

$$\left.\begin{aligned}
\theta_2 &\in C^0\big([0,T],H^2(\Omega)\cap H^1_0(\Omega)\big)\\
\partial_t\theta_2 &\in C^0\big([0,T],L^2(\Omega)\big)\\
\partial_t\nabla\theta_2 &\in L^2\big([0,T],L^2(\Omega)\big)
\end{aligned}\right\}. \tag{4.34}$$

Proof. We can use the Faedo-Galerkin method and the proof follows directly from the consideration in [18] ∎

Now, we formulate a theorem about the regularity of the solution to problem (4.22) - (4.25).

Theorem 4.6 (Regularity to problem (4.22) - (4.25)). *Let the following conditions be satisfied:*

1° $s\geq\left[\frac{3}{2}\right]+4=5$ *is an arbitrary but fixed integer.*

2° *For* $i,j=1,2$*,*

$$\begin{aligned}
a^{ij}_{\alpha\beta} &\in C^0([0,T]\times\bar{\Omega})\cap L^\infty\big([0,T],L^\infty(\Omega)\big)\\
\mathcal{D}_x a^{ij}_{\alpha\beta} &\in L^\infty\big([0,T],H^{s-2}(\Omega)\big)\\
\partial^k_t a^{ij}_{\alpha\beta} &\in L^\infty\big([0,T],H^{s-1-k}(\Omega)\big) \quad (1\leq k\leq s-2)\\
\partial^{s-1}_t a^{ij}_{\alpha\beta} &\in L^2\big([0,T],L^2(\Omega)\big).
\end{aligned}$$

3° *For all* $\theta_1, \theta_2 \in H_0^1(\Omega)$ *and all* $t \in [0,T]$ *the inequality*

$$\|\theta_1\|_1^2 + \|\theta_2\|_1^2 \le \gamma_4 \left\{ \left(a_{\alpha\beta}^{ij} \frac{\partial \theta_i}{\partial x_\alpha}, \frac{\partial \theta_j}{\partial x_\beta} \right) + \|\theta_1\|^2 + \|\theta_2\|^2 \right\}$$

is satisfied for some constant $\gamma_4 > 0$.

4° *For* $t \in [0,T]$, $-a_{\alpha\beta}^{ij}(t) \frac{\partial^2 \theta_i}{\partial x_\alpha x_\beta} \in H^k(\Omega)$ *with* $\theta_1, \theta_2 \in H_0^1(\Omega)$ *implies that* $\theta_1, \theta_2 \in H^{k+2}(\Omega)$ *and*

$$\|V\|_{k+2} \le \gamma_3 \left(\left\| -a_{\alpha\beta}(t) \frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} \right\|_k + \|V\| \right)$$

where $V = (\theta_1, \theta_2)^*, 0 \le k \le s-2$ *and* $\gamma_3 > 0$ *is some constant.*

5°

$$\begin{aligned}
\partial_t^k \bar{g}_1 &\in C^0\big([0,T], H^{s-2-k}(\Omega)\big) \quad (0 \le k \le s-2) \\
\partial_t^{s-1} \bar{g}_1 &\in L^2\big([0,T], H^{-1}(\Omega)\big) \\
\partial_t \bar{g}_2 &\in C^0\big([0,T], H^{s-2-k}(\Omega)\big) \quad (0 \le k < s-2) \\
\partial_t^{s-1} \bar{g}_2 &\in L^2\big([0,T], H^{-1}(\Omega)\big).
\end{aligned}$$

Then there exists a unique solution $V = (\theta_1, \theta_2)^*$ *of problem* (4.22) – (4.25) *with the properties*

$$\left.\begin{aligned}
\partial_t^k \theta_1 &\in C^0\big([0,T], H^{s-2}(\Omega) \cap H_0^1(\Omega)\big) \quad (0 \le k \le s-2) \\
\partial_t^{s-1} \theta_1 &\in C^0\big([0,T], L^2(\Omega)\big) \\
\partial_t^{s-1} \nabla\theta_1 &\in L^2\big([0,T] L^2(\Omega)\big) \\
\partial_t^k \theta_2 &\in C^0\big([0,T], H^{s-2}(\Omega) \cap H_0^1(\Omega)\big) \quad (0 \le k \le s-2) \\
\partial_t^{s-1} \theta_2 &\in C^0\big([0,T], L^2(\Omega)\big) \\
\partial_t^{s-1} \nabla\theta_2 &\in L^2\big([0,T], L^2(\Omega)\big)
\end{aligned}\right\}. \tag{4.35}$$

Proof. It can be found in [16] ∎

Next we present an energy estimate for the solution of problem (4.22) - (4.25).

Theorem 4.7 (Energy estimate for parabolic system (4.22)). *Let the conditions of Theorem* 4.6 *be fulfilled. Then the solution* $V = (\theta_1, \theta_2)^*$ *to problem* (4.22) – (4.25) *established in Theorem* 3.2 *satisfies the inequality*

$$\begin{aligned}
\sum_{k=0}^{s-2} |\partial_t \theta_1|_{s-k,T}^2 + \sum_{k=0}^{s-2} |\partial_t^k \theta_2|_{s-k,T}^2 + |\partial_t^{s-1} \theta_1|_{0,T}^2 + |\partial_t^{s-1} \theta_2|_{0,T}^2 \\
+ \int_0^t \big[\|\partial_t^{s-1} \nabla\theta_1(\tau)\|^2 + \|\partial_t^{s-1} \nabla\theta_2(t)\|^2\big] d\tau \le N_1 R_0 e^{N_2 \zeta(T)}
\end{aligned} \tag{4.36}$$

