On Representations of Partial *-Algebras Based on 𝔅-Weights

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Abstract. A generalization of the GNS-representation is investigated that represents partial *-algebras as systems of operators acting on a partial inner product space (i.e., PIP-space). It is based on possibly indefinite 𝔅-weights which are closely related to the positive 𝔅-weights introduced by J.-P. Antoine, Y. Soulet and C. Trapani. Some additional assumptions had to be made in order to guarantee the GNS-construction. Different partial products of operators on a PIP-space are considered which allow the GNS-construction under suitable conditions. Several examples illustrate the argumentation and indicate inherent problems.

Keywords: *Partial algebras, GNS-construction, weights, partial inner product spaces*

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1. Introduction

The development of the theory of partial *-algebras has been motivated by the appearance of such structures in models of local quantum field theory and quantum statistical mechanics (see, e.g., [6, 13]). Lately, several standard results of the theory of *-algebras of operators were extended to a certain degree to partial *-algebras, for instance representation theory, modular theory of Tomita-Takesaki, and automorphism groups and *-derivations. For details and further references, we refer to the review [2] by J.-P. Antoine, A. Inoue and C. Trapani.

As the GNS-construction is one of the basic tools of the theory of *-algebras, there arises a particular interest in extending it to partial *-algebras. A promising approach to this problem has been made by J.-P. Antoine, A. Inoue and C. Trapani [3], starting with a positive sesquilinear form on a partial *-algebra, using a subspace of the space of all right multipliers to set up the representation, and taking into account the possible lack of (semi-)associativity. The result is a representation of the partial *-algebra into the partial O^*-algebra $L^\dagger(D, H)$. Nevertheless, this approach might be not general enough. Yet for *-algebras there exists a GNS-construction based on weights, that is, positive functionals defined on the positive cone of the *-algebra that do not necessarily take finite values. In order to give a GNS-construction for partial *-algebras that generalizes also the theory of weights, J.-P. Antoine, Y. Soulet and C. Trapani [5] introduced the notion of a (positive) 𝔅-weight. The GNS-construction based on a 𝔅-weight will lead

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to a representation of the given partial *-algebra as a system of operators acting on some partial inner product space (PIP-space); the basic ideas of PIP-spaces were developed by J.-P. Antoine and A. Grossmann in earlier papers [1].

Our treatment of the subject follows closely the program presented in [5], except that we do not require the $\mathfrak{B}$-weight to be positive. However, there will be included several examples which show that our results are also relevant to the case of positive $\mathfrak{B}$-weights. In Section 2 we give a sufficient and necessary condition that a $\mathfrak{B}$-weight determines the structure of a non-degenerate PIP-space. Sections 3 and 4 are devoted to the study of a generalized GNS-representation of a partial *-algebra as systems of operators acting on a non-degenerate PIP-space V. To do this, the linear space of all continuous linear operators on V (denoted by $Op(V)$) must be equipped with a multiplicative structure. In Section 3 the underlying multiplicative structure is based on a definition due to J.-P. Antoine and A. Grossmann [1]. It turns out that additional assumptions must be made in order to guarantee the GNS-representation. Under certain conditions some of the additional assumptions can be removed by introducing more general partial products on $Op(V)$. This is the central theme of Section 4. For the convenience of the reader, the necessary definitions concerning PIP-spaces, partial *-algebras and $\mathfrak{B}$-weights are included in Sections 2 and 3. For a more detailed study we refer to the references [1 - 5, 10].

2. Construction of non-degenerate PIP-spaces

In this section we investigate the problem whether, given a (possibly indefinite) $\mathfrak{B}$-weight Ω on a partial *-algebra, there exists a non-degenerate PIP-space associated with Ω in a natural way. It turns out that the existence of such a PIP-space may be characterized by additional conditions on the $\mathfrak{B}$-weight. Examples show that these conditions do not follow from the axioms of $\mathfrak{B}$-weights.

Let us summarize some notations and definitions concerning PIP-spaces, partial *-algebras and $\mathfrak{B}$-weights. These definitions are essentially equivalent to the original definitions in [1, 5], except that our forms are linear in the first argument and that the form Ω is not required to be positive semi-definite here. A weak linear compatibility on a $\mathbb{C}$-vector space V is a symmetric binary relation $\#$ on V such that all non-empty sets of the type

$$M^{\#} := \left\{ \varphi \in V : \varphi \# \psi \ \text{ for all } \psi \in M \right\} \qquad (M \subset V)$$

(called assaying subspaces) are linear subspaces of V. A linear compatibility on V is a weak linear compatibility such that all sets of the type $M^{\#}$ are non-empty. Suppose now that $\#$ is a linear compatibility on the $\mathbb{C}$-vector space V and let $\Gamma(\#)$ denote the graph of $\#$. A partial inner product on $(V, \#)$ is a mapping

$$\Gamma(\#) \ni (\varphi, \psi) \to \langle \varphi, \psi \rangle \in \mathbb{C}$$

which is linear in φ and satisfies $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$ whenever $\varphi \# \psi$. A triple $(V, \#, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is a partial inner product on $(V, \#)$, is called a PIP-space. It is said to be non-degenerate if $\langle \varphi, \psi \rangle = 0$ for all $\psi \in V^{\#}$ implies that $\varphi = 0$.

A partial *-algebra is a *-vector space $\mathfrak{A}$ together with a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ and a mapping $\Gamma \ni (x, y) \rightarrow x \cdot y = xy \in \mathfrak{A}$ such that $(x, y), (x, z) \in \Gamma$ and $\lambda, \mu \in \mathbb{C}$ imply that $(x, \lambda y + \mu z) \in \Gamma$, $(y^*, x^*) \in \Gamma$, and that

$$\left.\begin{array}{r} x \cdot (\lambda y + \mu z) = \lambda(x \cdot y) + \mu(x \cdot z) \\ (x \cdot y)^* = y^* \cdot x^* \end{array}\right\}.$$

The set of right multipliers of a subset $\mathfrak{N} \subset \mathfrak{A}$ is the set

$$R(\mathfrak{N}) := \big\{ x \in \mathfrak{A} : (y, x) \in \Gamma \ \text{ for all } y \in \mathfrak{N} \big\}.$$

GNS-constructions map abstract partial *-algebras into certain spaces of linear operators acting on linear spaces, for instance on unitary spaces or on PIP-spaces. Such spaces of linear operators represent also basic examples of spaces with partially defined products. Examples in [9 - 11] show that these spaces are not necessarily associative. For this reason, associativity is not included in the axioms of partial *-algebras.

Now (possibly indefinite) $\mathfrak{B}$-weights are defined as follows.

Definition 2.1. Suppose $\mathfrak{A}$ is a partial *-algebra, $\mathfrak{B}$ is a linear subspace of $R(\mathfrak{A})$, and $\sharp$ is a weak linear compatibility on $\mathfrak{A}$ with graph $\Gamma(\sharp)$. A mapping

$$\Omega : \Gamma(\sharp) \ni (x, y) \rightarrow \Omega(x, y) \in \mathbb{C}$$

which is linear in x and satisfies

$$\Omega(x, y) = \overline{\Omega(y, x)}$$

whenever $(x, y) \in \Gamma(\sharp)$ is said to be a $\mathfrak{B}$-*weight* if the following conditions are satisfied:

(i) $\mathfrak{B} \times \mathfrak{B} \cup \mathfrak{A}\mathfrak{B} \times \mathfrak{B} \subset \Gamma(\sharp)$ (where $\mathfrak{A}\mathfrak{B} := \{ab : a \in \mathfrak{A} \text{ and } b \in \mathfrak{B}\}$).

(ii) $\Omega(xb_1, b_2) = \Omega(b_1, x^*b_2)$ for all $x \in \mathfrak{A}$ and $b_1, b_2 \in \mathfrak{B}$.

(iii) If $x_1, x_2 \in \mathfrak{A}$ and $x_1 \in R(\{x_2\})$, then $(x_1b_1, x_2^*b_2) \in \Gamma(\sharp)$ for all $b_1, b_2 \in \mathfrak{B}$ and $\Omega(x_1b_1, x_2^*b_2) = \Omega((x_2x_1)b_1, b_2)$.

