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The Classical and the Modified Neumann Problems for the Inhomogeneous Pluriholomorphic System in Polydiscs

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Abstract. The classical Neumann problem for the inhomogeneous pluriholomorphic system in a polydisc is considered. Its solvability conditions and its solution are given. It is shown that the problem is not well-posed. To fix the solution the boundary condition is modified. For the modified problem the solvability conditions and the solution which is unique up to an arbitrary constant are explicitly given.

Keywords: *Pluriholomorphic functions, inhomogeneous pluriholomorphic systems, Neumann problem, polydiscs*

AMS subject classification: 35 J 25, 32 A 30, 30 E 25

1. Introduction

The Neumann problem for the inhomogeneous pluriholomorphic system in the unit ball was studied in [1]. However, about the Neumann problem even for the homogeneous pluriholomorphic system in the unit polydisc nothing can be found in the literature, although a great deal of research has been done about the $\overline{\partial}$-Neumann problem in polydiscs (see, e.c., [2, 3, 6]).

Let

$$\mathbb{D}^n = \Big\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \, |z_k| < 1 \quad (1 \le k \le n) \Big\}$$

be the unit polydisc, f_{kl} and γ be given functions with $f_{kl\overline{z}_j} \in L_1(\overline{\mathbb{D}^n}) \cap C(\overline{\mathbb{D}^n})$ and $\gamma \in C(\partial_0 \mathbb{D}^n)$. Consider the inhomogeneous system of $\frac{n(n+1)}{2}$ independent equations

$$\frac{\partial^2 u}{\partial \overline{z}_k \partial \overline{z}_\ell} = f_{k\ell}(z) \qquad (1 \le k, \, \ell \le n) \tag{1}$$

with given right-hand sides satisfying the conditions

$$f_{k\ell}(z) = f_{\ell k}(z) \qquad \text{and} \qquad \frac{\partial f_{k\ell}}{\partial \overline{z}_s} - \frac{\partial f_{ks}}{\partial \overline{z}_\ell} = 0 \;\; (1 \le s \le n).$$

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Problem ($\mathbf{N}_2$). Find a $C^1(\overline{\mathbb{D}^n})$-solution of system (1) satisfying the Neumann condition

$$\frac{\partial u}{\partial \nu_\zeta} = \gamma_0(\zeta) \qquad (\zeta \in \partial_0 \mathbb{D}^n) \tag{2}$$

where $\frac{\partial u}{\partial \nu_\zeta}$ denotes the outward normal derivative of u at the point $\zeta \in \partial_0 \mathbb{D}^n$.

By definition it is known (see [4]) that the Neumann condition (2) for the unit polydisc turns out to be

$$\sum_{j=1}^{n} \Big(z_j \frac{\partial u}{\partial z_j} + \overline{z}_j \frac{\partial u}{\partial \overline{z}_j} \Big)\Big|_\zeta = \gamma(\zeta) \qquad (\zeta \in \partial_0 \mathbb{D}^n) \tag{3}$$

with $\gamma(\zeta) = \gamma_0(\zeta)\sqrt{n}$. It is known that the general solution to system (1) is representable as

$$u(z) = \phi_0(z) + \langle \phi(z), z \rangle + u_0(z) \tag{4}$$

where $\phi(z) = \big(\phi_1(z), \ldots, \phi_n(z)\big)$, every $\phi_k \ \ (k = 0, \ldots, n)$ being an arbitrary funtion analytic in $\mathbb{D}^n$, and u_0 is a special solution to system (1) given by

$$u_0 = \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \le \ell_1 \le n \\ 1 \le \ell_2, \ldots, \ell_\mu \le n}} T_{\ell_\mu} \cdots T_{\ell_2} T_{\ell_1}^2 f_{\ell_1 \ell_1 \overline{\zeta}_{\ell_2} \cdots \overline{\zeta}_{\ell_\mu}}$$
$$+ \sum_{\nu=2}^{n} (-1)^{\nu} \sum_{1 \le \ell_1 < \ldots < \ell_\nu \le n} T_{\ell_\nu} \cdots T_{\ell_1} f_{\ell_1 \ell_2 \overline{\zeta}_{\ell_3} \cdots \overline{\zeta}_{\ell_\nu}}$$

(see [5]).

It is well known that for any given real-valued continuous function γ on $\partial \mathbb{D}$ there exists an analytic function w in $\mathbb{D}$, the real part of which has the boundary values γ on $\partial \mathbb{D}$, $\operatorname{Re} w = \gamma$. A solution can be given by the Schwarz integral $S\gamma$ which is the complex counterpart of the Poisson integral $P\gamma$. Hence γ turns out to be the boundary values of a harmonic function in $\mathbb{D}$. For two complex variables in order that a given real-valued function on the distinguished boundary $\partial_0 \mathbb{D}^2$ of the unit bidisc $\mathbb{D}^2$ is the boundary value function of the real part of an analytic function in $\mathbb{D}^2$ it has to belong to the space $\partial Ph_{\mathbb{D}^2}$ of boundary values of pluriharmonic functions in $\mathbb{D}^2$. It is known that not any function defined on $\partial_0 \mathbb{D}^2$ is in $\partial Ph_{\mathbb{D}^2}$ (see [1]). However, for our discussion we need to look at the problem a litle bit further.

