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# Existence Results for the Equation $-\Delta u = f(x, u)$ in $R^n$

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**Abstract.** Strong solutions to the class of semilinear elliptic equations $-\Delta u = f(x, u)$ on the entire space and with possibly supercritical growth for $f(x, \cdot)$ are obtained by mainly using fixed points arguments. The case of discontinuous non-linearities is then examined.

**Keywords:** *Elliptic equations in the whole space, discontinuous non-linearities, strong solutions*

**AMS subject classification:** 35 J 60, 35 R 70

## 0. Introduction

Let $n$ be a positive integer, $n \geq 3$, and let $p \in (1, \frac{n}{2})$. According to [14], write $p^* = \frac{np}{n-p}$ and $p^{**} = \frac{np}{n-2p}$, and denote by $\hat{H}_0^{2,p}(\mathbb{R}^n)$ the space of all $u \in L^{p^{**}}(\mathbb{R}^n)$ such that $u_{x_i} \in L^{p^*}(\mathbb{R}^n)$ and $u_{x_i x_j} \in L^p(\mathbb{R}^n)$ $(1 \leq i, j \leq n)$ where, as usual, derivatives are understood in weak sense.

Given a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfying the standard Carathéodory conditions, consider the semilinear elliptic problem

$$u \in \hat{H}_0^{2,p}(\mathbb{R}^n), \qquad -\Delta u = f(x, u) \quad \text{in } \mathbb{R}^n. \tag{P$_1$}$$

Equations of the above type have been widely investigated in the last decade, for the most part through variational techniques and often assuming that $z \mapsto f(x, z)$ has a subcritical growth or is of a very special form. Let us mention the recent papers [6, 9] as a general reference on the subject. A different approach, which chiefly employs fixed point arguments, is taken here. Indeed, exploiting the bijectivity of the operator $-\Delta : \hat{H}_0^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ (see [14: Chapter III]), we first reduce problem (P$_1$) to the following

$$v \in L^p(\mathbb{R}^n), \qquad v = f(x, G(v)) \quad \text{in } \mathbb{R}^n \tag{P$_2$}$$

where $G = (-\Delta)^{-1}$. Next, we apply a simple result (Lemma 1.1) concerning the weak convergence of sequences in the space $\hat{H}_0^{2,p}(\mathbb{R}^n)$ together with a modified version

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(Lemma 1.3) of the classical Schauder-Tychonoff fixed point theorem to solve problem ($P_2$). This method yields hypotheses on the non-linearity $f$ that are rather different from those usually adopted; vide Theorem 2.1 below. As an example, no subcritical growth condition for $f(x,\cdot)$ is required, but only that the growth rate turn out less than $\frac{p^{**}}{p}$. Such exponent is obviously greater than the critical one (namely, $p^*$) every time that $n \le 5$ or $n > 5$ and $p > \frac{1}{4}(n+1+\sqrt{n^2-6n+1})$. For the sake of completeness we then examine, in Theorem 2.2 and Remark 2.1, the case when the function $(x,z) \mapsto f(x,z)$ is bounded, but not necessarily continuous, with respect to $z$. Finally, the existence of non-negative solutions is also investigated through a representation formula (proved in Lemma 1.2) for the linear equation corresponding to problem ($P_1$).

## 1. Preliminaries

Let $n$ be a positive integer, $n \ge 3$. Given a number $r > 0$, the symbol $B_r$ indicates the open ball in $\mathbb{R}^n$ of radius $r$ centered at zero. Moreover, 'measurable' always means Lebesgue measurable and $|E|$ stands for the measure of the set $E$.