(iv) If $x \in \mathfrak{B}$ and $\Omega(y, x) = 0$ for all $y \in \mathfrak{B}$, then $\Omega(y, x) = 0$ for all $y \in \mathfrak{B}^\sharp$.

If $\mathfrak{A}$ is a *-algebra and ω a linear functional on $\mathfrak{A}$ satisfying $\omega(x^*x) \in \mathbb{R}$ for all $x \in \mathfrak{A}$, then one obtains a sesquilinear form Ω on $\mathfrak{A}$ by setting $\Omega(x, y) = \omega(y^*x)$. It arises the question if we can start with a linear functional on a partial *-algebra $\mathfrak{A}$ and construct a $\mathfrak{B}$-weight in a similar way. Naturally, additional assumption must be made. In view of Definition 2.1/(ii) and (iii), we impose on $\mathfrak{A}$ semi-associativity: $\mathfrak{A}$ is called semi-associative if $x \in R(\{y\})$ implies $xb \in R(\{y\})$ for all $b \in R(\mathfrak{A})$ and $y(xb) = (yx)b$.

The following proposition describes a situation where a linear functional determines the structure of a $\mathfrak{B}$-weight.

Proposition 2.2. *Suppose $\mathfrak{A}$ is a semi-associative partial $*$-algebra, $\mathfrak{B}$ is a linear subspace of $R(\mathfrak{A})$, D is a linear subspace of $\mathfrak{A}$, and $\omega : D \to \mathbb{C}$ is a linear functional such that the following conditions are satisfied:*

(i) *$D = D^*$ $(:= \{x^* : x \in D\})$ and $\omega(x^*) = \overline{\omega(x)}$ for all $x \in D$.*

(ii) *$\mathfrak{B}^*\mathfrak{B} \subset D$ and $\mathfrak{B}^*\mathfrak{A}\mathfrak{B}$ $(= \mathfrak{B}^*(\mathfrak{A}\mathfrak{B})) \subset D$.*

(iii) *If $x \in \mathfrak{B}$ and $\mathfrak{B}^*x \subset \ker(\omega)$, then $\mathfrak{B}'^*x \subset \ker(\omega)$ where $\mathfrak{B}' := \{y \in \mathfrak{A} : \mathfrak{B}^*y \subset D\}$.*

Define $x \, \sharp \, y$ if $x \in R(\{y^*\})$ and $y^*x \in D$. Then $\sharp$ is a linear compatibility on $\mathfrak{A}$, and Ω, defined by $\Omega(x,y) = \omega(y^*x)$ whenever $x \, \sharp \, y$, is a $\mathfrak{B}$-weight on $\mathfrak{A}$ in the sense of Definition 2.1.

Proof. Since $(y^*, x) \in \Gamma$ and $y^*x \in D$ imply that $(x^*, y) \in \Gamma$ and $x^*y \in D$, $\sharp$ is symmetric. Since the partial product on $\mathfrak{A}$ is distributive and D is a linear subspace, $\sharp$ defines a linear compatibility. Using the hypothesis, one proves easily that Ω satisfies the conditions of Definition 2.1. For instance, if $x_1 \in R(\{x_2\})$, then repeated application of semi-associativity gives for all $b_1, b_2 \in \mathfrak{B}$

$$D \ni b_2^*((x_1x_2)b_1) \; = \; b_2^*(x_1(x_2b_1)) \; = \; (b_2^*x_1)(x_2b_1) \,,$$

which implies Definition 2.1/(iii). To verify Definition 2.1/(iv), observe that $\mathfrak{B}^\sharp = \mathfrak{B}'$ and apply condition (iii) ∎

The following proposition is the main result of this section. It characterizes $\mathfrak{B}$-weights to which there is associated a non-degenerate PIP-space in a natural way. PIP-spaces obtained in this manner will serve in the following sections as representation spaces for a generalized GNS-construction.

Proposition 2.3. *Let subspaces of $\mathfrak{B}^\sharp$ be defined by*

$$\mathfrak{N}_1 \; = \; \left\{x \in \mathfrak{B}^{\sharp\sharp} : \Omega(x,y) = 0 \quad \text{for all } y \in \mathfrak{B}^\sharp \right\}$$
$$\mathfrak{N}_2 \; = \; \left\{x \in \mathfrak{B}^\sharp : \Omega(x,y) = 0 \quad \text{for all } y \in \mathfrak{B}^{\sharp\sharp} \right\}.$$

Then for a linear subspace $\mathfrak{N} \subset \mathfrak{B}^\sharp$ the following conditions are equivalent:

(i) *On $V := \mathfrak{B}^\sharp / \mathfrak{N}$ there exists a non-degenerate PIP-space structure such that*

$$(x + \mathfrak{N}) \, \# \, (y + \mathfrak{N}) \;\Leftrightarrow\; x \, \sharp \, y \qquad and \qquad x \, \sharp \, y \;\Rightarrow\; \langle x + \mathfrak{N}, y + \mathfrak{N} \rangle = \Omega(x,y).$$

(ii) *$\mathfrak{N} = \mathfrak{N}_1 = \mathfrak{N}_2$.*

Proof. (ii) $\Rightarrow$ (i): Let $x_1, x_2 \in \mathfrak{B}^\sharp$ and $n_1, n_2 \in \mathfrak{N}$. If $(x_1, x_2) \in \Gamma(\sharp)$, then we observe that $(x_1 + n_1, x_2) \in \Gamma(\sharp)$; using firstly $n_1 \in \mathfrak{B}^{\sharp\sharp}$ since $\mathfrak{N} = \mathfrak{N}_1$ and secondly the linearity of $\sharp$. Continuing in this way gives $(x_1 + n_1, x_2 + n_2) \in \Gamma(\sharp)$. Hence $(x_1 + n_1, x_2 + n_2) \in \Gamma(\sharp)$ for all $n_1, n_2 \in \mathfrak{N}$ if and only if $(x_1, x_2) \in \Gamma(\sharp)$; thus $\#$ is well defined. By the properties of $\Gamma(\sharp)$, it follows that $\#$ defines a linear compatibility on V. The above arguments allow us to write

$$\Omega(x_1 + n_1, x_2 + n_2) = \Omega(x_1, x_2) + \Omega(x_1, n_2) + \Omega(n_1, x_2) + \Omega(n_1, n_2)$$

whenever $(x_1, x_2) \in \Gamma(\sharp)$. Since $\mathfrak{N} = \mathfrak{N}_1$, we get $\Omega(x_1 + n_1, x_2 + n_2) = \Omega(x_1, x_2)$. Hence $\langle \cdot, \cdot \rangle$ is well defined. That $\langle \cdot, \cdot \rangle$ defines a partial inner product follows from the assumed properties of Ω. By $\mathfrak{N} = \mathfrak{N}_2$, V is non-degenerate.

(i) $\Rightarrow$ (ii): Clearly, $\mathfrak{N} \subset \mathfrak{B}^{\sharp\sharp}$ is necessary, otherwise $\#$ would not be well defined. Suppose $n \in \mathfrak{N}$, $y \in \mathfrak{B}^\sharp$ and $\Omega(n, y) \neq 0$. Then for any $x \in \mathfrak{B}^{\sharp\sharp}$ we get $\Omega(x + n, y) \neq \Omega(x, y)$, so $\langle \cdot, \cdot \rangle$ is not well defined; therefore $\mathfrak{N} \subset \mathfrak{N}_1$. Note that $V^\# = \{x + \mathfrak{N} : x \in \mathfrak{B}^{\sharp\sharp}\}$. As V is non-degenerate, we have $\mathfrak{N}_2 \subset \mathfrak{N}$. Finally, $\mathfrak{N}_1 \subset \mathfrak{N}_2$ since $\mathfrak{B}^{\sharp\sharp} \subset \mathfrak{B}^\sharp$. Thus we have proved $\mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \mathfrak{N} \subset \mathfrak{N}_1$ which yields (ii) ∎

The following two examples illustrate that for arbitrary $\mathfrak{B}$-weights the assertions of Proposition 2.3 are not necessarily satisfied. The first example shows that the assertion $\mathfrak{N}_1 = \mathfrak{N}_2$ depends strongly on the weak linear compatibility $\sharp$. The second example is especially designed to show that there is no natural generalization of the Schwarz inequality for positive $\mathfrak{B}$-weights.

