Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 2, 561–573

On the Existence of Almost Periodic Lyapunov Functions for Impulsive Differential Equations

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Abstract. In the paper the existence of almost periodic piecewise continuous functions of Lyapunov's type for impulsive differential equations is considered. The impulses take place at fixed moments of time.

Keywords: *Almost periodic functions, impulsive differential equations*

AMS subject classification: 34 A 37

1. Introduction

Many processes by physics, biology, etc. are characterized by the fact that at certain moments they change their state by jumps. These processes during their evolution are subject to short time perturbations whose duration is negligible in comparision with the duration of the process. That is why we can assume that these perturbations are carried out "instantly", in the form of impulses. Adequate mathematical models of such processes are systems of impulsive differential equations. In the recent years these equations have been the object of numerous investigations [1 - 5, 7].

In this paper we shall prove convers theorem of the type of Massera's theorem [6], i.e. that for impulsive differential equations there exists a piecewise continuous almost periodic Lyapunov function with certain properties.

2. Preliminary notes

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$, $D \subset \mathbb{R}^n$ be a compact subset and $B_\alpha = \{x \in \mathbb{R}^n : \|x\| < \alpha\}$ for $\alpha = \text{const} > 0$. By $B = \big\{\{\tau_k\}_{k=-\infty}^{\infty} \subset \mathbb{R}^n : \tau_k < \tau_{k+1}\big\}$ we denote the set of all sequences, unbounded and strictly increasing, with distance $\rho(\{\tau_k^{(1)}\}, \{\tau_k^{(2)}\}) = \inf_{\varepsilon > 0}\{|\tau_k^{(1)} - \tau_k^{(2)}| < \varepsilon \ (k \in \mathbb{Z})\}$.

We shall consider the system of impulsive differential equations

$$\left.\begin{aligned} \dot{x} &= f(t, x) && (t \neq \tau_k) \\ \Delta x(\tau_k) &= I_k(x(\tau_k)) && (k \in \mathbb{Z}) \\ x(t_0 + 0) &= x_0 && (t_0 \in \mathbb{R}) \end{aligned}\right\} \tag{1}$$

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ISSN 0232-2064 / $ 2.50

where

$$f : \mathbb{R} \times D \to \mathbb{R}^n$$
$$I_k : D \to \mathbb{R}^n$$
$$\{\tau_k\}_{k=-\infty}^{\infty} \in B$$
$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k).$$

We denote by $x(t) = x(t; t_0, x_0)$ the solution of system (1) with the initial condition $x(t_0 + 0; t_0, x_0) = x_0$.

We introduce the following assumptions:

(H1) $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ and $f(t, 0) = 0$ for $t \in \mathbb{R}$.

(H2) The function f is Lipschitz continuous with respect to its second argument in $\mathbb{R} \times D$, uniformly on $t \in \mathbb{R}$, with constant $L_1(D) > 0$, i.e. $\|f(t, \overline{x}) - f(t, x)\| \leq L_1(D)\|\overline{x} - x\|$ for $\overline{x}, x \in D$.

(H3) $I_k \in C(D, \mathbb{R}^n)$ and $I_k(0) = 0$ $(k \in \mathbb{Z})$.

(H4) The functions I_k $(k \in \mathbb{Z})$ are Lipschitz continuous in D with constant $L_2(D) > 0$, i.e. $\|I_k(\overline{x}) - I_k(x)\| \leq L_2(D)\|\overline{x} - x\|$ for $\overline{x}, x \in D$.

(H5) $(I + I_k) : D \to D$ $(k \in \mathbb{Z})$, where I is the identity in $\mathbb{R}^n$.

Recall (see [3: p. 46]) from assumptions (H1) - (H5) it follows that the solution $x(t)$ of system (1) is a piecewise continuous function with points of discontinuity at the moments τ_k $(k \in \mathbb{Z})$ at which it is continuous from the left.

We denote

$$\Gamma = \Big\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x \in B_\alpha \ (\alpha = \text{const} > 0)\Big\}$$

$$G_k = \Big\{(t, x) \in \Gamma : \tau_{k-1} < t < \tau_k\Big\} \quad \text{and} \quad G = \bigcup_{k \subset \mathbb{Z}} G_k$$

$$S_\alpha = \Big\{(t, x) \in \Gamma : x \in B_\alpha \text{ if } (t, x) \in G \text{ and } x + I_k(x) \in B_\alpha \text{ if } t = \tau_k\Big\}.$$

Definition 1. We shall say that a function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ belong to the *class* $\mathcal{V}$ if the following conditions are satidfied:

1. V is continuous in G and $V(t, 0) = 0$ for all $t \in \mathbb{R}$.

2. For each $k \subset \mathbb{Z}$ and each point $x_0 \in B_h$ the limits

$$V(\tau_k - 0, x_0) = \lim_{\substack{(t,x) \to (\tau_k, x_0) \\ (t,x) \in G_k}} V(t, x) \quad \text{and} \quad V(\tau_k + 0, x_0) = \lim_{\substack{(t,x) \to (\tau_k, x_0) \\ (t,x) \in G_{k+1}}} V(t, x)$$

exist and are finite, and the equality $V(\tau_k - 0, x_0) = V(t_k, x_0)$ holds.

3. V is locally Lipschitz in x, i.e. for $\overline{x}, x \in B_\alpha$ there exists a positive constant $h(\alpha)$ such that

$$\|V(t, \overline{x}) - V(t, x)\| \leq h(\alpha)\|\overline{x} - x\| \tag{2}$$

for $t \in \mathbb{R}$.