where

$$\begin{aligned}
R_0 = (1+T) \Bigg\{ \sum_{k=0}^{s-2} \big(\|\theta_1^k\|_{s-k}^2 + \|\theta_2^k\|_{s-k}^2\big) + |\mathcal{D}^{s-2} \bar{g}_1|_{0,T}^2 + |\bar{\mathcal{D}}^{s-2} \bar{g}_2|_{0,T}^2 \\
+ \int_0^T \big[\|\partial_t^{s-1} \bar{g}_1(\tau)\|_{H^{-1}}^2 + \|\partial_t^{s-1} \bar{g}_2(\tau)\|_{H^{-1}}^2\big] d\tau \Bigg\},
\end{aligned} \tag{4.37}$$

and $N_1 = N_1(S_0, \gamma_3, \gamma_4)$ *and* $N_2 = N_2(S, \gamma_4, \gamma_4)$ *are positve constants with*

$$S_0 = \sum_{i,j=1}^{2} \|a_{\alpha\beta}^{ij}(0)\|_{L^\infty} + \sum_{i,j=1}^{2} \|\mathcal{D}_x a_{\alpha\beta}^{ij}(0)\|_{s-3} \tag{4.38}$$

and

$$\begin{aligned} S = \sup_{0\le t\le T} \sum_{i,j}^{2} \|a_{\alpha\beta}^{ij}(t)\|_{L^\infty} + \sum_{i,j=1}^{2} |\mathcal{D}_x a_{\alpha\beta}^{ij}|_{s-2,T} \\ + \sum_{k=1}^{s-2} \sum_{i,j=1}^{2} |\partial_t^k a_{\alpha\beta}^{ij}|_{s-1-k,T} + \int_0^T \sum_{i,j=1}^{2} \|\partial_t^{s-1} a_{\alpha\beta}^{ij}(\tau)\|^2 d\tau, \end{aligned} \tag{4.39}$$

γ_3 *and* γ_4 *given in the assumptions of Theorem* 4.6 *and* $\zeta(T) = T(1+T)$.

Proof. It can be found in [6] ∎

5. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the Banach fixed point theorem. For this reason we define by $Z(N,T)$ the set of functions $(u, \varphi, \theta_1, \theta_2)$ which satisfy the conditions

$$\left.\begin{aligned} \partial_t^k \varphi_j,\ \partial_t^k u_j \in L^\infty([0,T], H^{s-k}(\Omega) \quad & (0 \le k \le s;\ j = 1,2,3) \\ \partial_t^k \theta_1 \in L^\infty\big([0,T], H^{s-k}(\Omega)\big) \quad & (0 \le k \le s-2) \\ \partial_t^{s-1} \nabla\theta_1 \in L^2\big([0,T], L^2(\Omega)\big) & \\ \partial_t^k \theta_2 \in L^\infty\big([0,T], H^{s-k}(\Omega)\big) \quad & (0 \le k \le s-2) \\ \partial_t^{s-1} \nabla\theta_2 \in L^2\big([0,T], L^2(\Omega)\big) & \end{aligned}\right\} \tag{5.1}$$

($s \ge \left[\frac{3}{2}\right] + 4 = 5$ being an arbitrary but fixed integer) with boundary and initial conditions of the form

$$u_j|_{\partial\Omega} = 0, \quad \varphi_j|_{\partial\Omega} = 0, \quad \theta_1|_{\partial\Omega} = 0, \quad \theta_2|_{\partial\Omega} = 0 \tag{5.2}$$

$$\partial_t^k u_j(0,x) = u_j^k(x) \quad (0 \le k \le s-1) \tag{5.3}$$

$$\begin{aligned} \partial_t^k \varphi_j(0,x) &= \varphi_j^k(x) \quad (0 \le k \le s-1;\ j = 1,2,3) \\ \partial_t^k \theta_1(0,x) &= \theta_1^k(x) \quad (0 \le k \le s-2) \\ \partial_t^k \theta_2(0,x) &= \theta_2^k(x) \quad (0 \le k \le s-2) \end{aligned} \tag{5.4}$$

and the inequality

$$\begin{aligned} |\bar{\mathcal{D}}^s u|_{0,T} + |\bar{\mathcal{D}}^s \varphi|_{0,T} \\ + \sum_{k=0}^{s-2} |\partial_t^k \theta_1|_{s-k,T}^2 + |\partial_t^{s-1}\theta_1|_{0,T}^2 + \sum_{k=0}^{s-2} |\partial_t^k \theta_2|_{s-k,T}^2 + |\partial_t^{s-2}\theta_2|_{0,T}^2 \\ + \int_0^t \big[\|\partial_t^{s-1}\nabla\theta_1(\tau)\|^2 + \|\partial_t^{s-1}\nabla\theta_2(\tau)\|^2\big] d\tau \le N^2 \end{aligned} \tag{5.5}$$

for N large enough.

Proof of Theorem 3.1. Let

$$(\bar u, \bar\varphi, \bar\theta_1, \bar\theta_2) \in Z(N,T).$$

We consider:

1° System (4.1) with

$$\bar c_{i\alpha j\beta} = c_{i\alpha j\beta}(\nabla\bar u, \nabla\bar\varphi, \bar\theta_1, \bar\theta_2) \tag{5.6}$$

$$\bar h_i = \alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\varepsilon_{jlk}\frac{\partial\bar\phi_k}{\partial x_l} + f_i - \bar c^1_{i\alpha}(\nabla\bar u, \bar\theta_1, \bar\theta_2)\frac{\partial\bar\theta_1}{\partial x_\alpha} - \bar c^2_{i\alpha}(\nabla\bar u, \bar\theta_1, \bar\theta_2)\frac{\partial\bar\theta_2}{\partial x_\alpha} \tag{5.7}$$

for $i = 1, 2, 3$.

2° System (4.4) with

$$\bar d_{i\alpha j\beta} = d_{i\alpha j\beta}(\nabla\bar u, \nabla\bar u, \bar\theta_1, \bar\theta_2) \tag{5.8}$$

$$\bar k_i = -\bar\alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\bar\varphi_j + \alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\varepsilon_{jlk}\frac{\partial\bar u_k}{\partial x_l} + Y_i \tag{5.9}$$

for $i = 1, 2, 3$.