Example 2.4. There exist a partial *-algebra $\mathfrak{A}$, a linear subspace $\mathfrak{B} \subset \mathfrak{A}$, and a $\mathfrak{B}$-weight Ω on $\mathfrak{A}$ such that $\mathfrak{N}_1 \neq \mathfrak{N}_2$.

Indeed, consider the *-algebra $\mathfrak{A} := L_\infty([0, 2])$ of all measurable, bounded, complex functions on the interval $[0, 2]$. Set $\mathfrak{B} = \chi_{[0,1]} \mathfrak{A}$, where $\chi_{[0,1]}$ denotes the characteristic function of the interval $[0, 1]$. Define $\sharp$ by setting $\Gamma(\sharp) = (\mathfrak{A} \times \mathfrak{B}) \cup (\mathfrak{B} \times \mathfrak{A})$ and define $\Omega(f, g) = \int_0^2 f \bar{g} dt$ whenever $(f, g) \in \Gamma(\sharp)$. Using $\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}$, one easily verifies that Ω is a $\mathfrak{B}$-weight in the sense of Definition 2.1. Now $\mathfrak{B}^\sharp = \mathfrak{A}$, $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B}$, and thus $\mathfrak{N}_1 = \{0\} \neq \chi_{[1,2]} \mathfrak{A} = \mathfrak{N}_2$. Notice that if we had defined $\Gamma(\sharp) = \mathfrak{A} \times \mathfrak{A}$, $\mathfrak{N}_1 = \mathfrak{N}_2$ would hold.

Incidentally, Example 2.4 yields an example of Proposition 2.2, too; just set $D = \mathfrak{B}$ and define $\omega(f) = \int_0^2 f dt$ for all $f \in D$.

We call a $\mathfrak{B}$-weight Ω positive if $\Omega(x, x) \geq 0$ whenever $(x, x) \in \Gamma(\sharp)$. One might try to obtain a better result as Proposition 2.3 by employing the Schwarz inequality. If, for instance, Ω is a positive semi-definite sesquilinear form on $\mathfrak{A}$, then it is sufficient to consider the set $\mathfrak{N} = \{x \in \mathfrak{B}^\sharp ; \Omega(x, x) = 0\}$ since the Schwarz inequality implies $\Omega(y, n) = 0$ for all $y \in \mathfrak{B}^\sharp$ and $n \in \mathfrak{N}$. Unfortunately, this approach is useless for PIP-spaces with positive $\mathfrak{B}$-weights. It can, namely, happen that $\Omega(y, n) \neq 0$ although $n \in \mathfrak{B}^{\sharp\sharp}$ and $\Omega(n, n) = 0$. We shall present an explicit example.

Example 2.5. There exist a partial *-algebra $\mathfrak{A}$, a linear subspace $\mathfrak{B} \subset \mathfrak{A}$, and a positive $\mathfrak{B}$-weight Ω on $\mathfrak{A}$ such that the following statements hold: There are $b \in \mathfrak{B}^{\sharp\sharp}$ and $y \in \mathfrak{B}^\sharp$ such that $\Omega(b, b) = 0$ but $\Omega(y, b) \neq 0$, $\mathfrak{N}_1, \mathfrak{N}_2 \subset \mathfrak{B}^{\sharp\sharp}$ and $\mathfrak{N}_1 \neq \mathfrak{N}_2$.

Indeed, let $\mathfrak{B}$ be the vector space φ of all complex sequences $(x_n) = (x_n)_{n \in \mathbb{N}}$ with a finite number of non-zero entries. Consider the complex vector space

$$\mathfrak{A} := \mathbb{C}(n) + \mathbb{C}(\tfrac{1}{n}) + \varphi ,$$

where (n) and $(\tfrac{1}{n})$ denote the sequences $(n)_{n \in \mathbb{N}}$ and $(\tfrac{1}{n})_{n \in \mathbb{N}}$, respectively. $\mathfrak{A}$ becomes a partial *-algebra by restricting the pointwise multiplication of sequences to $\Gamma := (\mathfrak{A} \times \mathfrak{B}) \cup (\mathfrak{B} \times \mathfrak{A})$ and defining an involution by complex conjugation. We introduce a

linear compatibility $\sharp$ and a positive $\mathfrak{B}$-weight Ω on $\mathfrak{A}$ by setting

$$\Gamma(\sharp) = (\mathfrak{A} \times (\mathbb{C}(\tfrac{1}{n}) + \mathfrak{B})) \cup ((\mathbb{C}(\tfrac{1}{n}) + \mathfrak{B}) \times \mathfrak{A})$$

$$\Omega((x_n), (y_n)) = \lim_{k \to \infty} x_k \overline{y_k} \quad \text{whenever} \quad ((x_n), (y_n)) \in \Gamma(\sharp).$$

Notice that $\Omega((x_n), (v_n)) = 0$ for all $(x_n) \in \mathfrak{A}$, $(v_n) \in \mathfrak{B}$ and that the product of any pair $((x_n), (y_n)) \in \Gamma$ lies in $\mathfrak{B}$. Combining these two facts, one proves easily that Ω is indeed a $\mathfrak{B}$-weight. Moreover, $\mathfrak{B}^\sharp = \mathfrak{A}$ and $\mathfrak{B}^{\sharp\sharp} = \mathbb{C}(\tfrac{1}{n}) + \mathfrak{B}$.

Given $(v_n), (w_n) \in \mathfrak{B}$ and $\alpha, \beta_1, \beta_2 \in \mathbb{C}$, we calculate

$$\Omega\big(\beta_1(\tfrac{1}{n}) + (v_n), \alpha(n) + \beta_2(\tfrac{1}{n}) + (w_n)\big) = \beta_1 \overline{\alpha}.$$

As an example, the choice $b = (\tfrac{1}{n}) \in \mathfrak{B}^{\sharp\sharp}$, $y = (n) \in \mathfrak{B}^\sharp$ gives $\Omega(b, b) = 0$ and $\Omega(y, b) = 1 \neq 0$. Furthermore, it follows $\mathfrak{N}_1 = \mathfrak{B}$ and $\mathfrak{N}_2 = \mathfrak{B}^{\sharp\sharp}$, hence $\mathfrak{N}_1 \neq \mathfrak{N}_2$ as asserted. Note that $\mathfrak{N}_1, \mathfrak{N}_2 \subset \mathfrak{B}^{\sharp\sharp}$ is satisfied.

3. Representations of partial *-algebras using factorization products

We turn now to the problem of constructing a representation of partial *-algebras as systems of operators acting on a non-degenerate PIP-space. The non-degenerate PIP-space in question will be, of course, the one constructed in the preceding section. For non-degenerate PIP-spaces we can adopt the concepts of dual pairings; if $(V, \#, \langle \cdot, \cdot \rangle)$ is a non-degenerate PIP-space and $X \subset V$ an assaying subspace, then the restriction of $\langle \cdot, \cdot \rangle$ to $X \times \overline{X^\#}$, where $\overline{X^\#}$ denotes the associated conjugate linear space of $X^\#$, is a dual pairing. From now on, each assaying subspace X will be equipped with the Mackey topology $\tau(X, \overline{X^\#})$, unless it is otherwise stated. The set of all continuous linear operators from X into another assaying subspace Y will be denoted by $\mathcal{L}(X, Y)$.

J.-P. Antoine and A. Grossmann [1] introduced the operator spaces $Op(V)$ (and more generally, $Op(V, W)$) of operators acting on PIP-spaces. They also defined products for certain n-tuples of those operators. In their paper J.-P. Antoine and A. Grossmann showed that, given a non-degenerate PIP-space $(V, \#, \langle \cdot, \cdot \rangle)$, the space $Op(V)$ is linearly isomorphic in a natural way to the space $\mathcal{L}(V^\#, V)$ of all continuous linear operators mapping $V^\#$ into V. Using this isomorphism, we identify here $Op(V)$ with $\mathcal{L}(V^\#, V)$. Then the following definition of products on $Op(V)$ is equivalent to the special case of the definition in [1], where all operators to be multiplied belong to $Op(V) = Op(V, V)$ for a fixed PIP-space $(V, \#, \langle \cdot, \cdot \rangle)$.