Let the real-valued function γ on $\partial_0 \mathbb{D}^2$ be representable by a Fourier series

$$\gamma(z_1, z_2) = \sum_{i,k=-\infty}^{+\infty} a_{ik} z_1^i z_2^k \quad \big((z_1, z_2) \in \partial_0 \mathbb{D}^2\big)$$
$$a_{ik} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \overline{\zeta}_1^i \overline{\zeta}_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} \quad (a_{-i,-k} = \overline{a_{ik}}).$$

Thus for the given γ we have two real pluriharmonic functions in $\mathbb{C}^2$: one in $\mathbb{D}^{++} = \mathbb{D}^2$ ($\mathbb{D}^{--} = \{z = (z_1, z_2) : |z_1| > 1 \text{ and } |z_2| > 1\}$), i.e.

$$\sum_{i,k=0}^{+\infty} \left\{ a_{ik} z_1^i z_2^k + a_{-i,-k} \overline{z}_1^i \overline{z}_2^k \right\} - a_{0,0},$$

and one in $\mathbb{D}^{+-} = \{z = (z_1, z_2) : |z_1| < 1 \text{ and } |z_2| > 1\}$ ($\mathbb{D}^{-+}$), i.e.

$$\sum_{i,k=1}^{+\infty} \left\{ a_{i,-k} z_1^i z_2^{-k} + a_{-i,k} z_1^{-i} z_2^k \right\}.$$

Clearly, if $\gamma \in \partial Ph_{\mathbb{D}^2}$, then obviuosly $a_{-i,k} = a_{i,-k} = 0$ for $i, k \in \mathbb{N}$, i.e.

$$a_{i,-k} = \frac{-1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \overline{\zeta}_1^i \zeta_2^k \frac{d\overline{\zeta}_1}{\overline{\zeta}_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad (i, k \in \mathbb{N})$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{z_1 \overline{\zeta}_1}{1 - z_1 \overline{\zeta}_1} \frac{\overline{z}_2 \zeta_2}{1 - \overline{z}_2 \zeta_2} \frac{d\overline{\zeta}_1}{\overline{\zeta}_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad \big((z_1, z_2) \in \mathbb{D}^2\big). \tag{5}$$

If $\gamma \in \partial Ph_{\mathbb{D}^{+-}}$, then $a_{i,k} = a_{-i,-k} = 0$ for $i, k \in \{0\} \cup \mathbb{N}$. This means γ satisfies

$$a_{i,k} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \overline{\zeta}_1^i \overline{\zeta}_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad (i, k \in \{0\} \cup \mathbb{N})$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{1}{1 - z_1 \overline{\zeta}_1} \frac{1}{1 - z_2 \overline{\zeta}_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad \big((z_1, z_2) \in \mathbb{D}^2\big).$$

Evidently, it is easy to see that $\partial Ph_{\mathbb{D}^2} = \partial Ph_{\mathbb{D}^{--}}$ and $\partial Ph_{\mathbb{D}^{+-}} = \partial Ph_{\mathbb{D}^{-+}}$. Further, if γ belongs to $\partial H_{\mathbb{D}^2}$ (the space of boundary values of functions, holomorphic in $\mathbb{D}^2$), then γ satisfies condition (5) and

$$a_{-i,-k} = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \zeta_1^i \zeta_2^k \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad (i, k \in \{0\} \cup \mathbb{N};\ i + k \neq 0)$$

as well, i.e.

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \Big(\frac{1}{1 - \overline{z}_1 \zeta_1} \frac{1}{1 - \overline{z}_2 \zeta_2} - 1 \Big) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \qquad \big((z_1, z_2) \in \mathbb{D}^2\big)$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \Big(\frac{\overline{z}_1 \zeta_1}{1 - \overline{z}_1 \zeta_1} + \frac{\overline{z}_2 \zeta_2}{1 - \overline{z}_2 \zeta_2} - \frac{\overline{z}_1 \zeta_1}{1 - \overline{z}_1 \zeta_1} \frac{\overline{z}_2 \zeta_2}{1 - \overline{z}_2 \zeta_2} \Big) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0.$$

On the basis of [1: Theorem 5.1] and from our discussion above we can get the following conclusion about the boundary values of holomorphic functions in polydiscs.

Lemma 1. *Let γ be a real-valued continuous function on $\partial_0\mathbb{D}^n$ satisfying $\gamma \in \partial H_{\mathbb{D}^n}$:*

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\ldots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\ldots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0.$$

Then

$$\phi(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\frac{d\zeta}{\zeta-z}$$

is analytic in $\mathbb{D}^n$ satisfying $\phi(\zeta)=\gamma(\zeta)$ on $\partial_0\mathbb{D}^n$.