Now, let $p \in (1, \frac{n}{2})$. Write $p^* = \frac{np}{n-p}$ and $p^{**} = \frac{np}{n-2p}$, and denote by $\hat{H}_0^{2,p}(\mathbb{R}^n)$ the space of all $u \in L^{p^{**}}(\mathbb{R}^n)$ such that

$$\frac{\partial u}{\partial x_i} \in L^{p^*}(\mathbb{R}^n) \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^n) \qquad (1 \le i,j \le n)$$

where, as usual, derivatives are understood in weak sense. If $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$, define

$$|u|_{2,p} = \left( \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_p^p \right)^{\frac{1}{p}}.$$

Owing to [14: Chapter III/Theorem 2.21], the following assertions hold:

**($a_1$)** $(\hat{H}_0^{2,p}(\mathbb{R}^n), |\cdot|_{2,p})$ *is a reflexive Banach space.*
**($a_2$)** $C_0^\infty(\mathbb{R}^n)$ *turns out dense in* $(\hat{H}_0^{2,p}(\mathbb{R}^n), |\cdot|_{2,p})$.
**($a_3$)** *There exists* $a > 0$ *satisfying* $\|u\|_{p^{**}} + \|\nabla u\|_{p^*} \le a\,|u|_{2,p}$ *for all* $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$.

**Remark 1.1.** Indicate with $c_q$ ($1 \le q < n$) the best value of the constant that appears in Sobolev's inequality [3: Theorem IX.9]. It is possible to set $a = c_p(1 + c_{p^*})\, n^{\frac{1}{p^*}}$, as a simple computation shows. Thus, since $c_q$ is known [15: Theorem], the same holds regarding $a$.

**Lemma 1.1.** *If* $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$, $\{u_h\} \subseteq \hat{H}_0^{2,p}(\mathbb{R}^n)$, *and* $\lim_{h\to\infty} u_h = u$ *weakly in* $\hat{H}_0^{2,p}(\mathbb{R}^n)$, *then* $\{u_h\}$ *has a subsequence which converges almost everywhere in* $\mathbb{R}^n$ *to the function* $u$.

**Proof.** From assertion ($a_3$) we easily infer that $\{u_h\}$ is bounded in $W^{1,p^*}(B_r)$ for any $r > 0$. Pick $r = 1$. Using the Rellich-Kondrachov theorem [3: Theorem IX.16] yields a subsequence $\{u_h^{(1)}\}$ of $\{u_h\}$ such that $\lim_{h\to\infty} u_h^{(1)}(x) = u(x)$ at almost all points $x \in B_1$. We now apply this argument again, with 1 replaced by 2, to obtain a subsequence $\{u_h^{(2)}\}$ of $\{u_h^{(1)}\}$ satisfying $\lim_{h\to\infty} u_h^{(2)}(x) = u(x)$ almost everywhere in $B_2$. And so on. The sequence $\{u_h^{(h)}\}$ clearly complies with the conclusion ∎

By assertion ($a_2$) and the Calderón-Zygmund inequality [5: p. 413/Corollary 2] there exists $b > 0$ such that $|u|_{2,p} \le b\,\|\Delta u\|_p$ for all $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$. Since, owing to [14: Chapter III/Theorem 4.6], the operator $-\Delta : \hat{H}_0^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a continuous bijection, the inverse operator $G = (-\Delta)^{-1}$ turns out linear, bijective, and continuous. Moreover, bearing in mind the preceding inequality, $\|G\| \le b$.

**Remark 1.2.** The constant $b$ can be explicitly estimated. Indeed, it is related to the norm of M. Riesz's transformation in $L^p(\mathbb{R}^n)$, which has been evaluated in [13: p. 177 – 180].