Definition 3.1. Let $(V, \#, \langle \cdot, \cdot \rangle)$ be a non-degenerate PIP-space. The *factorization product $T_n \circ \ldots \circ T_1$* of elements of $Op(V)$ $(= \mathcal{L}(V^\#, V))$ is said to be defined if there are assaying subspaces $E_0, \ldots, E_n$ of V and continuous extensions $S_j \in \mathcal{L}(E_{j-1}, E_j)$ of T_j. In this case

$$T_n \circ \ldots \circ T_1 \varphi = S_n(\ldots(S_1 \varphi)\ldots).$$

On $Op(V)$ there is defined an involution $A \mapsto A^*$ such that $\langle A\varphi, \psi \rangle = \langle \varphi, A^*\psi \rangle$ for all $\varphi, \psi \in V^\#$, i.e., A^* is the dual $A' \in \mathcal{L}(\overline{V^\#}, \overline{V})$ of A, considered as an element of $\mathcal{L}(V^\#, V)$.

Our aim is to construct a linear mapping $\pi : \mathfrak{A} \to Op(V)$ that respects adjoints and products, i.e., $\pi(x^*) = \pi(x)^*$ for all $x \in \mathfrak{A}$ and $\pi(xy) = \pi(x) \circ \pi(y)$ whenever $y \in R(\{x\})$. As in [5], the basic idea is to define the operator $\pi(x)$ on the set $\{b+\mathfrak{N} : b \in \mathfrak{B}\} \subset V$ (see Proposition 2.3 for notations) by setting $\pi(x)(b + \mathfrak{N}) = xb + \mathfrak{N}$. Unfortunately, we are facing two difficulties: In general, $\pi(x)$ is not yet defined on $V^{\#}$ since $\{b + \mathfrak{N} : b \in \mathfrak{B}\}$ is not necessarily equal to $V^{\#}$, and for $y \in R(\{x\})$ we must find an assaying subspace $X \subset V$ such that $\pi(y) \in \mathcal{L}(V^{\#}, X)$ and that $\pi(x)$ has an extension belonging to $\mathcal{L}(X, V)$. Under these circumstances it seems natural that we need further assumptions. For this purpose we state the following definition.

Definition 3.2. Suppose $\mathfrak{A}$ is a partial *-algebra, $\mathfrak{B}$ is a linear subspace of $R(\mathfrak{A})$, and Ω is $\mathfrak{B}$-weight on $\mathfrak{A}$. If X and Y are linear subspaces of $\mathfrak{B}^{\sharp}$ and $X \subset Y^{\sharp}$, then $\Sigma(X, Y)$ denotes the topology on X that is generated by the family of semi-norms $\{p_y\}_{y \in Y}$, where $p_y(x) := |\Omega(x, y)|$. We say the partial product of $\mathfrak{A}$ is Ω-*hypocontinuous with respect to* $\mathfrak{B}$, if the linear functionals

$$\mathfrak{B} \ni b \longmapsto \Omega(xb, w) \in \mathbb{C}$$

are continuous for all $x \in \mathfrak{A}$ and $w \in (x\mathfrak{B})^{\sharp}$ with respect to the topology $\Sigma(\mathfrak{B}, \mathfrak{B}^{\sharp})$, and if for each $x \in \mathfrak{A}$ there exists a family $\mathcal{M}$ of $\Sigma(\mathfrak{B}^{\sharp\sharp}, \mathfrak{B}^{\sharp})$-bounded subsets of $\mathfrak{B}$ such that $\mathfrak{B}^{\sharp\sharp} = \cup_{M \in \mathcal{M}} \overline{M}$, where $\overline{M}$ denotes the closure of M with respect to $\Sigma(\mathfrak{B}^{\sharp\sharp}, \mathfrak{B}^{\sharp})$, and the closed, convex, circled hull of xM is quasi-compact with respect to $\Sigma((x\mathfrak{B})^{\sharp\sharp}, (x\mathfrak{B})^{\sharp})$ for each $M \in \mathcal{M}$.

Remarks.

1. In general, the topology $\Sigma(X, Y)$ does not separate the points of X. That is why we require the closed, convex, circled hull of xM to be quasi-compact; the term "compact" we reserve for compact Hausdorff spaces.

2. The terminology "hypocontinuous" alludes to the concept of $\mathfrak{S}$-hypocontinuous bilinear forms as defined by Bourbaki [7: Chapter III, §5, 3.]. How Ω-hypocontinuous partial products are related to $\mathfrak{S}$-hypocontinuous sesquilinear forms will become clear in the proof of the next proposition.

Now we are in a position to prove the following version of a generalized *GNS*-representation.

Proposition 3.3. *Let Ω be a $\mathfrak{B}$-weight on a partial *-algebra $\mathfrak{A}$. Suppose that $\mathfrak{N} := \{x \in \mathfrak{B}^{\sharp} : \Omega(x, y) = 0$ for all $y \in \mathfrak{B}\}$ satisfies the assertions of Proposition 2.3. Let $(V, \#, \langle \cdot, \cdot \rangle)$ denote the non-degenerate PIP-space defined in Proposition 2.3. Assume that the partial product of $\mathfrak{A}$ is Ω-hypocontinuous with respect to $\mathfrak{B}$. Then there exists a unique linear mapping $\pi : \mathfrak{A} \to Op(V)$ such that $\pi(x)(b + \mathfrak{N}) = xb + \mathfrak{N}$ for all $b \in \mathfrak{B}$ and $x \in \mathfrak{A}$. Furthermore, π satisfies $\pi(x^*) = \pi(x)^*$ for all $x \in \mathfrak{A}$ and $\pi(xy) = \pi(x) \circ \pi(y)$ whenever $y \in R(\{x\})$.*

It is useful to introduce some temporary notations. Let $\iota : \mathfrak{B}^{\sharp} \to \mathfrak{B}^{\sharp}/\mathfrak{N}$ denote the canonical mapping. For an element $\iota(a) \in \mathfrak{B}^{\sharp}/\mathfrak{N} \ (= V)$ we write synonymously $a + \mathfrak{N}$ and $\hat{a}$; similarly we write $\widehat{M}$ in place of $\iota(M)$, where $M \subset \mathfrak{B}^{\sharp}$. The proof of Proposition 3.3 requires four steps: We shall

– define a linear mapping $\tilde{\pi}(x) : \widehat{\mathfrak{B}} \to V$

– give a unique extension $\pi(x) \in \mathcal{L}(V^\#, V)$ of $\tilde{\pi}(x)$

– prove that $\pi(x^*) = \pi(x)^*$, and finally

– show that $\pi(xy) = \pi(x) \circ \pi(y)$ whenever $y \in R(\{x\})$.

Proof of Proposition 3.3. *First step:* As mentioned above, for each $x \in \mathfrak{A}$ we define a linear mapping $\tilde{\pi}(x) : \widehat{\mathfrak{B}} \to V$ by setting

$$\tilde{\pi}(x)(b + \mathfrak{N}) = xb + \mathfrak{N}.$$

We have to show that this mapping is well defined. By Definition 2.1/(i), $xb \in \mathfrak{B}^\sharp$ for all $x \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Given $b_1, b_2 \in \mathfrak{B}$ such that $b_1 - b_2 \in \mathfrak{N}$, it follows from Definition 2.1/(ii) and $\mathfrak{N} = \mathfrak{N}_1$ that

$$\Omega(xb_1 - xb_2, b) = \Omega(b_1 - b_2, x^*b) = 0 \qquad \text{for all } b \in \mathfrak{B} .$$

This implies $xb_1 - xb_2 \in \mathfrak{N}$, hence $\tilde{\pi}(x)$ does not depend on the choice of representatives.