2. The classical problem

From (4) it follows that

$$u_{\overline{z}_k}=\phi_k(z)+u_{0\overline{z}_k},\qquad u_{z_k}=\frac{\partial\phi_0}{\partial z_k}+\sum_{\mu=1}^{n}\overline{z}_\mu\frac{\partial\phi_\mu}{\partial z_k}+\frac{\partial u_0}{\partial z_k}$$

where

$$u_{0\overline{z}_k}=\sum_{\nu=1}^{n}(-1)^{\nu+1}\sum_{1\le k_1<\ldots<k_\nu\le n}T_{k_\nu}\cdots T_{k_1}f_{k_1k\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}\qquad(1\le k\le n).$$

Substituting these expressions into (3), we obtain an equality for $\zeta\in\partial_0\mathbb{D}^n$:

$$\sum_{k=1}^{n}\overline{\zeta}_k\left(\phi_k(\zeta)+\sum_{j=1}^{n}\zeta_j\frac{\partial\phi_k}{\partial\zeta_j}+\frac{\zeta_k}{n}\sum_{j=1}^{n}\zeta_j\frac{\partial\phi_0}{\partial\zeta_j}\right)$$
$$=\sum_{k=1}^{n}\overline{\zeta}_k\left(\frac{\zeta_k}{n}\gamma(\zeta)-\frac{\partial u_0}{\partial\overline{\zeta}_k}-\frac{\zeta_k}{n}\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right).$$

Evidently, this equality is satisfied if

$$\phi_k(\zeta)+\sum_{j=1}^{n}\zeta_j\frac{\partial\phi_k}{\partial\zeta_j}+\frac{\zeta_k}{n}\sum_{j=1}^{n}\zeta_j\frac{\partial\phi_0}{\partial\zeta_j}=\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]-\frac{\partial u_0}{\partial\overline{\zeta}_k}\tag{6}$$

holds for any $\zeta\in\partial_0\mathbb{D}^n$ and $1\le k\le n$. Since the left-hand side represents the boundary values of a holomorphic function in $\mathbb{D}^n$, the right-hand side does too. Thus according to Lemma 1, the problem is solvable if and only if the conditions

$$\begin{aligned}&\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\ldots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\ldots<k_\nu\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left\{\frac{\langle\zeta,z\rangle}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]-\langle\operatorname{grad}_{\overline{\zeta}}u_0,z\rangle\right\}\\&\qquad\times\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta}=0\qquad(z\in\mathbb{D}^n)\end{aligned}\tag{7}$$

are satisfied. Then

$$\begin{aligned}&\phi_k(z)+\sum_{j=1}^{n} z_j\frac{\partial\phi_k}{\partial z_j}+\frac{z_k}{n}\sum_{j=1}^{n} z_j\frac{\partial\phi_0}{\partial z_j}\\&\quad=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left\{\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]-\frac{\partial u_0}{\partial\overline{\zeta}_k}\right\}\frac{d\zeta}{\zeta-z}\qquad(z\in\mathbb{D}^n)\end{aligned}$$

is analytic in $\mathbb{D}^n$ and satisfies condition (7).

To derive the solution of problem (N_2) we apply the Cauchy formula to (6), and by taking into account

$$\frac{1}{2\pi i}\int_{\partial\,\mathbb{D}} Tf(\zeta)\frac{d\zeta}{\zeta-z}=0,\qquad\text{i.e.}\ \ \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n} u_{0\overline{z}_k}\frac{d\zeta}{\zeta-z}=0$$

we get the partial differential equations for $z\in\mathbb{D}^n$

$$\begin{aligned}\phi_k(z)+\sum_{j=1}^{n} z_j\frac{\partial\phi_k}{\partial z_j}&=-\frac{z_k}{n}\sum_{j=1}^{n} z_j\frac{\partial\phi_0}{\partial z_j}\\&\quad+\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\frac{d\zeta}{\zeta-z}.\end{aligned}\tag{8}$$

By the transformation

$$\left.\begin{aligned}\omega_1&=z_1\\ \omega_2&=\frac{z_1}{z_2}\\ &\ \vdots\\ \omega_n&=\frac{z_1}{z_n}\end{aligned}\right\}$$

we obtain for (8) the equations

$$\begin{aligned}\omega_1\frac{\partial\phi_1}{\partial\omega_1}+\phi_1&=-\frac{\omega_1^2}{n}\frac{\partial\phi_0}{\partial\omega_1}+\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_1}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\\&\quad\times\frac{d\zeta_1}{\zeta_1-\omega_1}\frac{d\zeta_2}{\zeta_2-\frac{\omega_1}{\omega_2}}\cdots\frac{d\zeta_n}{\zeta_n-\frac{\omega_1}{\omega_n}}\\ \omega_1\frac{\partial\phi_k}{\partial\omega_1}+\phi_k&=-\frac{\omega_1^2}{n\omega_k}\frac{\partial\phi_0}{\partial\omega_1}+\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^{n}\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\\&\quad\times\frac{d\zeta_1}{\zeta_1-\omega_1}\frac{d\zeta_2}{\zeta_2-\frac{\omega_1}{\omega_2}}\cdots\frac{d\zeta_n}{\zeta_n-\frac{\omega_1}{\omega_n}}\quad(k=2,\dots,n).\end{aligned}$$

