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A Non-Differentiability Result for the Inversion Operator between Sobolev Spaces

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Abstract. The order of differentiability of the inversion operator $\mathcal{J}$ between certain spaces or manifolds of distributionally differentiable functions is shown to be sharp in the following sense. Up to a certain order k guaranted by inverse function arguments, the operator $\mathcal{J}$ is everywhere differentiable and $\mathcal{J}^{(k)}$ is continuous. On the other hand, $\mathcal{J}$ is nowhere $k+1$ times differentiable.

Keywords: *Inversion operators, differentiability, Sobolev spaces and manifolds*

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1. Introduction

Let $\mathcal{M} = \mathcal{M}_1$ and $\mathcal{M}_2$ be compact C^∞ manifolds without boundary and let n_i be the dimension of $\mathcal{M}_i$ $(i = 1, 2)$; $n = n_1$. For $\frac{n}{2} < s \in \mathbb{N}$ define

$$H^s(\mathcal{M}_1, \mathcal{M}_2) = \left\{ \mathcal{F} : \mathcal{M}_1 \to \mathcal{M}_2 \;\middle|\; \begin{array}{l} \text{for any } x \in \mathcal{M}_1, \psi \circ \mathcal{F} \circ \varphi^{-1} \in H^s(\varphi(U), \mathbb{R}^{n_2}) \\ \text{for any chart } (U, \varphi) \text{ containing } x \\ \text{and any chart } (V, \psi) \text{ at } \mathcal{F}(x) \text{ in } \mathcal{M}_2 \end{array} \right\}$$

and note that $H^s(\mathcal{M}_1, \mathcal{M}_2) \subset C(\mathcal{M}_1, \mathcal{M}_2)$. Following Marsden [7] we briefly recall the manifold structure of $H^s(\mathcal{M}_1, \mathcal{M}_2)$. Denote the tangent bundle of $\mathcal{M}_1$ and $\mathcal{M}_2$ by $T\mathcal{M}_1$ and $T\mathcal{M}_2$, respectively. For each $\mathcal{F} \in H^s(\mathcal{M}_1, \mathcal{M}_2)$ define

$$T_{\mathcal{F}} H^s(\mathcal{M}_1, \mathcal{M}_2) = \left\{ \chi \in H^s(\mathcal{M}_1, T\mathcal{M}_2) : \pi_{\mathcal{M}_2} \circ \chi = \mathcal{F} \right\}$$

and

$$\overline{\exp}_{\mathcal{F}} : T_{\mathcal{F}} H^s(\mathcal{M}_1, \mathcal{M}_2) \to H^s(\mathcal{M}_1, \mathcal{M}_2), \quad \overline{\exp}_{\mathcal{F}}(\mathcal{X}) = \exp \circ \chi$$

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where $\pi_{\mathcal{M}_2} : T\mathcal{M}_2 \to \mathcal{M}_2$ is the canonical projection and $\exp : T\mathcal{M}_2 \to \mathcal{M}_2$ is an exponential map. The properties of the exponential map we need in the sequel are that, for any $y \in \mathcal{M}_2$, $\exp_y = \exp|_{T_y\mathcal{M}_2}$ is a C^∞ diffeomorphism of $T_y\mathcal{M}_2$ onto a neighborhood of y in $\mathcal{M}_2$, and $\exp_y(0) = y$, $(\exp_y)'(0) = \mathrm{id}_{T_y\mathcal{M}_2}$. It can be shown that $\{\overline{\exp}_{\mathcal{F}}\}_{\mathcal{F}\in H^s(\mathcal{M}_1,\mathcal{M}_2)}$ gives rise to a C^∞ atlas on $H^s(\mathcal{M}_1,\mathcal{M}_2)$. For any $\mathcal{F} \in H^s(\mathcal{M}_1,\mathcal{M}_2)$ fixed, $\overline{\exp}_{\mathcal{F}}$ is a C^∞ diffeomorphism of (the Banachable space) $T_{\mathcal{F}}H^s(\mathcal{M}_1,\mathcal{M}_2)$ onto a neighborhood of $\mathcal{F}$ in $H^s(\mathcal{M}_1,\mathcal{M}_2)$, and $\overline{\exp}_{\mathcal{F}}(0) = \mathcal{F}$, $(\overline{\exp}_F)'(0) = \mathrm{id}_{T_{\mathcal{F}}H^s(\mathcal{M}_1,\mathcal{M}_2)}$.

Define

$$\mathcal{D}^s(\mathcal{M}) = \left\{ \mathcal{F} \in H^s(\mathcal{M},\mathcal{M}) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is one-to-one, orientation} \\ \text{preserving and } \mathcal{F}^{-1} \in H^s(\mathcal{M},\mathcal{M}) \end{array} \right\}.$$

From now on we assume that $s > \frac{n}{2} + 1$, which guarantees that $\mathcal{D}^s(\mathcal{M})$ is open in $H^s(\mathcal{M},\mathcal{M})$ and henceforth $\mathcal{D}^s(\mathcal{M})$ is a manifold. Given $\mathcal{F} \in \mathcal{D}^s(\mathcal{M})$ arbitrarily, consider the mapping

$$\mathcal{A}_{\mathcal{F}} : T_{\mathrm{id}}H^s(\mathcal{M},\mathcal{M}) \to T_{\mathcal{F}}H^s(\mathcal{M},\mathcal{M}), \quad \mathcal{A}_F(v) = v \circ \mathcal{F}.$$

For brevity, we write $\mathcal{X}^s(\mathcal{M}) = T_{\mathrm{id}}H^s(\mathcal{M},\mathcal{M})$, the (Banachable) space of H^s vector fields on $\mathcal{M}$. The norm on $\mathcal{X}^s(\mathcal{M})$ will be denoted by $|\cdot|_s$. This norm comes from the standard Sobolev norm $\|\cdot\|_{H^s} = |\cdot|_s$ on coordinate charts. Anticipating Lemma 1/(ii) we see that $\mathcal{A}_{\mathcal{F}}$ is a C^∞ diffeomorphism of $\mathcal{X}^s(\mathcal{M})$ onto $T_{\mathcal{F}}H^s(\mathcal{M},\mathcal{M})$. Hence $\mathcal{D}^s(\mathcal{M})$ is a C^∞ manifold over $\mathcal{X}^s(\mathcal{M})$ and $\{\overline{\exp}_{\mathcal{F}} \circ \mathcal{A}_{\mathcal{F}}\}_{\mathcal{F}\in\mathcal{D}^s(\mathcal{M})}$ gives rise to a C^∞ atlas on $\mathcal{D}^s(\mathcal{M})$.

In the following lemma we collect some basic results on $\mathcal{D}^s(\mathcal{M})$ (for similar results in the category of continuously differentiable functions we refer to Franks [3] and Irwin [4: Appendix]).

Lemma 1.