Second step: Our next goal is to show that $\tilde{\pi}(x)$ admits a unique continuous extension $\pi(x) : V^\# \to \widehat{(x\mathfrak{B})}^{\#\#}$. Let $\mathcal{S}(V^\#, \widehat{(x\mathfrak{B})}^\#)$ denote the set of all separately continuous sesquilinear forms $\beta : V^\# \times \widehat{(x\mathfrak{B})}^\# \to \mathbb{C}$. The proof hinges on the (real linear) isomorphisms

$$\mathcal{S}(V^\#, \widehat{(x\mathfrak{B})}^\#) \cong \mathcal{L}(V^\#, \widehat{(x\mathfrak{B})}^{\#\#}) \cong \mathcal{L}(\widehat{(x\mathfrak{B})}^\#, V) ,$$

where $\mathcal{L}(V^\#, \widehat{(x\mathfrak{B})}^{\#\#}) \cong \mathcal{L}(\widehat{(x\mathfrak{B})}^\#, V)$ consists in taking adjoints, that is, given a $\beta \in \mathcal{S}(V^\#, \widehat{(x\mathfrak{B})}^\#)$, there exist unique $\mathrm{B} \in \mathcal{L}(V^\#, \widehat{(x\mathfrak{B})}^{\#\#})$ and $\mathrm{C} \in \mathcal{L}(\widehat{(x\mathfrak{B})}^\#, V)$ such that

$$\beta(\hat{v}, \widehat{w}) = \langle \mathrm{B}\,\hat{v}, \widehat{w} \rangle = \langle \hat{v}, \mathrm{C}\,\widehat{w} \rangle \qquad \text{for all } \hat{v} \in V^\# \text{ and } \widehat{w} \in \widehat{(x\mathfrak{B})}^\#.$$

The isomorphisms can be obtained by identifying the sesquilinear forms on $V^\# \times \widehat{(x\mathfrak{B})}^\#$ with the bilinear forms on $V^\# \times \overline{\widehat{(x\mathfrak{B})}}^\#$ and applying the appropriate results for bilinear forms (see Köthe [8: §40, 1.]).

It is also known that a linear mapping is continuous with respect to the corresponding Mackey topologies if and only if it is continuous with respect to the corresponding weak topologies. If we refer to the latter case, we shall say "weakly continuous".

With the notations established in Definition 3.2, we observe that the weak topology $\sigma(\widehat{X}, \widehat{X}^\#)$ is the quotient topology of $\Sigma(X, X^\#)$, where X is an assaying subspace of $\mathfrak{B}^\sharp$. To see this, note that

$$\mathfrak{N} = \{x \in \mathfrak{B}^{\sharp\sharp} : \Omega(x, y) = 0 \ \text{ for all } y \in \mathfrak{B}^\sharp\}$$
$$\subset \{x \in X : \Omega(x, y) = 0 \ \text{ for all } y \in X^\sharp\}$$
$$\subset \{x \in \mathfrak{B}^\sharp : \Omega(x, y) = 0 \ \text{ for all } y \in \mathfrak{B}^{\sharp\sharp}\}$$
$$= \mathfrak{N},$$

hence

$$\mathfrak{N} = \left\{ x \in X : \Omega(x, y) = 0 \ \text{ for all } y \in X^\sharp \right\} = \bigcap_{y \in X^\sharp} p_y^{-1}(0).$$

As a consequence, $\iota : (X, \Sigma(X, X^\sharp)) \to (\widehat{X}, \sigma(\widehat{X}, \overline{\widehat{X^\#}}))$ is a continuous, open mapping. In addition, $(\widehat{\mathfrak{B}}, \sigma(\widehat{\mathfrak{B}}, \overline{V}))$ is a topological subspace of $(V^\#, \sigma(V^\#, \overline{V}))$.

Next, for $x \in \mathfrak{A}$ define

$$\tilde{\beta}_x : \widehat{\mathfrak{B}} \times (\widehat{x\mathfrak{B}})^\# \ni (\hat{b}, \widehat{w}) \longmapsto \langle \tilde{\pi}(x)\,\hat{b}, \widehat{w} \rangle \in \mathbb{C}.$$

Since $\tilde{\pi}(x)\,\hat{b} \in (\widehat{x\mathfrak{B}})^{\#\#}$, $\tilde{\beta}_x$ is weakly continuous in the second argument. To prove continuity in the first argument, consider the linear functionals

$$\left. \begin{array}{l} \Phi_{x,w} : \mathfrak{B} \ni b \longmapsto \Omega(xb, w) \in \mathbb{C} \\[4pt] \phi_{x,w} : \widehat{\mathfrak{B}} \ni \hat{b} \longmapsto \langle \tilde{\pi}(x)\,\hat{b}, \widehat{w} \rangle \in \mathbb{C} \end{array} \right\} \quad \left(x \in \mathfrak{A}, w \in (x\mathfrak{B})^\sharp \right).$$

Since $\langle \tilde{\pi}(x)\,\hat{b}, \widehat{w} \rangle = \Omega(xb, w)$, we have $\Phi_{x,w} = \phi_{x,w} \circ \iota$ and $\phi_{x,w}^{-1}(A) = \iota(\Phi_{x,w}^{-1}(A))$, where $A \subset \mathbb{C}$. But $\Phi_{x,w}$ is continuous by the Ω-hypocontinuity of the partial product, and ι is open, hence $\phi_{x,w}^{-1}(U)$ is open for every open set $U \subset \mathbb{C}$. This implies the weak continuity of $\phi_{x,w}$ and, moreover, the weak continuity of $\tilde{\beta}_x$ in the first argument. Let $\mathcal{M}$ be a family of subsets of $\mathfrak{B}$ satisfying the assumptions of Definition 3.2. Set $\widehat{\mathcal{M}} = \{\widehat{M} : M \in \mathcal{M}\}$. As $\sigma(V^\#, \overline{V})$ is the quotient topology of $\Sigma(\mathfrak{B}^{\#\#}, \mathfrak{B}^\#)$, we have $V^\# = \cup_{\widehat{M} \in \widehat{\mathcal{M}}} \overline{\widehat{M}}$, where the closure $\overline{\widehat{M}}$ of $\widehat{M}$ is taken with respect to $\sigma(V^\#, \overline{V})$, and all $\widehat{M} \in \widehat{\mathcal{M}}$ are bounded. Applying the facts that the closed, convex, circled hull of xM is quasi-compact with respect to $\Sigma((x\mathfrak{B})^{\#\#}, (x\mathfrak{B})^\#)$ (see Definition 3.2) and that ι is continuous, one verifies readily that the closed, convex, circled hull of $\widehat{(xM)}$ is compact with respect to $\sigma((\widehat{x\mathfrak{B}})^{\#\#}, (\widehat{x\mathfrak{B}})^\#)$; hence the polar

$$(\widehat{xM})^\circ = \left\{ \hat{y} \in (\widehat{x\mathfrak{B}})^\# : |\tilde{\beta}_x(\widehat{m}, \hat{y})| \leq 1 \text{ for all } \widehat{m} \in \widehat{M} \right\}$$

is a 0-neighbourhood in $(\widehat{x\mathfrak{B}})^\#$. Now, collecting the properties of $\tilde{\beta}_x$ and $\widehat{\mathcal{M}}$, we observe that $\tilde{\beta}_x$ is $\widehat{\mathcal{M}}$-hypocontinuous as defined by Bourbaki [7]. This is the crucial observation. It follows by a theorem concerning hypocontinuous bilinear mappings (see [7: Chapter III, §5, 4.]) that $\tilde{\beta}_x$ has a unique separately continuous extension

$$\beta_x : V^\# \times (\widehat{x\mathfrak{B}})^\# \longrightarrow \mathbb{C}.$$

By the above mentioned isomorphisms, this implies that there exist unique linear operators

$$\mathrm{X} \in \mathcal{L}(V^\#, (\widehat{x\mathfrak{B}})^{\#\#}) \qquad \text{and} \qquad \mathrm{Y} \in \mathcal{L}((\widehat{x\mathfrak{B}})^\#, V)$$

such that

$$\beta_x(\hat{v}, \widehat{w}) = \langle \mathrm{X}\,\hat{v}, \widehat{w} \rangle = \langle \hat{v}, \mathrm{Y}\,\widehat{w} \rangle \qquad \text{for all } \hat{v} \in V^\#, \widehat{w} \in (\widehat{x\mathfrak{B}})^\#. \tag{1}$$