3° System (4.7) - (4.8) with

$$\left.\begin{aligned}
a^{11}_{\alpha\beta} &= \bar a^{11}(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2)\\
a^{12}_{\alpha\beta} &= \bar a^{12}_{\alpha\beta}(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2)\\
a^{21}_{\alpha\beta} &= \bar a^{21}_{\alpha\beta}(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2)\\
a^{22}_{\alpha\beta} &= \bar a^{22}(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2)
\end{aligned}\right\} \tag{5.10}$$

and

$$\left.\begin{aligned}
\bar g_1 &= \bar{\mathbf C}^1_{j\alpha}(\nabla\bar u, \bar\theta_1, \bar\theta_2)\frac{\partial^2 u_j}{\partial x_\alpha \partial t} + g_1(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2, t, x)\\
\bar g_2 &= \bar{\mathbf C}^2_{j\alpha}(\nabla\bar u, \bar\theta_1, \bar\theta_2)\frac{\partial^2 u_j}{\partial x_\alpha \partial t} + g_2(\nabla\bar u, \bar\theta_1, \bar\theta_2, \nabla\bar\theta_1, \nabla\bar\theta_2, t, x)
\end{aligned}\right\}. \tag{5.11}$$

We rewrite these systems in the form

$$\begin{aligned}
&\partial_t^2 u_i - c_{i\alpha j\beta}(\nabla\bar u, \nabla\bar\varphi, \bar\theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta}\\
&\qquad = \alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\varepsilon_{jlk}\frac{\partial\bar\varphi_k}{\partial x_l} - \bar c^j_{i\alpha}(\nabla\bar u, \bar\theta)\frac{\partial\theta_j}{\partial x_\alpha} + fi
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&\partial_t^2 \varphi_i - d_{i\alpha j\beta}(\nabla\bar u, \bar\varphi, \bar\theta)\frac{\partial^2 \bar\varphi_j}{\partial x_\alpha \partial x_\beta}\\
&\qquad = -\bar\alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\bar\varphi_j + \alpha_{ij}(\nabla\bar u, \nabla\bar\varphi)\varepsilon_{jlk}\frac{\partial\bar u_k}{\partial x_l} + Y_i
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&\partial_t \theta_i - \bar a_{i\alpha j\beta}(\nabla\bar u, \bar\theta, \nabla\bar\theta)\frac{\partial\theta_j}{\partial x_\alpha \partial x_\beta}\\
&\qquad = \bar{\mathbf C}^i_{j\alpha}(\nabla\bar u, \bar\theta, \nabla\bar\theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial t} + \bar g_i(\nabla\bar u, \bar\theta, \nabla\bar\theta, x, t)
\end{aligned} \tag{5.14}$$

for $i = 1, 2$ where $\theta = (\theta_1, \theta_2)^*$ and

$$a_{i\alpha j\beta} = \bar{a}^{ij}_{\alpha\beta}(\nabla\bar{u}, \bar{\theta}, \nabla\bar{\theta}) \qquad (i, j = 1, 2;\ \alpha, \beta = 1, 2, 3) \tag{5.15}$$

with boundary and initial conditions (4.2) - (4.3), (4.5) - (4.6) and (4.9) - (4.10), respectively. The function u appearing in (5.14) is the solution of system (5.12) with conditions (4.2) - (4.3). Taking into account the class of functions $(\bar{u}, \bar{\varphi}, \bar{\theta}) \in Z(N, T)$, we notice that to the systems (5.12), (5.13) ans (5.14) with conditions (4.2) - (4.3), (4.4) - (4.6) and (4.9) - (4.10) we can apply Theorems 4.1, 4.3 and 4.6, respectively. In view of this fact it follows that for arbitrary $(\bar{u}, \bar{\varphi}, \bar{\theta}) \in Z(N, T)$ there exists a unique solution (u, φ, θ) to problem (5.12) - (5.14) with initial-boundary conditions (4.2) - (4.3), (4.5) - (4.6) and (4.9) - (4.10), respectively. This means there exists a mapping

$$\sigma :\ Z(N, T) \ni (\bar{u}, \bar{\varphi}, \bar{\theta}) \ \longrightarrow\ \sigma(\bar{u}, \bar{\varphi}, \bar{\theta}) = (u, \varphi, \theta). \tag{5.16}$$

Statement I. *σ maps the set $Z(N, T)$ into itself under the condition that N is larg and T small enough.*

For this, we introduce the notation

$$\begin{aligned} E_0 = &\sum_{k=0}^{s} \|u^k\|^2_{s-k} + \sum_{k=0}^{s} \|\varphi^k\|^2_{s-k} + \sum_{k=0}^{s-2} \|\theta^k\|^2_{s-k} + \|\theta^{s-1}\|^2 \\ &+ \sum_{k=0}^{s-2} |\partial_t^k(\bar{h}, \bar{k}, \bar{g})|^2_{s-2-k,T} + \int_0^T \|\partial_t^{s-1}(\bar{h}, \bar{k}, \bar{g})(\tau)\|^2 d\tau. \end{aligned} \tag{5.17}$$

Taking into account properties of the elements from the set $Z(N, T)$, applying a Sobolev inequality and some theorems from [19, 20] we get for the function $\bar{h}$ defined by (5.7) the estimates

$$\|\partial_k^{s-1}\bar{h}\|^2 \le 2\left(C\sum_{i=1}^{s-1} \left\|\bar{\mathcal{D}}^{s-1}(\nabla\bar{u}, \nabla\bar{\varphi}, \bar{\theta})\right\|^2\right)^2 + 2C\|\partial_t^{s-1}\bar{h}\|^2 \le C(N) + C\|\partial_t^{s-1}\bar{h}\|^2$$

and

$$\int_0^T \|\partial_k^{s-1}\bar{h}\|\, d\tau \le C(N)(1 + T) + CE_0. \tag{5.18}$$