4. For any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^n$

$$V(t+0, x + I_k(x)) \le V(t,x) \qquad (k \in \mathbb{Z}) \tag{3}$$

follows.

Let $V \in \mathcal{V}$, $t > t_0$ with $t \neq \tau_k$ and $x \in \mathbb{R}^n$. We introduce the function

$$D^+V(t, x(t)) = \liminf_{\delta \to 0^+} \frac{V(t+\delta, x(t+\delta; t, x)) - V(t,x)}{\delta}. \tag{4}$$

In the further considerations we also use the functions classes

$$PC(\mathbb{R}, \mathbb{R}^n) = \left\{ x : \mathbb{R} \to \mathbb{R}^n \;\middle|\; \begin{array}{l} x \text{ is piecewise continuous with points} \\ \text{of discontinuity of the first kind } \tau_k \\ (k \in \mathbb{Z}) \text{ and } x(\tau_k - 0) = x(\tau_k) \end{array} \right\}$$

and

$$\mathcal{P} = \Big\{ a \in C(\mathbb{R}^+, \mathbb{R}^+) : a \text{ is monotone increasing in } \mathbb{R}^+ \text{ and } a(0) = 0 \Big\}.$$

Definition 2 (see [8: Theorem 6.3]). The function $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ is said to be *almost periodic* in t uniformly with respect to $x \in D$ if for every sequence of real numbers $\{s'_m\}$ there exists a subsequence $\{s_n\}$, $s_n = s'_{m_n}$ such that the sequence $\{f(t+s_n, x)\}$ converges uniformly with respect to $t \in \mathbb{R}$ and $x \in D$.

Definition 3 (see [1]). The sequence $\{I_k(x)\}_{k \in \mathbb{Z}}$, $I_k \in C(D, \mathbb{R}^n)$ is said to be *almost periodic uniformly with respect to* $x \in D$ if for every sequence of integer numbers $\{m'\}$ there exists a subsequence $\{m_n\}$ such that the sequence $I_{k+m_n}(x)$ converges uniformly for $n \to \infty$.

Let $H \subset \mathbb{R}$. Introduce the sets

$$\theta_\varepsilon(H) = \{t + \varepsilon : t \in H \text{ and } \varepsilon \in \mathbb{R}\} \qquad \text{and} \qquad F_\varepsilon(H) = \cap_{\varepsilon > 0} \theta_\varepsilon(H)$$

and let, for $T, P \in B$, $s(T \cup P) : B \to B$ be a map such that the set $s(T \cup P)$ form a strictly increasing sequence. By $\phi = (\varphi(t), T)$ we denote the elements from the space $PC \times B$, and for every sequence of real number $\{s_n\}_{n \ge 1}$ denote $\theta_{s_n}\phi = \{\varphi(t+s_n), T + s_n\} \subset PC \times B$, where $T + s_n = \{\tau_k + s_n : k \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$.

Definition 4 (see [1]). The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$ $(k, j \in \mathbb{Z})$ is said to be *uniformly almost periodic* if for any $\varepsilon > 0$ there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods common for all the sequences $\{\tau_k^j\}$.

Lemma 1 (see [1]). *The set of sequences* $\{\tau_k^j\}$ *is uniformly almost periodic if and only if from each infinite sequence of shifts* $\{\tau_k + \alpha_m\}$ $(k \in \mathbb{Z}, m \in \mathbb{N}, \alpha_m \in \mathbb{R})$ *we can choose a subsequence, which is convergent in* B.

Definition 5. The sequence $\{\phi_n\}$, $\phi_n = (\varphi_n(t), T_n) \in PC \times B$ is convergent to ϕ, $\phi = (\varphi(t), T) \in PC \times B$ if for any $\varepsilon > 0$ there exists an $n_0 > 0$ such that, for $n \ge n_0$, $\rho(T, T_n) < \varepsilon$ and $|\varphi_n(t) - \varphi(t)| < \varepsilon$ uniformly for $t \in R \setminus F_\varepsilon(s(T \cup T_n))$.

Definition 6. $\varphi \in PC(\mathbb{R}, D)$ is said to be an *almost periodic piecewise continuous function* with points of discontinuity of the first kind from the set T if for every sequence of real numbers $\{s'_m\}$ there exists a subsequence $\{s_n\}$, $s_n = s'_{m_n}$ such that $\theta_{s_n}(\phi)$ is compact in $PC \times B$.

We introduce the following assumptions:

(H6) The function $f(t, x)$ is almost periodic in t uniformly with respect to $x \in D$.

(H7) The sequence $\{I_k(x)\}$ is almost periodic uniformly with respect to $x \in D$.

(H8) The sequences $\{\tau_k^j\}_{k,j \subset \mathbb{Z}}$, $\tau_k^j = \tau_{k+j} - \tau_k$ are uniformly almost periodic.

Remark 1. From [1] it follows that condition (H8) is satisfied if and only if $\tau_k = kp + c_k$ where $\{c_k\}$ is an almost periodic sequence of real numbers and $p \neq 0$. Then $\lim_{q\to\infty} \frac{i(a,a+q)}{q} = \frac{1}{p}$ where $i(a, a+q)$ is the number of points τ_k lying in the interval $(a, a+q)$.

