Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 2, 453–468

Non-Existence Results for a Semilinear Hyperbolic Problem with Boundary Condition of Memory Type

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Abstract. We consider a problem which models the evolution of sound in a compressible fluid with reflection of sound at the surface of the material. Different methods such as the concavity method of Levine, the potential well method and an argument due to Tsutsumi are used to derive global non-existence theorems.

Keywords: *Semilinear hyperbolic equations, blow-up in finite time*

AMS subject classification: 35 L 05, 35 L 70

1. Introduction

In this paper we shall consider the semilinear problem

$$\left.\begin{aligned} & u_{tt}(t,x) + \alpha u_t(t,x) = \Delta u(t,x) + f(t,u) && (t>0, x\in\Omega) \\ & \frac{\partial u}{\partial \nu}(t,x) + \int_0^t k(t-s,x)u_s(s,x)\,ds = 0 && (t>0, x\in\Gamma_0) \\ & u(t,x) = 0 && (t>0, x\in\Gamma_1) \\ & u(0,x) = u_0(x), u_t(0,x) = u_1(x) && (x\in\Omega) \end{aligned}\right\} \tag{1}$$

where

- Ω is a bounded domain in $\mathbb{R}^n$ with a boundary $\Gamma = \partial\Omega$ of class C^2
- (Γ_0, Γ_1) is a partition of Γ such that $\operatorname{int}\Gamma_1 \neq \emptyset$
- $\nu(x)$ denotes the outward normal vector to Γ at $x \in \Gamma$
- $\frac{\partial}{\partial \nu}$ is the normal derivative on Γ
- α is a real number the sign of which is to be precised later
- f, k, u_0, u_1 are given functions
- subscript t denotes differentiation with respect to t
- Δ is the Laplacian.

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ISSN 0232-2064 / $ 2.50

This problem models the evolution of sound in a compressible fluid with reflection of sound at the surface of the material. The linear case was derived and studied by Propst and Prüss in [15]. They proved existence of a strong regular solution u in $C(R_+; H^2(\Omega)) \cap C^1(R_+; V) \cap C^2(R_+; L^2(\Omega))$ with $\mathbb{R}_+$ the set of all non-negative real numbers, provided that (among other conditions) $u_0 \in H^2(\Omega) \cap V$ and $u_1 \in V$, for the problem

$$\left.\begin{array}{ll} u_{tt}(t,x) = \Delta u(t,x) + g(t,x) & (t>0, x \in \Omega) \\ \dfrac{\partial u}{\partial \nu}(t,x) + \displaystyle\int_0^t k(t-s,x) u_s(s,x)\, ds = h(t,x) & (t>0, x\in\Gamma_0) \\ u(t,x) = 0 & (t>0, x\in\Gamma_1) \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & (x\in\Omega). \end{array}\right\} \tag{2}$$

Many boundary conditions such as

$$\left.\begin{array}{rl} \dfrac{\partial p}{\partial \nu}(t,x) + \zeta(x) p_t(t,x) = 0 & (t>0, x\in\Gamma) \\ \dfrac{\partial p}{\partial \nu}(t,x) + \beta(x) p_t(t,x) + \alpha(x) p(t,x) = 0 & (t>0, x\in\Gamma) \\ m(x)\delta_{tt}(t,x) + d(x)\delta_t(t,x) + K(x)\delta(t,x) = -p(t,x) & \\ \dfrac{\partial p}{\partial \nu}(t,x) + \delta_{tt}(t,x) = 0 & (t>0, x\in\Gamma) \end{array}\right\}$$

may be regarded as special forms of the boundary condition $(1)_2$. See [15] for the physical meaning of these constraints and references therein for investigations of problems with these boundary conditions.

In [6], the authors have considered problem (2) and proved a uniform stabilization result provided the equation contains a mild dissipation inside the domain (or in the boundary condition). A blow up result has been proved by the same authors in [7] for problem (1) with the boundary condition $\frac{\partial u}{\partial \nu} = -p(x)g(u_t)$ on Γ_0 and a mild (or strong) antidissipation (i.e. $\alpha < 0$) inside the domain. It was shown that the above mentioned boundary dissipation may have no effect on the energy and blow up occurs for certain types of sources.

It is the purpose of this work to establish some non-existence results for problem (1). The plan of the paper is as follows: in Section 2 we prove a blow up result for $\alpha \geq 0$, using the concavity method of Levine [8, 10, 11] as formulated in Kalantarov and Ladyzhenskaya [5]. The result is then enlarged to another class of nonlinearities in Section 3 via a technique by Sleeman [17] (which is in fact a modification of the concavity method) and an analogue to the Kalantarov and Ladyzhenskaya theorem for the new class of non-linearities. We consider negative initial energy, vanishing initial energy as well as positive initial energy. For the latter case we combine the concavity method with the potential well method. In the case $k(x,t) = p(x)e^{-t}$ another proof based on an argument due to Tsutsumi [19] and an appropriate energy functional is given. This is established in Section 4.

2. The mildly damped problem

In this section and Section 3 we need the following theorem which may be found in [3] (see also [18]).