In particular,

$$\langle \tilde{\pi}(x)\,\hat{b}, \widehat{w} \rangle \;=\; \langle \mathrm{X}\,\hat{b}, \widehat{w} \rangle \qquad \text{for all } \hat{b} \in \hat{\mathfrak{B}},\ \widehat{w} \in (\widehat{x\mathfrak{B}})^{\#}\,,$$

which gives $\mathrm{X}\!\restriction_{\hat{\mathfrak{B}}} = \tilde{\pi}(x)$ since $(\widehat{x\mathfrak{B}})^{\#}$ separates the points of $(\widehat{x\mathfrak{B}})^{\#\#}$. Furthermore,

$$\mathfrak{N} = \left\{ x \in \mathfrak{B}^{\sharp} : \Omega(x,y) = 0 \ \text{ for all } y \in \mathfrak{B} \right\}$$

implies that the polar of $\widehat{\mathfrak{B}}$ taken in V is $\{0\}$, and this is equivalent to the density of $\widehat{\mathfrak{B}}$ in $V^{\#}$. Thus the continuous extension of $\tilde{\pi}(x)$ is unique. As the embedding $(\widehat{x\mathfrak{B}})^{\#\#} \hookrightarrow V$ is continuous, X can also be considered as an element of $\mathcal{L}(V^{\#}, V)$. Setting $\pi(x)\,\hat{v} = \mathrm{X}\,\hat{v}$ for all $\hat{v} \in V^{\#}$, we obtain the desired continuous extension $\pi(x) : V^{\#} \to V$ of $\tilde{\pi}(x)$.

Third step: We observe that

$$\langle \pi(x)^*\,\hat{b}_1, \hat{b}_2 \rangle \;=\; \langle \hat{b}_1, \pi(x)\,\hat{b}_2 \rangle \;=\; \langle \pi(x^*)\,\hat{b}_1, \hat{b}_2 \rangle \qquad \text{for all } \hat{b}_1, \hat{b}_2 \in \hat{\mathfrak{B}},\ x \in \mathfrak{A}.$$

The first equality follows from the definition of the involution on $Op(V)$; the second equality follows by using $\pi(x)(b + \mathfrak{N}) = xb + \mathfrak{N}$ for all $b \in \mathfrak{B}$, the definition of $\langle \cdot, \cdot \rangle$ and Definition 2.1/(ii). Since $\widehat{\mathfrak{B}}$ separates the points of V and is dense in $V^{\#}$, the above equation implies $\pi(x)^* = \pi(x^*)$.

Fourth step: Suppose $x, y \in \mathfrak{A}$ and $y \in R(\{x\})$. An application of Definition 2.1/(iii) shows that $(\widehat{y\mathfrak{B}}) \subset (\widehat{x^*\mathfrak{B}})^{\#}$ which gives $(\widehat{y\mathfrak{B}})^{\#\#} \subset (\widehat{x^*\mathfrak{B}})^{\#}$. Furthermore, the embedding $(\widehat{y\mathfrak{B}})^{\#\#} \hookrightarrow (\widehat{x^*\mathfrak{B}})^{\#}$ is continuous. Replacing x by y in equation (1) and using the continuity of the embedding $(\widehat{y\mathfrak{B}})^{\#\#} \hookrightarrow (\widehat{x^*\mathfrak{B}})^{\#}$, we find an operator $\mathrm{S}_1 \in \mathcal{L}(V^{\#}, (\widehat{x^*\mathfrak{B}})^{\#})$ such that $\mathrm{S}_1\,\hat{v} = \pi(y)\,\hat{v}$ for all $\hat{v} \in V^{\#}$ (see the final part of the second step). Replacing x by x^* in Equation (1), we find an operator $\mathrm{S}_2 \in \mathcal{L}((\widehat{x^*\mathfrak{B}})^{\#}, V)$ such that

$$\langle \pi(x^*)\,\hat{v}, \widehat{w} \rangle \;=\; \langle \hat{v}, \mathrm{S}_2\,\widehat{w} \rangle \qquad \text{for all } \hat{v} \in V^{\#},\ \widehat{w} \in (\widehat{x^*\mathfrak{B}})^{\#}. \tag{2}$$

Moreover, equation (2) implies $\pi(x^*)^* = \mathrm{S}_2\!\restriction_{V^{\#}}$ which gives, by the third step, $\pi(x)\,\hat{v} = \mathrm{S}_2\,\hat{v}$ for all $\hat{v} \in V^{\#}$. Hence the factorization product $\pi(x) \circ \pi(y)$ is defined and satisfies $\pi(x) \circ \pi(y)\,\hat{v} = \mathrm{S}_2(\mathrm{S}_1\,\hat{v})$ for all $\hat{v} \in V^{\#}$.

Now, for all $\hat{b}_1, \hat{b}_2 \in \widehat{\mathfrak{B}}$ we have

$$\langle \pi(xy)\,\hat{b}_1, \hat{b}_2 \rangle \;=\; \langle \pi(y)\,\hat{b}_1, \pi(x^*)\,\hat{b}_2 \rangle \;=\; \langle \mathrm{S}_2(\mathrm{S}_1\,\hat{b}_1), \hat{b}_2 \rangle \;=\; \langle \pi(x) \circ \pi(y)\,\hat{b}_1, \hat{b}_2 \rangle\,;$$

the first identity is obtained by using Definition 2.1/(iii), and the second identity follows from equation (2). Since $\widehat{\mathfrak{B}}$ separates the points of V and is dense in $V^{\#}$, this implies $\pi(xy) = \pi(x) \circ \pi(y)$, and the proof is complete ∎

Remarks.

1. Let E and F be locally convex spaces. A theorem by Bourbaki [7: Chapter III, §5, 3.] asserts that if F is barrelled, then every separately continuous bilinear form from $E \times F$ into $\mathbb{C}$ is $\mathfrak{S}$-hypocontinuous for any family $\mathfrak{S}$ of bounded subsets of E. An examination of the proof of Proposition 3.3 shows that if the spaces $(\widehat{x\mathfrak{B}})^{\#}$ are

barrelled for all $x \in \mathfrak{A}$, then we can replace the hypothesis that the partial product on $\mathfrak{A}$ is Ω-hypocontinuous with respect to $\mathfrak{B}$ by the statement that all linear functionals $\mathfrak{B} \ni b \longmapsto \Omega(xb, w) \in \mathbb{C}$ are continuous for all $x \in \mathfrak{A}$ and $w \in (x\mathfrak{B})^\sharp$ with respect to $\Sigma(\mathfrak{B}, \mathfrak{B}^\sharp)$.

2. Let $\mathfrak{N}$ satisfy the assertions of Proposition 2.3. If $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B} + \mathfrak{N}$, and if the linear functionals $\mathfrak{B} \ni b \longmapsto \Omega(xb, w) \in \mathbb{C}$ are continuous for all $x \in \mathfrak{A}$ and $w \in (x\mathfrak{B})^\sharp$ with respect to $\Sigma(\mathfrak{B}, \mathfrak{B}^\sharp)$, then the partial product on $\mathfrak{A}$ is at once Ω-hypocontinuous with respect to $\mathfrak{B}$. To see this, set $\mathcal{M} = \{b\}_{b \in \mathfrak{B}}$ and note that $\overline{\{b\}} = \{b + n : n \in \mathfrak{N}\}$.

Using the preceding remarks, we can restate Proposition 3.3 in the following way.

Proposition 3.4. *Let Ω be a $\mathfrak{B}$-weight on a partial *-algebra $\mathfrak{A}$ and suppose that $\mathfrak{N} := \{x \in \mathfrak{B}^\sharp : \Omega(x, y) = 0 \text{ for all } y \in \mathfrak{B}\}$ satisfies the assertions of Proposition 2.3. Let $(V, \#, \langle \cdot, \cdot \rangle)$ denote the non-degenerate PIP-space defined in Proposition 2.3. Assume that the linear functionals $\mathfrak{B} \ni b \longmapsto \Omega(xb, w) \in \mathbb{C}$ are continuous for all $x \in \mathfrak{A}$ and $w \in (x\mathfrak{B})^\sharp$ with respect to $\Sigma(\mathfrak{B}, \mathfrak{B}^\sharp)$. Suppose that one of the following conditions is satisfied:*

(i) $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B} + \mathfrak{N}$.