(i) *$\mathcal{D}^s$ is a group under composition.*

(ii) *If $\mathcal{F} \in \mathcal{D}^s(\mathcal{M})$, then the map*

$$\mathcal{N}(\cdot,\mathcal{F}) : \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^s(\mathcal{M}), \quad \mathcal{N}(\mathcal{G},\mathcal{F}) = \mathcal{G} \circ \mathcal{F}$$

is of class C^∞ (α-lemma).

(iii) *If $\mathcal{F} \in \mathcal{D}^s(\mathcal{M})$, then the map*

$$\mathcal{N}(\mathcal{F},\cdot) : \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^s(\mathcal{M}), \quad \mathcal{N}(\mathcal{F},\mathcal{G}) = \mathcal{F} \circ \mathcal{G}$$

is continuous (ω-lemma global).

(iv) *More generally, the composition or Nemytskii operator*

$$\mathcal{N}(\cdot,\cdot) : \mathcal{D}^{s+k}(\mathcal{M}) \times \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^{s-q}(\mathcal{M}), \quad \mathcal{N}(\mathcal{F},\mathcal{G}) = \mathcal{F} \circ \mathcal{G}$$

is of class C^{k+q}, for any $k \in \mathbb{N}$ and $q \in [0, s - \frac{n}{2} - 1] \cap \mathbb{N}$.

(v) $\mathcal{D}^s(\mathcal{M})$ *is a topological group. In particular, the inversion operator*

$$\mathcal{J} : \mathcal{D}^s(\mathcal{M}) \to H^s(\mathcal{M}, \mathcal{M}), \quad \mathcal{J}(\mathcal{F}) = \mathcal{F}^{-1}$$

is continuous.

(vi) *More generally, the map*

$$\mathcal{J} : \mathcal{D}^{s+k}(\mathcal{M}) \to H^s(\mathcal{M}, \mathcal{M}), \quad \mathcal{J}(\mathcal{F}) = \mathcal{F}^{-1}$$

is of class C^k for any $k \in \mathbb{N}$.

Proof. Proofs for statements (i) - (iii), the case $q = 0$ of statement (iv), and statement (v) can be found in Marsden [7]. The case $q \neq 0$ of statement (iv) can be proved similarily. A simple proof (in the case $k > 0$) of statement (vi) is presented here for convenience. Set

$$\begin{aligned} &\mathcal{R} : \mathcal{X}^{s+k}(\mathcal{M}) \times \mathcal{X}^s(\mathcal{M}) \to \mathcal{X}^s(\mathcal{M}) \\ &\mathcal{R}(v, w) = (\overline{\exp}_{\mathrm{id}} \circ \mathcal{A}_{\mathrm{id}})^{-1} \mathcal{N}\Big(\overline{\exp}_{\mathcal{F}} \circ \mathcal{A}_{\mathcal{F}}(v), \overline{\exp}_{\mathcal{F}^{-1}} \circ \mathcal{A}_{\mathcal{F}^{-1}}(w)\Big). \end{aligned}$$

By a direct calculation, $\mathcal{R}(0,0) = 0$ and

$$\begin{aligned} ([\mathcal{R}'_w(0,0)]w)(x) &= \Big(\exp_x^{-1}\big(\exp_{\mathcal{F}(\exp_{\mathcal{F}^{-1}(x)} w(\mathcal{F}^{-1}(x)))} 0\big)\Big)'_w(x) \\ &= \mathrm{id}_{T_x\mathcal{M}} \cdot \mathcal{F}'(\mathcal{F}^{-1}(x)) \cdot \mathrm{id}_{T_{\mathcal{F}^{-1}(x)}\mathcal{M}} \cdot w(\mathcal{F}^{-1}(x)) \\ &= \mathcal{F}'(\mathcal{F}^{-1}(x)) \cdot w(\mathcal{F}^{-1}(x)) \end{aligned}$$

which shows that $\mathcal{R}'_w(0,0)$ is an invertible element of $L(\mathcal{X}^s(\mathcal{M}), \mathcal{X}^s(\mathcal{M}))$. Differentiability properties of $\mathcal{N}$ (given in the $q = 0$ case of statement (iv)) and the implicit function theorem yield a C^k coordinate representation of $\mathcal{J}$ at $\mathcal{F}$ ∎

The aim of this paper is to show that statement (vi) of Lemma 1 is optimal in the sense that

$$\mathcal{J} : \mathcal{D}^{s+k}(\mathcal{M}) \to H^s(\mathcal{M}, \mathcal{M}), \quad \mathcal{J}(\mathcal{F}) = \mathcal{F}^{-1}$$

is nowhere $k+1$ times differentiable. The corresponding result on manifolds of continuously differentiable functions was proved in our earlier paper [2]. Though the main line of argumentation remains the same, several technical modifications are needed throughout and, reflecting the difference between L^2 and maximum norms, the construction in the later Lemma 8 for proving the result is new.

Theorem 1. *Let $s, k \in \mathbb{N}$ and assume that $s > \frac{n}{2} + 2$. Then the operator*

$$\mathcal{J} : \mathcal{D}^{s+k}(\mathcal{M}) \to H^s(\mathcal{M}, \mathcal{M}), \quad \mathcal{J}(\mathcal{F}) = \mathcal{F}^{-1}$$

is nowhere $k+1$ times differentiable.

The proof is postponed to Section 3 below. Its core is a reductio ad absurdum argument. Assuming $\mathcal{J}$ is $k+1$ times differentiable at some $\mathcal{F}_0 \in \mathcal{D}^{s+k}(\mathcal{M})$, a formula for (coordinate representations of) $\mathcal{J}^{(k+1)}(\mathcal{F}_0)$ is derived. We do this by exploiting the assumption $s > \frac{n}{2} + 2$ to point out that the formula for $\mathcal{J}^{(j)}(\mathcal{F}_0), j = k$ (obtained in Section 2 by standard methods of combinatorial enumeration) remains then valid for $j = k+1$, too. The last step is to demonstrate that the formula for $\mathcal{J}^{(k+1)}(\mathcal{F}_0)$ leads to a contradiction.

We do not know whether the assumption $s > \frac{n}{2} + 2$ can be weakened to $s > \frac{n}{2} + 1$. Topological properties alone do not seem to make inequality $s > \frac{n}{2} + 2$ necessary.

Our basic references for composition, inversion and differentiation are [1, 7, 8, 10]. It is a challenging question to characterize those pairs /scales of Banach/Fréchet spaces/manifolds in which Theorem 1 holds true. It is worth mentioning here that Lemmata 1 - 7 remain valid for Sobolev spaces of fractional order. Thus the extension of Theorem 1 for $s \notin \mathbb{N}$ requires only a construction in proving Lemma 8 that works for any real s satisfying $s > \frac{n}{2} + 2$. (On the other hand, extensions for fractional differentiation seem to be much harder. The $k \notin \mathbb{N}$ version of Lemma 1 seems to be unknown, too.)

Of course, differentiation and differentiability are understood in the sense of Fréchet throughout.