Theorem 1. *If $k \in C(\mathbb{R}_+)$ is non-negative, non-increasing and convex, then*

$$|(k * u)(t)|^2 \leq 2k(0+)\, \Re\int_0^t u(\tau)\overline{(k * u)(\tau)}\, d\tau. \tag{3}$$

Next we prepare some material necessary to our investigation. By a positive definite function $a \in L^1_{loc}(\mathbb{R}_+)$ it is meant a function satisfying

$$\int_0^T v(t) \int_0^t a(t-s)v(s)\, dsdt \geq 0 \tag{4}$$

for all $v \in C(\mathbb{R}_+)$ and every $T > 0$. It is known that if a is a twice differentiable function such that

$$(-1)^n a^{(n)}(t) \geq 0 \quad (t > 0, n \in \mathbb{N}_0) \qquad \text{and} \qquad a' \neq 0,$$

then a is positive definite (see [13]). We set

$$V = \big\{u \in H^1(\Omega) :\, u|_{\Gamma_1} = 0\big\}.$$

The Poincaré inequality and the trace inequality yield the existence of constants $\delta > 0$ and $\beta > 0$ such that

$$\|v\|_2^2 \leq \delta\, \|\nabla v\|_2^2 \tag{5}$$

and

$$\int_{\Gamma_0} v^2 d\sigma \leq \beta \left(\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx\right), \tag{6}$$

respectively, for all $v \in V$. We will also need the existence of a constant $\eta > 0$ such that the following Sobolev-Poincaré inequality (cf. [9] for instance)

$$\|v\|_p \leq \eta\, \|\nabla v\|_2 \tag{7}$$

holds for all $v \in V$ where $1 \leq p \leq \frac{2n}{n-2}$ for $n \geq 3$ and $1 \leq p < \infty$ for $n = 1, 2$. We shall use repeatedly the algebraic inequality

$$ab \leq \mu a^2 + \tfrac{1}{4\mu} b^2 \tag{8}$$

for all $a, b \in \mathbb{R}$ and $\mu > 0$.

Firstly, the function $f(t, u)$ is assumed to satisfy the following rather general assumptions:

- **(H1)** $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function such that there exists a continuously differentiable function $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfying

  $$\mathcal{F}'(t, u) = \mathcal{F}_t(t, u) + u_t f(t, u) \qquad \text{for all } t \geq 0$$

  where the prime denotes Fréchet differentiation with respect to t and the subscript t denotes partial differentiation with respect to t the first variable.
- **(H2)** $\mathcal{F}_t(t, u) \geq 0$ for all u and $t \geq 0$.

We begin by a lemma which assembles some results from [8] and [10] in the form given in [5].

Lemma 2. *Suppose that there exists a twice differentiable positive function φ satisfying the inequality*

$$\varphi(t)\varphi''(t)-(1+\gamma)(\varphi'(t))^2 \geq -2C_1\varphi(t)\varphi'(t)-C_2\varphi(t)^2 \qquad (t\geq 0) \tag{9}$$

for some $\gamma>0$ and $C_1, C_2\geq 0$.

(a) *Let $C_1+C_2>0$ and $\varphi(0)>0$.*

(i) *If $\varphi'(0)>-\gamma_2\gamma^{-1}\varphi(0)$, then $\varphi(t)\to+\infty$ as*

$$t\to t_1\leq T=\frac{1}{2\sqrt{C_1^2+\gamma C_2}}\ln\left\{\frac{\gamma_1\varphi(0)+\gamma\varphi'(0)}{\gamma_2\varphi(0)+\gamma\varphi'(0)}\right\} \tag{10}$$

where $\gamma_1=-C_1+\sqrt{C_1^2+\gamma C_2}$ and $\gamma_2=-C_1-\sqrt{C_1^2+\gamma C_2}$.

(ii) *If $\varphi'(0)=-\gamma_2\gamma^{-1}\varphi(0)$, then either the solution blows up in finite time or else $\varphi(t)=\varphi(0)e^{-\gamma^{-1}\gamma_2 t}$.*

(iii) *If $\varphi'(0)<-\gamma_2\gamma^{-1}\varphi(0)$, then either the solution blows up in finite time or else $\varphi(t)\leq\varphi(0)e^{-\gamma^{-1}\gamma_2 t}$.*

(b) *Let $C_1=C_2=0$ and $\varphi(0)>0$. This case is the concavity method with a blow up time $t_2\leq\frac{\varphi(0)}{\gamma\varphi'(0)}$.*

Proof. (a) Let us define the new function $\Phi(t)=\varphi^{-\gamma}(t)$. It is easy to see that Φ satisfies the second order differential equation

$$\Phi''(t)+2C_1\Phi'(t)-\gamma C_2\Phi(t)=l(t)\leq 0 \tag{11}$$

and the solution is given by

$$\Phi(t)=\beta_1e^{\gamma_1 t}+\beta_2e^{\gamma_2 t}+\tfrac{1}{\gamma_1-\gamma_2}\int_0^t l(s)\left\{e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}\right\}ds$$

with

$$\left.\begin{aligned}\beta_1+\beta_2&=\Phi(0)\\ \beta_1\gamma_1+\beta_2\gamma_2&=\Phi'(0)\end{aligned}\right\}.$$

Solving for β_1 and β_2, we find

$$\begin{aligned}\beta_1&=-(\gamma_1-\gamma_2)^{-1}\left\{\gamma_2\varphi(0)+\gamma\varphi'(0)\right\}\varphi^{-1-\gamma}(0)\\ \beta_2&=(\gamma_1-\gamma_2)^{-1}\left\{\gamma_1\varphi(0)+\gamma\varphi'(0)\right\}\varphi^{-1-\gamma}(0).\end{aligned}$$

(i) Clearly, we have $\beta_1<0$ and $\beta_2>0$. As $l(t)\leq 0$ we deduce that $\Phi(t)$ must vanish for some finite time t_1 estimated by (10). Consequently, $\varphi(t)\to+\infty$ as t approaches t_1.

(ii) If $\varphi'(0)=-\gamma_2\gamma^{-1}\varphi(0)$, then $\beta_1=0$ and $\beta_2=\varphi^{-\gamma}(0)$. Hence by (11), $\varphi(t)\geq\varphi(0)e^{-\gamma^{-1}\gamma_2 t}$. We set $\chi(t)=e^{\gamma^{-1}\gamma_2 t}\varphi(t)$. Therefore, either $\chi(t)$ is constant or else $\chi'(\tilde{t})>0$ for some $\tilde{t}$ and the argument in (i) will apply for $\tilde{t}$ in place of 0.