Integrating these equations we get

$$
\begin{aligned}
\omega_1\phi_1 = &-\int_0^{\omega_1}\frac{t^2}{n}\frac{\partial\phi_0}{\partial t}\,dt + \int_0^{\omega_1}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_1}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\\
&\times\frac{d\zeta_1}{\zeta_1-t}\frac{d\zeta_2}{\zeta_2-\frac{t}{\omega_2}}\cdots\frac{d\zeta_n}{\zeta_n-\frac{t}{\omega_n}}\,dt + C_1\\
\omega_1\phi_k = &-\int_0^{\omega_1}\frac{t^2}{n\omega_k}\frac{\partial\phi_0}{\partial t}\,dt + \int_0^{\omega_1}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\\
&\times\frac{d\zeta_1}{\zeta_1-t}\frac{d\zeta_2}{\zeta_2-\frac{t}{\omega_2}}\cdots\frac{d\zeta_n}{\zeta_n-\frac{t}{\omega_n}}\,dt + C_k \quad (k=2,\dots,n).
\end{aligned}
$$

Substituting $\omega_1 = 0$ on both sides we see that $C_k = 0 \ \ (1 \le k \le n)$. Returning to the original variables we have

$$
\begin{aligned}
z_1\phi_1(z) = &\int_0^1\frac{z_1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_1}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\frac{d\zeta}{\zeta-sz}\,ds\\
&- z_1\int_0^1\frac{sz_1}{n}\sum_{j=1}^n(sz_j)\frac{\partial\phi_0(sz)}{\partial(sz_j)}\,ds\\
z_1\phi_k(z) = &\int_0^1\frac{z_1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\frac{d\zeta}{\zeta-sz}\,ds\\
&- z_1\int_0^1\frac{sz_k}{n}\sum_{j=1}^n(sz_j)\frac{\partial\phi_0(sz)}{\partial(sz_j)}\,ds \quad (k=2,\dots,n),
\end{aligned}
$$

i.e. for $k = 1,\dots,n$ we have

$$
\phi_k(z) = \int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\frac{d\zeta}{\zeta-sz}\,ds - \int_0^1\frac{s^2z_k}{n}\,d\phi_0(sz).
$$

Hence representation (4) gets the form

$$
\begin{aligned}
u(z) = &\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle\zeta,z\rangle}{n}\left[\gamma(\zeta)-\sum_{j=1}^n\zeta_j\frac{\partial u_0}{\partial\zeta_j}\right]\frac{d\zeta}{\zeta-sz}\,ds\\
&+ u_0(z) + \phi_0(z) - \int_0^1\frac{\langle sz,sz\rangle}{n}\,d\phi_0(sz).
\end{aligned}
$$

If we take $\phi_0(z) = \sum_{|\kappa|\geq 0} a_\kappa z^\kappa$ $(z \in \mathbb{D}^n)$, then

$$\begin{aligned}
\phi_0(z) &- \int_0^1 \frac{\langle sz, sz\rangle}{n}\, d\phi_0(sz) \\
&= \sum_{|\kappa|\geq 0} a_\kappa z^\kappa - \int_0^1 \frac{s^2|z|^2}{n}\left[\sum_{j=1}^n \left(\sum_{|\kappa|\geq 1} a_\kappa \frac{\kappa_j (sz)^\kappa}{s z_j}\right) z_j ds\right] \\
&= \sum_{|\kappa|\geq 0} a_\kappa z^\kappa - \int_0^1 \frac{s|z|^2}{n} \sum_{|\kappa|\geq 1} a_\kappa |\kappa| (sz)^\kappa ds \\
&= a_0 + \sum_{|\kappa|\geq 1} a_\kappa \left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)}\right] z^\kappa.
\end{aligned}$$

Thus

$$\begin{aligned}
u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n}\left[\gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j}\right] \frac{d\zeta}{\zeta - sz}\, ds + u_0(z) \\
+ \sum_{|\kappa|\geq 0} a_\kappa \left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)}\right] z^\kappa \qquad (z \in \mathbb{D}^n).
\end{aligned} \tag{9}$$

Theorem 1. *Problem* (N_2) *is solvable if and only if its right-hand sides satisfy condition* (7) *on* $\partial_0\mathbb{D}^n$. *The general solution can be given by* (9). *The corresponding homogeneous problem has infinitely many linearly independent non-trivial solutions*

$$\left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)}\right] z^\kappa \qquad (|\kappa| > 0,\ z \in \mathbb{D}^n).$$

Problem (N_2) *is not well-posed.*

3. The modified problem

Since solution (9) includes a free analytic function, clearly to get a fixed solution only a Schwarz problem is needed to be solved. So we introduce an additional boundary condition.

