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Multiple Solutions for a System of (n_i, p_i) Boundary Value Problems

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Abstract. We consider the system of boundary value problems

$$\left.\begin{aligned} u_i^{(n_i)}(t) + f_i(t, u_1(t), \dots, u_m(t)) &= 0 \\ u_i^{(j)}(0) &= 0 \\ u_i^{(p_i)}(1) &= 0 \end{aligned}\right\}$$

for $t \in [0,1]$, $i = 1, \dots, m$ and $0 \le j \le n_i - 2$ where $n_i \ge 2$ and $1 \le p_i \le n_i - 1$. Several criteria are offered for the existence of single and twin solutions of the system that are of fixed signs.

Keywords: *Solutions of fixed signs, systems of boundary value problems*

AMS subject classification: 34 B 15

1. Introduction

In this paper we shall consider the system of boundary value problems

$$\left.\begin{aligned} u_i^{(n_i)}(t) + f_i(t, u_1(t), \dots, u_m(t)) &= 0 \\ u_i^{(j)}(0) &= 0 \\ u_i^{(p_i)}(1) &= 0 \end{aligned}\right\} \tag{1.1}$$

for $t \in [0,1]$, $i = 1, \dots, m$ and $0 \le j \le n_i - 2$. Throughout, for each i, it is assumed that $n_i \ge 2$ and $1 \le p_i \le n_i - 1$. A solution $u = (u_1, \dots, u_m)$ of system (1.1) will be sought in $B = (C[0,1])^m = C[0,1] \times \cdots \times C[0,1]$ (m times). We say that u is a solution of *fixed sign* if for each $1 \le i \le m$ we have $\gamma_i u_i \ge 0$ on $[0,1]$ where $\gamma_i \in \{1, -1\}$. Throughout, with $\gamma_i \in \{1, -1\}$ given, we define

$$K = \left\{ u = (u_1, \dots, u_m) \in B \,\middle|\, \gamma_i u_i \ge 0 \text{ for all } 1 \le i \le m \right\}$$

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and

$$K_+ = K\backslash 0 = \Big\{ u = (u_1, \ldots, u_m) \in K \Big| \gamma_j u_j > 0 \text{ for some } j \in \{1, \ldots, m\} \Big\}.$$

For each $1 \le i \le m$, it is assumed that f_i is continuous on $[0,1] \times K$.

The aim of this paper is to provide various conditions on the nonlinearities f_i $(1 \le i \le m)$ so that system (1.1) has single as well as twin solutions that are of fixed signs. Specifically, we shall consider two cases. The first is when f_i $(1 \le i \le m)$ satisfy certain 'fixed-sign' condition, namely,

$$\textbf{(A)} \quad \begin{cases} \gamma_i f_i(t, u_1, \ldots, u_m) \ge 0 & \text{if } (t,u) \in [0,1] \times K \\ \gamma_i f_i(t, u_1, \ldots, u_m) > 0 & \text{if } (t,u) \in [0,1] \times K_+ \end{cases}$$

and the second is when condition (A) is relaxed.

There are numerous recent investigations on the existence of solutions of boundary value problems, these are well documented in the monographs [1, 2, 4, 5]. In fact, particular cases of system (1.1) when $m = 1$ arise in various physical phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, infectious diseases, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems [7 - 10, 14, 15, 18, 19]. Our present work extends the vast literature on boundary value problems to a *system* of boundary value problems. For other related work on systems of boundary value problems, we refer to recent contributions of [3, 6, 20 - 23]. It is noted that in all these works, the criteria developed are *different* from our current work, and some of the systems are *not* as general as what we are considering here.

The outline of the paper is as follows. In Section 2 we shall state Krasnosel'skii's fixed point theorem in a cone and present some inequalities for a certain Green's function which are needed later. Under the assumption of condition (A), the existence of single and twin fixed-sign solutions of system (1.1) is treated in Sections 3 and 4, respectively. Finally, in Sections 5 and 6 we discuss the case when condition (A) is removed.

2. Preliminaries

In this section we shall state Krasnosel'skii's fixed point theorem in a cone which is used later to establish existence criteria for the solution of system (1.1). Certain inequalities involving Green's function related to system (1.1) are also included. These inequalities are important in defining an appropriate cone which is essential in Krasnosel'skii's fixed point theorem.

Theorem 2.1 (see [17]). *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that either

(a) $\|Su\| \le \|u\| \quad (u \in C \cap \partial\Omega_1)$ *and* $\|Su\| \ge \|u\| \quad (u \in C \cap \partial\Omega_2)$

or

(b) $\|Su\| \ge \|u\| \quad (u \in C \cap \partial\Omega_1)$ *and* $\|Su\| \le \|u\| \quad (u \in C \cap \partial\Omega_2)$.

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To obtain a solution of system (1.1), we require a mapping whose kernel $G_i(t,s)$ is Green's function of the (n_i, p_i) boundary value problem

$$\left.\begin{aligned} -y^{(n_i)}(t) &= 0 \\ y^{(j)}(0) &= 0 \\ y^{(p_i)}(1) &= 0 \end{aligned}\right\} \tag{2.1}$$

for $t \in [0,1]$ and $0 \le j \le n_i - 2$. It is known (see [4: p. 191]) that

$$G_i(t,s) = \frac{1}{(n_i-1)!} \begin{cases} t^{n_i-1}(1-s)^{n_i-p_i-1} - (t-s)^{n_i-1} & \text{if } s \in [0,t] \\ t^{n_i-1}(1-s)^{n_i-p_i-1} & \text{if } s \in [t,1] \end{cases} \tag{2.2}$$

and

$$\frac{\partial^j}{\partial t^j} G_i(t,s) \ge 0 \tag{2.3}$$

for $0 \le j \le p_i$ and $(t,s) \in [0,1] \times [0,1]$.

Lemma 2.1 (see [4: p. 192]). *For $(t,s) \in [\frac{1}{4}, \frac{3}{4}] \times [0,1]$ we have*

$$G_i(t,s) \ge \left(\tfrac{1}{4}\right)^{n_i-1} \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}].$$

Lemma 2.2 (see [4: p. 191]). *For $p_i \ge 1$ and $(t,s) \in [0,1] \times [0,1]$ we have*

$$G_i(t,s) \le \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}].$$

Throughout, we shall denote the interval $I = [\frac{1}{4}, \frac{3}{4}]$.