(iii) Condition $\varphi'(0) < -\gamma_2\gamma^{-1}\varphi(0)$ means that $\chi'(0) < 0$. We deduce that $\chi'(t) < 0$ for all $t \geq 0$ unless there exists a finite time $\hat{t}$ for which $\chi'(\hat{t}) = 0$. The first possibility leads to the relation $\varphi(t) \leq \varphi(0)e^{-\gamma^{-1}\gamma_2 t}$. The second possibility with the help of the argument in (ii) implies the blow up in finite time, the alternative $\varphi(t) = \varphi(0)e^{-\gamma^{-1}\gamma_2 t}$ being excluded since it leads to the case $\varphi'(0) = -\gamma_2\gamma^{-1}\varphi(0)$.

(b) If $C_1 = C_2 = 0$, we meet the concavity method. Indeed, (11) yields $\Phi''(t) \leq 0$ and by integration we find

$$\Phi(t) \leq \Phi(0) + \Phi'(0)t \qquad \text{or} \qquad \varphi^\gamma(t) \geq \varphi^\gamma(0)\left\{1 - \gamma\varphi^{-1}(0)\varphi'(0)t\right\}^{-1}.$$

We conclude as in the first case (a) ∎

Let us define the energy functional by

$$E(t) = \tfrac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)\, dx - \int_\Omega \mathcal{F}(t,u)\, dx \tag{12}$$

and for $C_1 = \frac{1}{2}\alpha, C_2 = \frac{1}{2}k(0)\beta$ and γ_2 as in Lemma 2 we set

$$\mathcal{A}(u_0,u_1) = 2\int_\Omega u_0 u_1 dx + \tfrac{\gamma_2}{\gamma}\int_\Omega u_0^2 dx$$

and assume that $\|u_0\|_2 \neq 0$.

We are now ready to state and prove our first theorem. For simplicity we shall consider kernels which are independent of the spatial variable x.

Theorem 3. *Suppose that $k = k(t)$ satisfies the assumptions in Theorem* 1. *Besides assumptions* (H1) *and* (H2) *assume that $f(t,u)$ satisfies assumption*

(H3) $uf(t,u) \geq (1 + \sqrt{1 + 2k(0)\beta})\mathcal{F}(t,u)$ *for some $\gamma > 0$ and all u and $t \geq 0$ where β is the best constant in* (6).

(a) *If $E(0) < 0$ and $\mathcal{A}(u_0,u_1) \geq 0$, then any regular solution of problem* (1) *blows up in finite time estimated by* (10).

(b) *If $E(0) = 0$ and*

(i) $\mathcal{A}(u_0,u_1) > 0$, *then we have the same conclusion as in* (a)

(ii) $\mathcal{A}(u_0,u_1) = 0$, *then either the solution blows up in finite time or else* $\|u\|_2^2 = \|u_0\|_2^2 e^{-\gamma^{-1}\gamma_2 t}$

(iii) $\mathcal{A}(u_0,u_1) < 0$, *then either the solution blows up in finite time or else* $\|u\|_2^2 \leq \|u_0\|_2^2 e^{-\gamma^{-1}\gamma_2 t}$.

Proof. (a) Differentiating the energy functional $E(t)$ (see (12)) we get

$$E'(t) = -\alpha\int_\Omega u_t^2 dx - \int_{\Gamma_0} u_t\left(\int_0^t k(t-s)u_s(s)\, ds\right) d\sigma - \int_\Omega \mathcal{F}_t(t,u)\, dx. \tag{13}$$

Integrating (13) over $[0,t]$ we find

$$\begin{aligned} E(t)-E(0) = &-\int_0^t\int_{\Gamma_0} u_s\left(\int_0^s k(s-r)u_r(r)\,dr\right)d\sigma ds \\ &-\alpha\int_0^t\int_\Omega u_t^2 dxds - \int_0^t\int_\Omega \mathcal{F}_t(s,u)dxds. \end{aligned} \tag{14}$$

By assumption (H2) and (4) it follows that

$$E(t)\le E(0)<0 \qquad \text{for all } t\ge 0. \tag{15}$$

On the other hand, putting

$$G(t)=\int_\Omega u^2 dx \tag{16}$$

we see that $G'(t)=2\int_\Omega uu_t dx$ and $G''(t)=2\int_\Omega(u_t^2+uu_{tt})\,dx$. Using problem (1) we obtain

$$\begin{aligned} G''(t) = &-2\int_\Omega|\nabla u|^2dx - 2\int_{\Gamma_0} u\left(\int_0^t k(t-s)u_s(s)\,ds\right)d\sigma \\ &+2\int_\Omega u_t^2dx + 2\int_\Omega uf(t,u)\,dx - 2\alpha\int_\Omega uu_tdx, \end{aligned}$$

and by (6), (8) and (14) - (16) it follows that

$$\begin{aligned} G''(t)\ge &-2(1+\mu\beta)\int_\Omega|\nabla u|^2dx + 2\int_\Omega u_t^2dx - 2\mu\beta G(t) \\ &-\alpha G'(t) + \tfrac{2k(0)}{\mu}E(t) + 2\int_\Omega uf(t,u)\,dx. \end{aligned}$$

If we set $W:=G(t)G''(t)-(1+\gamma)(G'(t))^2$, then

$$\begin{aligned} W\ge G(t)\Big\{&-2(1+\mu\beta)\int_\Omega|\nabla u|^2dx + 2\int_\Omega u_t^2dx - 2\mu\beta G(t) \\ &-\alpha G'(t)+\tfrac{2k(0)}{\mu}E(t)+2\int_\Omega uf(t,u)\,dx\Big\} - 4(1+\gamma)\left\{\int_\Omega uu_tdx\right\}^2. \end{aligned}$$

