A New Method for
Obtaining Solutions of the Dirac Equation

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Abstract. The Dirac operator with pseudoscalar, scalar or electric potential and the Schrödinger operator are considered. For any potential depending on an arbitrary function ξ satisfying the equation

$$\Delta\xi - \gamma(\xi) \cdot \frac{d\gamma(\xi)}{d\xi} = 0 \qquad (*)$$

where $\gamma(\xi) = |\operatorname{grad}\xi|$ there are constructed special solutions of the Dirac and the Schrödinger equations, and in some cases the fundamental solutions are obtained also. The class of solutions of equation $(*)$ is sufficiently ample. For example, if 1) ξ is harmonic and 2) the gradient squared of ξ is constant, then ξ satisfies $(*)$. That is, in particular, any complex linear combination of three variables $\xi = ax_1 + bx_2 + cx_3 + d$ satisfies equation $(*)$, and the solutions may be obtained for any potential depending on such ξ. All results are obtained using some special biquaternionic projection operators constructed after having solved an eikonal equation corresponding to ξ.

Keywords: *Dirac equation, quaternionic analysis*

AMS subject classification: 30G35, 81Q05

1. Introduction

In this work we consider the following Dirac equations:

a) with pseudoscalar potential

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im + \gamma_0\gamma_5 g(x)\right]\Phi(t,x) = 0 \qquad (1)$$

b) with scalar potential

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im + g(x)\right]\Phi(t,x) = 0 \qquad (2)$$

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c) with electric potential

$$\left[\gamma_0 \partial_t - \sum_{k=1}^{3} \gamma_k \partial_k + im + i\gamma_0 g(x) \right] \Phi(t, x) = 0 \tag{3}$$

where the Dirac matrices have the standard (see, e.g., [6, 20]) form

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_k = \frac{\partial}{\partial x_k} \quad (m \in \mathbb{R})$$

g is a complex-valued function

Φ is a $\mathbb{C}^4 -$ valued function.

For some class of potentials g we will construct the exact time-harmonic solutions of these equations, that is solutions of the form

$$\Phi(t, x) = q(x)e^{i\omega t}$$

where $\omega \in \mathbb{R}$ and q is a $\mathbb{C}^4$-valued function depending on $x = (x_1, x_2, x_3)$. For example, in the case of pseudoscalar potential (equation (1)) we have the equation for q

$$\mathbb{D}_{\omega,m}^{ps} q(x) := \left[i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + im + \gamma_0\gamma_5 g(x) \right] q(x) = 0. \tag{4}$$

Moreover, the obtained solutions will be used for constructing exact solutions of the Schrödinger equation

$$\Delta u(x) + v(x)u(x) = 0. \tag{5}$$

Finally, some fundamental solutions for the Dirac operators will be constructed also.

There exist dozens of works on exact solutions of relativistic wave equations. The reader is referred to the encyclopaedic monograph [2] for a bibliography and review of known exact solutions up to the late 1980s. Let us notice that the method of separation of variables was always the main tool for obtaining exact solutions of relativistic wave

equations and here it is substituted by an essentially different technique which takes into account other characteristics of the potential. That is why the class of potentials considered in this work is so different from the potentials for which there were known exact solutions.

Of course, the main question is for which potentials g from (1) - (3) and v from (5) we are able to realize the program described above. The definition of such a class of potentials will be introduced with all necessary details in Section 4. Here we describe it not rigorously.

Let us consider formally the equation

$$\Delta\xi - \gamma(\xi) \cdot \gamma'(\xi) = 0 \tag{6}$$

where $\gamma(\xi) = |\operatorname{grad}\xi|$. Equation (6) has an ample class of solutions. For instance, if

1) $\Delta\xi = 0$
2) $(\operatorname{grad}\xi)^2 = C^2$ ($C \in \mathbb{C}$ constant),

then ξ satisfies equation (6). In particular, any linear combination of three variables $\xi = ax_1 + bx_2 + cx_3 + d$, where $a, b, c, d \in \mathbb{C}$ are arbitrary constants, satisfies conditions 1) and 2), and consequently (6) also. Moreover, we can say the same about any function $\xi(x) = \zeta(z) + c\,x_3$, where $\zeta(z)$ is an arbitrary analytic function of the complex variable $z = x_1 + ix_2$. In Section 4 we prove also a quite curious proposition which gives an idea about the largeness of the Ξ-class.

All the technique described in the present article works perfectly for potentials g and v being arbitrary functions of ξ which in its turn satisfies equation (6), that is, for $g = g(\xi(x))$ and $v = v(\xi(x))$ we will obtain solutions of equations (1) - (3) and (5).

It should be emphasized that even for $\xi = x_1$, that is, for potentials depending only on x_1, the results of this work are not trivial and can not be obtained by other known methods.