Problem $(\mathbf{N}_2^*)$ Find a $C^1(\overline{\mathbb{D}^n})$ solution to system (1) satisfying the Neumann condition (2) and

$$\operatorname{Re} u(\zeta) = \gamma^*(\zeta) \qquad (\zeta \in \partial_0 \mathbb{D}^n). \tag{10}$$

We call this problem the *modified Neumann problem* for system (1).

Let $f_{k\ell} = 0$ in (1). Then the solvability condition (7) takes the form

$$\begin{aligned}
\sum_{\nu=1}^n \sum_{\lambda=0}^{\nu-1} \sum_{\substack{1\leq k_1<\ldots<k_\lambda\leq n \\ 1\leq k_{\lambda+1}<\ldots<k_\nu\leq n}} \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \\
\times \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \frac{d\zeta}{\zeta} = 0 \quad (\zeta \in \partial_0\mathbb{D}^n;\ z \in \mathbb{D}^n \cup \partial_0\mathbb{D}^n)
\end{aligned} \tag{11}$$

and it means that every $\zeta_k\gamma(\zeta)$ on $\partial_0\mathbb{D}^n$ $(1 \le k \le n)$ belongs to $\partial H_{\mathbb{D}^n}$. Actually, it is evident that $\gamma \in \partial H_{\mathbb{D}^n}$. Note if $\zeta_1\gamma(\zeta) = \varphi_1(\zeta)$ with $\varphi_1 \in \partial H_{\mathbb{D}^n}$, then $\gamma(\zeta) = \overline{\zeta_1}\varphi_1(\zeta)$. If $\gamma \notin \partial H_{\mathbb{D}^n}$, then $\zeta_2\overline{\zeta_1}\varphi_1(\zeta) \notin \partial H_{\mathbb{D}^n}$. But by the condition above $\zeta_2\gamma(\zeta) \in \partial H_{\mathbb{D}^n}$. This is a contradiction. Hence condition (11) becomes

$$\sum_{\nu=1}^{n}\sum_{\lambda=0}^{\nu-1}\sum_{\substack{1\le k_1<\ldots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\ldots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta} = 0. \quad (12)$$

Substituting (10) into (9) shows

$$\sum_{|\kappa|\ge 0}\frac{\overline{a}_\kappa\overline{\zeta}^\kappa + a_\kappa\zeta^\kappa}{2+|\kappa|} = \gamma^*(\zeta) - \mathrm{Re}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle\eta,\zeta\rangle}{n}\gamma(\eta)\frac{d\eta}{\eta - s\zeta}\,ds =: 2\Gamma(\zeta),$$

i.e.

$$\mathrm{Re}\sum_{|\kappa|\ge 0}\frac{a_\kappa\zeta^\kappa}{2+|\kappa|} = \Gamma(\zeta) \qquad (\zeta\in\partial_0\mathbb{D}^n). \quad (13)$$

Due to the character of the left-hand side of (13), the right-hand side Γ on $\partial_0\mathbb{D}^n$ is also the boundary value of a function, pluriharmonic in $\mathbb{D}^n$. This means the given function Γ on $\partial_0\mathbb{D}^n$ must satisfy the condition

$$\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\ldots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\ldots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\Gamma(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta} = 0. \quad (14)$$

In fact, due to $\gamma \in \partial H_{\mathbb{D}^n}$, it follows that

$$2\Gamma(\zeta) = \gamma^*(\zeta) - \mathrm{Re}\int_0^1\frac{\langle s\zeta,\zeta\rangle}{n}\gamma(s\zeta)\,ds = \gamma^*(\zeta) - \mathrm{Re}\int_0^1 s\gamma(s\zeta)\,ds.$$

Hence $\mathrm{Re}\int_0^1 s\gamma(s\zeta)\,ds \in \partial Ph_{\mathbb{D}^n}$ and condition (14) implies that $\gamma^* \in \partial Ph_{\mathbb{D}^n}$, i.e.

$$\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\ldots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\ldots<k_\nu\le n}} \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta} = 0. \quad (15)$$

So if this condition is satisfied, then the Schwarz problem (13) is solvable and the solution is given by

$$\begin{aligned}\sum_{|\kappa|\ge 0}\frac{a_\kappa z^\kappa}{|\kappa|+2} &= \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\Gamma(\zeta)\Big[2\frac{\zeta}{\zeta - z} - 1\Big]\frac{d\zeta}{\zeta} + iC^0\\ &= \sum_{|\kappa|>0}\frac{2}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\Gamma(\zeta)(z\overline{\zeta})^\kappa\frac{d\zeta}{\zeta} + \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\Gamma(\zeta)\frac{d\zeta}{\zeta} + iC^0\end{aligned}$$

with an arbitrary real constant C^0, it is analytic in $\mathbb{D}^n$ and satisfies equation (13) (see [1]). One can see that

$$a_0 = \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Gamma(\zeta) \frac{d\zeta}{\zeta} + i2C^0$$
$$a_\kappa = \frac{2(2+|\kappa|)}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Gamma(\zeta) \overline{\zeta}^\kappa \frac{d\zeta}{\zeta} \quad (|\kappa| > 0).$$