Let assumptons (H1) - (H8) be fulfilled and let $\{s_m{}'\}$ be an arbitrary sequence of real numbers. Then there exists a subsequence $\{s_n\}$, $s_n = s_{m_n}{}'$ such that the sequence $\{f(t + s_n, x)\}$ converges uniformly for $x \in D$ to a function $f^s(t, x)$ and the set of sequences $\{\tau_k - s_n\}$ $(k \in \mathbb{Z})$ is convergent to the sequence τ_k^s uniformly with respect to $k \in \mathbb{Z}$ as $n \to \infty$.

By $\{k_{n_i}\}$ we denote a sequence of integer numbers such that the subsequence $\{t_{k_{n_i}}\}$ converges to τ_k^s, uniformly with respect to k, as $i \to \infty$. From condition (H8) it follows that there exists a subsequence of the sequence $\{k_{n_i}\}$ such that the sequence $\{I_{k_{n_i}}(x)\}$ converges uniformly to the limit denoted by $I_k^s(x)$. Then for every sequence $\{s'_m\}$ system (1) moves to a system E^s in the form

$$\left.\begin{aligned} \dot{x}(t) &= f^s(t, x) && (t \neq \tau_k^s) \\ \Delta x(\tau_k^s) &= I_k(x(\tau_k^s)) && (k \in \mathbb{Z}) \\ x(t_0 + 0) &= x_0 && (t_0 \in \mathbb{R}). \end{aligned}\right\} \tag{5}$$

Definition 7. The set of all systems in the form (5) is said to be *module* of system (1) and we denote this set by $\operatorname{mod}(f, I_k, \tau_k)$.

Definition 8 (see [7: Definition 12.3]). The zero solution $x(t) \equiv 0$ of system (1) is said to be:

8.1 *stable*, if

$$(\forall \varepsilon > 0)(\forall t_0 \in \mathbb{R})(\exists \delta > 0)(\forall x_0 \in B_\delta)(\forall t \in \mathbb{R}, t > t_0) :$$
$$\|x(t; t_0, x_0)\| < \varepsilon;$$

8.2 *uniformly stable*, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})(\forall x_0 \in B_\delta)(\forall t \in \mathbb{R}, t > t_0) :$$
$$\|x(t; t_0, x_0)\| < \varepsilon;$$

8.3 *asymptotically stable*, if it is stable and

$$(\forall t_0 \in \mathbb{R})(\exists \lambda = \lambda(t_0) > 0)(\forall x_0 \in B_\delta : (t_0, x_0) \in S_\lambda) :$$
$$\lim_{t \to \infty} x(t; t_0, x_0) = 0.$$

8.4 *asymptotically stable in the large*, if it is stable and every solutions of system (1) tends to zero as $t \to \infty$.

8.5 *quasi-equi-asymptotically stable in the large*, if

$$(\forall \alpha > 0)(\forall \varepsilon > 0)(\forall t_0 \in \mathbb{R})(\exists T > 0)(\forall x_0 \in B_\alpha)(\forall t \geq t_0 + T(t_0, \varepsilon, \alpha)) :$$
$$\|x(t; t_0, x_0)\| < \varepsilon.$$

Definition 9. The solution $x(t; t_0, x_0)$ of system (1) is called *equi-bounded* if

$$(\forall \alpha > 0)(\forall t_0 \in \mathbb{R})(\exists \beta > 0)(\forall x_0 \in B_\alpha)(\forall t \in \mathbb{R}, t \geq t_0) :$$
$$\|x(t; t_0, x_0)\| < \beta.$$

Definition 10. The solution $x(t; t_0, x_0)$ of system (1) is said to be *perfectly uniform-asymptotically stable in the large* if δ in Definition 8.1, T in Definition 8.5 and β in Definition 9 are independent on t_0 for all $t_0 \in \mathbb{R}$.

In the further considerations we also use the following lemmas.

Lemma 2 (see [6: Lemma 1]). *Given any real function $A(r, \varepsilon)$ of real variables, defined, continuous and positive in $Q = \{(r, \varepsilon) : r \in \mathbb{R}^+ \text{ and } \varepsilon > 0\}$, there are two continuous functions $h = h(r), h(r) > 0$ and $g = g(\varepsilon), g(\varepsilon) > 0, g(0) = 0$ such that $h(r)g(\varepsilon) \leq A(r, \varepsilon)$ in Q.*

Lemma 3 (see [3: Lemma 4.2]). *Let the inequality*

$$u(t) \leq c + \int_{t_0}^{t} u(s)v(s)\, ds + \sum_{t_0 < \tau_k < t} \beta_k u(\tau_k)$$

hold where $c > 0$ and $\beta_k \geq 0$ $(k \in \mathbb{Z})$, u is a piecewise continuous function with points of discontinuity of the first kind τ_k $(k \in \mathbb{Z})$ and $v(t) \geq 0$ is a locally integrable function. Then

$$u(t) \leq c \prod_{t_0 < \tau_k < t} (1 + \beta_k) \exp\left(\int_{t_0}^{t} v(s)\, ds \right)$$

for $t \geq t_0$.

3. Main results

We shall prove a convers theorem of the type of Massera's theorem.