2. Local formulae for the derivatives of $\mathcal{J}$

Let $U \neq \emptyset$ be a bounded open subset of $\mathbb{R}^n$. Define

$$D_U^s = \Big\{ f \in H^s(U, \mathbb{R}^n) : \, f \text{ is invertible with a } H^s \text{ inverse} \Big\}$$

and consider an $f_0 \in D_U^s$ arbitrarily choosen. Assuming ∂U smooth enough, our standing assumption $s > \frac{n}{2} + 1$ implies that $f_0 \in C^1(U, \mathbb{R}^n)$ with $f_0(U)$ open and $\mathrm{cl}(f_0(U))$ compact in $\mathbb{R}^n$. Finally, let V and W be open subsets of $\mathbb{R}^n$ satisfying

$$\emptyset \neq W \subset \mathrm{cl}(W) \subset f_0(U) \subset \mathrm{cl}(f_0(U)) \subset V.$$

Then there is an $\varepsilon > 0$ with the properties as follows. For any $f \in D_U^s$ with $\|f - f_0\|_{H^s} < \varepsilon$ and $g \in D_V^s$, the composition function $g \circ f$ is defined and belongs to D_U^s. We write $g \circ f = N(g, f)$ (the local version of Lemma 1/(iv)). Moreover, for any $f \in D_U^s$ with $\|f - f_0\|_{H^s} < \varepsilon$, there exists a unique $h \in H^s(W, \mathbb{R}^n)$ such that $f \circ h = \mathrm{id}_W$. We write $h = J(f)$ (the local version of Lemma 1/(vi)).

This section is devoted to local properties where domains of the underlying H^s functions play no rule and are omitted. For brevity, we write D^s and H^s instead of D_U^s and $H^s(U, \mathbb{R}^n)$.

Formulae for higher order derivatives of J contain an exponentially growing number of summation terms. In order to write them in a compact form we follow Rybakowski's version [9] of the method of equating coefficients in Taylor expansions for implicitly defined maps and use graphs as summation indices. This approach has been worked out in [2] for the operator J between spaces of continuously differentiable functions. Formulae obtained in [2: Section 2] remain valid in the Sobolev space setting as well. Lemmata 2 - 5 below contain formulae for the higher order derivatives of the operator J between

Sobolev spaces. From the view-point of combinatorial enumeration, Lemmata 2 - 5 are restatements of some results of [2: Section 2] established within the framework of continuously differentiable functions. Fortunately, the original proofs when combined with Theorem 2.1/(ii) (closedness with respect to pointwise multiplication) and Theorem 2.6/(iii) (chain rule formula) of [6] can be repeated in the Sobolev setting. The extra considerations are needed to ensure that the individual summation terms in Lemmata 2 and 3 below make sense and belong to H^s. The final reason for this is the fact that, with pointwise addition and multiplication, $H^s(U,\mathbb{R})$ is a Banach algebra (of continuous functions) for $s > \frac{n}{2}$. This makes density arguments for products possible.

Throughout this paper, graphs are understood as finite "vertex" sets carrying abstract structures like "labelling" and "system of directed edges". Thus different geometric realizations of the same graph are identified. We present the detailed definition of Cayley trees but otherwise use basic terminology of graph theory without any further notice. A Cayley tree of type $\mathcal{C}_j$ is simply a labelled rooted tree on j vertices ($j \in \mathbb{N}$). In other words, a Cayley tree of type $\mathcal{C}_j$ is a triplet $\tau = (V, \lambda, E)$ consisting of an abstract vertex set $V = \{v_1, \ldots, v_j\}$, a labelling $\lambda : V \to \mathbb{N}$ and the set of directed edges $E = \{e_1, \ldots, e_{j-1}\} \subset V \times V$. The defining requirements on λ and E are as follows. By definition, λ is injective. For a fixed $r \in V$ it is required that, given a vertex $v \in V \backslash \{r\}$ arbitrarily, r is the starting point of a directed path that terminates at v. It is easily seen that vertex r, the root of τ, is uniquely determined. Two Cayley trees $\tau = (V, \lambda, E)$ and $\tilde{\tau} = (\tilde{V}, \tilde{\lambda}, \tilde{E})$ of type $\mathcal{C}_j$ are isomorphic if there are bijections $B : V \to \tilde{V}$ and $b : \{\lambda(v_i)\}_{i=1}^{j} \to \{\tilde{\lambda}(\tilde{v}_i)\}_{i=1}^{j}$ such that $\tilde{\lambda}(B(v)) = b(\lambda(v))$ for each $v \in V$ as well as $(v, w) \in E$ if and only if $(B(v), B(w)) \in \tilde{E}$. From now on, letter $\mathcal{C}_j$ stands for a maximal collection of pairwise non-equivalent Cayley trees on j vertices ($j \in \mathbb{N}$). The labelling set is chosen for $\{1, 2, \ldots, j\}$. Consider also

$$\mathcal{R}^1_{1+j} = \Big\{\tau \in \mathcal{C}_{j+1} : \text{ root } r \text{ labelled by } j+1\Big\},$$

the set of Cayley trees on $1 + j$ vertices with a fixed (labelling of the) root.

Consider a Cayley tree $\tau = (V, \lambda, E) \in \mathcal{C}_j$ with $j \leq k$ and let $h_1, \ldots, h_j \in (D^{s+k})^j$. Labelling λ gives rise to a differential assignment according to the rules as follow:

(C-DA1) If the root r is of degree d, then the differential monomial $h^{(d)}_{\lambda(r)}$ is assigned to r.

(C-DA2) If the vertex $v \in V \backslash \{v\}$ is of degree d, then the differential monomial $h^{(d-1)}_{\lambda(v)}$ is assigned to v.

In particular, vertices of degree one are associated with h_l for some $l \in \{1, 2, \ldots, j\}$ as above. This gives the possibility of assigning differential expressions to subtrees via the inductive bracket rules.

(C-DA3) If a subtree τ' with root $v \neq r$ is chosen in such a way that the components $\tau_1, \ldots, \tau_{d-1}$ of the forest $\tau' \backslash \{v \text{ plus adjoint edges}\}$ are already associated with the differential expressions $E_1, \ldots, E_{d-1}$, then the differential expression

$$[h^{(d-1)}_{\lambda(v)}](E_1, \ldots, E_{d-1})$$

is assigned to τ'.

(C-DA4) If components $\tau_1, \dots, \tau_d$ of a forest $\tau \backslash \{r$ plus adjacent edges$\}$ are already associated with the differential expressions $E_1, \dots, E_d$, then the differential expression

$$[d_\tau^{\mathcal{C}}(J; \mathrm{id})](h_1, \dots, h_j) = [h_{\lambda(r)}^{(d)}](E_1, \dots, E_d)$$

is assigned to the tree τ itself.

Now we have

Lemma 2. *Consider the operator* $J : D^{s+k} \to H^s,\ J(f) = f^{-1}$. *Then*

$$[J^{(j)}(\mathrm{id})](h_1, \dots, h_j) = (-1)^j \sum_{\tau \in \mathcal{C}_j} [d_\tau^{\mathcal{C}}(J; \mathrm{id})](h_1, \dots, h_j)$$

whenever $(h_1, \dots, h_j) \in (D^{s+k})^j \quad (j = 0, 1, \dots, k)$.