The main tool of this work are some specially constructed biquaternionic projection operators which in some sense allow us to reduce the problem to a simpler one. To develop all this technique we need to rewrite the Dirac equations (1) - (3) in the corresponding biquaternionic form. This is done in Section 2 with the help of a quite simple matrix transformation introduced in [11] (see also [14] and [15: Section 12]). The obtained iquaternionic "images" of equations (1) - (3) have some differences related with the different positions of potentials in (1) - (3). So we show in Section 3 how with the aid of some additional tricks they can be reduced to the same biquaternionic equation. Then we introduce the projection operators corresponding to this biquaternionic equation. The construction of projection operators is based on the solution of an eikonal equation. In Section 4 we define the above mentioned class of combinations ξ of independent variables which we call Ξ-class, and in all following sections we work with potentials depending on $\xi \in \Xi$. In Section 5 we obtain solutions of the Dirac equation in quaternionic form which then are used in Section 6 to obtain solutions of the Schrödinger equation. The solution of the multi-dimensional Schrödinger equation is reduced, in the sense which will become clear later, to the solution of the ordinary Riccati

equation. In Section 7 we discuss the possibility of factorization of the Schrödinger operator. We show that this problem is equivalent to the finding of one particular solution of the Schrödinger equation and therefore for the class of potentials under consideration. Using the results of Section 6, it is a solved problem when it is possible to solve the corresponding Riccati equation. In Section 8 we consider the possibility of application of the technique described in the preceding sections to the problem of construction of fundamental solutions. Here only the case of linear combination of independent variables $\xi = ax_1 + bx_2 + cx_3 + d$ is discussed. In Section 9 we show how the solutions of the quaternionic images of Dirac equations can be transformed into solutions of the Dirac equations in traditional form (1) - (3). We consider in detail only the case of pseudoscalar potential – for the scalar and electric potentials the procedure is similar to that for the pseudoscalar potential.

This work is a continuation of the article [12], where the technique of biquaternionic projection operators based on the solution of eikonal equation was proposed for the first time, as well as of the paper [13], where it was applied to the Dirac equations (1) - (3) with harmonic potentials g the gradients squared of which are constant. In the present article we considerably enlarge the class of potentials which can be treated using the new technique and obtain also solutions of the Schrödinger equation. Nevertheless, the work [13] is recommended as a more detailed description of the procedure of transformation of the solutions of the Dirac equation in quaternionic form into solutions of equations (1) - (3). It also contains the explanation why the tool of complex quaternions was chosen for the analysis of the Dirac equation.

2. Preliminaries

We denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (= biquaternions). The elements of $\mathbb{H}(\mathbb{C})$ are represented in the form $\rho = \sum_{k=0}^{3} \rho_k i_k$, where $\rho_k \in \mathbb{C}$, i_0 is the unit and i_1, i_2, i_3 are standard quaternionic imaginary units: $i_1^2 = i_2^2 = i_3^2 = -1$ and $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$. We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition, i commutes with i_k $(k = 0, \ldots, 3)$. We will use also the vector representation of $\rho \in \mathbb{H}(\mathbb{C})$: $\rho = \mathrm{Sc}(\rho) + \mathrm{Vec}(\rho)$, where $\mathrm{Sc}(\rho) = \rho_0$ and $\mathrm{Vec}(\rho) = \vec{\rho} = \sum_{k=1}^{3} \rho_k i_k$. The complex quaternions of the form $\rho = \vec{\rho}$ are called *purely vectorial* and identified with vectors from $\mathbb{C}^3$. The quaternion $\overline{\rho} = \rho_0 - \vec{\rho}$ is called *conjugate* to ρ.

Let us denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. Note that

$$\rho \in \mathfrak{S} \quad \Longleftrightarrow \quad \rho\overline{\rho} = 0 \quad \Longleftrightarrow \quad \rho^2 = 2\rho_0\rho \quad \Longleftrightarrow \quad \rho_0^2 = (\vec{\rho})^2 \tag{7}$$

(see [15: p. 28]). As usual, zero is not included to $\mathfrak{S}$.

We will consider $\mathbb{H}(\mathbb{C})$-valued functions given in a domain $\Omega \subset \mathbb{R}^3$. On the set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the well-known Moisil-Theodoresco operator D is defined by the expression $D = \sum_{k=1}^{3} i_k \partial_k$. It was introduced in [16, 17] (see also, e.g., [5, 7 - 9]). The equation $Df = 0$ is equivalent to the system

$$\left. \begin{array}{r} \mathrm{div}\, \vec{f} = 0 \\ \mathrm{grad}\, f_0 + \mathrm{rot}\, \vec{f} = 0 \end{array} \right\} .$$

The following Leibniz rule holds (see, e.g., [9: p. 24] and [15: p. 63]): Let $\{f, g\} \subset C^1(\Omega; \mathbb{H}(\mathbb{C}))$. Then

$$D[f \cdot g] = D[f] \cdot g + \bar{f} \cdot D[g] + 2(\mathrm{Sc}(fD))[g] \qquad (8)$$

where $(\mathrm{Sc}(fD))[g] = -\sum_{k=1}^{3} f_k \partial_k g$. Note that if $f \equiv f_0$ (Vec $(f) \equiv 0$), then $D[f_0 \cdot g] = \mathrm{grad}\, f_0 \cdot g + f_0 \cdot D[g]$ where $\mathrm{grad}\, f_0 = i_1 \partial_1 f_0 + i_2 \partial_2 f_0 + i_3 \partial_3 f_0$.