By the Schwarz inequality it appears that

$$\begin{aligned} W\ge G(t)\Big\{&-2(1+\mu\beta)\int_\Omega|\nabla u|^2dx - 2(1+2\gamma)\int_\Omega u_t^2dx + \tfrac{2k(0)}{\mu}E(t) \\ &-2\mu\beta G(t) - \alpha G'(t) + 2\int_\Omega uf(t,u)\,dx\Big\}. \end{aligned}$$

Let $\mu=2\gamma$. Then by the definition of $E(t)$ we have

$$\begin{aligned} W\ge G(t)\Big\{&-2\big(2+4\gamma-\tfrac{k(0)\beta}{2\gamma}\big)E(t) - 2\mu\beta G(t) - \alpha G'(t) \\ &+2\int_\Omega uf(t,u)\,dx - 4(1+2\gamma)\int_\Omega \mathcal{F}(t,u)\,dx - 2a(1+2\gamma)\Big\}. \end{aligned}$$

Taking $\gamma = \frac{1}{4}\big(-1 + \sqrt{1 + 2k(0)\beta}\big)$, we see that assumption (H3) implies

$$W \geq G(t)\left\{-\sqrt{1+2k(0)\beta}G(t) - \alpha G'(t)\right\}. \tag{17}$$

Applying Lemma 2 with the constants $C_1 = \frac{1}{2}\alpha$ and $C_2 = \sqrt{1+2k(0)\beta}$, we conclude. An estimation of the escape time is given in Lemma 2.

(b) If $E(0) = 0$, then (i) when the initial data satisfy $\mathcal{A}(u_0, u_1) > 0$ the argument in (a) still holds giving the same conclusion. Statements (ii) and (iii) follow readily from (ii) and (iii) of Lemma 2 ∎

For the next proposition let us consider the case $f(t,u) = |u|^{p-1}u \quad (p > 1)$ (easy assumptions may be found for general functions $f(t,u)$).

Proposition 4. *The set of initial data u_0, u_1 satisfying $E(u_0,u_1) < 0$ and $\mathcal{A}(u_0,u_1) \geq 0$ is not empty.*

Proof. Recall that

$$E(u_0,u_1) = \tfrac{1}{2}\int_\Omega (u_1^2 + |\nabla u_0|^2)\,dx - \int_\Omega \mathcal{F}(0,u_0)\,dx$$

and

$$\mathcal{A}(u_0,u_1) = 2\int_\Omega u_0 u_1 dx + C_0 \int_\Omega u_0^2 dx$$

with $C_0 = -\frac{\gamma_2}{\gamma}$. First observe that we can always find a sufficiently large δ such that $u_0 = \delta v_0$ satisfies

$$\tfrac{1}{8}C_0^2 \int_\Omega u_0^2 dx + \tfrac{1}{2}\int_\Omega |\nabla u_0|^2 dx < \tfrac{1}{p+1}\int_\Omega |u_0|^{p+1} dx.$$

Suppose on the contrary that

$$\tfrac{1}{p+1}\int_\Omega |u_0|^{p+1} dx \leq \tfrac{C_0^2}{8}\int_\Omega u_0^2 dx + \tfrac{1}{2}\int_\Omega |\nabla u_0|^2 dx$$

for all $\delta > 0$. Then

$$\tfrac{\delta^{p-1}}{p+1}\int_\Omega |v_0|^{p+1} dx \leq \tfrac{C_0^2}{8}\int_\Omega v_0^2 dx + \tfrac{1}{2}\int_\Omega |\nabla v_0|^2 dx$$

for all $\delta > 0$. This is a contradiction.

We choose u_1 such that $u_1(x) < \frac{C_0}{2}u_0(x)$ and

$$\tfrac{1}{8}C_0^2 \int_\Omega u_0^2 dx < \tfrac{1}{2}\int_\Omega u_1^2 dx < \tfrac{1}{p+1}\int_\Omega |u_0|^{p+1} dx - \tfrac{1}{2}\int_\Omega |\nabla u_0|^2 dx.$$

Then, clearly, $E(u_0,u_1) < 0$ and $2\int_\Omega u_0 u_1 dx > C_0 \int_\Omega u_0^2 dx$, that is $\mathcal{A}(u_0,u_1) > 0$ ∎

Remark 1. It is sufficient to require that hypothesis concerning $f(t,u)$ be satisfied only for the solution of the problem. The same remark is valid for the hypothesis below.

Remark 2. Note that Theorem 3 gives the blow up in finite time for the solution in the L^2-norm.

Remark 3. Theorem 3 is interesting from the theoretical point of view. In practice it is enough to use the concavity method of Levine provided we find the appropriate functional. In our case, the choice of the functional

$$G(t) = \|u\|_2^2 + \alpha\left\{\int_0^t \|u(s)\|_2^2 ds + (T_0 - t)\|u_0\|_2^2\right\} + a(t+t_0)^2$$

would simplify the term $\alpha\int_\Omega uu_t dx$ and thereafter $G'(t)$ in (17). As for $G(t)$ we combine the Poincaré and Trace inequalities instead of (6).

Hypothesis (H3) is satisfied by the class of functions $f(t,u) = g(t)\,|u|^{p-1}u$ with

$$p \geq \sqrt{1+2k(0)\beta}. \tag{18}$$

Next we shall see that even for positive initial energy it is possible to obtain a blow up result. We consider the non-linearity

$$f(t,u) \equiv f(u) = |u|^{p-1}u.$$

Let us introduce the sets

$$\mathcal{P} = \left\{v \in H^2(\Omega)\cap V :\, P(v) := \|\nabla v\|_2^2 - \|v\|_{p+1}^{p+1} < 0\right\}$$

and for $\gamma > \frac{1}{2}\sqrt{\beta k(0)}$, the best constant η in (7) and $q = 2\frac{p+1}{p-1}$,

$$\mathcal{E} = \left\{(v,w) \in H^2(\Omega)\cap V \times V :\, E(v,w) := E(t) < \frac{4\gamma^2 - \beta k(0)}{4\gamma(1+2\gamma)}\eta^{-q}\right\}.$$

In the next proposition it will be proved, for solutions of problem (1), that $\mathcal{P}$ and $\mathcal{E}$ are invariant sets. That is, if we start in these sets we remain inside these sets as long as the solution exists (see [14] and references therein for the method of proof).