Theorem 1. *Let assumptions* $(H1)-(H8)$ *be fulfilled and suppose that the zero solution of system* (1) *is perfectly uniform-asymptotically stable in the large. Then there exists a Lyapunov function* $V \in \mathcal{V}$ *which is almost periodic in* t *uniformly with respect to* $x \in D$*,* D *a compact subset in* $\mathbb{R}^n$*, and such that*

$$a(\|x\|) \le V(t,x) \le b(\|x\|) \tag{6}$$

for $a, b \in \mathcal{P}$ *where* $a(r), b(r) \to \infty$ *as* $r \to \infty$ *and*

$$D^+V(t,x) \le -cV(t,x) \tag{7}$$

for $(t,x) \in \Gamma$ *and* $c = \text{const} > 0$.

Proof. Let $\Omega(\sigma,\alpha) = \{(t,x) : t \in (-\sigma,\sigma) \text{ and } x \in B_\alpha\}$. By equi-asympotic stability in the large, the solutions of system (1) are equi-bounded, i.e. there exists a constant $\beta = \beta(\alpha) > 0$ such that, for $(t_0,x_0) \in \Omega(\sigma,\alpha)$, $\|x(t;t_0,x_0)\| < \beta(\alpha)$ for $t \ge t_0$. Moreover, there exists a $T(\alpha,\varepsilon) > 0$ such that from $(t_0,x_0) \in \Omega(\sigma,\alpha)$ we obtain $\|x(t;t_0,x_0)\| < \varepsilon$ for $t \ge t_0 + T(\alpha,\varepsilon)$. If $\varepsilon > 1$ we set $T(\alpha,\varepsilon) = T(\alpha,1)$.

From conditions (H2) and (H4), there exist $L_1(\alpha,\varepsilon) > 0$ and $L_2(\alpha,\varepsilon) > 0$ such that if $0 \le t \le \sigma + T(\alpha,\varepsilon), \overline{x} \in B_{\beta(\alpha)}$ and $x \in B_{\beta(\alpha)}$, we get

$$\begin{aligned} \|f(t,\overline{x}) - f(t,x)\| &\le L_1(\alpha,\varepsilon)\|\overline{x} - x\| \\ \|I_k(\overline{x}) - I_k(x)\| &\le L_2(\alpha,\varepsilon)\|\overline{x} - x\| \quad (k \in \mathbb{Z} \setminus \{0\}). \end{aligned}$$

Let

$$\begin{aligned} f^* &= 1 + \max \|f(t,x)\| \qquad \big(0 \le t \le T(\alpha,\varepsilon), x \in B_{\beta(\alpha)}\big) \\ I^* &= \max \|I_k(x)\| \qquad \big(x \in B_{\beta(\alpha)}, k \in \mathbb{Z}\big) \end{aligned}$$

and let $c = \text{const} > 0$. We set

$$\begin{aligned} A(\alpha,\varepsilon) &= e^{cT(\alpha,\varepsilon)} \\ &\quad \times \left\{ \left(2\Big(f^* + I^*\frac{1}{p}\Big) + \Big(\frac{1}{p}+1\Big)T(\alpha,\varepsilon)\right) e^{L_1(\alpha,\varepsilon)+\frac{1}{p}\ln(1+L_2(\alpha,\varepsilon))} + \beta(\alpha) \right\}. \end{aligned} \tag{8}$$

From Lemma 2 it follows that there exist two functions $h(\alpha) > 0$ and $g(\varepsilon) > 0$ for $\varepsilon > 0, g(0) = 0$ such that

$$g(\varepsilon)A(\alpha,\varepsilon) \le h(\alpha). \tag{9}$$

For $i \in \mathbb{N}$ we define $V_i(t,x)$ by

$$\begin{aligned} V_i(t,x) &= g\Big(\frac{1}{i}\Big) \sup_{\tau \ge 0} G_i\big(\|x(t+\tau,t,x)\|\big) e^{c\tau} \quad (t \ne \tau_k) \\ V_i(\tau_k,x) &= V(\tau_k - 0, x) \end{aligned} \tag{10}$$

where

$$G_i(z) = \begin{cases} z - \frac{1}{i} & \text{if } z \geq \frac{1}{i} \\ 0 & \text{if } 0 \leq z \leq \frac{1}{i}. \end{cases}$$

Clearly, $G_i(z) \to \infty$ as $z \to \infty$, for each i, and

$$|G(z_1) - G_i(z_2)| \leq |z_1 - z_2| \tag{11}$$

where $z_1, z_2 \geq 0$. From the definition of $V_i(t,x)$ it is clear that

$$g\Big(\frac{1}{i}\Big) G_i(\|x\|) \leq V_i(t,x) \qquad \text{and} \qquad V_i(t,0) \equiv 0 \tag{12}$$

as $(t,x) \in \Omega(\sigma,\alpha)$.

On the other hand, from (8) and (9),

$$V_i(t,x) \leq g\Big(\frac{1}{i}\Big) G_i(\beta(\alpha)) e^{cT(\alpha,\frac{1}{i})} \leq g\Big(\frac{1}{i}\Big)\beta(\alpha) e^{cT(\alpha,\frac{1}{i})} \leq h(\alpha) \tag{13}$$

follows. Then from (12) and (13) for $V_i(t,x)$ $(i \in \mathbb{N})$ at $t \neq \tau_k$ it follows that (6) holds. For $(t',x'), (t,x) \in \Omega(\sigma,\alpha)$ and $t < t'$ we get