Let us introduce the integral operator

$$(Tf)(x) = \int_{\Omega} \mathcal{K}(x - y) f(y)\, d\Omega_y \qquad (x \in \mathbb{R}^3)$$

which is the analog corresponding to D of the complex T-operator. Here $\mathcal{K}(x) = -\frac{\vec{x}}{4\pi |x|^3}$. For any $f \in C^1(\Omega) \cap C(\overline{\Omega})$ we have $DTf = f$ in Ω (see, e.g., [9: Chapter 2]).

Denote $\tilde{q}(x) = q(x_1, x_2, -x_3)$. The domain $\tilde{\Omega}$ is assumed to be obtained from the domain $\Omega \subset \mathbb{R}^3$ by the reflection $x_3 \to -x_3$. In [11] (see also [14] and [15: Section 12]) a map $\mathcal{A}$ was introduced which transforms a function $q : \tilde{\Omega} \subset \mathbb{R}^3 \to \mathbb{C}^4$ into a function $\rho : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule

$$\rho = \mathcal{A}[q] := \frac{1}{2}\Big[-(\tilde{q}_1 - \tilde{q}_2)i_0 + i(\tilde{q}_0 - \tilde{q}_3)i_1 - (\tilde{q}_0 + \tilde{q}_3)i_2 + i(\tilde{q}_1 + \tilde{q}_2)i_3 \Big].$$

Note that $\mathcal{A}$ is a $\mathbb{C}$-linear transformation. The corresponding inverse transformation is defined by the equality

$$\mathcal{A}^{-1}[\rho] = \Big(-i\tilde{\rho}_1 - \tilde{\rho}_2, -\tilde{\rho}_0 - i\tilde{\rho}_3, \tilde{\rho}_0 - i\tilde{\rho}_3, i\tilde{\rho}_1 - \tilde{\rho}_2 \Big).$$

The transformations $\mathcal{A}$ and $\mathcal{A}^{-1}$ may be represented in matrix form by

$$\rho = \mathcal{A}[q] = \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix}$$

and

$$q = \mathcal{A}^{-1}[\rho] = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\rho}_0 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{pmatrix}.$$

We will need some algebraic properties of this pair of transforms shown in [14]:

1. $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_1\mathcal{A}^{-1}[\rho] = i_1\rho$
2. $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_2\mathcal{A}^{-1}[\rho] = i_2\rho$
3. $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_3\mathcal{A}^{-1}[\rho] = -i_3\rho$
4. $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_0\mathcal{A}^{-1}[\rho] = \rho i_1$
5. $\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}[\rho] = -i\rho i_2$.

Let us consider the biquaternionic operator

$$R_\alpha^{ps} = D - i\tilde{g}(x)I + M^\alpha$$

where I is the identity operator, M^α is the operator of multiplication by the complex quaternion α from the right-hand side: $M^\alpha f = f\alpha$ and $\alpha = -(i\omega i_1 + mi_2)$. The operator R_α^{ps} is equivalent to the operator $\mathbb{D}_{\omega,m}^{ps}$ due to the equality

$$R_\alpha^{ps} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_{\omega,m}^{ps}\mathcal{A}^{-1}. \tag{9}$$

The proof of this equality is a direct corollary of the algebraic properties 1 - 5 of the transformation $\mathcal{A}$. In other words, a function q belongs to $\ker \mathbb{D}_{\omega,m}^{ps}(\Omega)$ if and only if $u := \mathcal{A}[q] \in \ker R_\alpha^{ps}(\tilde{\Omega})$.

Similar equalities we obtain for the operators

$$\mathbb{D}_{\omega,m}^{sc} = i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k\partial_k + im + g(x)$$

$$\mathbb{D}_{\omega,m}^{el} = i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k\partial_k + im + i\gamma_0 g(x).$$

Namely,

$$R_\alpha^{sc} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_{\omega,m}^{sc}\mathcal{A}^{-1} \quad \text{where} \quad (R_\alpha^{sc} = D + M^{i\tilde{g}(x)i_2} + M^\alpha) \tag{10}$$

$$R_\alpha^{el} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_{\omega,m}^{el}\mathcal{A}^{-1} \quad \text{where} \quad (R_\alpha^{el} = D + M^{-i\tilde{g}(x)i_1} + M^\alpha). \tag{11}$$

3. Projection operators

In the first part of this section we will show how the operators $R_\alpha^{ps}, R_\alpha^{sc}, R_\alpha^{el}$ can be reduced to the operator $D - f_0 I$ where f_0 is a scalar function. Then for the operator $D - f_0 I$ we will construct a corresponding pair of projection operators Q^+ and Q^- which will be used in the following sections.