**Lemma 1.2.** *For every $v \in L^p(\mathbb{R}^n)$ one has*

$$G(v)(x) = \frac{1}{n(n-2)\,\omega_n} \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^{n-2}}\, dy \qquad (x \in \mathbb{R}^n)$$

*where $\omega_n$ indicates the volume of the unit ball in $\mathbb{R}^n$.*

**Proof.** From [12: Theorem 71.II] it follows that $G(v) \in L^{p^{**}}(\mathbb{R}^n)$. Define, whenever $1 \le i \le n$,

$$w_i(x) = -\frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x_i - y_i)v(y)}{|x-y|^n}\, dy \qquad (x \in \mathbb{R}^n).$$

[12: Theorem 72.I] produces $w_i \in L^{p^*}(\mathbb{R}^n)$, while [12: Theorem 77.III] leads to $\frac{\partial w_i}{\partial x_j} \in L^p(\mathbb{R}^n)$ $(1 \le j \le n)$. Let $\{v_h\} \subseteq C_0^\infty(\mathbb{R}^n)$ satisfy

$$\lim_{h\to\infty} \|v_h - v\|_p = 0 \tag{1}$$

and let

$$u_h(x) := \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{v_h(y)}{|x-y|^{n-2}}\, dy \qquad (h \in \mathbb{N}, x \in \mathbb{R}^n). \tag{2}$$

Through the above-mentioned results we also get $\lim_{h\to\infty} \|u_h - G(v)\|_{p^{**}} = 0$ as well as $\lim_{h\to\infty} \|\frac{\partial u_h}{\partial x_i} - w_i\|_{p^*} = 0$. Now, if $\phi \in C_0^\infty(\mathbb{R}^n)$, one has

$$\left\langle \frac{\partial G(v)}{\partial x_i}, \phi \right\rangle = -\lim_{h\to\infty} \int_{\mathbb{R}^n} u_h(x) \frac{\partial \phi(x)}{\partial x_i}\, dx = \lim_{h\to\infty} \int_{\mathbb{R}^n} \frac{\partial u_h(x)}{\partial x_i} \phi(x)\, dx = \langle w_i, \phi \rangle,$$

namely $\frac{\partial G(v)}{\partial x_i} = w_i$ for all $i$. Consequently, $G(v) \in \hat{H}_0^{2,p}(\mathbb{R}^n)$.

To achieve the conclusion we note that, by the Calderón-Zygmund inequality and the completeness of $L^p(\mathbb{R}^n)$, for any fixed $i, j \in \{1, 2, \dots, n\}$ the sequence $\{\frac{\partial^2 u_h}{\partial x_i \partial x_j}\}$ converges in $L^p(\mathbb{R}^n)$ to a function $w_{ij}$, while arguing as before yields $\frac{\partial^2 G(v)}{\partial x_i \partial x_j} = w_{ij}$. Therefore, because of (1) and (2), $-\Delta G(v)(x) = v(x)$ at almost all $x \in \mathbb{R}^n$ ∎

**Remark 1.3.** Owing to the preceding lemma, the conditions $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$ and $\Delta u(x) \le 0$ almost everywhere in $\mathbb{R}^n$ imply $u(x) \ge 0 \ \ (x \in \mathbb{R}^n)$.

The next useful version [1: Theorem 1] of the classical Schauder-Tychonoff fixed point theorem will applied.

**Lemma 1.3.** *Let $X$ be a metrizable locally convex topological vector space and let $K$ be a non-empty, weakly compact, convex subset of $X$. Suppose $T : K \to K$ is a function with weakly sequentially closed graph. Then there exists $x_0 \in K$ such that $x_0 = T(x_0)$.*

We shall also employ the following result, which represents a very special case of [2: Theorem 3.1]. As usual, $M(\mathbb{R}^n)$ denotes the family of all (equivalence classes of) measurable functions $w : \mathbb{R}^n \to \mathbb{R}$; a multifunction $F$ from $\mathbb{R}^n$ into $\mathbb{R}$ is called *measurable* provided for each open set $A \subseteq \mathbb{R}$ the set $\{x \in \mathbb{R}^n : F(x) \cap A \neq \emptyset\}$ is measurable; we say that $F$ has a closed graph if the set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \in F(x)\}$ is closed in $\mathbb{R}^n \times \mathbb{R}$.