$$\begin{aligned} &\big|V_i(t',x') - V_i(t,x)\big| \\ &\quad \leq g\Big(\frac{1}{i}\Big) \sup_{\tau \geq 0} \Big| G_i(\|x(t'+\tau;t',x')\|) - G_i(\|x(t+\tau;t,x)\|)\Big| e^{c\tau} \\ &\quad \leq g\Big(\frac{1}{i}\Big) \sup_{0<\tau<T(\alpha,\frac{1}{i})} e^{c\tau} \Big\| x(t'+\tau;t',x') - x(t+\tau;t,x)\Big\| \\ &\quad \leq g\Big(\frac{1}{i}\Big) \sup_{\tau\geq 0} e^{c\tau} \Big\{ \Big\| x(t'+\tau;t',x') - x(t'+\tau;t,x)\Big\| \\ &\qquad + \Big\| x(t'+\tau;t,x) - x(t+\tau;t,x)\Big\| \Big\}. \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} &\Big\| x(t'+\tau;t,x) - x(t+\tau;t,x)\Big\| \\ &\quad \leq \int_{t+\tau}^{t'+\tau} \|f(s,x(s))\|\, ds + \sum_{t+\tau<\tau_k<t'+\tau} \|I_k(x(\tau_k))\| \\ &\quad \leq \max_{\substack{s\in[t+\tau,t'+\tau] \\ x \in B_{\beta(\alpha)}}} \|f(s,x(s))\| + \max_{\tau_k \in [t+\tau,t'+\tau]} \|I_k(x(\tau_k))\|\, i(t+\tau,t'+\tau) \\ &\quad \leq \Big(f^* + I^* \frac{1}{p}\Big)(t'-t). \end{aligned} \tag{15}$$

Let $X = x(t';t,x)$. From Lemma 3 we obtain

$$\begin{aligned} &\Big\| x(t'+\tau;t,x) - x(t'+\tau;t',x')\Big\| \\ &\quad \leq \|X - x'\| \exp\Big\{ L_1\Big(\alpha,\frac{1}{i}\Big) + \frac{1}{p}\ln\Big(1 + L_2\Big(\alpha,\frac{1}{i}\Big)\Big) T\Big(\alpha,\frac{1}{i}\Big)\Big\} \\ &\quad \leq (\|X - x\| + \|x - x'\|) \exp\Big\{ L_1\Big(\alpha,\frac{1}{i}\Big) + \frac{1}{p}\ln\Big(1 + L_2\Big(\alpha,\frac{1}{i}\Big)\Big) T\Big(\alpha,\frac{1}{i}\Big)\Big\} \\ &\quad \leq \Big(\Big(f^* + I^*\frac{1}{p}\Big)(t'-t) + \|x - x'\|\Big) \\ &\qquad \times \exp\Big\{ L_1\Big(\alpha,\frac{1}{i}\Big) + \frac{1}{p}\ln\Big(1 + L_2\Big(\alpha,\frac{1}{i}\Big)\Big) T\Big(\alpha,\frac{1}{i}\Big)\Big\}. \end{aligned} \tag{16}$$

Then from (15), (16) and (8) for (14) we get

$$\begin{aligned}
&\big|V_i(t,x)-V_i(t',x')\big| \\
&\quad\le g\Big(\frac{1}{i}\Big)\sup_{0\le\tau\le T(\alpha,\frac{1}{i})} e^{c\tau}\Big(\Big(f^*+I^*\frac{1}{p}\Big)+\Big(f^*+I^*\frac{1}{p}\Big)(t'-t)+\|x-x'\|\Big) \\
&\quad\times \exp\Big\{L_1\Big(\alpha,\frac{1}{i}\Big)+\frac{1}{p}\ln\Big(1+L_2\Big(\alpha,\frac{1}{i}\Big)\Big)T\Big(\alpha,\frac{1}{i}\Big)\Big\} \\
&\quad\le g\Big(\frac{1}{i}\Big)2\Big(f^*+I^*\frac{1}{p}\Big)\exp\Big\{L_1\Big(\alpha,\frac{1}{i}\Big)+\frac{1}{p}\ln\Big(1+L_2\Big(\alpha,\frac{1}{i}\Big)\Big)T\Big(\alpha,\frac{1}{i}\Big)\Big\} \\
&\qquad\times\big(|t'-t|+\|x-x'\|\big) \\
&\quad\le h(\alpha)\big(|t'-t|+\|x-x'\|\big).
\end{aligned}\tag{17}$$

As $x\in B_{\beta(\alpha)}$ and $t=\tau_k$, from here it follows that $V_i(t,x)$ is continuous and for $t=t'$ we obtain (2), i.e. the function $V_i(t,x)$ is locally Lipschiz continuous.

Let $\tau_k\in R$ and $x\in B_{\beta(\alpha)}$ be fixed, and let $t',t''\in(\tau_k,\tau_{k+1}]$, $x',x''\in B_{\beta(\alpha)}$ and $u'=x(t';\tau_k,x,)$, $u''=x(t'';\tau_k,x,)$. Then

$$\begin{aligned}
&\big|V_i(t',x')-V_i(t'',x'')\big| \\
&\quad\le\big|V_i(t',x')-V_i(t',u')\big|+\big|V_i(t'',x'')-V_i(t'',u'')\big|+\big|V_i(t',u')-V_i(t'',u'')\big|.
\end{aligned}\tag{18}$$

By the fact that the functions $V_i(t,x)$ and $f(t,x)$ are Lipschiz continuous we obtain the estimates

$$|V_i(t',x')-V_i(t',u')|\le h(\alpha)\|x'-u'\|$$

$$\|x'-u'\|\le\|x'-x\|+\|u'-x\|$$

$$\|u'-x\|\le\int_{\tau_k}^{t'}L_1\exp\Big\{\int_{\tau_k}^{s}L_1d\tau\Big\}ds\|x\|\equiv N(t')\|x\|.$$