Acting similarly and using the fact $\gamma(t) = \gamma(0) + \int_0^t \partial_t\gamma(t)\, dt$ we get

$$\begin{aligned} \sum_{k=0}^{s-2} \left\{ |\partial_t^k\bar{h}|^2_{s-2-k,T} + \left|\partial_t^k \alpha_{ij}(\nabla\bar{u}, \nabla\bar{\varphi})\varepsilon_{jlk}\frac{\partial\bar{\varphi}_k}{\partial x_\alpha}\right|^2_{s-2-k,T} + \left|\partial_t^k c^j_{i\alpha}(\nabla\bar{u}, \bar{\theta})\frac{\partial\bar{\theta}_j}{\partial x_\alpha}\right|^2_{s-2-k,T} \right\} \\ \le C(E_0) + C(N)T(1 + T). \end{aligned} \tag{5.19}$$

Using the same estimation we get

$$\int_0^T \|\partial_t^{s-1}\bar{k}\|\, d\tau \le C(N)(1 + T) + CE_0 \tag{5.20}$$

and

$$\sum_{k=0}^{s-1}\left\{|\partial_t^k\bar{k}|^2_{s-2-k,T}+|\partial_t^k\bar{\alpha}_{ij}(\nabla\bar{u},\nabla\bar{\varphi})\bar{\varphi}_j|^2_{s-2-k,T}+\left|\partial_t^k\alpha_{ij}(\nabla\bar{u},\nabla\bar{\varphi})\varepsilon_{jlk}\frac{\partial\bar{u}_k}{\partial x_i}\right|^2_{s-2-k,T}\right\}$$
$$\leq C(E_0)+C(N)T(1+T). \tag{5.21}$$

Putting (5.18) and (5.19) into energy estimate (4.14) in Theorem 4.2 we obtain

$$\begin{aligned}|\bar{\mathcal{D}}^s u|^2_{0,T}\leq \bar{K}_1(E_0,\gamma_0,\gamma_1)\Big\{&1+C(N)\\ &\times T^{\frac12}\left[1+T^{\frac12}+T+T^{\frac32}+T^2+T^{\frac52}+T^3\right]\Big\}\\ &\times e^{C(N)T^{\frac12}(1+T^{\frac12}+T^{\frac32}+T^2+T^{\frac32})}\end{aligned} \tag{5.22}$$

(since $K_2(L,\gamma_2,\gamma_2)\leq C(N)$). Putting (5.21) and (5.22) into energy estimate (4.20) in Theorem 4.4 we get

$$\begin{aligned}|\bar{\mathcal{D}}^s\varphi|^2_{0,T}\leq{}&\bar{K}_1'(\varepsilon_0,\gamma_0',\gamma_0')\\ &\times(1+C(N))T^{\frac12}\left[1+T^{\frac12}+T+T^{\frac32}+T^2+T^{\frac52}+T^3\right]\\ &\times e^{C(N)T^{\frac12}(1+T^{\frac12}+T+T^{\frac32}+T^2+T^{\frac52})}.\end{aligned} \tag{5.23}$$

Now, we estimate the term $\bar{G}$. After some calculations, we get

$$\begin{aligned}\int_0^T\|\partial_t^{s-1}\bar{G}\|^2_{-1}d\tau\leq{}&C(E_0)\\ &+\left(\sup_{0\leq t\leq T}\eta_1\big(\|(\bar{u},\bar{\theta})\|_{s-1},\ldots,\|\partial_t^{s-2}(\bar{u},\bar{\theta})\|_1\big)+C(N)\right)\\ &\times(1+|\bar{\mathcal{D}}^s u|^2_{0,T})\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\sum_{k=0}^{s-1}|\partial_t^k\bar{G}|^2_{s-2-k,T}\leq{}&C(E_0)\\ &+\left(\sup_{0\leq t\leq \tau}\eta_2\big(\|(\bar{u},\bar{\theta})\|_{s-1},\ldots,\|\partial_t^{s-2}(\bar{u},\bar{\theta})\|_1\big)+C(N)\right)\\ &\times(1+T)T(1+|\bar{\mathcal{D}}^s u|^2_{0,T}).\end{aligned} \tag{5.25}$$

From (5.24) and (5.25) we have

$$\begin{aligned}\sum_{k=0}^{s-2}&|\partial_t^k\bar{G}|^2_{s-2-k,T}+\int_0^T\|\partial_t^{s-1}\bar{G}\|^2_{-1}d\tau\\ &\leq\sup_{0\leq t\leq T}\eta_3\big(E_0,\|(\bar{u},\bar{\theta})\|_{s-1},\ldots,\|\partial_t^{s-2}(\bar{u},\bar{\theta})\|_1\big)\\ &\quad\times\big(1+C(N)T(1+T)\big)(1+|\bar{\mathcal{D}}u|^2_{0,T}).\end{aligned} \tag{5.26}$$

Putting (5.26) into energy estimate (4.30) in Theorem 4.7, we get

$$\begin{aligned}&\sum_{k=0}^{s-2}|\partial_t^k\theta|^2_{s-k,T}+\int_0^T\|\partial_t^{s-1}\nabla\theta\|^2d\tau\\&\quad\le\sup_{0\le t\le T}\bar K_2\big(E_0,\|(\bar u,\theta)\|_{s-1},\dots,\|\partial_t^{s-1}(\bar u,\bar\theta)\|_1\big)\\&\quad\quad\times\big(1+C(N)T(1+T)^2\big)(1+|\bar{\mathcal D}^su|^2_{0,T})e^{C(N)T(1+T)}.\end{aligned}\tag{5.27}$$