First, let us consider the Dirac operator with pseudoscalar potential in quaternionic form R_α^{ps} with the additional condition $\omega^2 \neq m^2$. In order to "remove" the term M^α from R_α^{ps} we will apply the following scheme. We will associate with the vector α a scalar γ the square of which is equal to α^2. This scalar γ will "substitute" the vector α in the differential equation. The "missing information" on the vector α will be saved in some special projection operators $P^\pm$ which will establish the necessary relationship between the original equation containing the vector α and the new one in which α is substituted by the scalar γ.

Thus, we denote by γ any complex square root from α^2: $\gamma^2 = \alpha^2$. Let us consider the pair of mutually complementary projection operators

$$P^\pm = \frac{1}{2\gamma}M^{(\gamma\pm\alpha)} \tag{12}$$

acting on the set of $\mathbb{H}(\mathbb{C})$-valued functions. The operators P^+ and P^- obviously commute with the operator R_α^{ps} but the most important fact is that

$$P^\pm M^\alpha = \frac{1}{2\gamma} M^{(\gamma\alpha\pm\alpha^2)} = \frac{1}{2\gamma} M^{(\gamma\alpha\pm\gamma^2)} = \pm\gamma P^\pm.$$

That is, the projection operators $P^\pm$ transform the vector α into the scalars $\pm\gamma$ and vice versa. Then the operator R_α^{ps} can be rewritten in the form

$$R_\alpha^{ps} = P^+ \big(D + (-i\tilde{g}(x) + \gamma)I\big) + P^- \big(D + (-i\tilde{g}(x) - \gamma)I\big). \tag{13}$$

Consequently, $u \in \ker R_\alpha^{ps}$ if and only if $u = P^+ u^+ + P^- u^-$ where u^+ and u^- satisfy the equations

$$\big(D + (-i\tilde{g}(x) + \gamma)\big)u^+(x) = 0 \tag{14}$$
$$\big(D + (-i\tilde{g}(x) - \gamma)\big)u^-(x) = 0. \tag{15}$$

Each of these equations is of the form

$$(D - f_0(x))u(x) = 0. \tag{16}$$

Thus, the equation

$$R_\alpha^{ps} u = 0 \tag{17}$$

is reduced to equation (16). The same scheme using some special projection operators converting vectors into scalars and vice versa we will use many times in this article.

Now let us consider the case $\omega^2 = m^2$. This equality is equivalent to the inclusion $\alpha \in \mathfrak{S}$ for $\alpha = -(i\omega i_1 + mi_2)$. Note that if $v \in \ker(D - i\tilde{g}I)$, then the function $u = v\alpha$ belongs to $\ker R_\alpha^{ps}$. Thus, in this case also for obtaining exact solutions of equation (17) we can consider an equation of form (16). Of course, not any solution u of equation (17) can be represented in the form $u = v\alpha$ where $v \in \ker(D - i\tilde{g}I)$. It can be seen that the general solution of equation (17) when $\omega^2 = m^2$ has the form $u = v - \theta\alpha$ where v is an arbitrary function from $\ker(D - i\tilde{g}I)$ and θ is any solution of the equation $(D - i\tilde{g})\theta = v$.

Now let us consider the operator R_α^{sc}. Denote

$$S^\pm = \frac{1}{2\eta(x)} M^{(\eta(x)\pm(i\tilde{g}(x)i_2+\alpha))}$$

where

$$\eta(x) = \sqrt{(i\tilde{g}(x)i_2 + \alpha)^2} = \sqrt{\omega^2 - (i\tilde{g}(x) - m)^2}.$$

Then we obtain

$$R_\alpha^{sc} = S^+(D + \eta(x)I) + S^-(D - \eta(x)I) = (D + \eta(x))S^+ + (D - \eta(x))S^-.$$

Here, in contrast to the previous case of the operator R_α^{ps}, the projection operators S^+ and S^- do not commute with the operators in parentheses. Thus, in general, it is not true that $u \in \ker R_\alpha^{sc}$ if and only if $u = S^+ u^+ + S^- u^-$ where $u^+ \in \ker(D + \eta I)$

and $u^- \in \ker(D - \eta I)$, but in the case when $\omega = 0$ the operators $S^\pm$ take the form $S^\pm = \frac{1}{2}M^{(1\pm ii_2)}$ and, consequently, commute with the operators $D \pm \eta I$. Thus, if $\omega = 0$, then $u \in \ker R_\alpha^{sc}$ if and only if $u = S^+ u^+ + S^- u^-$ where $u^+ \in \ker(D + (\tilde{g} + im)I)$ and $u^- \in \ker(D - (\tilde{g} + im)I)$. Thus the problem is reduced to equation (16) also.