3. Existence of a fixed-sign solution under condition (A)

In this section we shall tackle the existence of a fixed-sign solution when f_i $(1 \le i \le m)$ fulfil condition (A). To begin, let the Banach space $B = (C[0,1])^m$. For $u = (u_1, \ldots, u_m) \in B$ define the norm

$$\|u\| = \max_{1 \le i \le m} \sup_{t \in [0,1]} |u_i(t)| = \max_{1 \le i \le m} |u_i|_0$$

where we denote $|u_i|_0 = \sup_{t \in [0,1]} |u_i(t)| \quad (1 \le i \le m)$. Define the operator $S: B \to B$ by

$$Su(t) = \big(S_1 u(t), \ldots, S_m u(t)\big) \tag{3.1}$$

for $t \in [0,1]$ where

$$S_i u(t) = \int_0^1 G_i(t,s)\, f_i\big(s, u_1(s), \ldots, u_m(s)\big) ds \tag{3.2}$$

for $t \in [0,1]$ and $1 \le i \le m$. Clearly, a fixed point of the operator S is a solution of system (1.1). Let

$$C = \left\{ u = \big(u_1, \ldots, u_m\big) \in B \;\middle|\; \begin{array}{l} \text{for each } 1 \le i \le m,\ \gamma_i u_i(t) \ge 0 \text{ for } t \in [0,1] \\ \text{and } \min_{t \in I} \gamma_i u_i(t) \ge \big(\tfrac{1}{4}\big)^{n_i - 1} |u_i|_0 \end{array} \right\}. \tag{3.3}$$

It is noted that C is a cone in B. Further, $C \subset K$. If $u \in C$ is a solution of system (1.1), then obviously u is a fixed-sign solution of that system.

Lemma 3.1. *The operator S maps C into itself.*

Proof. Let $u \in C \quad (\subset K)$. In view of condition (A) and (2.3), we obtain for $t \in [0,1]$ and $1 \le i \le m$

$$\gamma_i S_i u(t) = \int_0^1 G_i(t,s) \gamma_i f_i\big(s, u_1(s), \ldots, u_m(s)\big) ds \ge 0. \tag{3.4}$$

Next, application of (3.4) and Lemma 2.2 yields

$$\begin{aligned} |S_i u(t)| &= \gamma_i S_i u(t) \\ &\le \int_0^1 \tfrac{1}{(n_i-1)!} (1-s)^{n_i - p_i - 1} [1 - (1-s)^{p_i}]\, \gamma_i f_i\big(s, u_1(s), \ldots, u_m(s)\big) ds \end{aligned}$$

for all $t \in [0,1]$ and $1 \le i \le m$. Consequently,

$$\begin{aligned} |S_i u|_0 \le \int_0^1 & \tfrac{1}{(n_i-1)!} (1-s)^{n_i - p_i - 1} [1 - (1-s)^{p_i}] \\ & \times \gamma_i f_i\big(s, u_1(s), u_2(s), \ldots, u_m(s)\big) ds \end{aligned} \tag{3.5}$$

for all $1 \le i \le m$. Now, using (3.4), Lemma 2.1 and (3.5), for each $1 \le i \le m$ and $t \in I$ we find

$$\begin{aligned} \gamma_i S_i u(t) &\ge \int_0^1 \big(\tfrac{1}{4}\big)^{n_i - 1} \tfrac{1}{(n_i-1)!} (1-s)^{n_i - p_i - 1} [1 - (1-s)^{p_i}]\, \gamma_i f_i\big(s, u_1(s), \ldots, u_m(s)\big) ds \\ &\ge \big(\tfrac{1}{4}\big)^{n_i - 1} |S_i u|_0. \end{aligned}$$

Hence

$$\min_{t \in I} \gamma_i S_i u(t) \ge \big(\tfrac{1}{4}\big)^{n_i - 1} |S_i u|_0 \tag{3.6}$$

for $1 \le i \le m$. Coupling (3.4) and (3.6), we obtain $S(C) \subseteq C$. Also, the standard arguments yield that S is completely continuous ∎

Theorem 3.1. *Suppose there exist two constants λ and η $(\neq \lambda)$ such that the following conditions are satisfied:*

(C1) *For each $1 \leq i \leq m$, we have*

$$\gamma_i f_i(t, u_1, \ldots, u_m) \leq \lambda a_i$$

for $(t, |u_1|, \ldots, |u_m|) \in [0,1] \times [0,\lambda]^m$ where

$$a_i = \left\{ \int_0^1 \tfrac{1}{(n_i-1)!} \, (1-x)^{n_i-p_i-1}[1-(1-x)^{p_i}] \, dx \right\}^{-1}. \tag{3.7}$$

(C2) *For some $1 \leq i \leq m$, we have*

$$\gamma_i f_i(t, u_1, \ldots, u_m) \geq \eta b_i$$

for all $(t, |u_1|, \ldots, |u_m|) \in I \times K_j$ and $j = 1, \ldots, m$ where

$$K_j = \left\{ (v_1, \ldots, v_m) \middle| \; v_j \in \left[\left(\tfrac{1}{4}\right)^{n_j-1} \eta, \eta \right] \text{ and } v_k \in [0, \eta] \text{ for } k \neq j \right\} \tag{3.8}$$

and

$$b_i = \left[\int_{x \in I} G_i\left(\tfrac{1}{4}, x\right) dx \right]^{-1}. \tag{3.9}$$

Then system (1.1) *has a fixed-sign solution u^* such that*

$$\min\{\lambda, \eta\} \leq \|u^*\| \leq \max\{\lambda, \eta\}. \tag{3.10}$$

Proof. We shall employ Theorem 2.1. For this, let

$$\Omega_1 = \{ u \in B : \|u\| < \lambda \} \qquad \text{and} \qquad \Omega_2 = \{ u \in B : \|u\| < \eta \}.$$

We shall show that

(i) $\|Su\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$
(ii) $\|Su\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$.

To justify statement (i), let $u \in C \cap \partial\Omega_1$. So $\|u\| = \lambda$. Applying (3.4), Lemma 2.2 and condition (C1), we get for $t \in [0,1]$ and $1 \leq i \leq m$

$$\begin{aligned} |S_i u(t)| &= \gamma_i S_i u(t) \\ &\leq \int_0^1 \tfrac{1}{(n_i-1)!} (1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}] \, \gamma_i f_i\big(s, u_1(s), \ldots, u_m(s)\big) ds \\ &\leq \int_0^1 \tfrac{1}{(n_i-1)!} \, (1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}] \lambda a_i \, ds \\ &= \lambda \\ &= \|u\|. \end{aligned}$$

Hence, $|S_iu|_0 \le \|u\|$ for $1 \le i \le m$ and so $\|Su\| = \max_{1\le i\le m} |S_iu|_0 \le \|u\|$.

Next, to verify statement (ii), let $u \in C \cap \partial\Omega_2$. Then $\|u\| = \eta$. Suppose that $\|u\| = |u_j|_0$ for some $j \in \{1, \ldots, m\}$. Since $u \in C$, it is clear that $|u_j(t)| \in \left[\left(\frac{1}{4}\right)^{n_j-1}\eta, \eta\right]$ for $t \in I$. Further, $|u_k(t)| \in [0, \eta]$ for $k \ne j$ and $t \in I$. Now, using condition (C2), we find for some $i \in \{1, \ldots, m\}$

$$\begin{aligned}
\left|S_iu\left(\tfrac{1}{4}\right)\right| &= \gamma_i S_iu\left(\tfrac{1}{4}\right)\\
&\ge \int_{s\in I} G_i\left(\tfrac{1}{4}, s\right)\gamma_i f_i\big(s, u_1(s), \ldots, u_m(s)\big)ds\\
&\ge \int_{s\in I} G_i\left(\tfrac{1}{4}, s\right)\eta b_i\, ds\\
&= \eta\\
&= \|u\|.
\end{aligned}$$

Consequently, $|S_iu|_0 \ge \|u\|$ and so $\|Su\| \ge \|u\|$.