Proposition 5. *Let $u(t)$ be a solution of problem* (1), *$p \geq 3+4\gamma$, $u_0 \in \mathcal{P}$ and $(u_0,u_1) \in \mathcal{E}$. If $p \leq 5$ when $n = 3$ (in this case $k(0)$ must satisfy $k(0) < \frac{1}{\beta}$) and $p < +\infty$ when $n = 1,2$, then $u \in \mathcal{P}$ and $(u,u_t) \in \mathcal{E}$. Moreover,*

$$\|\nabla u\|_2^2 > \eta^{-q} \qquad \textit{for all}\;\; t \geq 0. \tag{19}$$

Proof. From $E(t) \leq E(0)$ (first inequality of (15)) and $(u_0,u_1) \in \mathcal{E}$ we have $(u,u_t) \in \mathcal{E}$. Let $u_0 \in \mathcal{P}$ and

$$T := \sup\left\{t \in [0,+\infty) :\, P(u(s)) < 0 \;\text{ for }\; 0 \leq s < t\right\}.$$

As $\mathcal{P}$ is an open set, it is clear that $T > 0$. Suppose for contradiction that $T < +\infty$. Then

$$E(u(T)) \geq \tfrac{1}{2}\|\nabla u(T)\|_2^2 - \tfrac{1}{p+1}\|u(T)\|_{p+1}^{p+1} = \tfrac{p-1}{2(p+1)}\|\nabla u(T)\|_2^2. \tag{20}$$

On the other hand, using the Sobolev-Poincaré inequality (7) we find

$$\|\nabla u(T)\|_2^{p+1} \geq \eta^{-(p+1)}\|u(T)\|_{p+1}^{p+1} = \eta^{-(p+1)}\|\nabla u(T)\|_2^2. \tag{21}$$

Therefore $\|\nabla u(T)\|_2^{p-1} \geq \eta^{-(p+1)}$. From (20) and (21) we deduce that $E(u(T)) \geq \frac{p-1}{2(p+1)}\eta^{-q}$. Observing that $p \geq 3+4\gamma$ implies the relation $\frac{p-1}{2(p+1)} \geq \frac{4\gamma^2-\beta k(0)}{4\gamma(1+2\gamma)}$ we obtain $E(u(t)) \geq \frac{4\gamma^2-\beta k(0)}{4\gamma(1+2\gamma)}\eta^{-q}$. This is a contradiction. Consequently, $T = +\infty$.

Finally, if $u \in \mathcal{P}$, then (19) holds from the Sobolev-Poincaré inequality (7) and the fact $P(u) < 0$. Indeed, $\|\nabla u\|_2^{p+1} \geq \eta^{-(p+1)}\|u\|_{p+1}^{p+1} > \eta^{-(p+1)}\|\nabla u\|_2^2$. This completes the proof ∎

Let us consider $\mathcal{A}(u_0,u_1)$ with $C_1 = \frac{1}{2}\alpha$ and $C_2 = k(0)\frac{\beta}{\gamma}$.

Theorem 6. *Assume that $u \in \mathcal{P}$, $(u_0,u_1) \in \mathcal{E}$, $\mathcal{A}(u_0,u_1) > 0$ and $p \geq 3+4\gamma$ holds. If $p \leq 5$ when $n = 3$ and $p < +\infty$ when $n = 1,2$, then the solution of problem* (1) *blows up in a finite time estimated by* (10).

Proof. From (12), (14), (16) and the assumption $p \geq 3+4\gamma$ we have

$$\begin{aligned} W \geq G(t)\Big\{&-2(1+\mu\beta)\|\nabla u\|_2^2 - 4(1+2\gamma)E(t) + \tfrac{2k(0)}{\mu}E(t) \\ &-\tfrac{2k(0)}{\mu}E(0) + 2(1+2\gamma)\|\nabla u\|_2^2\Big\} - 2\mu\beta G^2(t) - \alpha G(t)G'(t) \end{aligned}$$

or

$$\begin{aligned} W \geq\ & G(t)\Big\{2(2\gamma-\mu\beta)\|\nabla u\|_2^2 - 2(2+4\gamma-\tfrac{k(0)}{\mu})E(t) - 2\tfrac{k(0)}{\mu}E(0)\Big\} \\ &- 2\mu\beta G^2(t) - \alpha G(t)G'(t). \end{aligned}$$

We choose $\mu = \frac{k(0)}{2\gamma}$. It follows that

$$W \geq G(t)\Big\{2\Big(2\gamma - \beta\tfrac{k(0)}{2\gamma}\Big)\|\nabla u\|_2^2 - 4(1+2\gamma)E(0)\Big\} - \tfrac{k(0)}{\gamma}\beta G^2(t) - \alpha G(t)G'(t)$$

or

$$W \geq 4(1+2\gamma)G(t)\Big\{\tfrac{4\gamma^2-\beta k(0)}{4\gamma(1+2\gamma)}\|\nabla u\|_2^2 - E(0)\Big\} - \tfrac{k(0)}{\gamma}\beta G^2(t) - \alpha G(t)G'(t).$$

Hence $W \geq -\frac{k(0)}{\gamma}\beta G^2(t) - \alpha G(t)G'(t)$. The rest is clear from Lemma 2 ∎

Remark 4. For large values of the initial data u_0 it may be worthy to use the argument in Knops, Levine and Payne [8: p. 65].