In the same way we obtain the equality

$$R_\alpha^{el} = Z^+(D + \nu(x)I) + Z^-(D - \nu(x)I) = (D + \nu(x))Z^+ + (D - \nu(x))Z^-$$

where

$$\nu(x) = \sqrt{(-i\tilde{g}(x)i_1 + \alpha)^2} = \sqrt{(\tilde{g}(x) + \omega)^2 - m^2}$$

and

$$Z^\pm = \frac{1}{2\nu(x)}M^{(\nu(x)\pm(-i\tilde{g}(x)i_1+\alpha))}.$$

For $m = 0$ we have $Z^\pm = \frac{1}{2}M^{(1\mp ii_1)}$, and in this case $u \in \ker R_\alpha^{el}$ if and only if $u = Z^+ u^+ + Z^- u^-$ where $u^+ \in \ker(D + (\tilde{g} + \omega)I)$ and $u^- \in \ker(D - (\tilde{g} + \omega)I)$.

Thus, we obtained that the Dirac equation with pseudoscalar potential (4) is reduced to an equation of form (16). The Dirac equation with scalar potential $\mathbb{D}_{0,m}^{sc}q(x) = 0$ is reduced to (16) also and we can say the same about the Dirac equation with electric potential $\mathbb{D}_{\omega,0}^{el}q(x) = 0$. So now we will concentrate on solutions of equation (16), but first let us consider the equation

$$(D - \operatorname{grad}\phi)u = 0 \tag{18}$$

where ϕ is some scalar complex-valued differentiable function and $u \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$. Note that this equation may be rewritten in the form

$$pDp^{-1}u = 0 \tag{19}$$

where $p = e^\phi$ (which was noticed in [19]) and hence the function $\frac{u}{p}$ is simply a null-solution of the Moisil-Theodoresco operator D. This fact, which is an echo of the gauge transformations, signifies that equation (18) does not represent any independent interest (it is quite interesting that the equation $(D - M^{\operatorname{grad}\phi})u = 0$ is much more difficult and has no such clear relation with the Moisil-Theodoresco equation).

Now the following simple idea seems to be attractive. On the one hand we have equation (16) with scalar potential and we do not know how to solve it. On the other hand we have equation (18) with vector potential which does not represent any interest because it is already solved. Besides, we can always construct some projection operators based on the corresponding biquaternionic zero divisors which are able to transform a scalar into a vector and vice versa if $(\text{scalar})^2 = (\text{vector})^2$. Let us consider the equation

$$(\operatorname{grad}\mu)^2 = f_0^2 \tag{20}$$

which is called *eikonal* (for its solution see, e.g., [1: Section 2.3] and [18: Section 3.1]), and let us introduce the operators of multiplication

$$Q^\pm = \frac{1}{2f_0}(f_0 \pm \operatorname{grad}\mu)I$$

defined on the set of $\mathbb{H}(\mathbb{C})$-valued functions. Here we assume that $f_0(x) \neq 0$ for all $x \in \Omega$. Q^+ and Q^- are mutually complementary and orthogonal projection operators. We have the equalities

$$D - f_0 I = Q^+(D - \operatorname{grad}\mu I) + Q^-(D + \operatorname{grad}\mu I)$$
$$= (D - \operatorname{grad}\mu)Q^+ + (D + \operatorname{grad}\mu)Q^-.$$

In other words, equation (16) is equivalent to the pair of equations

$$Q^+(D - \operatorname{grad}\mu)u = 0 \tag{21}$$
$$Q^-(D + \operatorname{grad}\mu)u = 0. \tag{22}$$

The general solution of each of these equations can be constructed quite easily (see [12]). The main problem is to find the intersection

$$\ker Q^+(D - \operatorname{grad}\mu I) \cap \ker Q^-(D + \operatorname{grad}\mu I)$$

which is a difficult task because the projection operators $Q^\pm$ do not commute with the operators in parentheses. Nevertheless, in [12] it was already shown that in some special cases the application of the operators $Q^\pm$ allows us to obtain some particular solutions. Here we considerably enlarge the class of potentials f_0 for which it can be done slightly modifying the projection operators $Q^\pm$ but not changing the principal idea described above.

4. Ξ-class

Let us introduce the notation $\gamma(\xi(x)) = |\operatorname{grad}\xi(x)|$ and define the following function class which will be called Ξ-*class*. We say that a scalar function ξ belongs to the Ξ-class in some domain $\Omega \subseteq \mathbb{R}^3$ and write $\xi \in \Xi(\Omega)$ if the following conditions are fulfilled:

1. $\xi \in C^2(\Omega)$.
2. γ and $\Delta\xi$ can be written as functions of ξ.
3. The equation

$$\Delta\xi - \gamma(\xi) \cdot \gamma'(\xi) = 0 \tag{23}$$

is satisfied. Note that this equation can be rewritten also as

$$\Delta\xi - \frac{1}{2}\frac{\partial}{\partial\xi}(\gamma^2(\xi)) = 0. \tag{24}$$

Let us consider some examples of functions from Ξ-class.