Adding inegualities (5.22), (5,23) and (5.27), we get

$$\begin{aligned}&|\bar{\mathcal D}^su|^2_{0,T}+|\bar{\mathcal D}^s\varphi|^2_{0,T}+\sum_{k=0}^{s-2}|\partial_t^k\theta|^2_{s-k,T}+\int_0^T\|\partial_t^{s-1}\nabla\theta\|^2d\tau+|\partial_t^{s-1}\theta|^2_{0,T}\\&\quad\le\sup_{0\le t\le T}K\big(E_0,\|(\bar u,\bar\theta)\|_{s-1},\dots,\|\partial_t^{s-2}(\bar u,\bar\theta)\|_1\big)\\&\quad\times\left(1+C(N)T^{\frac12}\sum_{i=0}^{12}T^{\frac i2}\right)e^{C(N)T^{\frac12}(1+T^{\frac12}+T+T^{\frac32})}.\end{aligned}\tag{5.28}$$

Now we notice that for $(\bar u,\bar\varphi,\bar\theta)\in Z(N,T)$ we get

$$\|\partial_t^k(\bar u,\bar\varphi,\bar\theta)\|_{s-1-k}\le\|(u^k,\varphi^k,\theta^k)\|_{s-1-k}+T^{\frac12}(1+T^{\frac12})<N.\tag{5.29}$$

Let N be large enough that

$$8K\big(E_0,\|(u^s,\theta^s)\|_{s-1},\dots,\|(u^{s-2},\theta^{s-2})\|_1\big)\le N^2.\tag{5.30}$$

Since K is a continuous function, so in view of (5.30) there exists T such that

$$\begin{aligned}&\sup_{0\le k\le T}K\big(E_0,\|(\bar u,\bar\theta)\|_{s-1},\dots,\|\partial_t^{s-2}(\bar u,\bar\theta)\|_1\big)\\&\quad\le 4K\big(E_0,\|(u^0,\theta^0)\|_{s-1},\dots,\|(u^{s-2},\theta^{s-2})\|_1\big)\end{aligned}\tag{5.31}$$

and

$$\bar\xi(T)=\left(1+C(N)T^{\frac12}\sum_{i=0}^{12}T^{\frac i2}\right)e^{C(N)T^{\frac12}(1+T^{\frac12}+T+T^{\frac32})}<2\tag{5.32}$$

(because $\bar\xi(T)$ is a continuous function and $\bar\xi(0)=1$). So in view of this fact we get from (5.28) the inequality

$$\begin{aligned}&|\bar{\mathcal D}^su|^2_{0,T}+|\bar{\mathcal D}^s\varphi|^2_{0,T}+|\partial_t^{s-1}\theta|^2_{0,T}+\sum_{k=0}^{s-2}|\partial_t^k\theta|^2_{s-k,T}+\int_0^T\|\partial_t^{s-1}\nabla\theta\|^2d\tau\\&\quad\le 4K\big(E_0,\|(u^0,\theta^0)\|_{s-1},\dots,\|(u^{s-2},\theta^{s-2})\|_1\big)\cdot 2\\&\quad\le N^2.\end{aligned}\tag{5.33}$$

From here it follows that $(u,\varphi,\theta)\in Z(N,T)$.

Statement II. *We prove that the mapping* $\sigma : Z(N,T) \to Z(N,T)$ *is a contraction under the condition that* T *is small enough.*

For this let W denote the complete metric space given by

$$W = \Big\{(\bar{u},\bar{\varphi},\bar{\theta}) : \bar{\mathcal{D}}^1\bar{u}, \bar{\mathcal{D}}^1\bar{\varphi}, \bar{\theta} \in L^\infty\big([0,T], L^2(\Omega)\big),\ \nabla\bar{\theta} \in L^2\big([0,T], L^2(\Omega)\big)\Big\} \tag{5.34}$$

with metric ρ given by

$$\begin{aligned} &\rho\big((\bar{u},\bar{\varphi},\bar{\theta}),(u,\varphi,\theta)\big) \\ &\quad = |\bar{\mathcal{D}}^1(\bar{u}-u)|^2_{0,T} + |\bar{\mathcal{D}}^1(\bar{\varphi}-\varphi)|^2_{0,T} + |\bar{\theta}-\theta|^2_{0,T} + \int_0^T \|\nabla(\bar{\theta}-\theta)\|^2 d\tau. \end{aligned} \tag{5.35}$$

$Z(N,T)$ is a closed subset of $\bar{W}$. Let $(\bar{u},\bar{\varphi},\bar{\theta}), (\bar{u}^*,\bar{\varphi}^*,\bar{\theta}^*) \in Z(N,T)$. Then

$$\left.\begin{aligned} \sigma(\bar{u},\bar{\varphi},\bar{\theta}) = (u,\varphi,\theta) \in Z(N,T) \\ \sigma(\bar{u}^*,\bar{\varphi}^*,\bar{\theta}^*) = (u^*,\varphi^*,\theta^*) \in Z(N,T) \end{aligned}\right\}. \tag{5.36}$$

In view of (5.12) - (5.14) we see that $u-u^*, \varphi-\varphi^*, \theta-\theta^*$ satisfy the system

$$\begin{aligned} &\partial_t(u_i-u_i^*) - c_{i\alpha j\beta}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta})\frac{\partial^2(u_j-u_j^*)}{\partial x_\alpha \partial x_\beta} \\ &\quad = \Big(c_{i\alpha j\beta}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*) - c_{i\alpha j\beta}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta})\Big)\frac{\partial^2 u_j^*}{\partial x_\alpha \partial x_\beta} \\ &\qquad + \Big(\alpha_{ij}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta}) - \alpha_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\Big)\varepsilon_{jlk}\frac{\partial\bar{\varphi}_k}{\partial x_l} \\ &\qquad + \alpha_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\varepsilon_{jlk}\left(\frac{\partial\bar{\varphi}_k}{\partial x_l} - \frac{\partial\bar{\varphi}_k^*}{\partial x_l}\right) \\ &\qquad + \Big(\bar{c}^j_{i\alpha}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta}) - \bar{c}^j_{i\alpha}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\Big)\frac{\partial\bar{\theta}_j}{\partial x_\alpha} \\ &\qquad + \bar{c}^j_{i\alpha}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\left(\frac{\partial\bar{\theta}_j}{\partial x_\alpha} - \frac{\partial\bar{\theta}_j^*}{\partial x_\alpha}\right) \end{aligned} \tag{5.37}$$