Similarly, consider a rooted tree $\tau = (V, \lambda, E) \in \mathcal{R}_{1+j}^1$ with $j \le k$ and let $f \in D^{s+k}$ and $(h_1, \dots, h_j) \in (D^{s+k})^j$. For brevity, we write $H_i = h_i \circ f^{-1}$ $(i = 1, \dots, j)$. Labelling λ gives rise to a differential assignment according to the rules as follow:

(R-DA1) If a root r is of degree d, then the differential monomial $(f^{-1})^{(d)}$ is assigned to r.

(R-DA2) If a vertex $v \in V \backslash \{r\}$ is of degree d, then the differential monomial $H_{\lambda(v)}^{(d-1)}$ is assigned to v.

(R-DA3) If a subtree τ' with root $v \ne r$ is chosen in such a way that the components $\tau_1, \dots, \tau_{d-1}$ of forest $\tau' \backslash \{v$ plus adjoint edges$\}$ are already associated with the differential expressions $E_1, \dots, E_{d-1}$, then the differential expression

$$[H_{\lambda(v)}^{(d-1)}](E_1, \dots, E_{d-1})$$

is assigned to τ'.

(R-DA4) If components $\tau_1, \dots, \tau_d$ of a forest $\tau \backslash \{r$ plus adjoint edges$\}$ are already associated with the differential expressions $E_1, \dots, E_d$, then the differential expression

$$[d_\tau^{\mathcal{R}}(J; f)](h_1, \dots, h_j) = [(f^{-1})^{(d)}](E_1, \dots, E_d)$$

is assigned to the tree τ itself.

Lemma 3. *Let* $f \in D^{s+k}$. *Then*

$$[J^{(j)}(f)](h_1, \dots, h_j) = (-1)^j \sum_{\tau \in \mathcal{R}_{1+j}^1} [d_\tau^{\mathcal{R}}(J; f)](h_1, \dots, h_j) \tag{1}$$

whenever $(h_1, \dots, h_j) \in (D^{s+k})^j \quad (j = 0, 1, \dots, k)$.

The case of real functions has the peculiarity that both $J^{(j)}(\mathrm{id})$ and $J^{(j)}(f)$ can be written in a more compact form.

Lemma 4. *Let $n = 1$. Then*

$$[J^{(j)}(\mathrm{id})](h_1, \ldots, h_j) = (-1)^j (h_1 \cdots h_j)^{(j-1)}$$

for $j = 1, 2, \ldots, k$.

Lemma 5. *Let $n = 1$. Then*

$$[J^{(j)}(f)](h_1, \ldots, h_j) = (-1)^j ((f^{-1})' \cdot h_1 \circ f^{-1} \cdots h_j \circ f^{-1})^{(j-1)},$$

for $j = 1, 2, \ldots, k$.

3. Proof of Theorem 1

We prove Theorem 1 by means of a series of lemmas.

Lemma 6. *Assume that $\mathcal{J}$ is $k+1$ times differentiable at some $\mathcal{F}_0 \in \mathcal{D}^{s+k}(\mathcal{M})$. Then $\mathcal{F}_0 \in \mathcal{D}^{s+k+1}(\mathcal{M})$.*

Proof. Let $x_0 \in \mathcal{M}$ be chosen arbitrarily. We will show that $\mathcal{F}_0^{-1}$ is of class H^{s+k+1} near x_0.

By the Whitney embedding theorem we may assume that $\mathcal{M} \subset \mathbb{R}^{2n+1}$. Moreover, by choosing a suitable embedding we may assume that there is an open neighborhood V of x_0 in $\mathcal{M}$ such that $V \subset \mathbb{R}^n \subset \mathbb{R}^{2n+1}$ and $\mathcal{F}_0^{-1}(V) \subset \mathbb{R}^n \subset \mathbb{R}^{2n+1}$, and that our atlas on $\mathcal{M}$ contains the special charts (id_U, U) and $(\mathrm{id}_{\mathcal{F}_0^{-1}(U)}, \mathcal{F}_0^{-1}(U))$ where U is an open subset of V with $x_0 \in U \subset \mathrm{cl}(U) \subset V$. The exponential map is choosen in such a way that

$$\begin{aligned} \exp_y w &= w + y \qquad \text{whenever } y, w + y \in U \\ \exp_y^{-1} w &= w - y \qquad \text{whenever } y, w \in \mathcal{F}_0^{-1}(U). \end{aligned}$$

Consider the coordinate representation of $\mathcal{J}$ at $\mathcal{F}_0$

$$\tilde{\mathcal{J}} : \mathcal{X}^{s+k}(\mathcal{M}) \to \mathcal{X}^s(\mathcal{M}), \quad \tilde{\mathcal{J}}(v) = (\overline{\exp}_{\mathcal{F}_0^{-1}} \circ \mathcal{A}_{\mathcal{F}_0^{-1}})^{-1} \mathcal{J}\ \overline{exp}_{\mathcal{F}_0} \circ \mathcal{A}_{\mathcal{F}_0}(v).$$

By the indirect hypothesis $\tilde{\mathcal{J}}$ is $k+1$ times differentiable at 0.

Next consider

$$\mathcal{X}^{s+k}_{B,\delta}(\mathcal{M}) := \left\{ v \in \mathcal{X}^{s+k}(\mathcal{M}) : v(x) = 0 \text{ if } x \in \mathcal{M} \backslash B, |v|_{s+k} < \delta \right\}$$

and the natural chart representation of $\tilde{\mathcal{J}}|_{\mathcal{X}^{s+k}_{B,\delta}(\mathcal{M})}$, where B is a fixed compact ball in U centered at x_0 and δ is a small positive number we specify below. Writing out the details, set

$$\mathcal{G}(v) = \exp_{\mathcal{F}_0(\cdot)} v(\mathcal{F}_0(\cdot)) \qquad \text{for each } v \in \mathcal{X}^{s+k}(\mathcal{M}).$$

Note that $\mathcal{G}(0) = \mathcal{F}_0$ and $\mathcal{G}(v) \in \mathcal{D}^{s+k}(\mathcal{M})$ for $|v|_{s+k}$ small enough. Using continuity we see there is a positive δ for which $v \in \mathcal{X}^{s+k}_{B,\delta}(\mathcal{M})$ implies

$$(\tilde{\mathcal{J}}(v))(x) = \exp_x^{-1}(\mathcal{G}(v))^{-1}(\mathcal{F}_0(x)) \quad \text{whenever } x \in \mathcal{M}$$

$$\left.\begin{aligned} (\mathcal{G}(v))(x) &= \mathcal{F}_0(x) \\ (\tilde{\mathcal{J}}(v))(x) &= 0 \end{aligned}\right\} \quad \text{whenever } x \in \mathcal{M}\backslash\mathcal{F}_0^{-1}(B)$$

$$\left.\begin{aligned} \mathcal{F}_0(x) + v(\mathcal{F}_0(x)) &\in U \\ (\mathcal{G}(v))^{-1}(\mathcal{F}_0(x)) &\in \mathcal{F}_0^{-1}(U) \end{aligned}\right\} \quad \text{whenever } x \in \mathcal{F}_0^{-1}(B).$$