**Lemma 1.4.** *Let $U$ be a non-empty set, let $\Phi : U \to M(\mathbb{R}^n)$ and $\Psi : U \to L^p(\mathbb{R}^n)$ be two operators, let $F$ be a multifunction from $\mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}$ with non-empty, convex, closed values. Assume the following:*

- **($\mathbf{i}_1$)** *$\Psi$ is bijective and, whenever $\lim_{h\to\infty} v_h = v$ weakly in $L^p(\mathbb{R}^n)$, there exists a subsequence of $\{\Phi(\Psi^{-1}(v_h))\}$ which converges to $\Phi(\Psi^{-1}(v))$ at almost all points of $\mathbb{R}^n$.*
- **($\mathbf{i}_2$)** *The set $\{z \in \mathbb{R} : F(\cdot, z)$ is measurable$\}$ is dense in $\mathbb{R}$.*
- **($\mathbf{i}_3$)** *$F(x, \cdot)$ has a closed graph for almost every $x \in \mathbb{R}^n$.*
- **($\mathbf{i}_4$)** *There exists a function $m \in L^p(\mathbb{R}^n)$ such that $F(x,z) \subseteq [-m(x), m(x)]$ for almost all $x \in \mathbb{R}^n$ and each $z \in \mathbb{R}$.*

*Then the problem $u \in U, \Psi(u)(x) \in F(x, \Phi(u)(x))$ almost everywhere in $\mathbb{R}^n$ possesses at least one solution.*

## 2. Existence theorems

Keep the notation of Section 1 and set $\hat{q} = \frac{np}{n-q(n-2p)}$ if $q \in (0, \frac{n}{n-2p})$. The main result of the present paper is the following.

**Theorem 2.1.** *Let $f$ be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}$. Suppose the following:*

- **($\mathbf{b}_1$)** *$f(\cdot, z)$ is measurable for all $z \in \mathbb{R}$.*
- **($\mathbf{b}_2$)** *$f(x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^n$.*
- **($\mathbf{b}_3$)** *There exist $\alpha \in L^p(\mathbb{R}^n)$, $q \in (0, \frac{n}{n-2p})$, and $\beta \in L^{\hat{q}}(\mathbb{R}^n)$ such that*

$$|f(x,z)| \le \alpha(x) + \beta(x)\,|z|^q \tag{3}$$

*for almost all $x \in \mathbb{R}^n$ and each $z \in \mathbb{R}$. Furthermore, when both $\|\beta\|_{\hat{q}} > 0$ and $q \ge 1$, it results $a\|G\|\|\beta\|_{\hat{q}} < 1$ or*

$$\|\alpha\|_p \le \frac{q-1}{q}\Big(\frac{1}{q(a\|G\|)^q\|\beta\|_{\hat{q}}}\Big)^{\frac{1}{q-1}} \tag{4}$$

*according to whether $q = 1$ or $q > 1$.*

*Then the equation $-\Delta u = f(x, u)$ in $\mathbb{R}^n$ has at least one solution $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$, which turns out non-negative provided $f$ is so.*

**Proof.** Define, for every $v \in L^p(\mathbb{R}^n)$,

$$T(v)(x) = f(x, G(v)(x)) \qquad (x \in \mathbb{R}^n).$$

Assumptions ($b_1$) and ($b_2$) guarantee that the function $T(v)$ is measurable; hypethesis ($b_3$) combined with the Hölder inequality [3: p. 57/Remarque 2] immediately lead to $T(v) \in L^p(\mathbb{R}^n)$. Therefore, $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