$$\begin{aligned} &\partial_t^2(\varphi_i-\varphi_i^*) - d_{i\alpha j\beta}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta})\frac{\partial^2(\varphi_j-\varphi_j^*)}{\partial x_\alpha \partial x_\beta} \\ &\quad = \Big(d_{i\alpha j\beta}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*) - d_{i\alpha j\beta}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta})\Big)\frac{\partial^2\varphi_j^*}{\partial x_\alpha \partial x_\beta} \\ &\qquad + (\bar{\alpha}_{ij}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta}) - \bar{\alpha}_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\bar{\varphi}_j \\ &\qquad + \bar{\alpha}_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)(\varphi_j - \bar{\varphi}_j^*) \\ &\qquad + (\alpha_{ij}(\nabla\bar{u},\nabla\bar{\varphi},\bar{\theta}) - \alpha_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\varepsilon_{jlk}\frac{\partial\bar{\varphi}_k}{\partial x_l} \\ &\qquad + \alpha_{ij}(\nabla\bar{u}^*,\nabla\bar{\varphi}^*,\bar{\theta}^*)\varepsilon_{jlk}\left(\frac{\partial\bar{\varphi}_k}{\partial x_l} - \frac{\partial\bar{\varphi}_k^*}{\partial x_l}\right) \end{aligned} \tag{5.38}$$

$$
\begin{aligned}
&\partial_t(\theta_i - \theta_i^*) + \bar{a}^{ij}_{\alpha\beta}(\nabla\bar{u}, \nabla\bar{\varphi}, \bar{\theta}_1, \bar{\theta}_2, \nabla\bar{\theta}_1, \nabla\bar{\theta}_2)\frac{\partial^2(\theta_j - \theta_j^*)}{\partial x_\alpha \partial x_\beta} \\
&\quad = \Big(\bar{a}^{ij}_{\alpha\beta}(\nabla\bar{u}^*, \nabla\bar{\varphi}^*, \bar{\theta}_1^*, \bar{\theta}_2^*, \nabla\theta_1^*, \nabla\bar{\theta}_2^*) \\
&\qquad - \bar{a}^{ij}_{\alpha\beta}(\nabla\bar{u}, \nabla\bar{\varphi}, \bar{\theta}_1, \bar{\theta}_2, \nabla\bar{\theta}_1\nabla\bar{\theta}_2)\Big)\frac{\partial^2\theta_j^*}{\partial x_\alpha\partial x_\beta} \\
&\qquad + \bar{\mathbf{C}}^i_{j\alpha}(\nabla\bar{u}\nabla\bar{\varphi}\bar{\theta}_1, \bar{\theta}_2\nabla\bar{\theta}_1\nabla\bar{\theta}_2)\frac{\partial^2(u_j - u_j^*)}{\partial x_\alpha\partial t} \\
&\qquad + \Big(\bar{\mathbf{C}}^i_{j\alpha}(\nabla\bar{u}, \nabla\bar{\varphi}, \bar{\theta}_1, \bar{\theta}_2, \nabla\bar{\theta}_1, \nabla\bar{\theta}_2) \\
&\qquad - \bar{\mathbf{C}}^i_{j\alpha}(\nabla\bar{u}^*, \nabla\bar{\varphi}^*, \bar{\theta}_1^*, \bar{\theta}_2^*, \nabla\bar{\theta}_1^*, \nabla\bar{\theta}_2^*)\Big)\frac{\partial^2 u_j}{\partial x_\alpha\partial t} \\
&\qquad + g_i(\nabla\bar{u}, \nabla\bar{\varphi}, \bar{\theta}_1, \bar{\theta}_2, \nabla\bar{\theta}_1, \nabla\bar{\theta}_2)(x,t) \\
&\qquad - g_i(\nabla\bar{u}^*, \nabla\bar{\varphi}^*, \bar{\theta}_1^*, \bar{\theta}_2^*, \nabla\bar{\theta}_1^*, \nabla\bar{\theta}_2^*)(x,t)
\end{aligned} \tag{5.39}
$$

It follows from Theorem (5.1) and the Sobolev inequality that

$$
\sup_{0\le t\le T}\left\|\bar{\mathcal{D}}^2(\bar{u}, \bar{\varphi}, \bar{\theta}, \bar{u}^*, \bar{\varphi}^*, \bar{\theta}^*, u, \varphi, \theta)\right\| \le CN. \tag{5.40}
$$

Multiplying (5.36) - (5.38) by $\partial_t(u-u^*), \partial_t(\varphi-\varphi^*), \partial_t(\theta_i-\theta_i^*)$ $(i=1,2)$, respectively, and integrating then over $[0,T]\times\Omega$, performing partial integration with respect to x, taking into account that

$$
\begin{aligned}
(u_i - u_i)^*|_{\partial\Omega} &= 0 \qquad & \partial_t^k(u_i - u_i^*)(0,x) &= 0 \quad (k=0,1) \\
(\varphi_i - \varphi_i^*)|_{\partial\Omega} &= 0 \qquad & \partial_t^k(\varphi_i - \varphi_i^*)(0,x) &= 0 \quad (k=0,1)
\end{aligned}
$$

and using (5.39), the mean value theorem and the Schwarz inequality, after some calculation we get