3. A modification of the concavity method

The following is a generalization of an idea by Sleeman [17] on a modification of the concavity method.

Theorem 7. *Let Ψ be a twice differentiable positive function such that*

$$\Psi(t)\Psi''(t) + (\gamma - 1)(\Psi'(t))^2 \leq -2C_1\Psi(t)\Psi'(t) + C_2\Psi(t)^2 \qquad (t \geq 0)$$

for some $\gamma > 1$ and constants $C_1, C_2 \geq 0$.

(a) *Let $C_1 + C_2 > 0$ and $\Psi(0) > 0$.*

(i) *If $\Psi'(0) < \frac{\gamma_2}{\gamma}\Psi(0)$, then $\Psi(t)$ cannot exist beyond the time*

$$T_1 = \frac{1}{2\sqrt{C_1^2 + \gamma C_2}} \ln\left\{\gamma_1\Psi(0) - \gamma\Psi''(0)\gamma_2\Psi(0) - \gamma\Psi''(0)\right\} \tag{22}$$

where $\gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2}$ and $\gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2}$.

(ii) *If $\Psi'(0) = \frac{\gamma_2}{\gamma}\Psi(0)$, then either $\Psi(t)$ cannot exist beyond a finite time or else $\Psi(t) = \Psi(0)e^{\gamma^{-1}\gamma_2 t}$.*

(iii) *If $\Psi'(0) > \frac{\gamma_2}{\gamma}\Psi(0)$, then either $\Psi(t)$ cannot exist beyond a finite time or else $\Psi(t) \geq \Psi(0)e^{\gamma^{-1}\gamma_2 t}$.*

(b) *If $C_1 = C_2 = 0$, $\Psi(0) > 0$ and $\Psi'(0) < 0$, we meet Sleeman's modified concavity method with the finite time $T_2 = -\gamma\frac{\Psi(0)}{\Psi'(0)}$.*

Proof. The method of proof is similar to that of Lemma 2. We use however the intermediate function $\Phi = \Psi^\gamma$ instead of the first one. The details are omitted ∎

Let us set

$$\mathcal{D}(u_0, u_1) = 2\int_\Omega u_0u_1\,dx - \frac{\gamma_2}{\gamma}\int_\Omega u_0^2\,dx$$

where $C_1 = \frac{1}{2}\alpha$, $C_2 = 4\gamma$ and γ_2 is as in Lemma 2.

Theorem 8. *Assume that hypotheses* (H1), (H2) *and*

(H4) *$uf(t,u) \leq 2(1-2\gamma)\mathcal{F}(t,u)$ for some $\gamma > \max\left\{1, \frac{1}{4}\left(1 + \sqrt{1 + 2k(0)\beta}\right)\right\}$ for all u and $t \geq 0$*

hold. If $E(0) \leq 0$ and for γ satisfying assumption (H4)

(i) *$\mathcal{D}(u_0, u_1) < 0$, then no regular solution can exist beyond the time T_1 given by* (22)*;*

(ii) *$\mathcal{D}(u_0, u_1) = 0$, then either the solution cannot exist beyond a finite time or else $\|u\|_2^2 = \|u_0\|_2^2 e^{\gamma^{-1}\gamma_2 t}$;*

(iii) *$\mathcal{D}(u_0, u_1) > 0$, then either the solution cannot exist beyond a finite time or else $\|u\|_2^2 \geq \|u_0\|_2^2 e^{\gamma^{-1}\gamma_2 t}$.*

Moreover, $\lim_{t\to T_1^-}\int_\Omega u_t^2\,dx = +\infty$.

Proof. Let $G(t) = \int_\Omega u^2 dx$. Then

$$G'(t) = 2\int_\Omega u u_t dx \qquad \text{and} \qquad G''(t) = 2\int_\Omega (u u_{tt} + u_t^2)\, dx.$$

Forming the expression $W = GG'' + (\gamma - 1)(G')^2$ we obtain

$$W = G(t)\left\{-2\int_\Omega |\nabla u|^2 dx - 2\int_{\Gamma_0} u \left(\int_0^t k(t-s) u_s(s)\, ds\right) d\sigma \right.$$
$$\left. -2\alpha\int_\Omega u u_t dx + 2\int_\Omega u_t^2 dx + 2\int_\Omega u f(t,u)\, dx\right\} + 4(\gamma - 1)\left\{\int_\Omega u u_t dx\right\}^2.$$

Using the Schwarz inequality, (6), (3) and (8) we find

$$W \leq G(t)\left\{-2\int_\Omega |\nabla u|^2 dx + 2\beta\mu\int_\Omega u^2 dx + 2\beta\mu\int_\Omega |\nabla u|^2 dx \right.$$
$$+\tfrac{2k(0)}{\mu}\int_0^t \int_{\Gamma_0} u_s \int_0^s k(s-z) u_z(z)\, dz d\sigma ds - \alpha G'(t)$$
$$\left. +2(2\gamma - 1)\int_\Omega u_t^2 dx + 2\int_\Omega u f(t,u)\, dx\right\}.$$

With the aid of (12), (14) and assumption (H4) we obtain

$$W \leq G(t)\left\{2\left(2\beta\mu - 2 - \tfrac{k(0)}{\mu}\right) E(t) + 2(2\gamma - \beta\mu)\int_\Omega u_t^2 dx + 2\beta\mu G(t) - \alpha G'(t)\right\}.$$

Choosing $\mu = 2\frac{\gamma}{\beta}$, we get

$$W \leq G(t)\left\{2\left(4\gamma - 2 - \tfrac{k(0)\beta}{2\gamma}\right) E(t)\right\} + 4\gamma G^2(t) - \alpha G'(t) G(t).$$

From our assumption $\gamma > \frac{1}{4}\left(1 + \sqrt{1 + 2k(0)\beta}\right)$ it follows that $4\gamma - 2 - \frac{k(0)\beta}{2\gamma} \geq 0$. Hence $W \leq 4\gamma G^2(t) - \alpha G'(t) G(t)$. We next apply Theorem 7.