Hence if conditions (12) and (15) are satisfied, i.e. if $\gamma \in \partial H_{\mathbb{D}^n}$ and $\gamma^* \in \partial Ph_{\mathbb{D}^n}$, then problem ($N_2^*$) with $f_{k\ell} = 0$ is solvable and the solution is given by

$$u(z) = \sum_{|\kappa| \geq 0} a_\kappa \left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)} \right] z^\kappa + \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z \rangle}{n} \gamma(\zeta) \frac{d\zeta}{\zeta - sz} ds$$

for $z \in \mathbb{D}^n$. But from

$$\begin{aligned}
&\sum_{|\kappa| \geq 0} a_\kappa \left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)} \right] z^\kappa \\
&\quad = a_0 + \sum_{|\kappa|>0} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[2 + |\kappa| \frac{n - |z|^2}{n} \right] (z\overline{\zeta})^\kappa \frac{d\zeta}{\zeta} \\
&\quad = a_0 + \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \\
&\qquad + \sum_{|\kappa|>0} \frac{n - |z|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) |\kappa| (z\overline{\zeta})^\kappa \frac{d\zeta}{\zeta} \\
&\quad = a_0 + \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \\
&\qquad + \sum_{|\kappa|>0} \frac{n - |z|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \frac{\partial}{\partial t} (tz\overline{\zeta})^\kappa \bigg|_{t=1} \frac{d\zeta}{\zeta} \\
&\quad = a_0 + \frac{n - |z|^2}{n} \frac{\partial}{\partial t} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - tz\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \bigg|_{t=1} \\
&\qquad + \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta}
\end{aligned}$$

we get

$$\begin{aligned}
u(z) = iC_0 &+ \frac{n - |z|^2}{n} \frac{\partial}{\partial t} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - tz\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \bigg|_{t=1} \\
&+ \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta}
\end{aligned}$$

where C_0 is an arbitrary real constant.

Next we make some simplifications. Let

$$I_1 = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \qquad (z \in \mathbb{D}^n).$$

Then

$$\begin{aligned} I_1 &= \frac{-1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \left\{ \operatorname{Re} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta}\, ds \right\} \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma^*(\zeta) \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \\ &=: -I_{1a} - I_{1b} + I_{1c} \end{aligned}$$

where

$$2I_{1a} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta}\, ds \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta}.$$

By changing the order of integration, we get

$$2I_{1a} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \int_0^1 \gamma(\eta) \left\{ \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \eta, \zeta \rangle}{n} \frac{1}{1 - s\zeta\overline{\eta}} \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \right\} ds \frac{d\eta}{\eta},$$

but

$$\begin{aligned} &\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \eta, \zeta \rangle}{n} \frac{1}{1 - s\zeta\overline{\eta}} \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta} \\ &\quad = \sum_{k=1}^n \frac{1}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\eta_k \overline{\zeta}_k}{1 - s\zeta_k \overline{\eta}_k} \\ &\quad \times \prod_{\tau=1 atop \tau \neq k}^n \frac{1}{1 - s\zeta_\tau \overline{\eta}_\tau} \frac{2d\zeta}{\zeta - z} - \overline{\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, \eta \rangle}{n} \frac{d\zeta}{\zeta - s\eta}} \\ &\quad = \sum_{k=1}^n \frac{2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \eta_k \left[\overline{\zeta}_k + \frac{s\overline{\eta}_k}{1 - s\zeta_k \overline{\eta}_k} \right] \\ &\quad \times \prod_{\substack{\tau=1 \\ \tau \neq k}}^n \frac{1}{1 - s\zeta_\tau \overline{\eta}_\tau} \frac{1}{1 - z\overline{\zeta}} \frac{d\zeta}{\zeta} - \frac{\overline{\langle s\eta, \eta \rangle}}{n} \\ &\quad = \sum_{k=1}^n \frac{2}{n} \eta_k \frac{s\overline{\eta}_k}{1 - sz_k \overline{\eta}_k} \prod_{\substack{\tau=1 \\ \tau \neq k}}^n \frac{1}{1 - sz_\tau \overline{\eta}_\tau} - s \\ &\quad = s \left[\frac{2}{1 - sz\overline{\eta}} - 1 \right] \end{aligned}$$

leads to

$$2I_{1a} = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\eta) \left[2\frac{\eta}{\eta - sz} - 1 \right] \frac{d\eta}{\eta} s\, ds.$$