Then

$$\big|V_i(t',x')-V_i(t',u')\big|\le h(\alpha)\|x'-x\|+h(\alpha)N(t')\|x\|.\tag{19}$$

By analogy,

$$\big|V_i(t'',x'')-V_i(t'',u'')\big|\le h(\alpha)\|x''-x\|+h(\alpha)N(t'')\|x\|.\tag{20}$$

Since $a_i(\delta)=\sup_{\tau>\delta}G_i(\|x(\tau_k+\tau,t_k,x)\|)e^{c\tau}$ is non-increasing and $\lim_{\delta\to0^+}a_i(\delta)=a_i(0)$, it follows that

$$\begin{aligned}
&\big|V_i(t',u')-V_i(t'',u'')\big| \\
&\quad=g\Big(\frac{1}{i}\Big)\Big|\sup_{s>0}G\big(\|x(t'+s;t',u')\|\big)e^{cs}-\sup_{s>0}G\big(\|x(t''+s;t'',u'')\|\big)e^{cs}\Big| \\
&\quad=g\Big(\frac{1}{i}\Big)\Big|a(t'-\ tau_k)e^{-c(t'-\tau_k)}-a(t''-\tau_k)e^{-c(t''-\tau_k)}\Big| \\
&\quad\to 0
\end{aligned}$$

as $t' \to \tau_k + 0$ and $t'' \to \tau_k + 0$. From (18) - (20) we obtain that there exists the limit $V_i(\tau_k + 0, x)$. The proof of the existence of the limit $V_i(\tau_k - 0, x)$ follows by analogy.

Let $\eta(t; t_0, x_0)$ be the solution of the initial value problem

$$\left.\begin{aligned} \dot{\eta} &= f(t, \eta) \\ \eta(t_0) &= x_0. \end{aligned}\right\}$$

Since $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$ and $s > \mu$ it follows that

$$x\big(s; \mu, \eta(\mu; \tau_k, x + I_k(x))\big) = x\big(s; \lambda, \eta(\lambda, \tau_k, x)\big).$$

Then

$$V_i\big(\mu, \eta(\mu; \tau_k, x + I_k(x))\big) \le V_i\big(\lambda, \eta(\lambda; \tau_k, x)\big)$$

and passing to the limits as $\mu \to \tau_k + 0$ and $\lambda \to \tau_k - 0$ we obtain

$$V_i\big(\tau_k + 0, x + I_k(x)\big) \le V_i(\tau_k - 0, x) = V_i(\tau_k, x). \tag{21}$$

From here we obtain inequality (3) for the functions $V_i(t, x)$.

Let $x \in B_{\beta(\alpha)}, t \in \mathbb{R}$ with $t \ne \tau_k$, $h > 0$ and $x' = x(t + h; t, x)$. Then

$$\begin{aligned} V_i(t + h, x') &= g\Big(\frac{1}{i}\Big) \sup_{s \ge 0} G_i\big(\|x(t + h + s, t + h, x')\|\big) e^{cs} \\ &= g\Big(\frac{1}{i}\Big) \sup_{\tau > h} G_i\big(\|x(t + \tau, t + h, x')\|\big) e^{c\tau} e^{-ch} \\ &\le V_i(t, x) e^{-ch} \end{aligned}$$

or

$$\frac{1}{h}\big(V_i(t + h, x') - V_i(t, x)\big) \le \frac{1}{h}(e^{-ch} - 1) V_i(t, x).$$

Consequently $D^+ V_i(t, x) \le -c V_i(t, x)$. From this inequality we obtain (7) for the function $V_i(t, x)$.

Now we define the desired function $V(t, x)$ by setting

$$\left.\begin{aligned} V(t, x) &= \sum_{i=1}^{\infty} \frac{1}{2^i} V_i(t, x) \quad (t \ne \tau_k) \\ V(\tau_k, x) &= V(\tau_k - 0, x). \end{aligned}\right\} \tag{22}$$

Since (14) implies the uniform convergence of the series of (22) in $\Omega(\sigma, \alpha)$ where σ, α are arbitrary, $V(t, x)$ is defined on $\mathbb{R} \times \mathbb{R}^n$, piecewise continuous along t, with points of discontinuity at the moments τ_k $(k \in \mathbb{Z})$ and it is continuous along x. From (12) clearly $V(t, 0) \equiv 0$. For x such that $\|x\| \ge 1$ from (12) and (22) we obtain

$$V(t, x) > \frac{1}{2} V_1(t, x) \ge \frac{1}{2} g(1) G_1(\|x\|) \ge \frac{1}{2}(\|x\| - 1) \tag{23}$$

and for x such that $\frac{1}{i} \le \|x\| \le \frac{1}{i-1}$ we obtain

$$\begin{aligned} V(t,x) &\ge \frac{1}{2^{i+1}} V_{i+1}(t,x) \\ &\ge \frac{1}{2^{i+1}} g\Big(\frac{1}{i+1}\Big) G_{i+1}(\|x\|) \\ &\ge \frac{1}{2^{i+1}} g\Big(\frac{1}{i+1}\Big)\Big(\|x\| - \frac{1}{i+1}\Big) \\ &\ge \frac{1}{2^{i+1}} g\Big(\frac{1}{i+1}\Big)\frac{1}{i(i+1)}. \end{aligned} \tag{24}$$