Example 1. Let $\xi = ax_1 + bx_2 + cx_3 + d$ where $a, b, c, d \in \mathbb{C}$ are arbitrary constants. In this case, obviously, $\Delta\xi = 0$ and $\gamma(\xi) = \sqrt{a^2 + b^2 + c^2} =: C$. Thus equation (23) is satisfied and $\xi \in \Xi(\Omega)$.

Example 2. Let $\xi = \zeta(z) + cx_3$ where ζ is an arbitrary analytic function depending on the complex variable $z = x_1 + ix_2$ and c is a constant. We have that $\Delta\xi = 0$ and $\operatorname{grad}\xi = \frac{d\zeta}{dz}i_1 + i\frac{d\zeta}{dz}i_2 + ci_3$. Thus $\gamma(\xi) = c$ and such ξ also belongs to the Ξ-class.

Note that if $\Delta\xi = 0$ and $\gamma(\xi) = \text{const}$, then equation (23) is satisfied. Moreover, both considered examples represent this special case. Of course, equation (23) admits more solutions.

The following proposition gives an idea about the largeness of the Ξ-class.

Proposition 1. *Let ξ be a real- or complex-valued function belonging to $\Xi(\Omega)$ and $s = s(\xi)$ a function such that there exist a continuous derivative $s'(\xi)$ and an inverse differentiable function $\xi = r(s)$. Then $s \in \Xi(\Omega)$.*

Proof. Under the formulated assumptions we obtain that s fulfills conditions 1 - 3, thus we have to show only that s satisfies the equation $\Delta s - \frac{1}{2}\frac{\partial}{\partial s}(\gamma^2(s)) = 0$. We have $\operatorname{grad} s = s'(\xi)\operatorname{grad}\xi$. Consequently, $\gamma^2(s) = (s'(\xi))^2\gamma^2(\xi)$ and

$$\Delta s = \operatorname{div}\operatorname{grad} s = \langle\operatorname{grad} s'(\xi), \operatorname{grad}\xi\rangle + s'(\xi)\Delta\xi = s''(\xi)\gamma^2(\xi) + s'(\xi)\Delta\xi.$$

Finally,

$$
\begin{aligned}
\frac{\partial}{\partial s}(\gamma^2(s)) &= 2\left(s'(\xi)\cdot\frac{\partial}{\partial s}(s'(\xi))\cdot\gamma^2(\xi) + (s'(\xi))^2\cdot\gamma(\xi)\cdot\frac{\partial}{\partial s}(\gamma(\xi))\right)\\
&= \frac{2}{s'(\xi)}\left(s'(\xi)\cdot s''(\xi)\cdot\gamma^2(\xi) + (s'(\xi))^2\cdot\gamma(\xi)\cdot\gamma'(\xi)\right)\\
&= 2\left(s''(\xi)\cdot\gamma^2(\xi) + s'(\xi)\cdot\gamma(\xi)\cdot\gamma'(\xi)\right).
\end{aligned}
$$

Thus, using the fact that $\xi \in \Xi(\Omega)$, we obtain the required equality:

$$
\begin{aligned}
\Delta s - \frac{1}{2}\frac{\partial}{\partial s}(\gamma^2(s)) &= s''(\xi)\gamma^2(\xi) + s'(\xi)\Delta\xi - s''(\xi)\cdot\gamma^2(\xi) - s'(\xi)\cdot\gamma(\xi)\cdot\gamma'(\xi)\\
&= s'(\xi)(\Delta\xi - \gamma(\xi)\cdot\gamma'(\xi))\\
&= 0
\end{aligned}
$$

and the statement is proved ∎

In what follows we will consider functions depending on $\xi \in \Xi(\Omega)$. As ξ may be a real- or complex-valued function, the differentiability with respect to ξ will be understood in real or complex sense, respectively, without additional comments. The expression $f'(\xi)$ will denote the real or complex derivative with respect to ξ, the second is for the case $\operatorname{Im}\xi \not\equiv 0$ in Ω.

5. Solution of the Dirac equation in quaternionic form

In this section we consider the equation

$$\left(D - f_0(\xi(x))\right)u(x) = 0 \tag{25}$$

in some domain $\Omega \subseteq \mathbb{R}^3$ where ξ is a scalar function of the independent variables x_1, x_2, x_3.