(ii) $(\widehat{x\mathfrak{B}})^\#$ is barrelled for all $x \in \mathfrak{A}$.

Then there exists a unique linear mapping π from $\mathfrak{A}$ into $Op(V)$ such that $\pi(x)(b + \mathfrak{N}) = xb + \mathfrak{N}$ for all $b \in \mathfrak{B}$ and $x \in \mathfrak{A}$. Furthermore, π satisfies $\pi(x^*) = \pi(x)^*$ for all $x \in \mathfrak{A}$ and $\pi(xy) = \pi(x) \circ \pi(y)$ whenever $y \in R(\{x\})$.

Example 4.4 below will show that we cannot dispense with a hypothesis that ensures in the preceding propositions the existence of the products $\pi(x) \circ \pi(y)$. Here we required the partial product of $\mathfrak{A}$ to be Ω-hypocontinuous with respect to $\mathfrak{B}$. It should be pointed out that this assumption is sufficient but we did not prove that it is necessary; it seems to be rather difficult to give a necessary condition. Observe that the factorization of the product $\pi(x) \circ \pi(y)$ was achieved by proving that $\pi(y) \in \mathcal{L}(V^\#, (\widehat{y\mathfrak{B}})^{\#\#})$ and that $\pi(x)$ admits an extension belonging to $\mathcal{L}((\widehat{x^*\mathfrak{B}})^\#, V)$, but any assaying subspace X such that $(\widehat{y\mathfrak{B}})^{\#\#} \subset X \subset (\widehat{x^*\mathfrak{B}})^\#$, that $\pi(y) \in \mathcal{L}(V^\#, X)$, and that $\pi(x)$ possesses an extension in $\mathcal{L}(X, V)$ factorizes the product $\pi(x) \circ \pi(y)$. A necessary condition would have to control all those assaying subspaces X.

4. Representations based on extended products

In this section, there will be considerd a second approach to GNS-representations of partial *-algebras which uses a more general partial product $T_1 * T_2$ of operators on PIP-spaces. This product is similar to the weak product defined in [4] for certain operators on Hilbert spaces. In contrary to the product considered in Section 3, it is defined only for certain pairs of operators, but not for n-tuples. However, it allows to construct a representation of a partial *-algebra $\mathfrak{A}$ based on a $\mathfrak{B}$-weight Ω, whenever besides the necessary conditions given in Section 2 also the quite general additional condition $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B} + \mathfrak{N}$ is satisfied. An example shows that it is impossible to define operators $\pi(x)$ $(x \in \mathfrak{A})$ as elements of $Op(V)$ in a natural way without any additional

condition. There will also be considered a further partial product $\bullet$, introduced in [14], which plays an intermediate role between the partial products $\circ$ and $*$. Several examples demonstrate properties of these products.

The products $\bullet$ and $*$ are defined as follows.

Definition 4.1. Let $(V, \#, \langle \cdot, \cdot \rangle)$ be a non-degenerate PIP-space. The product $T_2 * T_1$ of two elements of $Op(V)$ is defined if and only if the following two equivalent conditions are satisfied:

(i) There exists a $C \in Op(V)$ such that

$$\langle T_1 \varphi, T_2^* \psi \rangle \;=\; \langle C \varphi, \psi \rangle \qquad \text{(for all } \varphi, \psi \in V^\#\text{).}$$

(ii) There exist linear mappings $C, D : V^\# \to V$ such that

$$\langle T_1 \varphi, T_2^* \psi \rangle \;=\; \langle C \varphi, \psi \rangle \;=\; \langle \varphi, D \psi \rangle \qquad \text{(for all } \varphi, \psi \in V^\#\text{).}$$

In this case
$$T_2 * T_1 \;:=\; C \;.$$

Definition 4.2. Let $(V, \#, \langle \cdot, \cdot \rangle)$ be a non-degenerate PIP-space. The product $T_2 \bullet T_1$ of two elements of $Op(V)$ is defined if and only if there exist assaying subspaces X, Y of V such that the following four conditions are satisfied:

(i) $T_1(V^\#) \subset X$.
(ii) $T_2^*(V^\#) \subset Y$.
(iii) T_2 has a continuous extension $S : X \to V$.
(iv) T_1^* has a continuous extension $R : Y \to V$.

In this case
$$T_2 \bullet T_1 \varphi \;:=\; S(T_1 \varphi) \qquad \text{for } \varphi \in V^\#.$$

Note that $T_2 \bullet T_1$ belongs to $Op(V)$ since its adjoint is given by

$$(T_2 \bullet T_1)^* \varphi \;=\; R(T_2^* \varphi) \qquad \text{for } \varphi \in V^\#.$$

Note also that the existence of $T_2 \circ T_1$ implies $T_2 \bullet T_1 = T_2 \circ T_1$, and that the existence of $T_2 \bullet T_1$ implies $T_2 * T_1 = T_2 \bullet T_1$. Actually, the domains of definition of the three partial products are different. This may be seen by using Example 4.6 below and [12: Examples 3.5 and 4.4].

Now it is not difficult to construct a representation by using the partial product $*$ (cf. [12]). The construction of the representation and some of its properties are described in the following proposition.

Proposition 4.3. *Let Ω be a $\mathfrak{B}$-weight on a partial $*$-algebra $\mathfrak{A}$. Suppose that $\mathfrak{N}$ is a linear subspace of $\mathfrak{B}^\sharp$ such that the assertions of Proposition 2.3 are satisfied and that $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B} + \mathfrak{N}$. Let $(V, \#, \langle \cdot, \cdot \rangle)$ denote the non-degenerate PIP-space defined in Proposition 2.3. Then the formula*

$$\pi(x)(b + \mathfrak{N}) \;=\; xb + \mathfrak{N} \qquad (b \in \mathfrak{B})$$

defines a linear mapping $\pi : \mathfrak{A} \to Op(V)$ *such that* $\pi(x^*) = \pi(x)^*$ *for all* $x \in \mathfrak{A}$. *Moreover,*

$$\pi(x_2 x_1) = \pi(x_2) * \pi(x_1)$$

for all $x_1, x_2 \in \mathfrak{A}$ *with* $x_1 \in R(\{x_2\})$.

Note that, provided assertion (ii) in Proposition 2.3 is satisfied, condition (iv) in Definition 2.1 means that $\{b + \mathfrak{N} : b \in \mathfrak{B}\}$ is a dense linear subspace of $\mathfrak{B}^{\sharp\sharp}/\mathfrak{N} = V^\#$. By the following two examples, this does not imply that there exist operators $\pi(a) \in Op(V)$ satisfying $\pi(a)(b + \mathfrak{N}) = ab + \mathfrak{N}$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

Example 4.4. There exist a partial *-algebra $\mathfrak{A}$, a linear subspace $\mathfrak{B} \subset \mathfrak{A}$, and a $\mathfrak{B}$-weight Ω on $\mathfrak{A}$, such that the spaces $\mathfrak{N}_1$ and $\mathfrak{N}_2$ defined in Proposition 2.3 are equal to $\{0\}$ and such that for some $a \in \mathfrak{A}$ the mapping

$$\mathfrak{B} \ni x \to a\,x \in \mathfrak{B}^\sharp \ \ (= V)$$

does not extend to an element of $Op(V)$.

Indeed, let $\mathfrak{A}$ be the *-algebra ω of all complex valued sequences (with pointwise algebraic operations), let $\mathfrak{B}$ be the *-subalgebra φ of all sequences of finite support, and let Ω be the usual scalar product of $l_2 \subset \omega$. Then, in particular, $\Gamma(\sharp) = l_2 \times l_2$ and it is easy to see that Ω is a $\mathfrak{B}$-weight. Moreover, using the notations of Proposition 2.3, we have $\mathfrak{N}_1 = \mathfrak{N}_2 = \{0\}$ and $V (= \mathfrak{B}^\sharp) = V^\# = l_2$. Consequently, $Op(V)$ coincides with the space of all bounded linear operators on l_2 and the example is completed by taking $a = (a_j)_{j \in \mathbb{N}}$ to be an unbounded sequence.