For the second part of the theorem we recall the second order differential equation (11) satisfied by $\Phi = G^\gamma$

$$\Phi''(t) + 2C_1\Phi'(t) - \gamma C_2 \Phi(t) = h(t) \leq 0.$$

The proof will be carried out for statement (i). We change the starting point for the other cases. An integration of the last expression over $[0, t]$ yields

$$\Phi'(t) \leq \Phi'(0) + 2C_1\Phi'(0) + \gamma C_2\int_0^t \Phi(s)\, ds$$

or

$$G'(t) \leq G^{-\gamma+1}(t)\left\{G^{\gamma-1}(0) G'(0) + 2C_1\gamma^{-1} G^\gamma(0) + C_2\int_0^t \Phi(s)\, ds\right\}.$$

We know that $\Phi(t) \leq \beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t}$, therefore

$$\int_0^t \Phi(s)\,ds < \tfrac{\beta_1}{\gamma_1} e^{\gamma_1 t} + \tfrac{\beta_2}{\gamma_2} e^{\gamma_2 t} - \tfrac{\beta_1}{\gamma_1} - \tfrac{\beta_2}{\gamma_2} < -\tfrac{\beta_1}{\gamma_1} - \tfrac{\beta_2}{\gamma_2} \qquad (t > 0)$$

or

$$C_2 \int_0^t \Phi(s)\,ds < \left\{-2C_1\gamma^{-1}G(0) - G'(0)\right\} G^{\gamma-1}(0) \qquad (t > 0).$$

Hence

$$\left\{2C_1\gamma^{-1}G^{\gamma}(0) + G^{\gamma-1}(0)G'(0) + C_2\int_0^t \Phi(s)\,ds\right\} < 0.$$

Observe also that for $t > 0$ we may write

$$\left\{2C_1\gamma^{-1}G^{\gamma}(0) + G^{\gamma-1}(0)G'(0) + C_2\int_0^t \Phi(s)\,ds\right\} < -A.$$

We obtain

$$G'(t) \leq (-A)G^{-\gamma+1}(t). \tag{23}$$

On the other hand, it is clear that

$$G'(t) = 2\int_\Omega u u_t dx \geq -2\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}} \geq -2G^{\frac{1}{2}}(t)\left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}}. \tag{24}$$

From (23) and (24) it follows that

$$\left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}} \geq \tfrac{A}{2} G^{-\gamma+\frac{1}{2}}(t) \geq \tfrac{A}{2}\left(\beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t}\right)^{-1+\frac{1}{2\gamma}}.$$

Consequently, $\lim_{t\to T_1^-}\int_\Omega u_t^2 dx = +\infty$ with T_1 as in (22) ∎

Remark 5. A class of non-linearities satisfying assumption (H4) may be of the form $f(t,u) = h(t)u^{-(4\alpha-1)}\phi(u)$ $(u > 0)$ where ϕ is a monotone non-decreasing function.

Remark 6. Theorems 7 and 8 remain valid if $\gamma < 1$. The proofs are very much the same. After computation we remark that even the formulations of the theorems are similar. Indeed, it suffices only to replace γ by $2 - \gamma$.

4. The case $k(x,t) = p(x)e^{-t}$

In this section another result on the blow up in finite time is proved. It concerns the case where the boundary material is characterized by the kernel $k(x,t) = p(x)e^{-t}$. The function p is assumed to be non-negative and uniformly bounded by M. The method of proof is based on a suitable choice of the energy functional combined with an argument by Tsutsumi [19]. This method has the advantage of working even in the case $\alpha < 0$. The problem we consider is then

$$\left.\begin{array}{ll} u_{tt}(t,x) + \alpha u_t(t,x) = \Delta u(t,x) + f(t,u) & (t > 0, x \in \Omega) \\ \dfrac{\partial u}{\partial \nu}(t,x) + \displaystyle\int_0^t p(x)e^{-(t-s)}u_s(s,x)\,ds = 0 & (t > 0, x \in \Gamma_0) \\ u(t,x) = 0 & (t > 0, x \in \Gamma_1) \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & (x \in \Omega). \end{array}\right\} \tag{25}$$

Theorem 9. *Let assumptions* (H1) *and* (H2) *hold. Further, suppose that the following hypothesis hold:*

(H5) $\int_\Omega u_0 u_1 dx > 0$ *and* $\int_\Omega (u_1^2 + |\nabla u_0|^2)\,dx < 2\int_\Omega \mathcal{F}(0,u_0)dx$

(H6) $\int_\Omega \{uf(t,u) - (D+M)\mathcal{F}(t,u)\}dx \geq C_1\|u\|_2^{2p}$ *for all* u *and* $t \geq 0$, *for some constant* $C_1 > 0$, $p > 1$, $D = \max\{\alpha - 2, \alpha\delta + \tilde{\beta} + 2\}$ *and* $\tilde{\beta} = \beta(\delta + 1)$

or

(H7) $\int_\Omega \{uf(t,u) - (D+1)\mathcal{F}(t,u)\}dx \geq C_2\|u\|_2^{2p}$ *for all* u *and* $t \geq 0$, *for some constant* $C_2 > 0$, $p > 1$, $D = max\{\alpha - 3, \alpha\delta + M\tilde{\beta} + 1\}$ *and* $\tilde{\beta} = \beta(\delta + 1)$.