Having obtained statements (i) and (ii), we conclude from Theorem 2.1 that S has a fixed point $u^* \in C \cap (\overline{\Omega}_2\backslash\Omega_1)$ or $C \cap (\overline{\Omega}_1\backslash\Omega_2)$. Therefore, (3.10) holds ∎

Let $M = \{1, 2, \ldots, m\}$. For $1 \le i, j \le m$, we introduce the following definitions:

$$\begin{aligned}
\max f_0^{i,j} &= \lim_{\max_{1\le k\le m} |u_k|\to 0^+} \ \max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|}\\
\min f_0^{i,j} &= \lim_{|u_j|\to 0^+} \ \min_{\substack{t\in[0,1]\\ |u_k|\in[0,\infty),\, k\in M\backslash\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|}\\
\max f_\infty^{i,j} &= \lim_{\min_{1\le k\le m} |u_k|\to\infty} \ \max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|}\\
\min f_\infty^{i,j} &= \lim_{|u_j|\to\infty} \ \min_{\substack{t\in[0,1]\\ |u_k|\in[0,\infty),\, k\in M\backslash\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|}.
\end{aligned}$$

Lemma 3.2. *Suppose, for each $1 \le i \le m$ and some $1 \le j \le m$, that one of the conditions*

$$\max f_0^{i,j} \in [0, a_i) \tag{3.11}$$

or

$$\max f_\infty^{i,j} \in [0, a_i) \tag{3.12}$$

is satisfied. Then condition (C1) *holds for some $\lambda > 0$.*

Proof. First, we shall show that (3.11) leads to (C1). Let $\varepsilon = a_i - \max f_0^{i,j}$ (> 0). Clearly, there exists $\lambda > 0$ (λ can be chosen arbitrarily small) such that

$$\max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|} \le \max f_0^{i,j} + \varepsilon = a_i$$

for all $(|u_1|, \ldots, |u_m|) \in [0, \lambda]^m$. For each $1 \le i \le m$, this subsequently implies

$$\gamma_i f_i(t, u_1, u_2, \ldots, u_m) \le a_i|u_j| \le a_i\lambda$$

for all $(t, |u_1|, \ldots, |u_m|) \in [0,1] \times [0,\lambda]^m$, i.e. condition (C1) holds.

Next, assume that (3.12) holds. Let $\delta = a_i - \max f_\infty^{i,j}$ (> 0). Then there exists $\theta > 0$ (θ can be chosen arbitrarily large) such that

$$\max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|} \le \max f_\infty^{i,j} + \delta = a_i \tag{3.13}$$

for all $(|u_1|, \ldots, |u_m|) \in [\theta, \infty)^m$. For each $1 \le i \le m$, there are two cases to consider.

Case 1: $\gamma_i f_i(t, u_1, \ldots, u_m)$ is bounded. So there exists $\Gamma > 0$ such that

$$\gamma_i f_i(t, u_1, \ldots, u_m) \le \Gamma$$

for all $(t, |u_1|, \ldots, |u_m|) \in [0,1]\times[0,\infty)^m$. Take $\lambda = \frac{\Gamma}{a_i}$ (since Γ can be chosen arbitrarily large, λ can be chosen arbitrarily large). It follows that

$$\gamma_i f_i(t, u_1, u_2, \ldots, u_m) \le \lambda a_i$$

for all $(t, |u_1|, \ldots, |u_m|) \in [0,1] \times [0,\lambda)^m \subseteq [0,1] \times [0,\infty)^m$.

Case 2: $\gamma_i f_i(t, u_1, \ldots, u_m)$ is unbounded. Then there exists $\lambda \ge \theta$ (λ can be chosen arbitrarily large) and $t_i \in [0,1]$ such that

$$\gamma_i f_i(t, u_1, \ldots, u_m) \le \max_{\rho_j\in\{1,-1\},1\le j\le m} \gamma_i f_i(t_i, \rho_1\lambda, \ldots, \rho_m\lambda)$$

for all $(t, |u_1|, \ldots, |u_m|) \in [0,1] \times [0,\lambda]^m$. In view of (3.13), the above inequality leads to

$$\gamma_i f_i(t, u_1, \ldots, u_m) \le a_i|\rho_j\lambda| = a_i\lambda$$

for all $(t, |u_1|, \ldots, |u_m|) \in [0,1] \times [0,\lambda]^m$.

Therefore, in both cases condition (C1) is fulfilled ∎

Lemma 3.3. *Suppose, for some* $1 \le i \le m$ *and each* $1 \le j \le m$*, that*

$$\min f_0^{i,j} \in (b_i 4^{n_j-1}, \infty] \tag{3.14}$$

or

$$\min f_\infty^{i,j} \in (b_i 4^{n_j-1}, \infty] \tag{3.15}$$

is satisfied. Then condition (C2) *holds for some* $\eta > 0$.

Proof. First, to show that (3.14) gives rise to condition (C2), we let $\varepsilon = \min f_0^{i,j} - b_i 4^{n_j-1}$ (> 0). Clearly, there exists $\eta > 0$ (η can be chosen arbitrarily small) such that

$$\min_{\substack{t\in[0,1] \\ |u_k|\in[0,\infty),k\in M\setminus\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|} \ge \min f_0^{i,j} - \varepsilon = b_i 4^{n_j-1}$$

for all $|u_j| \in [0,\eta]$. Hence, for some $1 \le i \le m$ and each $1 \le j \le m$, we find

$$\gamma_i f_i(t, u_1, \ldots, u_m) \ge b_i 4^{n_j-1}|u_j| \ge b_i 4^{n_j-1}\left(\tfrac{1}{4}\right)^{n_j-1}\eta = b_i\eta \tag{3.16}$$

for all $(t, |u_1|, \ldots, |u_m|) \in I \times K_j \subseteq [0,1] \times [0,\infty)^{j-1} \times [0,\eta] \times [0,\infty)^{m-j}$. So condition (C2) holds.