The second part of I_1 which has to be simplified is

$$2I_{1b} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \int_0^1 \overline{\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta}}\, ds \left[\frac{2}{1 - z\overline{\zeta}} - 1 \right] \frac{d\zeta}{\zeta}.$$

By changing the order of the integrals

$$2I_{1b} = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\int_0^1 \overline{\gamma(\eta)}\left\{\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle\zeta,\eta\rangle}{n}\frac{1}{1-s\eta\overline{\zeta}}\left[\frac{2}{1-z\overline{\zeta}}-1\right]\frac{d\zeta}{\zeta}\right\}ds\frac{d\eta}{\eta}$$

and from

$$\begin{aligned}
&\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle\zeta,\eta\rangle}{n}\frac{1}{1-s\eta\overline{\zeta}}\left[\frac{2}{1-z\overline{\zeta}}-1\right]\frac{d\zeta}{\zeta}\\
&\quad=\sum_{k=1}^{n}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\zeta_k\overline{\eta}_k}{n}\left[1+s\eta_k\overline{\zeta}_k+\frac{(s\eta_k\overline{\zeta}_k)^2}{1-s\eta_k\overline{\zeta}_k}\right]\\
&\qquad\times\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{1}{1-s\eta_\tau\overline{\zeta}_\tau}\left\{2\left[1+\frac{z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\right]\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{1}{1-sz_\tau\overline{\zeta}_\tau}-1\right\}\frac{d\zeta}{\zeta}\\
&\quad=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\,\mathbb{D}_k}\frac{1}{n}\left[\zeta_k\overline{\eta}_k+s+s\frac{s\eta_k\overline{\zeta}_k}{1-s\eta_k\overline{\zeta}_k}\right]\left\{2\left[1+\frac{z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\right]-1\right\}\frac{d\zeta_k}{\zeta_k}\\
&\quad=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\,\mathbb{D}_k}\frac{1}{n}\left[\zeta_k\overline{\eta}_k+s+s\frac{s\eta_k\overline{\zeta}_k}{1-s\eta_k\overline{\zeta}_k}\right]\left[1+\frac{2z_k\overline{\zeta}_k}{1-z_k\overline{\zeta}_k}\right]\frac{d\zeta_k}{\zeta_k}\\
&\quad=\sum_{k=1}^{n}\frac{1}{n}\left[s+2z_k\overline{\eta}_k\right]
\end{aligned}$$

we have

$$\begin{aligned}
2I_{1b} &= \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\int_0^1\overline{\gamma(\eta)}\left[2\frac{\langle z,\eta\rangle}{n}+s\right]ds\frac{d\eta}{\eta}\\
&= \frac{2}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle z,\eta\rangle}{n}\overline{\gamma(\eta)}\frac{d\eta}{\eta}+\frac{1}{2(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\overline{\gamma(\eta)}\frac{d\eta}{\eta}.
\end{aligned}$$

Thus we have got I_1 calculated as

$$\begin{aligned}
I_1 = &\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\left[2\frac{\zeta}{\zeta-z}-1\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\left[2\frac{\zeta}{\zeta-sz}-1\right]\frac{d\zeta}{\zeta}s\,ds\\
&-\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle z,\zeta\rangle}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}-\frac{1}{4(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}.
\end{aligned}$$

Now let

$$I_2 := \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}2\Gamma(\zeta)\left[\frac{1}{1-tz\overline{\zeta}}-1\right]\frac{d\zeta}{\zeta}.$$

Similar to I_1 it is easy to get

$$\begin{aligned}
I_2 = &\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\left[\frac{\zeta}{\zeta-tz}-1\right]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\left[\frac{\zeta}{\zeta-stz}-1\right]\frac{d\zeta}{\zeta}s\,ds\\
&-\frac{1}{2(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle tz,\zeta\rangle}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}.
\end{aligned}$$

So we have

$$
\begin{aligned}
u(z) = iC_0 &+ \frac{n-|z|^2}{n}\frac{\partial}{\partial t}\bigg\{\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\bigg[\frac{\zeta}{\zeta-tz}-1\bigg]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\bigg[\frac{\zeta}{\zeta-stz}-1\bigg]\frac{d\zeta}{\zeta}s\,ds\\
&-\frac{1}{2(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle tz,\zeta\rangle}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}\bigg\}\bigg|_{t=1}\\
&+\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma^*(\zeta)\bigg[2\frac{\zeta}{\zeta-z}-1\bigg]\frac{d\zeta}{\zeta}\\
&-\frac{1}{2}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\bigg[2\frac{\zeta}{\zeta-sz}-1\bigg]\frac{d\zeta}{\zeta}s\,ds\\
&-\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle z,\zeta\rangle}{n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}-\frac{1}{4(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\overline{\gamma(\zeta)}\frac{d\zeta}{\zeta}
\end{aligned}
\tag{16}
$$

where C_0 is an arbitrary real constant.