Then the solution of problem (25) *blows up in finite time.*

Proof. First assume that assumption (H6) holds. Suppose for contradiction that u exists globally in time. Let us define the energy functional by

$$E(t) = \tfrac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)\,dx + \tfrac{1}{2}\int_{\Gamma_0} p(x)\left(\int_0^t e^{-(t-s)}u_s(s)\,ds\right)^2 d\sigma - \int_\Omega \mathcal{F}(t,u)\,dx. \tag{26}$$

Then

$$\begin{aligned} E'(t) &= -\alpha\int_\Omega u_t^2 dx + \int_{\Gamma_0} u_t\frac{\partial u}{\partial \eta}d\sigma + \int_\Omega u_t f(t,u)\,dx \\ &\quad - \int_\Omega u_t f(t,u)\,dx - \int_\Omega \mathcal{F}_t(t,u)\,dx \\ &\quad + \int_{\Gamma_0} p(x)\left\{u_t - \int_0^t e^{-(t-s)}u_s(s)\,ds\right\}\left(\int_0^t e^{-(t-s)}u_t(s)\,ds\right)d\sigma \\ &= -\alpha\int_\Omega u_s^2 dx - \int_\Omega \mathcal{F}_t(t,u)\,dx - \int_{\Gamma_0} p(x)\left(\int_0^t e^{-(t-s)}u_s(s)ds\right)^2 d\sigma \\ &\leq 0. \end{aligned}$$

Thus $E(t) \le E(0) < 0$. Next introducing the function $G(t) = \int_\Omega u^2 dx$ it appears that

$$G''(t) = 2\|u_t\|_2^2 - 2\alpha \int_\Omega u u_t dx - 2\|\nabla u\|_2^2 + 2\int_\Omega u f(t,u)\, dx \\ - 2\int_{\Gamma_0} up(x) \int_0^t e^{-(t-s)} u_s(s)\, dx d\sigma.$$

Using (5), (6) and

$$\left| 2\int_{\Gamma_0} up(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right) d\sigma \right| \\ \le \tilde{\beta}\|\nabla u\|_2^2 + M \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma \tag{27}$$

we obtain

$$G''(t) \ge 2\|u_t\|_2^2 - \alpha\|u_t\|_2^2 - 2\|\nabla u\|_2^2 - \alpha\delta\|\nabla u\|_2^2 - \tilde{\beta}\|\nabla u\|_2^2 \\ + 2\int_\Omega u f(t,u)\, dx - M\int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s)\, ds \right)^2 d\sigma.$$

By the definition of $E(t)$ (26) we see that

$$G''(t) \ge (2-\alpha)\|u_t\|_2^2 - (2 + \tilde{\beta} + \alpha\delta)\|\nabla u\|_2^2 + 2\int_\Omega u f(t,u)\, dx \\ - 2M \left\{ E(t) - \tfrac{1}{2}\|u_t\|_2^2 - \tfrac{1}{2}\|\nabla u\|_2^2 + \int_\Omega \mathcal{F}(t,u)\, dx \right\}$$

or

$$G''(t) \ge (2 - \alpha + M)\|u_t\|_2^2 - (2 + \tilde{\beta} + \alpha\delta - M)\|\nabla u\|_2^2 \\ + 2\int_\Omega u f(t,u)\, dx - 2ME(t) - 2M\int_\Omega \mathcal{F}(t,u)\, dx.$$

Again, the definition of $E(t)$ yields

$$G''(t) \ge -2\max\{\alpha - 2, 2 + \tilde{\beta} + \alpha\delta\} \left(E(t) + \int_\Omega \mathcal{F}(t,u)\, dx \right) \\ + 2\int_\Omega u f(t,u)\, dx - 2ME(t) - 2M\int_\Omega \mathcal{F}(t,u)\, dx$$

or

$$G''(t) \ge -2(D+M)E(0) + 2\left\{ \int_\Omega u f(t,u)\, dx - (D+M)\int_\Omega \mathcal{F}(t,u)\, dx \right\}.$$

From assumption (H6) it results that

$$G''(t) \ge -2(D+M)E(0) + C_1 G(t)^p. \tag{28}$$

This leads us to an argument by Tsutsumi [19] (see also [1]). Clearly, $G'(t) > 0$ for all $t \geq 0$. Multiplying (28) by $G'(t)$ and integrating we find

$$\tfrac{1}{2}G'(t)^2 - \tfrac{1}{2}G'(0)^2 \geq -2(D+M)E(0)\{G(t) - G(0)\} + \tfrac{C_1}{p+1}\{G(t)^{p+1} - G(0)^{p+1}\}$$

or

$$G'(t) \geq \left\{C_3 - 4(D+M)E(0)G(t) + 2\tfrac{C_1}{p+1}G(t)^{p+1}\right\}^{\frac{1}{2}},$$

with $C_3 = G'(0)^2 + 4(D+M)E(0)G(0) - 2\frac{C_1}{p+1}G(0)^{p+1}$. Consequently, by integration we deduce

$$t_{\max} \leq T_0 = \int_{\|u_0\|_2^2}^{\infty} \left\{C_3 - 4(D+M)E(0)z + 2\tfrac{C_1}{p+1}z^{p+1}\right\}^{-\frac{1}{2}} dz. \tag{29}$$

As T_0 is finite, (29) provides a contradiction.

In the case of assumption (H7) the proof is similar, we use the estimate

$$\begin{aligned}&\left|2\int_{\Gamma_0} up(x)\left(\int_0^t e^{-(t-s)}u_s(s)\,ds\right)d\sigma\right|\\ &\qquad\leq \tilde{\beta}M\|\nabla u\|_2^2 + \int_{\Gamma_0} p(x)\left(\int_0^t e^{-(t-s)}u_s(s)\,ds\right)^2 d\sigma\end{aligned}$$

instead of (27) ∎

Acknowledgments. The authors wish to express their gratitude to the referee for valuable comments and suggestions.

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Received 04.06.1999; in revised form 07.12.1999