Next, assume that (3.15) is satisfied. Let $\delta = \min f_\infty^{i,j} - b_i 4^{n_j-1}$ (>0). Then there exists $\eta > 0$ (η can be chosen arbitrarily large) such that

$$\min_{\substack{t\in[0,1] \\ |u_k|\in[0,\infty), k\in M\setminus\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m)}{|u_j|} \geq \min f_\infty^{i,j} - \delta = b_i 4^{n_j-1}$$

for all $|u_j| \in \left[\left(\frac{1}{4}\right)^{n_j-1}\eta, \infty\right)$. Thus, for some $1 \leq i \leq m$ and each $1 \leq j \leq m$, (3.16) follows for $(t, |u_1|, \ldots, |u_m|) \in I \times K_j \subseteq [0,1] \times [0,\infty)^{j-1} \times \left[\left(\frac{1}{4}\right)^{n_j-1}\eta, \infty\right) \times [0,\infty)^{m-j}$. So condition (C2) is fulfilled ∎

Corollary 3.1. *Suppose one of the following conditions is satisfied:*

(a) (3.11) *holds for each* $1 \leq i \leq m$ *and some* $1 \leq j \leq m$*, and* (3.15) *holds for some* $1 \leq i \leq m$ *and each* $1 \leq j \leq m$

or

(b) (3.12) *holds for each* $1 \leq i \leq m$ *and some* $1 \leq j \leq m$*, and* (3.14) *holds for some* $1 \leq i \leq m$ *and each* $1 \leq j \leq m$.

Then system (1.1) *has a fixed-sign solution* u^*.

Proof. It follows from Theorem 3.1 and Lemmas 3.2 and 3.3 ∎

Remark 3.1. In [11 - 13, 16] $(m = 1)$, the existence criteria developed require $\max f_0, \min f_0, \max f_\infty, \min f_\infty \in \{0, \infty\}$. However, there are functions that do not satisfy this condition. Hence, our results generalize and extend all these recent investigations. To cite some examples, for $m = 2$ and $\gamma_1 = \gamma_2 = 1$, we have:

(a) $f_i(t, u_1, u_2) = \frac{e^{u_1+u_2}-1}{1+t^2}$, $\max f_0^{i,j} = 1$, $\min f_0^{i,j} = 0.5$, $\max f_\infty^{i,j} = \min f_\infty^{i,j} = \infty$ $(j = 1, 2)$.

(b) $f_i(t, u_1, u_2) = (t+1)\sinh(u_1 + u_2)$, $\max f_0^{i,j} = 2$, $\min f_0^{i,j} = 1$, $\max f_\infty^{i,j} = \min f_\infty^{i,j} = \infty$ $(j = 1, 2)$.

(c) $f_i(t, u_1, u_2) = u_1 + t^2 e^{-u_2}$, $\max f_0^{i,j} = \infty$, $\max f_\infty^{i,j} = 1$ $(j = 1, 2)$, $\min f_0^{i,1} = \min f_\infty^{i,1} = 1$, $\min f_0^{i,2} = \min f_\infty^{i,2} = 0$.

Example 3.1. Consider the system

$$\left.\begin{aligned} x^{(5)}(t) + \frac{e^{x+y} - 1}{1+t^2} &= 0 \\ y^{(4)}(t) + (t+1)\sinh(x+y) &= 0 \\ x^{(j)}(0) = x^{(p_1)}(1) &= 0 \\ y^{(k)}(0) = y^{(p_2)}(1) &= 0 \end{aligned}\right\} \tag{3.17}$$

for $t \in [0,1]$, $j = 0, 1, 2, 3$ and $k = 0, 1, 2$. Here $n_1 = 5, n_2 = 4, 1 \leq p_1 \leq 4, 1 \leq p_2 \leq 3, m = 2$, $f_1(t, x, y) = \frac{e^{x+y}-1}{1+t^2}$ and $f_2(t, x, y) = (t+1)\sinh(x+y)$. Fix $\gamma_1 = \gamma_2 = 1$.

Clearly, condition (A) is satisfied. Since $\min f_\infty^{i,j} = \infty$ for $i, j \in \{1, 2\}$, by Lemma 3.3 condition (C2) is fulfilled for some $\eta > 0$. Next, for $\lambda > 0$ it is clear that

$$f_1(t, x, y) = \frac{e^{x+y} - 1}{1 + t^2} \leq e^{2\lambda} - 1$$

$$f_2(t, x, y) = (t + 1)\sinh(x + y) \leq 2\sinh(2\lambda)$$

for $(t, |x|, |y|) \in [0, 1] \times [0, \lambda]^2$. Thus condition (C1) is satisfied if we can find some $\lambda > 0$ such that

$$\left.\begin{aligned} e^{2\lambda} - 1 &\leq \lambda a_1 \\ 2\sinh(2\lambda) &\leq \lambda a_2 \end{aligned}\right\}. \tag{3.18}$$

It can be checked by direct computation that (3.18) holds for $\lambda = 1$. Hence we conclude by Theorem 3.1 that system (3.17) has a positive solution $u^* = (x^*, y^*)$.

4. Existence of two fixed-sign solutions under condition (A)

By applying the results of Section 3, in this section we obtain criteria for the existence of at least two fixed-sign solutions when f_i $(1 \leq i \leq m)$ satisfy condition (A).

Theorem 4.1. *Suppose condition* (C1) *holds for some* $\lambda > 0$. *Further, let* (3.14) – (3.15) *be satisfied for some* $1 \leq i \leq m$ *and each* $1 \leq j \leq m$. *Then system* (1.1) *has two fixed-sign solutions* u^* *and* $\bar{u}$ *such that*

$$0 < \|u^*\| \leq \lambda \leq \|\bar{u}\|. \tag{4.1}$$

Proof. By Lemma 3.3, condition (3.14) leads to condition $(C2)|_{\eta=\eta_1}$ and (3.15) gives rise to condition $(C2)|_{\eta=\eta_2}$, where η_1 and η_2 can be chosen arbitrarily small and large, respectively. Therefore, it is clear that

$$\eta_1 < \lambda < \eta_2. \tag{4.2}$$

It now follows from Theorem 3.1 that system (1.1) has a solution u^* such that $\eta_1 \leq \|u^*\| \leq \lambda$, and another solution $\bar{u}$ with $\lambda \leq \|\bar{u}\| \leq \eta_2$. Hence, (4.1) is immediate ∎

Theorem 4.2. *Suppose condition* (C2) *holds for some* $\eta > 0$. *Further, let* (3.11) – (3.12) *be satisfied for each* $1 \leq i \leq m$ *and some* $1 \leq j \leq m$. *Then system* (1.1) *has two fixed-sign solutions* u^* *and* $\bar{u}$ *such that*

$$0 < \|u^*\| \leq \eta \leq \|\bar{u}\|. \tag{4.3}$$

Proof. Applying Lemma 3.2, we find that condition (3.11) implies $(C1)|_{\lambda=\lambda_1}$ and (3.12) leads to $(C1)|_{\lambda=\lambda_2}$, where λ_1 and λ_2 can be chosen arbitrarily small and large, respectively. Hence, it is clear that

$$\lambda_1 < \eta < \lambda_2. \tag{4.4}$$

We now conclude from Theorem 3.1 that system (1.1) has a solution u^* with $\lambda_1 \leq \|u^*\| \leq \eta$ and another solution $\bar{u}$ satisfying $\eta \leq \|\bar{u}\| \leq \lambda_2$. Thus, (4.3) follows immediately ∎

5. Existence of a fixed-sign solution

In this section the non-linearities $f_i \ (1 \le i \le m)$ are *not* required to fulfil condition (A). We shall consider the Banach space $(B, \|\cdot\|)$ as in Section 3.