From (23) and (24) we can find $a \in \mathcal{P}$ such that $a(r) \to \infty$ as $r \to \infty$ and $a(\|x\|) \le V(t,x)$. Let $(t,x), (t',x') \in \Omega(\sigma,\alpha)$ with $t < t'$. Then

$$\begin{aligned} \big|V(t,x) - V(t',x')\big| &\le \sum_{i=1}^{\infty} \frac{1}{2^i}\big|V_i(t,x) - V_i(t',x')\big| \\ &\le \sum_{i=1}^{\infty} \frac{1}{2^i} h(\alpha)\big(|t-t'| + \|x-x'\|\big) \\ &\le h(\alpha)\big(|t-t'| + \|x-x'\|\big). \end{aligned} \tag{25}$$

From here it follows that for $x \in B_{\beta(\alpha)}$ and $t \ne \tau_k$ the function $V(t,x)$ is continuous and for $t = t'$ we obtain (2).

Let $\tau_k \in \mathbb{R}, x \in B_{\beta(\alpha)}$ be fixed and $\theta_j \in (\tau_k, \tau_{k+1}], x_j \in B_{\beta(\alpha)}$ where $u_j = x(\theta_j;\tau_k,x_j)$ $(j=1,2)$. Then

$$\begin{aligned} &\big|V(\theta_j,x_j) - V(\theta_j,u_j)\big| \\ &\quad = \sum_{i=1}^{\infty} \frac{1}{2^i}\big|V_i(\theta_j,x_j) - V_i(\theta_j,u_j)\big| \\ &\quad \le \sum_{i=1}^{\infty} \frac{1}{2^i} g\Big(\frac{1}{2^i}\Big)\Big|a(\theta_1-\tau_k)e^{-c(\theta_1-\tau_k)} - a(\theta_2-\tau_k)e^{-c(\theta_2-\tau_k)}\Big| \\ &\quad \to 0 \end{aligned}$$

for $\theta_j \to \tau_k + 0 \quad (j=1,2)$, i.e. there exists the limit $V(\tau_k+0,x)$. The proof of the existence of the limit $V(\tau_k-0,x)$ follows by analogy.

Let $\eta(t;t_0,x_0)$ be the solution of the initial value problem

$$\left.\begin{aligned} \dot\eta &= f(t,\eta) \\ \eta(t_0) &= x_0. \end{aligned}\right\}$$

For $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$ and $s > \mu$ from (21) we get

$$\begin{aligned} V\big(\tau_k+0, x+I_k(x)\big) &= \sum_{i=1}^{\infty} \frac{1}{2^i} V_i\big(\tau_k+0, x+I_k(x)\big) \\ &\le \sum_{i=1}^{\infty} V(\tau_k - 0, x) \\ &\le \sum_{i=1}^{\infty} \frac{1}{2^i} V_i(\tau_k, x) \\ &= V(\tau_k, x). \end{aligned} \tag{26}$$

Let $x \in B_{\beta(\alpha)}, t \neq \tau_k$ and $h > 0$. Then from (22) we obtain

$$D^+V(t,x) = \limsup_{h\to 0^+} \frac{V_i(t+h, x(t+h;t,x)) - V_i(t,x)}{h}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{2^i}(-cV_i(t,x)) \le -cV(t,x).$$

Then

$$D^+V(t,x) \le -cV(t,x). \tag{27}$$

Consequently, from last inequality it follows that there exists $V(t,x)$ from $\mathcal{V}$ such that (6) and (7) are fulfilled. Here we shall show that $V(t,x)$ is almost periodic in t uniformly with respect to $x \in B_{\beta(\alpha)}$. From condition 2 of Theorem 1 it follows that if $x \in B_{\beta(\alpha)}$, then there exists $\beta(\alpha) > 0$ such that $\|x(\tau;t,x)\| \le \beta(\alpha)$ for any $t \ge \tau$. From assumptions (H6) - (H8) we get that for an arbitrary sequence $\{s_m{}'\}$ there exist a subsequence $\{s_n\},\ s_n = s_{m_n}{}'$ moving (1) in mod (f, I_k, τ_k). Then as $x \in B_{\beta(\alpha)}$ we obtain

$$\begin{aligned} &\big|V_i(t+s_n,x) - V_i(t+s_p)\big| \\ &\quad\le \sup_{\tau\ge 0} g\Big(\frac{1}{i}\Big)e^{c\tau}\Big|G_i\big(\|x(t+s_n+\tau)\|\big) - G_i\big(\|x(t+s_p+\tau;t+s_p,x)\|\big)\Big| \\ &\quad\le g\Big(\frac{1}{i}\Big) \sup_{0\le\tau\le T(\alpha,\frac{1}{i})} e^{c\tau}\Big\|x\big(t+s_n+\tau;t+s_n,x\big) - x\big(t+s_p+\tau;t+s_n,x\big)\Big\|. \end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned} x\big(t+s_n+\tau;t+s_n,x\big) &= x + \int_t^{t+\tau} f\big(\sigma+s_n, x(\sigma+s_n;t+s_n,x)\big)d\sigma \\ &\quad + \sum_{t<\sigma_i(s_n)<t+\tau} I_{i+i(s_n)}\big(x\big(\sigma_i(s_n)+s_n;t+s_n,x\big)\big) \end{aligned} \tag{29}$$