Example 4.5. There exist a partial *-algebra $\mathfrak{A}$, a linear subspace $\mathfrak{B} \subset \mathfrak{A}$, and a $\mathfrak{B}$-weight Ω on $\mathfrak{A}$, such that the equivalent conditions of Proposition 2.3 are satisfied and that for some $a \in \mathfrak{A}$ the space $\mathfrak{N}_1 = \mathfrak{N}_2$ defined in Proposition 2.3 is not invariant for the mapping

$$\mathfrak{B} \ni x \to a\,x \in \mathfrak{B}^\sharp.$$

This example will be constructed in the space

$$\mathfrak{A} := \mathbb{C}^5 = \big\{(x_1, x_2, x_3, x_4, x_5) : x_j \in \mathbb{C}\big\}$$

endowed with its usual structure of a *-vector space and with the (commutative but non-associative) partial product defined as follows:

$$\mathfrak{C} := \big\{(x_j)_{j=1}^5 \in \mathfrak{A} : x_5 = 0\big\}$$
$$\Gamma := (\mathfrak{C} \times \mathfrak{A}) \cup (\mathfrak{A} \times \mathfrak{C})$$

$$\left.\begin{aligned}
(x_j)_{j=1}^5 \cdot (y_j)_{j=1}^5 := {}& (x_1 y_1, x_2 y_2, x_3 y_3, x_4 y_4, 0) \\
& + (x_5 y_3, 0, x_5 y_1, 0, 0) \\
& + (x_3 y_5, 0, x_1 y_5, 0, 0)
\end{aligned}\right\} \quad \text{for } \big((x_j)_{j=1}^5, (y_j)_{j=1}^5\big) \in \Gamma.$$

Furthermore, we define a $\mathfrak{B}$-weight by setting

$$\mathfrak{B} = \left\{ (x_j)_{j=1}^5 \in \mathfrak{A} : x_3 = x_4 = x_5 = 0 \right\}$$

$$\Gamma(\sharp) = \mathfrak{C} \times \mathfrak{C}$$

$$\Omega\big((x_j)_{j=1}^5, (y_j)_{j=1}^5\big) = \sum_{j=2}^4 x_j \, \overline{y_j} \qquad \text{for } \big((x_j)_{j=1}^5, (y_j)_{j=1}^5\big) \in \Gamma(\sharp).$$

Easy computations show that all properties of Definition 2.1 are satisfied. E.g., it follows for $x = (x_j)_{j=1}^5 \in \mathfrak{A}$, $y = (y_j)_{j=1}^5 \in \mathfrak{C}$, and $b = (b_j)_{j=1}^5, c = (c_j)_{j=1}^5 \in \mathfrak{B}$ that $yc \in \mathfrak{B}$ and that

$$\Omega(xb, yc) \;=\; x_2 \, b_2 \, \overline{y_2} \, \overline{c_2} \;=\; \Omega(b, (x^*y)c).$$

By commutativity, this implies Definition 2.1/(iii).

Clearly, $\mathfrak{B}^\sharp = \mathfrak{B}^{\sharp\sharp} = \mathfrak{C}$ and $\mathfrak{N}_1 = \mathfrak{N}_2 = \mathbb{C} \cdot (1, 0, 0, 0, 0)$. Setting $a = (0, 0, 0, 0, 1)$, we have $a \cdot (1, 0, 0, 0, 0) \notin \mathfrak{N}_1$, which completes the example.

Our final example shows that the representation described in Proposition 4.3 cannot be constructed by using the product on $Op(V)$ defined in Definition 3.1, in general.

Example 4.6. There exist a partial *-algebra $\mathfrak{A}$, a linear subspace $\mathfrak{B} \subset \mathfrak{A}$, and a $\mathfrak{B}$-weight Ω on $\mathfrak{A}$, such that all assumptions of Proposition 4.3 are satisfied and that for some $a \in \mathfrak{A}$ with $a \in R(\{a\})$ the product $\pi(a) \circ \pi(a)$ does not exist in the sense of Definition 3.1.

Let again ω denote the *-algebra of all complex valued sequences $(x_j) = (x_j)_{j \in \mathbb{N}}$ (with pointwise operations). Let φ be the *-subalgebra of all sequences of finite support. Consider elements $a = (a_n)$ and $a^2 = ((a_n)^2)$ of ω, where $(a_n)_{n \in \mathbb{N}}$ is a fixed unbounded sequence of positive real numbers. Define now

$$\left.\begin{aligned}
\mathfrak{A} &= \varphi + \mathbb{C}a + \mathbb{C}a^2 \\
\Gamma &= (\varphi + \mathbb{C}a) \times (\varphi + \mathbb{C}a) \cup \mathfrak{A} \times \varphi \cup \varphi \times \mathfrak{A}.
\end{aligned}\right\}$$

Endowed with the linear operations and the involution induced from ω and with the partial product obtained as the restriction of the product of ω to Γ, $\mathfrak{A}$ is a partial *-algebra. Setting $\mathfrak{B} = \varphi$ and defining $\sharp$ and Ω by

$$\Gamma(\sharp) = \mathfrak{A} \times \varphi \cup \varphi \times \mathfrak{A}$$

$$\Omega((x_j), (y_j)) = \sum_{j=1}^\infty x_j \overline{y_j} \qquad \qquad ((x_j), (y_j)) \in \Gamma(\sharp)) \,,$$

we get a $\mathfrak{B}$-weight Ω on $\mathfrak{A}$. It is easy to see that the assertions of Proposition 2.3 are satisfied for $\mathfrak{N} = \{0\}$ and that $\mathfrak{B}^{\sharp\sharp} = \mathfrak{B}$. Consequently, the PIP-space $(V, \#, \langle \cdot, \cdot \rangle)$ constructed in Proposition 2.3 coincides here with $(\mathfrak{A}, \sharp, \Omega(\cdot, \cdot))$, and all assumptions of Proposition 4.3 are satisfied. Given $(x_j) \in \mathfrak{A}$, the operator $\pi((x_j))$ acts on $V^\# = \mathfrak{B}$ as multiplication operator with the sequence (x_j).

We show that $\pi(a) : V^{\#} \to V^{\#}$ is not continuous. Otherwise it would have a continuous adjoint $A : V \to V$ such that

$$\langle \pi(a)b_1, b_2 \rangle = \langle b_1, A\, b_2 \rangle \qquad \text{(for all } b_1 \in V^{\#}, \, b_2 \in V).$$

This would imply that $A\, a^2 = ((a_j)^3)$, which is impossible.

It can be seen in the same way that $\pi(a)$ does not have a continuous extension to an operator $A : V \to V$. Since the only assaying subspaces are V and $V^{\#}$, this implies that the product $\pi(a) \circ \pi(a)$ does not exist in the sense of Definition 3.1. Clearly, $\pi(a) * \pi(a) = \pi(a^2)$ by Proposition 4.3.

Remarks.

1. Since in the previous example all operators $\pi(x)$ $(x \in \mathfrak{A})$ satisfy $\pi(x)V^{\#} \subset V^{\#}$, it can be seen easily that in this example the equation

$$\pi(x_2 x_1) \;=\; \pi(x_2) \bullet \pi(x_1)$$

is satisfied for all $x_1, x_2 \in \mathfrak{A}$ with $x_1 \in R(\{x_2\})$. However, such a property is not satisfied in the general case, as it can be shown by using [12: Example 4.4].

2. In particular it follows from Example 4.6 that there exist operators T_1 and T_2 on some PIP-space such that $T_2 \bullet T_1$ exists, but $T_2 \circ T_1$ is not defined. Similarly, it can be shown by using [12: Examples 3.5 or 4.4] that there exist operators T_1 and T_2 on some PIP-space such that $T_2 * T_1$ exists, whereas $T_2 \bullet T_1$ is not defined.

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