Lemma 5.1. *Let $L_k \ (1 \le k \le m)$ be given non-negative constants. Then the system*

$$\left.\begin{aligned} u_i^{(n_i)}(t) + \gamma_i L_i = 0 \\ u_i^{(j)}(0) = 0 \\ u_i^{(p_j)}(1) = 0 \end{aligned}\right\} \tag{5.1}$$

for $t \in [0,1]$, $i = 1, \dots, m$ and $0 \le j \le n_i - 2$ has a fixed-sign solution $u^L \in C$ (see (3.3)). In particular, for $L_k = 0 \ (1 \le k \le m)$ we can take $u^L(t) = 0 \ (t \in [0,1])$.

Proof. It is immediate from Theorem 3.1 ∎

Theorem 5.1. *Suppose there exist non-negative constants $L_k \ (1 \le k \le m)$ and two positive constants λ and $\eta \ (\neq \lambda)$ such that the following conditions are satisfied:*

(D1) *For each $1 \le i \le m$ we have*

$$\gamma_i f_i(t, u_1, \dots, u_m) + L_i \ge 0 \tag{5.2}$$

for all $(t,u) \in [0,1] \times K$.

(D2) *For each $1 \le i \le m$ we have*

$$a_i(u) \equiv \int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}]\,[\gamma_i f_i(s,u_1,\dots,u_m) + L_i]\,ds \le \lambda$$

for all $(|u_1|, \dots, |u_m|) \in [0,\lambda]^m$.

(D3) *For some $1 \le i \le m$ we have*

$$b_i(u) \equiv \int_{s\in I} G_i\big(\tfrac14, s\big)[\gamma_i f_i(s,u_1,\dots,u_m) + L_i]\,ds \ge \eta$$

for all $(|u_1|, \dots, |u_m|) \in \cup_{j=1}^m K_j$ where

$$K_j = \Big\{(v_1,\dots,v_m)\Big|\, v_j \in \big[\big(\tfrac14\big)^{n_j-1}\eta_j^*, \eta\big],\ v_k \in [0,\eta] \ \text{for } k \ne j\Big\} \tag{5.3}$$

and

$$\eta_j^* = \begin{cases} \eta & \text{if } L_k = 0 \ \ (1 \le k \le m) \\ \big[1 - \tfrac12\big(\tfrac14\big)^{n_j-1}\big]\eta & \text{if } L_k \ne 0 \text{ for some } k \text{ and } \eta > 2(4)^{n_j-1}\|u^L\| > 0 \\ 0 & \text{if } L_k \ne 0 \text{ for some } k \text{ and } \eta \text{ is small enough.} \end{cases} \tag{5.4}$$

Then system (1.1) has a fixed-sign solution u^* such that

$$\min\{\lambda,\eta\} \le \|u^* + u^L\| \le \max\{\lambda,\eta\} \tag{5.5}$$

where u^L is as in Lemma 5.1.

Proof. It is clear that system (1.1) has a solution u if and only if $q = u + u^L$ is a solution of the operator equation

$$q = Tq \tag{5.6}$$

where $T : B \to B$ is defined by

$$Tq(t) = \big(T_1 q(t), \ldots, T_m q(t)\big) \qquad (t \in [0,1]) \tag{5.7}$$

$$T_i q(t) = \int_0^1 G_i(t,s) h_i\big(s, (q - u^L)(s)\big) ds \qquad (t \in [0,1], 1 \le i \le m) \tag{5.8}$$

$$h_i(t, x_1, \ldots, x_m) = f_i(t, \rho_1, \ldots, \rho_m) + \gamma_i L_i \qquad (1 \le i \le m) \tag{5.9}$$

and, for $1 \le i \le m$,

$$\rho_i = \begin{cases} x_i & \text{if } \gamma_i x_i \ge 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

From (5.9) we see that $h_i : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ is continuous and well defined. Further, T is continuous and completely continuous.

To show that equation (5.6) has a solution, we shall employ Theorem 2.1. First, we shall prove that T maps C (see (3.3)) into itself. For this, let $q \in C \ \ (\subset K)$. In view of condition (D1) and (2.3), we obtain, for $t \in [0,1]$ and $1 \le i \le m$,

$$\gamma_i T_i q(t) = \int_0^1 G_i(t,s) \gamma_i h_i\big(s, (q - u^L)(s)\big) ds \ge 0. \tag{5.11}$$

Next, an application of (5.11) and Lemma 2.2 yields

$$\begin{aligned} |T_i q(t)| &= \gamma_i T_i q(t) \\ &\le \int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i - p_i - 1}[1 - (1-s)^{p_i}] \gamma_i h_i\big(s, (q - u^L)(s)\big) ds \end{aligned}$$

for all $t \in [0,1]$ and $1 \le i \le m$. Hence

$$|T_i q|_0 \le \int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i - p_i - 1}[1 - (1-s)^{p_i}] \gamma_i h_i\big(s, (q - u^L)(s)\big) ds \tag{5.12}$$

for all $1 \le i \le m$. Now, using (5.11), Lemma 2.1 and (5.12), for each $1 \le i \le m$ and $t \in I$ we find

$$\begin{aligned} \gamma_i T_i q(t) &\ge \int_0^1 \big(\tfrac{1}{4}\big)^{n_i - 1} \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i - p_i - 1}[1 - (1-s)^{p_i}] \gamma_i h_i\big(s, (q - u^L)(s)\big) ds \\ &\ge \big(\tfrac{1}{4}\big)^{n_i - 1} |T_i q|_0. \end{aligned}$$

It follows that

$$\min_{t \in I} \gamma_i T_i q(t) \ge \big(\tfrac{1}{4}\big)^{n_i - 1} |T_i q|_0 \tag{5.13}$$

for all $1 \le i \le m$. Coupling (5.11) and (5.13), we obtain $T(C) \subseteq C$.

Next, define the set

$$C^L = \left\{ q \in C \,\middle|\, \begin{array}{l} \text{for each } 1 \le i \le m,\ \gamma_i(q_i - u_i^L)(t) \ge 0 \text{ for } t \in [0,1] \\ \text{and } \min_{t\in I} \gamma_i(q_i - u_i^L)(t) \ge \left(\frac{1}{4}\right)^{n_i-1} |q_i - u_i^L|_0 \end{array} \right\}. \tag{5.14}$$

Note that C^L contains the element $u^L + \gamma$ where $\gamma = (\gamma_1, \ldots, \gamma_m)$. Let

$$\Omega_1 = \{q \in C^L \mid \|q\| < \lambda\} \qquad \text{and} \qquad \Omega_2 = \{q \in C^L \mid \|q\| < \eta\}.$$

We claim that

(i) $\|Tq\| \le \|q\|$ for $q \in C \cap \partial\Omega_1$

(ii) $\|Tq\| \ge \|q\|$ for $q \in C \cap \partial\Omega_2$.