Let us slightly modify the definition of the projection operators $Q^{\pm}$ (compare with Section 3). Taking into account that f_0 depends on ξ we introduce $Q^{\pm}$ with respect to ξ. Namely,

$$Q^{\pm} = \frac{1}{2\xi}(\xi \pm \operatorname{grad}\mu)I \tag{26}$$

where

$$(\operatorname{grad}\mu(x))^2 = \xi^2(x) \qquad (x \in \Omega) \tag{27}$$

and $\xi(x) \neq 0$ for all $x \in \Omega$. Note that any solution μ of the eikonal equation (27) is convenient for us. Sometimes, when $Q^\pm$ will be considered simply as $\mathbb{H}(\mathbb{C})$-valued functions enjoying such important property as the equivalence of the function to its square, we will refer to $Q^\pm$ as idempotents. When the operational nature of $Q^\pm$ will be important we will follow the above used terminology calling them projection operators. We hope that it cannot provoke any misunderstanding because, in fact, the idempotent or projection operator is the same mathematical notion, the first variant is used more frequently in algebra while the second one – in functional analysis.

Let us prove the following

Lemma 1. *Let $\xi \in \Xi(\Omega)$. Then the idempotents $Q^\pm$ may be constructed in the form*

$$Q^\pm = \frac{1}{2}\left(1 \pm \frac{i\operatorname{grad}\xi}{|\operatorname{grad}\xi|}\right) \tag{28}$$

and

$$D[Q^\pm] = 0 \qquad in \ \ \Omega. \tag{29}$$

Proof. Comparing (28) with the general form (26) of $Q^\pm$ we find that $\operatorname{grad}\mu = \frac{i\xi\operatorname{grad}\xi}{|\operatorname{grad}\xi|}$. From this equality we find $\mu(\xi) = i\int \frac{\xi}{|\operatorname{grad}\xi|}\,d\xi$. In other words, μ is an antiderivative of the function $\frac{i\xi}{|\operatorname{grad}\xi|}$. Moreover, μ is a solution of the eikonal equation (27): $(\operatorname{grad}\mu)^2 = -\frac{\xi^2(\operatorname{grad}\xi)^2}{|\operatorname{grad}\xi|^2} = \xi^2$. Consequently, $Q^\pm$ are idempotents.

Let us verify (29). We have $D[Q^\pm] = \pm iD\left[\frac{\operatorname{grad}\xi}{|\operatorname{grad}\xi|}\right]$. Using the quaternionic Leibniz rule (8) we obtain

$$D\left[\frac{\operatorname{grad}\xi}{|\operatorname{grad}\xi|}\right] = D\left[\frac{\operatorname{grad}\xi}{\gamma(\xi)}\right]$$

$$= -\gamma^{-2}(\xi) \cdot \frac{\partial\gamma(\xi)}{\partial\xi} \cdot (\operatorname{grad}\xi)^2 - \gamma^{-1}(\xi)\Delta\xi$$

$$= \frac{\partial\gamma(\xi)}{\partial\xi} - \frac{1}{\gamma(\xi)}\Delta\xi.$$

Using condition 3 from the definition of Ξ-class we obtain that this expression is zero and $D[Q^\pm] = 0$ ∎

Using Lemma 1 we obtain the following main result of this section.

Theorem 1. *Let $\xi \in \Xi(\Omega)$, $Q^\pm$ be defined by (28), $A^\pm$ be arbitrary constant complex quaternions and*

$$h^\pm(\xi) = e^{\pm i\int \frac{f_0(\xi)}{\gamma(\xi)}\,d\xi}. \tag{30}$$

Then the function

$$u = Q^+ h^+(\xi)A^+ + Q^- h^-(\xi)A^- \tag{31}$$

is a solution of equation (25).

Proof. First, let us notice that the constant complex quaternions $A^{\pm}$ appear in (31) due to the right $\mathbb{H}(\mathbb{C})$-linearity of equation (25). Then it is sufficient to prove the theorem for the function $Q^+ h^+$ (for the second term $Q^- h^-$ the proof is completely analogous).

Let us apply the operator D to the function $u^+ = Q^+ h^+$ using the Leibniz rule (8):

$$Du^+ = \operatorname{grad} h^+ \cdot Q^+ + h^+ DQ^+ .$$

Due to Lemma 1 we obtain $Du^+ = \operatorname{grad} h^+ \cdot Q^+$. Note that $\operatorname{grad} h^+(\xi) = \frac{\partial h^+(\xi)}{\partial \xi} \cdot \operatorname{grad} \xi$ and

$$\operatorname{grad} \xi \cdot Q^+ = \frac{1}{2}\left(\operatorname{grad} \xi + \frac{i(\operatorname{grad} \xi)^2}{|\operatorname{grad} \xi|} \right) = -i|\operatorname{grad} \xi| \frac{1}{2}\left(\frac{i \operatorname{grad} \xi}{|\operatorname{grad} \xi|} + 1 \right) = -i\gamma(\xi)Q^+$$