and

$$\begin{aligned} x\big(t+s_p+\tau;t+s_p,x\big) &= x + \int_t^{t+\tau} f\big(\sigma+s_p, x(\sigma+s_p;t+s_p,x)\big)d\sigma \\ &\quad + \sum_{t<\sigma_i(s_p)<t+\tau} I_{i+i(s_p)}\big(x\big(\sigma_i(s_p)+s_p;t+s_p,x\big)\big) \end{aligned} \tag{30}$$

where $\sigma_i(s_j) = \tau_k - s_j$ $(j = n, p)$ and the numbers $i(s_n)$ and $i(s_p)$ are such that $i + i(s_j) = k$. From (29) and (30) it follows that

$$\begin{aligned}
&\Big\|x(t+s_n+\tau;t+s_n,x) - x(t+s_p+\tau,t+s_p,x)\Big\| \\
&\quad\le \int_t^{t+\tau} \Big\|f\big(\sigma+s_n, x(\sigma+s_n;t+s_n,x)\big) - f\big(\sigma+s_p, x(\sigma+s_n;t+s_n,x)\big)\Big\| d\sigma \\
&\qquad + \int_t^{t+\tau} \Big\|f\big(\sigma+s_p, x(\sigma+s_n;t+s_n,x)\big) - f\big(\sigma+s_p, x(\sigma+s_p;t+s_p,x)\big)\Big\| d\sigma \\
&\qquad + \sum_{t<\sigma_i<t+\tau} \Big\|I_{i+i(s_n)}\big(x(\sigma_i(s_n)+s_n;t+s_n,x)\ I_{i+i(s_p)}\big(x(\sigma_i(s_n)+s_n;t+s_n,x)\big)\Big\| \\
&\qquad + \sum_{t<\sigma_i<t+\tau} \Big\|I_{i+i(s_p)}\big(x(\sigma_i(s_n)+s_n;t+s_n,x)\big) - I_{i+i(s_p)}\big(x(\sigma_i(s_p)+s_p;t+s_p,x)\big)\Big\|.
\end{aligned}$$

From $x(\sigma + s_n; t + s_n, x) \in B_{\beta(\alpha)}$ it follows that for any $\varepsilon > 0$ there exists a number $N(\varepsilon) > 0$ such that as $n, p \ge N(\varepsilon)$ we obtain

$$\Big\|f\big(\sigma+s_n, x(\sigma+s_n;t+s_n,x)\big) - f\big(\sigma+s_p, x(\sigma+s_n;t+s_n,x)\big)\Big\| < \varepsilon \quad (31)$$

$$\Big\|I_{i+i(s_n)}\big(x(\sigma_i(s_n)+s_n;t+s_n,x)\big) - I_{i+i(s_p)}\big(x(\sigma_i(s_n)+s_n;t+s_n,x)\big)\Big\| < \varepsilon. \quad (32)$$

Then from (31) - (32) and assumptions (H2) and (H4) we obtain

$$\begin{aligned}
&\Big\|x(t+s_n+\tau;t+s_n,x) - x(t+s_p+\tau,t+s_p,x)\Big\| \\
&\quad\le \varepsilon\tau\Big(1+\frac{1}{p}\Big) + \int_t^{t+\tau} L_1(\alpha)\Big\|x(\sigma+s_n;t+\sigma_n,x) - x(\sigma+s_p,t+s_p,x)\Big\| d\sigma \qquad (33)\\
&\qquad + \sum_{t<\sigma_i<t+\tau} L_2(\alpha)\Big\|x(\sigma_i(s_n)+s_n;t+s_n,x) - x(\sigma_i(s_p)+s_p;t+s_p,x)\Big\|.
\end{aligned}$$

On the other hand, from Lemma 3 and (33) we obtain

$$\begin{aligned}
&\Big\|x(t+s_n+\tau;t+s_n,x) - x(t+s_p+\tau,t+s_p,x)\Big\| \\
&\quad\le \varepsilon\tau\Big(1+\frac{1}{p}\Big)e^{(L_1(\alpha,\frac{1}{i})+\frac{1}{p}\ln(1+L_2(\alpha,\frac{1}{i})))\tau}.
\end{aligned} \qquad (34)$$

From (34) and (28) we get that

$$\begin{aligned}
&\big|V_i(t+s_n,x) - V_i(t+s_p,x)\big| \\
&\quad\le g\Big(\frac{1}{i}\Big)\Big(1+\frac{1}{p}\Big)T\Big(\alpha,\frac{1}{i}\Big)e^{(c+L_1(\alpha,\frac{1}{i})+\frac{1}{p}\ln(1+L_2(\alpha,\frac{1}{i}))T(\alpha,\frac{1}{i}))}\varepsilon \qquad (35)\\
&\quad\le h(\alpha)\varepsilon.
\end{aligned}$$

From here it follows that $V_i(t+s_n, x)$ is uniformly convergent with respect to $t \in R$ and $x \in B_{\beta(\alpha)}$. Then $V_i(t, x)$ is almost periodic on t uniformly with respect to $x \in B_{\beta(\alpha)}$. Inequality (22) implies that for $n, p \in N(\varepsilon)$ and $x \in B_{\beta(\alpha)}$ we obtain

$$\big|V(t+s_n,x) - V(t+s_p,x)\big| \le h(\alpha)\varepsilon,$$

i.e. $V(t, x)$ is almost periodic in t with respect to $x \in B_{\beta(\alpha)}$, and the proof of Theorem 1 is complete ∎

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Received 12.07.1999; in revised form 04.01.00