Let us first show that there exists $r > 0$ fulfilling $T(K_r) \subseteq K_r$ where

$$K_\rho := \{v \in L^p(\mathbb{R}^n) : \|v\|_p \le \rho\} \qquad (\rho > 0).$$

If $v \in K_\rho$, then by (3), the Hölder inequality and assertion ($a_3$) of Section 1 we get

$$\begin{aligned} \|T(v)\|_p &\le \|\alpha\|_p + \|\beta\|_{\hat{q}} \|G(v)\|_{p^{**}}^q \\ &\le \|\alpha\|_p + (a|G(v)|_{2,p})^q \|\beta\|_{\hat{q}} \\ &\le \|\alpha\|_p + (a\|G\|)^q \|\beta\|_{\hat{q}}\, \rho^q. \end{aligned} \tag{5}$$

Thus, when $q < 1$ the conclusion follows at once from

$$\lim_{\rho \to +\infty} \big( \|\alpha\|_p + (a\|G\|)^q \|\beta\|_{\hat{q}} \rho^q - \rho \big) = -\infty$$

whereas for $q = 1$ it is a simple consequence of the hypothesis $a\|G\|\,\|\beta\|_{\hat{q}} < 1$. Suppose next $q > 1$ and $\|\beta\|_{\hat{q}} > 0$, and set

$$r = \Big( \frac{1}{q(a\|G\|)^q \|\beta\|_{\hat{q}}} \Big)^{\frac{1}{q-1}}. \tag{6}$$

Using (4) – (6) yields

$$\|T(v)\|_p \le \frac{q-1}{q} r + (a\|G\|)^q \|\beta\|_{\hat{q}}\, r^q = r,$$

namely $T(v) \in K_r$. Since $v$ was arbitrary, $T(K_r) \subseteq K_r$.

We now claim that the function $T|_{K_r}$ has a weakly sequentially closed graph. To prove this, pick two sequences $\{v_h\}, \{w_h\} \subseteq K_r$ satisfying the conditions

$$w_h = T(v_h) \;\; (h \in \mathbb{N}), \qquad \lim_{h \to \infty} v_h = v, \qquad \lim_{h \to \infty} w_h = w \;\; \text{weakly in } L^p(\mathbb{R}^n). \tag{7}$$

Evidently, $K_r$ is convex, closed, and bounded. Owing to the reflexivity of $L^p(\mathbb{R}^n)$, it turns out weakly (sequentially) compact. Hence, $v, w \in K_r$. The properties of $G$ guarantee that $\lim_{h \to \infty} G(v_h) = G(v)$ weakly in $\hat{H}_0^{2,p}(\mathbb{R}^n)$. Because of Lemma 1.1, and taking a subsequence if necessary, we obtain $\lim_{h \to \infty} G(v_h)(x) = G(v)(x)$ at almost all points $x \in \mathbb{R}^n$. Therefore, by assumptions ($b_1$) and ($b_2$), the sequence $\{T(v_h)\}$ converges almost everywhere in $\mathbb{R}^n$ to $T(v)$. Since $\{T(v_h)\} \subseteq K_r$, combining [7: Theorem 13.44] with (7) produces $w = T(v)$.

We have thus proved that $T|_{K_r}$ satisfies the hypotheses of Lemma 1.3. So, there exists $v \in K_r$ such that $v = T(v)$. The function $u := G(v)$ lies in $\hat{H}_0^{2,p}(\mathbb{R}^n)$ and one has $-\Delta u(x) = f(x, u(x))$ for almost all $x \in \mathbb{R}^n$. Finally, if $f$ is non-negative, then due to Lemma 1.2 the same holds regarding the solution $u$ ∎

The next result treats the case when $(x, z) \mapsto f(x, z)$ is bounded, but not necessarily continuous, with respect to $z$. Write, for $(x, z) \in \mathbb{R}^n \times \mathbb{R}$,

$$\underline{f}(x, z) = \lim_{\delta \to 0^+} \inf_{|\zeta - z| \leq \delta} f(x, \zeta) \quad \text{and} \quad \overline{f}(x, z) = \lim_{\delta \to 0^+} \sup_{|\zeta - z| \leq \delta} f(x, \zeta).$$

It is a simple matter to check that $\underline{f}(x, \cdot)$ and $\overline{f}(x, \cdot)$ are, respectively, lower semicontinuous and upper semicontinuous.