Since

$$\left.\begin{aligned} v(\mathcal{F}_0(x)) &= 0 \\ (\mathcal{G}(v))^{-1}(\mathcal{F}_0(x)) &= x \end{aligned}\right\} \quad \text{whenever } x \in \mathcal{F}_0^{-1}(U)\backslash\mathcal{F}_0^{-1}(B)$$

we conclude that

$$\begin{aligned} (\mathcal{G}(v))(x) &= \mathcal{F}_0(x) + v(\mathcal{F}_0(x)) \\ (\tilde{\mathcal{J}}(v))(x) &= (\mathcal{G}(v))^{-1}(\mathcal{F}_0(x)) - x \end{aligned}$$

or, equivalently,

$$(\tilde{\mathcal{J}}(v))(x) = \mathcal{F}_0^{-1}(\mathrm{id} + v)^{-1}\mathcal{F}_0(x) - x \qquad \text{for all } x \in \mathcal{F}_0^{-1}(U).$$

Now we pass from $\mathcal{X}^{s+k}_{B,\delta}(\mathcal{M})$ to (the open δ-ball of $C(B,\mathbb{R}^n) \supset$) $H_0^{s+k}(B,\mathbb{R}^n)$, a Sobolev space with vanishing trace on the boundary. By letting

$$(\mathcal{K}(v))(x) = \mathcal{F}_0^{-1}(\mathrm{id} + v)^{-1}\mathcal{F}_0(x) - x \qquad \text{if } x \in \mathcal{F}_0^{-1}(B),$$

a C^k mapping

$$\mathcal{K} : \ H_0^{s+k}(B,\mathbb{R}^n) \hookrightarrow H_0^s(\mathcal{F}_0^{-1}(B),\mathbb{R}^n)$$

is defined (for $|v|_{s+k}$ small enough). The mapping $\mathcal{K}$ decomposes as

$$\mathcal{K} = L_s \circ K \circ L_{s+k}$$

where

$$\begin{aligned} &K : \ H_0^{s+k}(\mathcal{F}_0^{-1}(B),\mathbb{R}^n) \hookrightarrow H_0^s(B,\mathbb{R}^n) \\ &(Kw)(x) = (\mathcal{F}_0 + w)^{-1}(x) - \mathcal{F}_0^{-1}(x) \quad \text{if } x \in B \end{aligned}$$

and

$$(L_j(v))(x) = v(\mathcal{F}_0(x)) \quad \text{whenever } x \in \mathcal{F}_0^{-1}(B), \ v \in H_0^j(B,\mathbb{R}^n) \text{ and } j = s, s+k.$$

Note that L_s and L_{s+k} are linear isomorphisms. In view of $K = L_s^{-1} \circ \mathcal{K} \circ L_{s+k}^{-1}$, the mapping K is of class C^k and the indirect hypothesis implies that K is $k+1$ times differentiable at 0.

Recall that $s > \frac{n}{2} + 2$. As a simple corollary of the case $q = 1$ of Lemma 1/(iv), the operator

$$\tilde{K} = (H_0^s(B, \mathbb{R}^n) \overset{inclusion}{\hookrightarrow} H_0^{s-1}(B, \mathbb{R}^n)) \circ K$$

is $k + 1$ times differentiable. In particular,

$$\tilde{K}^{(k+1)}(0) = (H_0^s(B, \mathbb{R}^n) \overset{inclusion}{\hookrightarrow} H_0^{s-1}(B, \mathbb{R}^n)) \circ K^{(k+1)}(0). \tag{2}$$

Consequently, Lemma 3 applies to $K^{(k+1)}(0)$ and

$$[K^{(k+1)}(0)](w_1, \ldots, w_{k+1}) \in H_0^s(B, \mathbb{R}^n)$$

or, equivalently, by passing to the leading term in (1),

$$[(\mathcal{F}_0^{-1})^{(k+1)}](w_1 \circ \mathcal{F}_0^{-1}, \ldots, w_{k+1} \circ \mathcal{F}_0^{-1}) \in H_0^s(B, \mathbb{R}^n)$$

whenever $w_1, \ldots, w_{k+1} \in H_0^{s+k}(\mathcal{F}_0^{-1}(B), \mathbb{R}^n)$. In fact, all other summation terms in (1) correspond to Cayley graphs for which $d(r)$, the degree of the root, is less than $k + 1$. Correspondingly, the order of each differentiation in those remaining summation terms is not greater than k. Since $H_0^s(B, \mathbb{R})$ is closed under pointwise multiplication, the smoothness properties $\mathcal{F}_0 \in \mathcal{D}^{s+k}(\mathcal{M})$ and $w_1 \circ \mathcal{F}_0^{-1}, \ldots, w_{k+1} \circ \mathcal{F}_0^{-1} \in H_0^{s+k}(B, \mathbb{R}^n)$ imply coordinatewise that all the remaining summation terms belong to $H_0^s(B, \mathbb{R}^n)$.

Next we apply a simplified version of the inverse method of Lanza [5, 6] and conclude that $(\mathcal{F}_0^{-1})^{(k+1)}$ is of class H^s at interior points of B. The $(k+1)$-linear symmetric operator $[(\mathcal{F}_0^{-1})^{(k+1)}]$ can be reconstructed from a carefully chosen finite collection of points on its graph. The $(k+1)$-th order mixed partial derivatives of the coordinate functions of $\mathcal{F}_0^{-1}$ satisfy a system of linear algebraic equations with coefficients in $H_0^{s+k}(\mathcal{F}_0^{-1}(B), \mathbb{R})$ and inhomogenities in $H_0^s(\mathcal{F}_0^{-1}(B), \mathbb{R})$. Locally, at each interior point of B, a density argument implies that all entries of the coefficient matrix can be made C^∞ smooth and the determinant can be made non-zero. Cramer's rule implies that $(\mathcal{F}_0^{-1})^{(k+1)}$ is of class H^s and, a fortiori, $\mathcal{F}_0^{-1}$ is of class H^{s+k+1} near x_0.