$$
\begin{aligned}
\|\bar{\mathcal{D}}_1(u - u_1^*)\|^2 \le C(N)\Bigg\{&\left(1 + \frac{1}{T^{\frac{1}{2}}}\right)\int_0^T \|\bar{\mathcal{D}}_1(u-u^*)\|^2 d\tau \\
&+ T^{\frac{1}{2}}(1+T)\Bigg[|\bar{\mathcal{D}}_1(\bar{u}-\bar{u}^*)|^2_{0,T} + |\bar{\mathcal{D}}_1(\bar{\varphi}-\bar{\varphi}^*)|^2_{0,T} \\
&+ |\bar{\theta}_1 - \bar{\theta}_1^*|^2_{0,T} + |\bar{\theta}_2 - \bar{\theta}_2^*|^2_{0,T} \\
&+ \int_0^T \|\nabla(\bar{\theta}_1 - \bar{\theta}_1^*)\|^2 d\tau + \int_0^T \|\nabla(\bar{\theta}_2 - \bar{\theta}_2^*)\|^2 d\tau\Bigg]\Bigg\}
\end{aligned} \tag{5.41}
$$

$$\begin{aligned}\|\bar{\mathcal{D}}_1(\varphi-\varphi^*)\|^2 \le C(N)\bigg\{&\Big(1+\frac{1}{T^{\frac{1}{2}}}\Big)\int_0^T \|\bar{D}_1(\varphi-\varphi^*)\|^2 d\tau \\ &+ T^{\frac{1}{2}}(1+T)\bigg[|\bar{\mathcal{D}}_1(\bar{u}-\bar{u}^*)|_{0,T}^2 + |\bar{\mathcal{D}}_1(\varphi-\varphi^*)|_{0,T}^2 \\ &+ |\bar{\theta}_1-\bar{\theta}_1^*|_{0,T}^2 + |\bar{\theta}_2-\bar{\theta}_2^*|_{0,T}^2 \\ &+ \int_0^T \|\nabla(\bar{\theta}_1-\bar{\theta}_1^*)\|^2 d\tau + \int_0^T \|\nabla(\bar{\theta}_2-\bar{\theta}_2^*)\|^2 d\tau\bigg]\bigg\}\end{aligned} \tag{5.42}$$

$$\begin{aligned}&\|\theta_1-\theta_1^*\|^2 + \int_0^t \|\nabla(\theta_1-\theta_1^*)\|^2 d\tau + \|\theta_2-\theta_2^*\|^2 + \int_0^t \|\nabla(\theta_2-\theta_2^*)\|^2 d\tau \\ &\le C(N)\bigg\{\Big(1+\frac{1}{T^{\frac{1}{2}}}\Big)\int_0^t \Big[\|\theta_1-\theta_1^*\|^2 + \|\theta_2-\theta_2^*\|^2 + \|\bar{\mathcal{D}}^1(u-u^*)\|^2\Big] d\tau \\ &+ T^{\frac{1}{2}}(1+T)\bigg[|\bar{\mathcal{D}}^1(\bar{u}-\bar{u}^*)|_{0,T}^2 + |\bar{\theta}_1-\bar{\theta}_1^*|_{0,T}^2 + |\bar{\theta}_2-\bar{\theta}_2^*|_{0,T}^2 \\ &+ \int_0^t \big(\|\nabla(\bar{\theta}_1-\bar{\theta}_1^*\|^2 + \|\nabla(\bar{\theta}_2-\bar{\theta}_2^*)\|^2\big)\bigg] d\tau.\bigg\}\end{aligned} \tag{5.43}$$

We deduce from (5.40) - (5.42) that

$$\begin{aligned}&\|\bar{\mathcal{D}}_1(u-u^*)\|^2 + \|\bar{\mathcal{D}}_1(\varphi-\varphi^*)\|^2 + \|\theta_1-\theta_2^*\|^2 + |\theta_2-\theta_2^*|^2 \\ &\quad + \int_0^T \big(\|\nabla(\theta_1-\theta_1^*)\|^2 + \|\nabla(\theta_2-\theta_2^*)\|^2\big) d\tau \\ &\le C(N)\bigg\{\Big(1+\frac{1}{T^{\frac{1}{2}}}\Big) \\ &\quad \times \int_0^T \Big[\|\bar{\mathcal{D}}_1(u-u^*)\|^2 + \|\bar{\mathcal{D}}_1(\varphi-\varphi^*)\|^2 + \|\theta_1-\theta_1^*\|^2 + \|\theta_2-\theta_2^*\|^2\Big] d\tau \\ &\quad + T^{\frac{1}{2}}(1+T)\bigg[|\bar{\mathcal{D}}^1(\bar{u}-\bar{u}^*)|_{0,T}^2 + |\bar{\mathcal{D}}_1(\bar{\varphi}-\bar{\varphi}^*)|_{0,T}^2 \\ &\quad + |\bar{\theta}_1-\bar{\theta}_1^*|_{0,T}^2 + |\bar{\theta}_2-\bar{\theta}_2^*|_{0,T}^* \\ &\quad + \int_0^T \big[\|\nabla(\bar{\theta}_1-\bar{\theta}_1^*)\|^2 + \|\nabla(\bar{\theta}_2-\bar{\theta}_2^*)\|^2\big] d\tau\bigg] \\ &\quad + \Big(1+\frac{1}{T^{\frac{1}{2}}}\Big)\int_0^T\int_0^s \big(\|\nabla(\theta_1-\theta_1^*)\|^2 + \|\nabla(\theta_2-\theta_2^*)\|^2\big) dt ds\bigg\}.\end{aligned} \tag{5.44}$$

Applying Gronwall's inequality to (5.43) we get

$$|\bar{\mathcal{D}}_1(u-u^*)|_{0,T}^2 + |\bar{\mathcal{D}}_1(\varphi-\varphi^*)|_{0,T}^2 + |\theta_1-\theta_1^*|_{0,T}^2 + |\theta_2-\theta_2^*|_{0,T}^2$$

$$
\begin{aligned}
&+\int_0^T \big(\|\nabla(\theta_1-\theta_1^*)\|^2+\|\nabla(\theta_2-\theta_2^*)\|^2\big)d\tau \\
&\le \varepsilon\bigg(|\bar{\mathcal{D}}_1(\bar u-\bar u^*)|_{0,T}^2+|\bar{\mathcal{D}}_1(\bar\varphi-\bar\varphi^*)|_{0,T}^2+|\bar\theta_1-\bar\theta_1^*|_{0,T}^2+|\bar\theta_2-\bar\theta_2^*|_{0,T}^2 \\
&\qquad +\int_0^T \big(\|\nabla(\bar\theta_1-\bar\theta_1^*)\|^2+\|\nabla(\bar\theta_1-\bar\theta_2^*)\|^2\big)d\tau\bigg)
\end{aligned}
\tag{5.45}
$$

where

$$
\varepsilon = C(N)T^{\frac{1}{2}}(1+T)\,e^{C(N)(T+T^{\frac{1}{2}})}. \tag{5.46}
$$