Lemma 2. *The modified Neumann problem* (N_2^*) *for pluriholomorphic functions in* $\mathbb{D}^n$ *is uniquely solvable if and only if conditions* (12) *and* (15) *are satisfied,i.e.* $\gamma \in \partial H_{\mathbb{D}^n}$ *and* $\gamma^* \in \partial Ph_{\mathbb{D}^n}$. *The solution unique up to an arbitrary real constant is given by* (16). *The problem is well-posed.*

Next we clarify the solution and the solvability conditions of the modified problem (N_2^*) for the inhomogeneous system (1). By substituting condition (10) into representation (9) we have

$$
\begin{aligned}
&\sum_{|\kappa|\geq 0}\left(\overline{a}_\kappa\overline{\zeta}^\kappa + a_\kappa\zeta^\kappa\right)\frac{1}{|\kappa|+2}\\
&\quad= \gamma^*(\zeta) - \operatorname{Re} u_0(\zeta)\\
&\qquad - \operatorname{Re}\int_0^1\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\frac{\langle \eta,\zeta\rangle}{n}\bigg[\gamma(\eta)-\sum_{j=1}^n\eta_j\frac{\partial u_0}{\partial\eta_j}\bigg]\frac{d\eta}{\eta-s\zeta}ds\\
&\quad=: 2F(\zeta)
\end{aligned}
$$

for $\zeta \in \partial_0\mathbb{D}^n$, i.e.

$$
\operatorname{Re}\sum_{|\kappa|\geq 0}\frac{a_\kappa\zeta^\kappa}{|\kappa|+2} = F(\zeta) \qquad (\zeta \in \partial_0\mathbb{D}^n). \tag{17}
$$

This means again $F \in \partial Ph_{\mathbb{D}^n}$ because the left-hand side belongs to $\partial Ph_{\mathbb{D}^n}$, i.e.

$$
\sum_{\nu=2}^n\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\leq k_1<\ldots<k_\lambda\leq n\\ 1\leq k_{\lambda+1}<\ldots<k_\nu\leq n}}\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}F(\zeta)\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\frac{d\zeta}{\zeta} = 0 \tag{18}
$$

for $z \in \mathbb{D}^n$. Then the Schwarz problem (17) is solvable and the solution can be given by

$$
\sum_{|\kappa|\geq 0}\frac{a_\kappa z^\kappa}{|\kappa|+2} = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}F(\zeta)\bigg[2\frac{\zeta}{\zeta-z}-1\bigg]\frac{d\zeta}{\zeta} + iC^1
$$

and from it one can derive that

$$a_0 = \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \frac{d\zeta}{\zeta} + i2C^1$$
$$a_\kappa = \frac{2(2+|\kappa|)}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta)\overline{\zeta}^\kappa \frac{d\zeta}{\zeta} \quad (|\kappa| > 0)$$

where C^1 is an arbitrary real constant. Substituting them into (9) we get

$$u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n} \left[\gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j}\right] \frac{d\zeta}{\zeta - sz}\, ds + u_0(z)$$
$$+ \sum_{|\kappa|\geq 0} \frac{2+|\kappa|}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \left[1 - \frac{|\kappa|\,|z|^2}{n(|\kappa|+2)}\right] F(\zeta)(z\overline{\zeta})^\kappa \frac{d\zeta}{\zeta} + i2C^1$$

for all $z \in \mathbb{D}^n$. Similarly to the case of the pluriholomorphic system we obtain

$$u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n} \left[\gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j}\right] \frac{d\zeta}{\zeta - sz}\, ds + u_0(z) + iC^*$$
$$+ \frac{\partial}{\partial t} \frac{n - |z|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \left[\frac{1}{1 - tz\overline{\zeta}} - 1\right] \frac{d\zeta}{\zeta}\bigg|_{t=1} \tag{19}$$
$$+ \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \left[\frac{2}{1 - z\overline{\zeta}} - 1\right] \frac{d\zeta}{\zeta}$$

where C^* is an arbitrary real constant.

Theorem 2. *The modified Neumann problem* (N_2^*) *for the inhomogeneous pluriholomorphic system* (1) *in* $\mathbb{D}^n$ *is solvable if and only if conditions* (7) *and* (18) *are satisfied. The solution which is unique up to an arbitrary real constant, is given by* (19). *The problem is well-posed.*

A simple application. Find the sums

$$\sum_{|k|>0} \frac{x^k}{|k|} \qquad \text{and} \qquad \sum_{|k|>0} |k|\, x^k \qquad (|x_1| < 1, \ldots, |x_n| < 1).$$

By the above method we get

$$\sum_{|k|>0} \frac{x_1^{k_1} \cdots x_n^{k_n}}{k_1 + \ldots + k_n} = \int_0^1 \Big(\frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1\Big) \frac{ds}{s}$$

and

$$\sum_{|k|>0} (k_1 + \ldots + k_n)\,(x_1^{k_1} \cdots x_n^{k_n}) = \frac{\partial}{\partial s} \Big(\frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1\Big)\Big|_{s=1}.$$

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