This holds true for any $x_0 \in \mathcal{M}$ implying $\mathcal{F}_0 \in \mathcal{D}^{s+k+1}(\mathcal{M})$ ∎

Lemma 7. *We may assume that $\mathcal{F}_0 = \mathrm{id}_{\mathcal{M}}$.*

Proof. This is an easy combination of Lemmata 1 and 6. In fact, consider the identity

$$\mathcal{J}(\mathcal{F}) = \mathcal{N}\big(\mathcal{F}_0, \mathcal{J}(\mathcal{N}(\mathcal{F}, \mathcal{F}_0))\big) \qquad \text{for each } \mathcal{F} \in \mathcal{D}^{s+k}(\mathcal{M}). \tag{3}$$

The inner composition operator $\mathcal{N}$ is understood as a mapping of $\mathcal{D}^{s+k}(\mathcal{M}) \times \{\mathcal{F}_0\}$ to $\mathcal{D}^{s+k}(\mathcal{M})$ and is of class C^∞ in $\mathcal{F}$. On both sides, the operator $\mathcal{J}$ is understood as a mapping of $\mathcal{D}^{s+k}(\mathcal{M})$ to $\mathcal{D}^s(\mathcal{M})$ and is of class C^k. However, the outer composition operator $\mathcal{N}$ is understood as a mapping of $\{\mathcal{F}_0\} \times \mathcal{D}^{s+k}(\mathcal{M})$ to $\mathcal{D}^s(\mathcal{M})$ and is of class C^{k+1}. Since $\mathcal{J}$ is $k+1$ times differentiable at $\mathcal{F}_0$, it follows that each side of (3) is $k+1$ times differentiable at $\mathcal{F} = \mathrm{id}_{\mathcal{M}}$ ∎

Lemma 8. *The operator*

$$\mathcal{J} : \mathcal{D}^{s+k}(\mathcal{M}) \to H^s(\mathcal{M}, \mathcal{M})$$

is not $k+1$ times differentiable at $\mathrm{id}_{\mathcal{M}}$.

Proof. It is enough to show that K, with the special choice $\mathcal{F}_0 = \mathrm{id}_{\mathcal{M}}$, is not $k+1$ times differentiable at 0. By (2), formula (1) applies to $K^{(k+1)}(0)$ just as to all preceding derivatives $K^{(j)}(w)$. We distinguish two cases according $k = 0$ or $k \neq 0$. There is no loss of generality in assuming that $x_0 = 0$ and $B = \{x \in \mathbb{R}^n : |x| \leq 4n\}$.

The Case $k = 0$: Thus $K'(0)w = -w$ for $w \in H_0^s(B, \mathbb{R}^n)$ and, by definition of the derivative as that of a multilinear mapping,

$$|(\mathrm{id} + w)^{-1} - \mathrm{id} + w|_s = o(|w|_s) \tag{4}$$

where (equivalently to the standard norm calculated on the basis of mixed partial derivatives)

$$|w|_s = \left(\sum_{i=0}^{s} \int_B \|w^{(i)}(x)\|_{L((\mathbb{R}^n)^i, \mathbb{R}^n)} dx \right)^{\frac{1}{2}}.$$

Using (2) again, we have

$$|(\mathrm{id} + w)^{-1} - \mathrm{id} + w|_{s-1} = o(|w|_s).$$

We arrive thus at a contradiction to (4) if we construct a sequence $\{W_l\} \subset H_0^s(B, \mathbb{R}^n)$ for which $|W_l|_s = O(a_l)$ and

$$\left\| \left((\mathrm{id} + W_l)^{-1} - \mathrm{id} + W_l \right)^{(s)} \right\|_{L_2(B, L((\mathbb{R}^n)^s, \mathbb{R}^n))} \geq c_1 \cdot a_l \tag{5}$$

where $\{a_l\}$ is some positive zero sequence and c_1 is a positive constant. The construction of W_l can be reduced to the case $n = 1$. In fact, it is enough to construct a sequence $\{w_l\} \subset H_0^s([-4, 4], \mathbb{R})$ satisfying

$$|w_l|_s = O(a_l) \tag{6}$$

and, with some positive constant c_2,

$$\left\| \left((\mathrm{id} + w_l)^{-1} - \mathrm{id} + w_l \right)^{(s)} \right\|_{L_2([-1,1], L(\mathbb{R}^s, \mathbb{R}))} \geq c_2 \cdot a_l. \tag{7}$$

Having done this, we can simply take $w_l = 0$ outside $[-4, 4]$ and set

$$W_l(x) = \mu(x) \cdot \left(w_l(x_1), w_l(x_2), \ldots, w_l(x_n) \right)$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ where $\mu : \mathbb{R}^n \to [0, 1]$ is a C^∞ function with

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \frac{1}{2}B \\ 0 & \text{if } x \notin \frac{3}{4}B. \end{cases}$$

For $l \in \mathbb{N}$ and $x \in [-4, 4]$, we define

$$\alpha_l(x) = \begin{cases} 1 & \text{if } x \in (\frac{i}{l}, \frac{i+1}{l}) \quad (i = -l + 2t, t = 0, 1, \ldots, l-1) \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_l(x) = \int_{-2}^{x} \int_{-2}^{t_1} \cdots \int_{-2}^{t_{s-1}} \alpha_l(t_s)\, dt_s \cdots dt_2 dt_1$$

$$\gamma_l(x) = \nu(x) \cdot \beta_l(x) \quad \left(\nu : [-4, 4] \to [0, 1] \text{ of } C^\infty - \text{type}, \nu(x) = \begin{cases} 1 & \text{if } |x| \le 3 \\ 0 & \text{if } |x| \ge \frac{7}{2}. \end{cases}\right)$$

By construction, $\gamma_l \in H_0^s([-4, 4], \mathbb{R})$. The s-th derivative of γ_l cannot be defined in the classical but only in the distributional sense. Nevertheless, the L_2-representation of $\gamma_l^{(s)}$ can be computed via the pointwise Leibniz rule of differentiation (valid for products of C_0^∞ and H^s functions) and we are justified in writing

$$\gamma_l^{(s)}(x) = \sum_{j=0}^{s-1} \binom{s}{j} \nu^{(s-j)}(x) \beta_l^{(j)}(x) + \nu(x)\alpha_l(x) \qquad (x \in [-4, 4])$$

and in particular, with a positive constant c_3,

$$|\gamma_l^{(j)}(x)| \le c_3 \qquad (x \in [-4, 4];\ j = 0, 1, \ldots, s). \tag{8}$$

In what follows $c_4, \ldots, c_{11}$ stay for positive constants that are independent of l and x (but may depend on s). Observe that $|\gamma_l|_s \le c_4$ and $c_5 \le \gamma_l(x) \le c_6$ whenever $|x| \le \frac{3}{2}$.

Now we are in a position to define w_l. For $1 \le l \in \mathbb{N}$ and $x \in [-4, 4]$, set

$$w_l(x) = \frac{\gamma_l(x)}{c_6 l}.$$

For l sufficiently large, say $l > l_0$, the case $j = 1$ of (8) implies $|w_l'(x)| \le \frac{1}{2}$ for each $x \in [-4, 4]$ and thus $\mathrm{id} + w_l$ is an increasing H^s self-diffeomorphism of $[-4, 4]$. Furthermore,

$$x + \frac{c_5}{c_6 l} \le (\mathrm{id} + w_l)(x) \le x + \frac{1}{l} \qquad \text{whenever } |x| \le \frac{3}{2}.$$

By taking the inverse functions, we obtain the crucial inequality

$$x - \frac{1}{l} \le (\mathrm{id} + w_l)^{-1}(x) \le x - \frac{c_5}{c_6 l} \qquad \text{whenever } |x| \le 1. \tag{9}$$

Choose $a_l = \frac{1}{l}$ and observe that property (6) is satisfied.