From (5.46) it follows that choosing T small enough, we get $\varepsilon < 1$. Therefore the mapping σ is a contraction. So, in view of the Banach fixed point theorem, it follows that the contraction mapping σ has a unique fixed point (u,φ,θ) in $Z(N,T)$. This implies that problem (1.1) - (1.4) with conditions (1.6) - (1.8) has a unique solution on $0 \le t \le T$. This completes the proof of Theorem 3.1 ∎

References

[1] Adams, R. A.: *Sobolev Spaces.* New York: Academic Press 1973.

[2] Agmon, I. A.: *Lectures on Elliptic Boundary-Value Problems.* Van Nostrand: Princeton 1965.

[3] Dafermos, C. M. and W. J. Hrusa: *Energy methods for quasilinear hyperbolic initial boundary-value problems. Applications to elektrodynamics.* Arch. Rat. Mech Anl. 87 (1985), 267 – 292.

[4] Gawinecki, J. and K. Sierpiński: *Existence, uniqueness and regularity of the solution of the first boundary initial value problem for the equations if thermodiffusion in a solid body.* Bull. Acad. Polon. Sci. Ser. Techn. 30 (1982), 163 – 171.

[5] Gawinecki, J. and P. Kacprzyk: *Existence, uniqueness and regularity of the solution to the first boundary-initial value problem of linear thermodiffusion in micropolar medium.* Bull. Acad. Polon. Sci., Ser. Techn. 42 (1994), 341 – 359.

[6] Gawinecki, J., Kacprzyk, P. and J. Jędrzejewski: *On energy estimate for some coupled parabolic systems of partial differential equations.* Biul. WAT XLV 12 (1996), 7 – 13.

[7] Kato, T.: *Abstract Differential Equations and Nonlinear Mixed Problems* Center for Pure and Appl. Math. Report Univ. of California, Berkely Published in Fermi Lectures Scuola Normale Sup., Pisa 1985.

[8] Kawashima, S.: *Systems of a Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics.* Thesis. Kyoto: University 1983.

[9] Kawashima, S. and M. Matsumura: *Mixed problems for symmetric hyperbolic-parabolic systems.* Manuscript 1989.

[10] Kawashima, S., Yanagisawa, T. and Y. Shizuta: *Mixed problem for quasilinear symmetric hyperbolic systems.* Proc. Japan Acad. Sci. (Ser. A) 63 (1987), 243 – 241.

[11] Ladyzhenskaya, O. A., Solonnikov, V. A. and N. N. Uralceva: *Linear and Quasi-Linear Equations of Parabolic Types* (Trans. Math. Monographs: Vol. 23). Providence (R.I.): Amer. Math. Soc. 1968.

[12] Li, D.: *The nonlinear initial-boundary value problem and the existence of multi-dimensional, shock wave for quasilinear hyperbolic-parabolic coupled systems.* Chin. Ann. Math. 8B(2) (1987), 252 – 280.

[13] Milani, A. J.: *A regularity result for strongly elliptic systems.* Bull. Uni. Math. Itali. (Ser. 2B) 6 (1983), 641 – 651.

[14] Naerlović-Vejlković, N. and M. Plavšić: *Thermodiffusion in elastic solids with microstructura.* Bull. Acad. Polon. Sci., Ser. Techn. 22 (1974), 623 – 632.

[15] Nowacki, W.: *Certains problems of thermodiffusion in solids.* Arch. Mech. Stos. 23 (1971), 731 – 754.

[16] Nowacki, W.: *Dynamic problems of thermodiffusion in solids.* Parts I - III. Bull. Acad. Polon. Sci., Ser. Techn. 23 (1974), 55 – 64, 205 – 211 and 257 – 266.

[17] Nowacki, W.: *Theory of Asymmetric Elasticity.* Warszawa: Pol. Sci. Publ. 1986.

[18] Plavšić, M. and N. Naerlović-Vejlković: *Field equations for thermodiffusion in elastic solids with microstructure.* Bull. Acad. Polon. Sci., Ser. Techn. 23 (1975), 483 – 492.

[19] Shibata, Y.: *On a local existence theorem for some quasilinear hyperbolic-parabolic coupled system with Neumann type boundary condition.* Manuscript.

[20] Shibata, Y. and Y. Tsutsumi: *Local existence of solution for the initial-boundary value problem of fully nonlinear equation.* Nonlin. Arch. Theory Math. Appl. 11 (1987), 335 – 365.

[21] Zajączkowski, W.: *Mixed problems for nonlinear symmetric hyperbolic system.* Math. Meth. Appl. Sci. 11 (1989), 139 – 168.

[22] Zheng, S.: *Initial boundary value problems for quasilinear hyperbolic-parabolic coupled systems in higher dimensonal spaces.* Chin. Ann. Math. 4B (1983), 443 – 462.

[23] Zheng, S. and W. Shen: *Global solutions to the Cauchy problem of quasilinear hyperbolic parabolic coupled systems.* Sci. Sinica Ser. 30 (1987), 1133 – 1149.

[24] Volpert, A. J. and S. T. Hudajev: *On the Cauchy problem for composite systems of nonlinear differential equations.* Math. USSR Sbornik 16 (1972), 517 – 544.

Received 17.07.1998; in revised form 13.03.2000