**Theorem 2.2.** *Suppose the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies assumption $(b_3)$ of Theorem 2.1 with $\beta = 0$ and, moreover,*

**(b$_4$)** *the set $\{z \in \mathbb{R} : \underline{f}(\cdot, z), \overline{f}(\cdot, z)$ are measurable$\}$ is dense in $\mathbb{R}$.*

*Then there exists $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$ such that*

$$\underline{f}(x, u(x)) \leq -\Delta u(x) \leq \overline{f}(x, u(x)) \qquad \text{for almost every } x \in \mathbb{R}^n. \tag{8}$$

*Furthermore, the function $u$ turns out non-negative provided $f$ is so.*

**Proof.** Let us verify briefly the hypotheses of Lemma 1.4. To this end, choose $U = \hat{H}_0^{2,p}(\mathbb{R}^n)$, $\Phi(u) = u$ and $\Psi(u) = -\Delta u$ $(u \in U)$ as well as $F(x, z) = [\underline{f}(x, z), \overline{f}(x, z)]$ for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}$. We already know that the operator $\Psi$ is bijective while $\Psi^{-1}$ is linear and continuous. Hence, by Lemma 1.1, condition (i$_1$) in Lemma 1.4 holds. Using assumption (b$_4$), [10: Proposition 1.1], and [8: Corollary 4.2] we then easily realize that the multifunction $F$ complies with (i$_2$). Since a standard argument (vide for instance [4: Example 1.3]) yields (i$_3$), while (i$_4$) comes immediately from the inequality $|f(x, z)| \leq \alpha(x)$ $((x, z) \in \mathbb{R}^n \times \mathbb{R})$, all the hypotheses of Lemma 1.4 are fulfilled. Thus, there exists a function $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$ such that $-\Delta u(x) \in F(x, u(x))$ almost everywhere in $\mathbb{R}^n$, which leads to (8).

Finally, when $f$ is non-negative, so is $\underline{f}$ and, through Lemma 1.2, we obtain $u(x) \geq 0$ for all $x \in \mathbb{R}^n$ ∎

**Remark 2.1.** Functions satisfying (8) are usually called ‘solutions in the multi-valued sense’ to the equation $-\Delta u = f(x, u)$ $(x \in \mathbb{R}^n)$. Evidently, each solution turns out also a solution in the multi-valued sense, whereas the converse is not true in general, unless further conditions are imposed. One of them is the following.

**(b$_5$)** *There exists $\Omega_0 \subseteq \mathbb{R}^n$ with $|\Omega_0| = 0$ such that the set*

$$D_f = \cup_{x \in \mathbb{R}^n \setminus \Omega_0} \{z \in \mathbb{R} : f(x, \cdot) \text{ is discontinuous at } z\}$$

*has measure zero. Moreover, for almost every $x \in \mathbb{R}^n$ and each $z \in D_f$, the inequality $\underline{f}(x, z) \leq 0 \leq \overline{f}(x, z)$ implies $f(x, z) = 0$.*

To show this, let us first note that if $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$, then $u \in W^{2,p}(B_r)$ for any $r > 0$ and consequently, by [11: Proposition 2.1], $\Delta u(x) = 0$ at almost all points $x \in u^{-1}(D_f)$. Now, suppose $u \in \hat{H}_0^{2,p}(\mathbb{R}^n)$ complies with (8). Exploiting the same technique employed in the proof of [11: Theorem 3.1] we easily infer that $u$ also solves the equation $-\Delta u = f(x, u)$ in $\mathbb{R}^n$.

Several functions $f$ enjoy properties (b$_4$) and (b$_5$). Here are two typical examples.