For $l > l_0$, define

$$S_l = \bigcup \left\{ \left(\frac{i}{l}, \frac{i}{l} + \frac{c_5}{c_6 l} \right) \Big| \, i = -l + 2t \text{ with } t = 0, 1, \ldots, l-1 \right\}.$$

Since $c_5 < c_6$, S_l is a collection of disjoint intervals in $[-1,1]$ and thus its measure is equal to $\frac{c_5}{c_6}$. Consequently, the proof of inequality (7) reduces to checking

$$\left|\left((\mathrm{id}+w_l)^{-1} - \mathrm{id} + w_l\right)^{(s)}(x)\right| \geq \frac{c_7}{l} \qquad \text{whenever } x \in S_l. \tag{10}$$

We begin by observing that

$$\begin{aligned}
&\left|\left((\mathrm{id}+w_l)^{-1} - \mathrm{id} + w_l\right)^{(s)}(x)\right| \\
&\quad = \left|\left(\left[1 + w_l'(\mathrm{id}+w_l)^{-1}\right]^{-1} - 1 + w_l'\right)^{(s-1)}(x)\right| \\
&\quad = \left|\left(\left[1 - w_l'(\mathrm{id}+w_l)^{-1} + (w_l'(\mathrm{id}+w_l)^{-1})^2 - \dots\right] - 1 + w_l'\right)^{(s-1)}(x)\right| \\
&\quad \geq \left|\left(-w_l'(\mathrm{id}+w_l)^{-1} + w_l'\right)^{(s-1)}(x)\right| - \sum_{k=2}^{\infty}\left|\left((w_l'(\mathrm{id}+w_l)^{-1})^k\right)^{(s-1)}(x)\right|
\end{aligned}$$

for each $x \in [-4,4]$ and $l > l_0$. Since

$$\left((w_l'(\mathrm{id}+w_l)^{-1})^k\right)^{(s-1)} = \frac{1}{(c_6 l)^2}\left((\gamma_l'(\mathrm{id}+w_l)^{-1})^2 \cdot (w_l'(\mathrm{id}+w_l)^{-1})^{k-2}\right)^{(s-1)}$$

for $2 \leq k \in \mathbb{N}$, the polynomial version of the Leibniz rule plus a repeated use of inequalities (8) and $|w_l'(x)| \leq \frac{1}{2}$ show that

$$\sum_{k=2}^{\infty}\left|\left((w_l'(\mathrm{id}+w_l)^{-1})^k\right)^{(s-1)}(x)\right| \leq \frac{c_8}{l^2}\sum_{k=2}^{\infty}\frac{k^{s-1}}{2^{(k-2)-(s-1)}}.$$

Furthermore, by standard manipulations with geometric series,

$$\begin{aligned}
&\left|\left((\mathrm{id}+w_l)^{-1} - \mathrm{id} + w_l\right)^{(s)}(x)\right| \\
&\quad \geq \left|\left(-w_l'(\mathrm{id}+w_l)^{-1} + w_l'\right)^{(s-1)}(x)\right| - \frac{c_9}{l^2} \\
&\quad = \left|\left(-w_l''(\mathrm{id}+w_l)^{-1} \cdot \left[1 + w_l'(\mathrm{id}+w_l)^{-1}\right]^{-1} + w_l''\right)^{(s-2)}(x)\right| - \frac{c_9}{l^2} \\
&\quad \geq \left|\left(-w_l''(\mathrm{id}+w_l)^{-1} + w_l''\right)^{(s-2)}(x)\right| - \frac{c_{10}}{l^2} - \frac{c_9}{l^2} \\
&\quad \vdots \qquad \textit{(inductively)} \\
&\quad \geq \left|-w_l^{(s)}\left((\mathrm{id}+w_l)^{-1}(x)\right) + w_l^{(s)}(x)\right| - \frac{c_{11}}{l^2}
\end{aligned}$$

for each $x \in [-4,4]$ and $l > l_0$. In particular, property (9) and the identification $\beta_l^{(s)} = \alpha_l$ lead to the inequality

$$\begin{aligned}
&\left|\left((\mathrm{id}+w_l)^{-1} - \mathrm{id} + w_l\right)^{(s)}(x)\right| \\
&\quad \geq \left|-w_l^{(s)}\left((\mathrm{id}+w_l)^{-1}(x)\right) + w_l^{(s)}(x)\right| - \frac{c_{11}}{l^2} \\
&\quad = \frac{1}{c_6 l} \cdot \left|-\alpha_l\left((\mathrm{id}+w_l)^{-1}(x)\right) + \alpha_l(x)\right| - \frac{c_{11}}{l^2} \\
&\quad = \frac{1}{c_6 l} - \frac{c_{11}}{l^2}
\end{aligned}$$

whenever $x \in S_l$. For l large enough, inequality (10) follows.

The case $k > 0$: As in the proof of the case $k = 0$ we can assume that $n = 1$. By definition,

$$\sup \frac{\left|\left[K^{(k)}(w) - K^{(k)}(0) - K^{(k+1)}(0)w\right](w_1, \dots, w_k)\right|_s}{|w_1|_{s+k} \cdots |w_k|_{s+k}} = o(|w|_{s+k}) \tag{11}$$

where the supremum is taken over all $0 \neq w_i \in H_0^{s+k}([-4,4], \mathbb{R})$. As in the case $k = 0$ above, (11) reduces to

$$\sup \frac{\left\|\left(\left[K^{(k)}(w) - K^{(k)}(0) - K^{(k+1)}(0)w\right](w_1, \dots, w_k)\right)^{(s)}\right\|_{L_2[-4,4]}}{|w_1|_{s+k} \cdots |w_k|_{s+k}} = o(|w|_{s+k}).$$

To arrive at a contradiction, it is enough to construct a sequence $\{z_l\} \subset H_0^{s+k}([-4,4], \mathbb{R})$ and to find non-zero elements $v_1, \dots, v_{k-1}, q_l \in H_0^{s+k}([-4,4], \mathbb{R})$ such that

$$|z_l|_{s+k} \to 0 \tag{12}$$

but

$$\frac{\left\|\left(\left[K^{(k)}(z_l) - K^{(k)}(0) - K^{(k+1)}(0)z_l\right](v_1, \dots, v_{k-1}, q_l)\right)^{(s)}\right\|_{L_2[-4,4]}}{|z_l|_{s+k} \cdot |q_l|_{s+k}} \not\to 0 \tag{13}$$

as $l \to \infty$.

In what follows $d_1, \dots, d_5$ stay for positive constants that are independent of l and x (but may depend on k and s).