To verify statement (i), let $q \in C \cap \partial\Omega_1$. So $\|q\| = \lambda$ which implies $\|q - u^L\| \le \lambda$. Using (5.11), Lemma 2.2 and (D2), we obtain for $t \in [0,1]$ and $1 \le i \le m$

$$\begin{aligned} |T_i q(t)| &= \int_0^1 G_i(t,s)\gamma_i h_i\big(s, (q-u^L)(s)\big) ds \\ &= \int_0^1 G_i(t,s)\gamma_i \big[f_i\big(s,(q-u^L)(s)\big) + \gamma_i L_i\big] ds \\ &\le \int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}]\,\big[\gamma_i f_i\big(s,(q-u^L)(s)\big) + L_i\big] ds \\ &= a_i(q-u^L) \\ &\le \lambda \\ &= \|q\|. \end{aligned}$$

As a result, $|T_i q|_0 \le \|q\|$ for $1 \le i \le m$ and so $\|Tq\| = \max_{1\le i\le m} |T_i q|_0 \le \|q\|$.

To show statement (ii), let $q \in C \cap \partial\Omega_2$. Then $\|q\| = \eta$ and $\|q - u^L\| \le \eta$. Suppose that $\|q - u^L\| = |q_j - u_j^L|_0$ for some $j \in \{1, \ldots, m\}$. Then for $t \in I$

$$|(q_j - u_j^L)(t)| \ge \left(\tfrac{1}{4}\right)^{n_j-1} \|q - u^L\| \ge \left(\tfrac{1}{4}\right)^{n_j-1} (\|q\| - \|u^L\|) \ge \left(\tfrac{1}{4}\right)^{n_j-1} \eta_j^*.$$

Thus $|(q_j - u_j^L)(t)| \in \left[\left(\frac{1}{4}\right)^{n_j-1}\eta_j^*, \eta\right]$ for $t \in I$. Further, $|(q_k - u_k^L)(t)| \in [0,\eta]$ for $k \ne j$ and $t \in I$. Now, applying condition (D3), we find that the following holds for some $i \in \{1, \ldots, m\}$:

$$\begin{aligned} \left|T_i q\left(\tfrac{1}{4}\right)\right| &= \gamma_i T_i q\left(\tfrac{1}{4}\right) \\ &\ge \int_{s\in I} G_i\left(\tfrac{1}{4}, s\right)\gamma_i\big[f_i\big(s,(q-u^L)(s)\big) + \gamma_i L_i\big] ds \\ &= b_i(q-u^L) \\ &\ge \eta \\ &= \|q\|. \end{aligned}$$

Consequently, $|T_i q|_0 \geq \|q\|$ and so $\|Tq\| \geq \|q\|$.

Now that we have established statements (i) and (ii), it follows from Theorem 2.1 that T has a fixed point $q^* \in C \cap (\overline{\Omega}_2 \backslash \Omega_1 \cup \overline{\Omega}_1 \backslash \Omega_2) \subseteq C^L$. Therefore, $\min\{\lambda, \eta\} \leq \|q^*\| \leq \max\{\lambda, \eta\}$. Since $q^* = u^* + u^L$, where u^* is a solution of system (1.1), and also $q^* \in C^L$, it is clear that u^* is also of fixed sign. The proof of the theorem is complete ∎

Let $M = \{1, \ldots, m\}$. For $1 \leq i, j \leq m$ we introduce the following definitions:

$$\begin{aligned}
\max f_0^{L,i,j} &= \lim_{\max_{1\leq k\leq m} |u_k| \to 0^+} \ \max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m) + L_i}{|u_j|} \\
\min f_0^{L,i,j} &= \lim_{|u_j| \to 0^+} \ \min_{\substack{t\in I \\ |u_k|\in[0,\infty), k\in M\backslash\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m) + L_i}{|u_j|} \\
\max f_\infty^{L,i,j} &= \lim_{\min_{1\leq k\leq m} |u_k| \to \infty} \ \max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m) + L_i}{|u_j|} \\
\min f_\infty^{L,i,j} &= \lim_{|u_j| \to \infty} \ \min_{\substack{t\in I \\ |u_k|\in[0,\infty), k\in M\backslash\{j\}}} \frac{\gamma_i f_i(t, u_1, \ldots, u_m) + L_i}{|u_j|}.
\end{aligned}$$

Further, for $1 \leq i \leq m$ we denote

$$\alpha_i = \left\{ \int_0^1 \tfrac{1}{(n_i-1)!} (1-s)^{n_i - p_i - 1}[1 - (1-s)^{p_i}]\, ds \right\}^{-1} \tag{5.15}$$

and

$$\beta_i = \left\{ \int_{s\in I} G_i\big(\tfrac{1}{4}, s\big) ds \right\}^{-1}. \tag{5.16}$$

Lemma 5.2. *Suppose there exist non-negative constants L_k $(1 \leq k \leq m)$ such that condition* (D1) *holds. If, for each $1 \leq i \leq m$ and some $1 \leq j \leq m$, one of the conditions*

$$\max f_0^{L,i,j} \in [0, \alpha_i) \tag{5.17}$$

or

$$\max f_\infty^{L,i,j} \in [0, \alpha_i) \tag{5.18}$$

is satisfied, then condition (D2) *holds for some $\lambda > 0$.*

Proof. First, we shall show that (5.17) implies (D2). Let $\varepsilon = \alpha_i - \max f_0^{L,i,j}$ (> 0). Clearly, there exists $\lambda > 0$ $(\lambda$ can be chosen arbitrarily small) such that

$$\max_{t\in[0,1]} \frac{\gamma_i f_i(t, u_1, \ldots, u_m) + L_i}{|u_j|} \leq \max f_0^{L,i,j} + \varepsilon = \alpha_i$$

for all $(|u_1|, \ldots, |u_m|) \in [0, \lambda]^m$. This subsequently provides

$$\gamma_i f_i(t, u_1, \ldots, u_m) + L_i \leq \alpha_i |u_j| \leq \alpha_i \lambda \tag{5.19}$$

for all $(t,|u_1|,\ldots,|u_m|)\in[0,1]\times[0,\lambda]^m$. Therefore, for each $1\le i\le m$ and $(|u_1|,\ldots,|u_m|)\in[0,\lambda]^m$, using (5.19) we get

$$\begin{aligned}a_i(u)&=\int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}]\left[\gamma_i f_i(s,u_1,\ldots,u_m)+L_i\right]ds\\&\le\int_0^1 \tfrac{1}{(n_i-1)!}\,(1-s)^{n_i-p_i-1}[1-(1-s)^{p_i}]\alpha_i\lambda\,ds\\&=\lambda\end{aligned}$$

which is (D2).