**Example 2.1.** Let $f(x, z) = \alpha(x)g(z)$ $((x, z) \in \mathbb{R}^n \times \mathbb{R})$ where $\alpha \in L^p(\mathbb{R}^n)$ while $g : \mathbb{R} \to \mathbb{R}$ has a finite variation and a positive infimum on $\mathbb{R}$. If $\alpha(x) \geq 0$ for almost every $x \in \mathbb{R}^n$, then the function $f$ fulfils the above-mentioned conditions. Indeed, ($b_4$) follows immediately from

$$\underline{f}(x, z) = \alpha(x) \lim_{\delta \to 0^+} \inf_{|\zeta - z| \leq \delta} g(\zeta), \qquad \overline{f}(x, z) = \alpha(x) \lim_{\delta \to 0^+} \sup_{|\zeta - z| \leq \delta} g(\zeta). \tag{9}$$

Since $D_f = \{z \in \mathbb{R} : g \text{ is discontinuous at } z\}$, the set $D_f$ is countable and so $|D_f| = 0$. Taking account of the assumption $\inf_{z \in \mathbb{R}} g(z) > 0$ besides (9), it is a simple matter to see that for almost all $x \in \mathbb{R}^n$ and each $z \in D_f$ the inequality $\underline{f}(x, z) \leq 0 \leq \overline{f}(x, z)$ forces $\alpha(x) = 0$, namely $f(x, z) = 0$. Therefore, ($b_5$) turns out true too.

**Example 2.2.** Pick $y^* > 0$ and choose a bounded sequence $\{y_h\} \subseteq \mathbb{R}$ satisfying

$$\inf_{h \in \mathbb{N}} y_h > 0, \qquad y^* \notin \left[\inf\nolimits_{h \in \mathbb{N}} y_h, \sup\nolimits_{h \in \mathbb{N}} y_h\right]. \tag{10}$$

Moreover, denote by $C$ a non-empty closed subset of $\mathbb{R}$ such that $|C| = 0$; as an example, $C$ could be the Cantor 'middle thirds' set, which is uncountable. The set $\mathbb{R} \setminus C$ is non-empty and open. Hence, it has at most countably many connected (open) components $A_h$ $(h \in \mathbb{N})$. We define, for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}$, $f(x, z) = \alpha(x)g(z)$, where $\alpha \in L^p(\mathbb{R}^n)$, $\alpha(x) \geq 0$ at almost all points of $\mathbb{R}^n$, and

$$g(z) = \begin{cases} y_h & \text{if } z \in A_h \ (h \in \mathbb{N}) \\ y^* & \text{if } z \in C. \end{cases}$$

The function $f$ complies with assertions ($b_4$) and ($b_5$). In fact, owing to (9), condition ($b_4$) holds. Elementary arguments, mainly based upon the identity $D_f = C$ and (10), then yield assertion ($b_5$).

**Remark 2.2.** Let $p \in (1, +\infty)$ and let $\Omega$ be a non-empty, bounded, open subset of $\mathbb{R}^n$. According to [14: Chapter III], denote by $\hat{H}^{2,p}(\mathbb{R}^n; \Omega)$ the space of all (equivalence classes of) measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that, for $1 \leq i, j \leq n$,

$$u, \ \frac{\partial u}{\partial x_i} \in L^p_{loc}(\mathbb{R}^n), \qquad \int_\Omega u(x)\, dx = \int_\Omega \frac{\partial u(x)}{\partial x_i}\, dx = 0, \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^n).$$

Arguing as before, but using in [14: Chapter III] Theorems 2.9 and 4.4 instead of Theorems 2.21 and 4.6, respectively, it is possible to show that, provided the space $\hat{H}^{2,p}_0(\mathbb{R}^n)$ is replaced by $\hat{H}^{2,p}(\mathbb{R}^n; \Omega)$, both Theorem 2.1, with $\beta = 0$, and Theorem 2.2 remain true for any $p \in (1, +\infty)$. However, whenever $p \geq \frac{n}{2}$, these results do not give informations about the sign of solutions.

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