For $1 \leq l \in \mathbb{N}$ and $|x| \leq 4$, we set $z_l(x) = \nu(x) \cdot \frac{1+x}{l}$ and observe that, for l sufficiently large, say $l > l_0$, $(\mathrm{id} + z_l)^{-1}$ is a C^∞ self-diffeomorphism of $[-4,4]$,

$$(\mathrm{id} + z_l)^{-1}(x) = \frac{x - \frac{1}{l}}{1 + \frac{1}{l}} \text{ if } |x| < 1 \qquad \text{and} \qquad \frac{d_1}{l} \leq |z_l|_{s+k} \leq \frac{d_2}{l}. \tag{14}$$

Thus (12) is satisfied. Similarly, set

$$v_i(x) = \nu(x) \qquad \text{whenever } |x| \leq 4 \text{ and } i = 1, \dots, k-1.$$

The definition of q_l requires a little more care. Consider a 3-periodic C^∞ function $Q : \mathbb{R} \to \mathbb{R}$ with the property that $Q(y) = y^{k+s+1}$ for each $y \in [-1,1]$ and set

$$q_l(x) = \nu(x) \cdot l^{-k-s} Q(lx) \qquad \text{whenever } |x| \leq 4 \text{ and } l > l_0.$$

It is readily checked that $\|q_l^{(j)}\|_{L_2[-4,4]} \to 0$ as $l \to \infty$ for $j = 0, \dots, k+s-1$ and, using the 3-periodicity of Q,

$$\|q_l^{(j)}\|^2_{L^2[-4,4]} \geq \int_{-3}^{3} |Q^{(j)}(lx)|^2 dx = \int_{-3}^{3} |Q^{(j)}(y)|^2 dy$$

for $j = k + s$. Together with an easy upper estimate, we conclude that

$$d_3 \le |q_l|_{s+k} \le d_4. \tag{15}$$

In view of (14) and (15) we see that (13) is implied by the property

$$l \cdot \Big\| \Big(\big[K^{(k)}(z_l) - K^{(k)}(0) - K^{(k+1)}(0) z_l \big] (v_1, \dots, v_{k-1}, q_l) \Big)^{(s)} \Big\|_{L_2[-1,1]} \not\to 0 \tag{16}$$

as $l \to \infty$.

For each $x \in [-1, 1]$, the particularly simple form of z_l and of $v_1, \dots, v_{k-1}$ implies via Lemmata 4 and 5 that

$$\begin{aligned}
&\Big([K^{(k)}(z_l)](v_1, \dots, v_{k-1}, q_l) \Big)^{(s)}(x) \\
&\quad = (-1)^k \Big(((\mathrm{id} + z_l)^{-1})' \cdot q_l((\mathrm{id} + z_l)^{-1}) \Big)^{(k-1+s)}(x) \\
&\quad = \frac{(-1)^k}{(1 + \frac{1}{l})^{k+s}} \, q_l^{(k+s-1)} \Big(\frac{x - \frac{1}{l}}{1 + \frac{1}{l}} \Big) \\
&\quad = \frac{(-1)^k}{(1 + \frac{1}{l})^{k+s}} \cdot \frac{1}{l} Q^{(k+s-1)} \Big(\frac{lx - 1}{1 + \frac{1}{l}} \Big)
\end{aligned}$$

and

$$\Big([K^{(k)}(0)](v_1, \dots, v_{k-1}, q_l) \Big)^{(s)}(x) = (-1)^k q_l^{(k-1+s)}(x) = (-1)^k \frac{1}{l} Q^{(k-1+s)}(lx)$$

and

$$\begin{aligned}
&\Big([K^{(k+1)}(0) z_l](v_1, \dots, v_{k-1}, q_l) \Big)^{(s)}(x) \\
&\quad = (-1)^{k+1} (z_l \cdot q_l)^{(k+s)}(x) \\
&\quad = (-1)^{k+1} \Big(\frac{1 + x}{l} \, q_l^{(k+s)}(x) + \frac{k + s}{l} \, q_l^{(k+s-1)}(x) \Big) \\
&\quad = (-1)^{k+1} \Big(\frac{1 + x}{l} \, Q^{(k+s)}(lx) + \frac{k + s}{l^2} \, Q^{(k+s-1)}(lx) \Big).
\end{aligned}$$

It follows immediately that (16) is a direct consequence of the slightly stronger property

$$\int_{-1}^{1} \Big| \frac{1}{(1 + \frac{1}{l})^p} \cdot Q^{(p-1)} \Big(\frac{lx - 1}{1 + \frac{1}{l}} \Big) - Q^{(p-1)}(lx) + (1 + x) Q^{(p)}(lx) \Big|^2 dx \ge d_5 \tag{17}$$

where $p = k + s > 3$ and $l > l_p = 10^p$.

We fix parameters $\gamma = \gamma(p) \in [\frac{9}{10}, 1)$ and $\Gamma = \Gamma(p) \in (0, \frac{1}{10}]$ in such a way that the inequality

$$\frac{\gamma^2}{2^p} - \Big(1 - \gamma + 3\Gamma + \frac{3}{l_p} \Big)^2 - 4 \Big(1 - \gamma + 3\Gamma + \frac{3}{l_p} \Big) \ge \frac{\frac{1}{2}}{2^p} \tag{18}$$

holds true. With $[\Gamma l]$ denoting the integer part of Γl, we define

$$T_l = \bigcup \Big\{ \Big(\frac{3i}{l} + \frac{3i-1}{l^2}, \frac{3i}{l} + \frac{1-\gamma}{l} + \frac{3i-\gamma}{l^2} \Big) \Big| \, i = 1, 2, \ldots, [\Gamma l] + 1 \Big\} \quad (l > l_p).$$

Observe that T_l is a collection of disjoint intervals in $[0,1]$ and that the measure of T_l is at least $([\Gamma l]+1)\frac{1-\gamma}{l} > \Gamma(1-\gamma)$. It is crucial that $x \in T_l$ is equivalent to

$$3i - 1 < \frac{lx-1}{1+\frac{1}{l}} < 3i - \gamma \qquad \text{for some } i \in \{1, 2, \ldots, [\Gamma l] + 1\}.$$

Consequently, for each $x \in T_l \;\; (l > l_p)$ we have

$$\begin{aligned}
&\Big| \frac{1}{(1+\frac{1}{l})^p} \cdot Q^{(p-1)}\Big(\frac{lx-1}{1+\frac{1}{l}}\Big) - Q^{(p-1)}(lx) + (1+x)Q^{(p)}(lx) \Big| \\
&\quad \geq \frac{1}{2^p} \Big| Q^{(p-1)}\Big(\frac{lx-1}{1+\frac{1}{l}}\Big)\Big| - |Q^{(p-1)}(lx)| - 2|Q^{(p)}(lx)| \\
&\quad \geq \frac{1}{2^p}\frac{(p+1)!}{2!}\gamma^2 - \frac{(p+1)!}{2!}\Big(\Big|1 - \gamma + \frac{3([\Gamma l]+1)}{l}\Big|^2 - 4\Big|1 - \gamma + \frac{3([\Gamma l]+1)}{l}\Big| \Big) \\
&\quad \geq \frac{(p+1)!}{2!}\Big(\frac{\gamma^2}{2^p} - \Big(1 - \gamma + 3\Gamma + \frac{3}{l_p}\Big)^2 - 4\Big(1 - \gamma + 3\Gamma + \frac{3}{l_p}\Big)\Big).
\end{aligned}$$

As a direct consequence of (18), inequality (17) and, a fortiori, (16) and (13) follow ∎

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