Next assume that (5.18) holds. Let $\delta=\alpha_i-\max f_\infty^{L,i,j}$ (>0). Then there exists $w>0$ (w can be chosen arbitrarily large) such that

$$\max_{t\in[0,1]}\frac{\gamma_i f_i(t,u_1,\ldots,u_m)+L_i}{|u_j|}\le\max f_\infty^{L,i,j}+\delta=\alpha_i \tag{5.20}$$

for all $(|u_1|,\ldots,|u_m|)\in[w,\infty)^m$. For each $1\le i\le m$ we shall consider two cases.

Case 1: $\gamma_i f_i(t,u_1,\ldots,u_m)+L_i$ is bounded. So there exists $R>0$ such that

$$\gamma_i f_i(t,u_1,\ldots,u_m)+L_i\le R$$

for all $(t,|u_1|,\ldots,|u_m|)\in[0,1]\times[0,\infty)^m$. Take $\lambda=\frac{R}{\alpha_i}$ (since R can be chosen arbitrarily large, λ can be chosen arbitrarily large). It follows that

$$\gamma_i f_i(t,u_1,u_2,\ldots,u_m)+L_i\le\lambda\alpha_i$$

for all $(t,|u_1|,\ldots,|u_m|)\in[0,1]\times[0,\lambda)^m\subseteq[0,1]\times[0,\infty)^m$. As seen earlier, this gives rise to condition (D2).

Case 2: $\gamma_i f_i(t,u_1,\ldots,u_m)+L_i$ is unbounded. Then there exists $\lambda\ge w$ (λ can be chosen arbitrarily large) and $t_i\in[0,1]$ such that

$$\gamma_i f_i(t,u_1,\ldots,u_m)+L_i\le\max_{\rho_j\in\{1,-1\},\ 1\le j\le m}\gamma_i f_i(t_i,\rho_1\lambda,\ldots,\rho_m\lambda)+L_i$$

for all $(t,|u_1|,\ldots,|u_m|)\in[0,1]\times[0,\lambda]^m$. In view of (5.20) this inequality leads to

$$\gamma_i f_i(t,u_1,\ldots,u_m)+L_i\le\alpha_i|\rho_j\lambda|=\alpha_i\lambda$$

for all $(t,|u_1|,\ldots,|u_m|)\in[0,1]\times[0,\lambda]^m$. Hence, condition (D2) is readily obtained ∎

Lemma 5.3. *Suppose there exist non-negative constants L_k $(1\le k\le m)$ such that condition* (D1) *holds. If, for some $1\le i\le m$ and each $1\le j\le m$, we have*

$$\min f_\infty^{L,i,j}\in\begin{cases}(\beta_i 4^{n_j-1},\infty] & \text{if } L_k=0,1\le k\le m\\ \left(\beta_i 4^{n_j-1}\left[1-\frac12\left(\frac14\right)^{n_j-1}\right]^{-1},\infty\right] & \text{if } L_k\ne0 \text{ for some } k,\end{cases} \tag{5.21}$$

then condition (D3) *holds for some $\eta>0$.*

Proof. Suppose that $L_k \neq 0$ for some k. Let $\varepsilon = \min f_{\infty}^{L,i,j} - \beta_i 4^{n_j-1}\big[1 - \frac{1}{2}\big(\frac{1}{4}\big)^{n_j-1}\big]^{-1}$ (> 0). Then there exists $\eta > 0$ (η can be chosen arbitrarily large) such that

$$\min_{\substack{t\in[0,1]\\ |u_k|\in[0,\infty),k\in M\setminus\{j\}}} \frac{\gamma_i f_i(t,u_1,\ldots,u_m)+L_i}{|u_j|} \geq \min f_{\infty}^{L,i,j} - \varepsilon = \beta_i 4^{n_j-1}\big[1-\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{n_j-1}\big]^{-1}$$

for all $|u_j| \in \big[\big(\frac{1}{4}\big)^{n_j-1}\big[1-\frac{1}{2}\big(\frac{1}{4}\big)^{n_j-1}\big]\eta, \infty\big)$. Thus, for some $1 \leq i \leq m$ and each $1 \leq j \leq m$, we find

$$\begin{aligned}\gamma_i f_i(t,u_1,\ldots,u_m)+L_i &\geq \beta_i 4^{n_j-1}\big[1-\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{n_j-1}\big]^{-1}|u_j| \\ &\geq \beta_i 4^{n_j-1}\big[1-\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{n_j-1}\big]^{-1}\big(\tfrac{1}{4}\big)^{n_j-1}\big[1-\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{n_j-1}\big]\eta \\ &= \beta_i\eta\end{aligned} \tag{5.22}$$

for all $(t,|u_1|,\ldots,|u_m|) \in I \times K_j \subseteq I \times [0,\infty)^{j-1} \times \big[\big(\frac{1}{4}\big)^{n_j-1}\big[1-\frac{1}{2}\big(\frac{1}{4}\big)^{n_j-1}\big]\eta,\infty\big) \times [0,\infty)^{m-j}$. Employing (5.22), for some $1 \leq i \leq m$ and $(|u_1|,\ldots,|u_m|) \in \cup_{j=1}^m K_j$ we get

$$b_i(u) = \int_{s\in I} G_i\big(\tfrac{1}{4},s\big)\big[\gamma_i f_i(s,u_1,\ldots,u_m)+L_i\big]ds \geq \int_{s\in I} G_i\big(\tfrac{1}{4},s\big)\beta_i\eta\, ds = \eta.$$

So condition (D3) is fulfilled. The case when $L_k = 0$ for all k can be similarly verified ∎

Lemma 5.4. *Let $L_k = 0$ $(1 \leq k \leq m)$ and let condition* (D1) *be satisfied. If, for some $1 \leq i \leq m$ and each $1 \leq j \leq m$, we have*

$$\min f_0^{L,i,j} \in (\beta_i 4^{n_j-1}, \infty], \tag{5.23}$$

then condition (D3) *holds for some $\eta > 0$.*

Proof. Let $\varepsilon = \min f_0^{L,i,j} - \beta_i 4^{n_j-1}$ (> 0). Clearly, there exists $\eta > 0$ (η can be chosen arbitrarily small) such that

$$\min_{\substack{t\in I\\ |u_k|\in[0,\infty),k\in M\setminus\{j\}}} \frac{\gamma_i f_i(t,u_1,\ldots,u_m)}{|u_j|} \geq \min f_0^{L,i,j} - \varepsilon = \beta_i 4^{n_j-1}$$

for all $|u_j| \in [0,\eta]$. Hence, for some $1 \leq i \leq m$ and each $1 \leq j \leq m$, we find

$$\gamma_i f_i(t,u_1,\ldots,u_m) \geq \beta_i 4^{n_j-1}|u_j| \geq \beta_i 4^{n_j-1}\big(\tfrac{1}{4}\big)^{n_j-1}\eta = \beta_i\eta$$

for all $(t,|u_1|,\ldots,|u_m|) \in I \times K_j \subseteq I \times [0,\infty)^{j-1} \times [0,\eta] \times [0,\infty)^{m-j}$. As seen in the proof of Lemma 5.3, this leads to condition (D3) ∎

Remark 5.1. In order to show that f_i satisfies condition (D3) $(\eta > 2(4)^{n_j-1}\|u^L\|)$, the condition $L_k = 0$ for all k in Lemma 5.4 is essential.

Corollary 5.1. *Suppose there exist non-negative constants L_k $(1 \leq k \leq m)$ such that condition* (D1) *holds. Let one of the conditions*

(a) (5.17) *holds for each $1 \leq i \leq m$ and some $1 \leq j \leq m$, and* (5.21) *holds for some $1 \leq i \leq m$ and each $1 \leq j \leq m$*

or

(b) $L_k = 0$ $(1 \leq k \leq m)$, (5.18) *holds for each $1 \leq i \leq m$ and some $1 \leq j \leq m$, and* (5.23) *holds for some $1 \leq i \leq m$ and each $1 \leq j \leq m$*

be fulfilled. Then system (1.1) *has a fixed-sign solution u^*.*

Proof. It is a direct consequence of Theorem 5.1 and Lemmas 5.2 - 5.4 ∎

Remark 5.2. A remark similar to Remark 3.1 applies. As an illustration, for $m = 2$ and $\gamma_1 = \gamma_2 = 1$ we have the following:

(a) $f_i(t, u_1, u_2) = \frac{e^{u_1+u_2}-1}{1+t^2} - 3$, $L_i = 3$, $\max f_0^{L,i,j} = 1$, $\min f_0^{L,i,j} = \frac{16}{25}$, $\max f_\infty^{L,i,j} = \min f_\infty^{L,i,j} = \infty$ $(j = 1, 2)$.

(b) $f_i(t, u_1, u_2) = (t + 1)\sinh(u_1 + u_2) - 2$, $L_i = 2$, $\max f_0^{L,i,j} = 2$, $\min f_0^{L,i,j} = \frac{5}{4}$, $\max f_\infty^{L,i,j} = \min f_\infty^{L,i,j} = \infty$ $(j = 1, 2)$.

(c) $f_i(t, u_1, u_2) = u_1 + t^2 e^{-u_2} - 5$, $L_i = 5$, $\max f_0^{L,i,j} = \infty$, $\max f_\infty^{L,i,j} = 1$ $(j = 1, 2)$, $\min f_0^{L,i,1} = \min f_\infty^{L,i,1} = 1$, $\min f_0^{L,i,2} = \infty$, $\min f_\infty^{L,i,2} = 0$.

Example 5.1. Consider the system

$$\left.\begin{aligned} x^{(5)}(t) + \frac{e^{x+y} - 6t^2 - 7}{1 + t^2} &= 0 \\ y^{(4)}(t) + [7\sinh(x + y) + t + 1]\sinh(x + y) - 7\cosh^2(x + y) &= 0 \\ x^{(j)}(0) = x^{(p_1)}(1) &= 0 \\ y^{(k)}(0) = y^{(p_2)}(1) &= 0 \end{aligned}\right\} \tag{5.24}$$

for $t \in [0, 1]$, $j = 0, 1, 2, 3$ and $k = 0, 1, 2$. Here $n_1 = 5$, $n_2 = 4$, $1 \le p_1 \le 4$, $1 \le p_2 \le 3$, $m = 2$ and

$$\begin{aligned} f_1(t, x, y) &= \frac{e^{x+y} - 6t^2 - 7}{1 + t^2} \\ f_2(t, x, y) &= [7\sinh(x + y) + t + 1]\sinh(x + y) - 7\cosh^2(x + y) \end{aligned}.$$

Fix $\gamma_1 = \gamma_2 = 1$, $L_1 = 6$ and $L_2 = 7$. Then we see that condition (D1) is satisfied. Since $\min f_\infty^{L,i,j} = \infty$ for $i, j \in \{1, 2\}$, by Lemma 5.3 condition (D3) is fulfilled for some $\eta > 0$. Next, it is clear that for $\lambda > 0$

$$\begin{aligned} f_1(t, x, y) + L_1 &= \frac{e^{x+y} - 1}{1 + t^2} \le \frac{e^{2\lambda} - 1}{1 + t^2} \\ f_2(t, x, y) + L_2 &= (t + 1)\sinh(x + y) \le (t + 1)\sinh(2\lambda) \end{aligned}$$

for all $(|x|, |y|) \in [0, \lambda]^2$. Thus, condition (D2) is fulfilled if we can find some $\lambda > 0$ such that

$$\int_0^1 \tfrac{1}{(n_1-1)!}\,(1 - s)^{n_1-p_1-1}[1 - (1 - s)^{p_1}]\,\frac{e^{2\lambda} - 1}{1 + s^2}\,ds \le \lambda \tag{5.25}$$

$$\int_0^1 \tfrac{1}{(n_2-1)!}\,(1 - s)^{n_2-p_2-1}[1 - (1 - s)^{p_2}]\,(t + 1)\sinh(2\lambda)\,ds \le \lambda. \tag{5.26}$$

It can be checked by direct computation that (5.25) and (5.26) are satisfied when $\lambda = 1$. Hence, we conclude by Theorem 5.1 that system (5.24) has a positive solution $u^* = (x^*, y^*)$.

6. Existence of two fixed-sign solutions

In this section, we apply the results of Section 5 to obtain criteria for the existence of at least two fixed-sign solutions. Once again, the non-linearities f_i $(1 \le i \le m)$ need *not* fulfil condition (A).

Theorem 6.1. *Let* $L_k = 0$ $(1 \le k \le m)$, *let condition* (D1) *be satisfied and suppose condition* (D2) *holds for some* $\lambda > 0$. *Further, let* (5.21) *and* (5.23) *be satisfied for some* $1 \le i \le m$ *and each* $1 \le j \le m$. *Then system* (1.1) *has two fixed-sign solutions* u^* *and* $\bar{u}$ *such that*

$$0 < \|u^*\| \le \lambda \le \|\bar{u}\|. \tag{6.1}$$

Proof. The proof uses Theorem 5.1, Lemmas 5.3 and 5.4, and is similar to that of Theorem 4.1 ∎

Theorem 6.2. *Suppose there exist non-negative constants* L_k $(1 \le k \le m)$ *such that condition* (D1) *is fulfilled and let condition* (D3) *hold for some* $\eta > 0$. *Further, let* (5.17) *and* (5.18) *be satisfied for each* $1 \le i \le m$ *and some* $1 \le j \le m$. *Then system* (1.1) *has two fixed-sign solutions* u^* *and* $\bar{u}$ *such that*

$$0 < \|u^* + u^L\| \le \eta \le \|\bar{u} + u^L\| \tag{6.2}$$

where u^L *is as in Lemma* 5.1.

Proof. The proof employs Theorem 5.1, Lemma 5.2, and a similar argument as in the proof of Theorem 4.2 ∎

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