(here once more the idempotent Q^+ allows us to turn a vector into a scalar). As can be seen easily, the function h^+ is a solution of the linear ordinary differential equation $\frac{dh^+(\xi)}{d\xi} + \frac{f_0(\xi)}{i\gamma(\xi)} h^+(\xi) = 0$. Then

$$Du^+ = \operatorname{grad} h^+ \cdot Q^+ = -i\gamma(\xi)\frac{\partial h^+(\xi)}{\partial \xi} \cdot Q^+ = f_0(\xi)h^+(\xi)Q^+ = f_0 \cdot u^+$$

and the statement is proved ∎

Example 3. Let $\xi = e^{x_1 + ix_2} + cx_3$ where c is constant. Then, obviously, ξ falls under Example 2 and belongs to the Ξ-class. In this case

$$\operatorname{grad} \xi = e^{x_1 + ix_2} i_1 + ie^{x_1 + ix_2} i_2 + ci_3$$

and $\gamma(\xi) = c$. The idempotents $Q^{\pm}$ take the form

$$Q^{\pm} = \frac{1}{2}\left(1 \pm \frac{i}{c}\left(e^{x_1 + ix_2} i_1 + ie^{x_1 + ix_2} i_2 + ci_3 \right) \right).$$

Thus, we obtain a solution of equation (25) in the form

$$
\begin{aligned}
u = {} & \left(1 + \frac{i}{c}\left(e^{x_1 + ix_2} i_1 + ie^{x_1 + ix_2} i_2 + ci_3 \right) \right) e^{\frac{i}{c}F_0(\xi)} A^+ \\
& + \left(1 - \frac{i}{c}\left(e^{x_1 + ix_2} i_1 + ie^{x_1 + ix_2} i_2 + ci_3 \right) \right) e^{-\frac{i}{c}F_0(\xi)} A^-
\end{aligned}
$$

where F_0 is an antiderivative of f_0.

6. Solution of the Schrödinger equation

In this section we consider the scalar three-dimensional Schrödinger equation

$$(\Delta + v(\xi(x)))u(x) = 0. \tag{32}$$

First, let us notice that applying the operator D to equation (25) we obtain that if u is a solution of (25), then

$$-\Delta u - D[f_0] \cdot u - f_0 Du = -\Delta u - \operatorname{grad} f_0 \cdot u - f_0^2 u = 0.$$

In other words, u is a solution of the Schrödinger equation with quaternionic potential

$$\left(\Delta + (f_0^2 + \operatorname{grad} f_0)\right)u = 0. \tag{33}$$

This observation will be used in the proof of the following

Theorem 2. *Let $\xi \in \Xi(\Omega)$, Q^+ defined by (28), $f_0(\xi)$ be a solution of the Riccati equation*

$$\frac{df_0(\xi)}{d\xi} + \frac{i}{\gamma(\xi)} f_0^2(\xi) = \frac{iv(\xi)}{\gamma(\xi)} \tag{34}$$

and $h(\xi) = e^{i \int \frac{f_0(\xi)}{\gamma(\xi)} d\xi}$. Then the function $u = Q^+ h$ is a solution of equation (32).

Proof. Note that $u = Q^+ h$ is a solution of equation (25) (see Theorem 1) and therefore it is a solution of equation (33) also. Taking into account that $\operatorname{grad} f_0 \cdot Q^+ = -i\frac{\partial f_0}{\partial \xi} \cdot \gamma(\xi) \cdot Q^+$ we obtain from (33) the equality

$$\left(\Delta + \left(f_0^2 - i\frac{\partial f_0}{\partial \xi}\gamma(\xi)\right)\right)u = 0. \tag{35}$$

The Riccati equation (34) gives us

$$v = -i\frac{\partial f_0}{\partial \xi}\gamma(\xi) + f_0^2(\xi). \tag{36}$$

Thus, from (35) and (36) we obtain that u is a solution of equation (32) ∎

Theorem 2 signifies that each component of the function $u = Q^+ h$ is a solution of equation (32). Consequently, any function of the form

$$\left(a_0 + \frac{1}{\gamma(\xi)}(a_1\partial_1\xi + a_2\partial_2\xi + a_3\partial_3\xi)\right)e^{i \int \frac{f_0(\xi)}{\gamma(\xi)} d\xi}$$

is a solution of equation (32), where f_0 is a solution of equation (34) and $a_k \in \mathbb{C}$ ($k = 0, \ldots, 3$) are arbitrary numbers.

Example 4. Let $\xi = ax_1 + bx_2 + cx_3 + d$ (Example 1) with $\sqrt{a^2 + b^2 + c^2} = 1$ and $v(\xi) = \xi(2i - \xi)$. Then the corresponding Riccati equation takes the form

$$\frac{df_0(\xi)}{d\xi} + if_0^2(\xi) = i\xi(2i - \xi). \tag{37}$$

To solve the Riccati equation it is sufficient to find any particular solution of it, then with its aid the Riccati equation may be reduced to a linear one. Here the term on the right-hand side prompts us that we can try to look for a particular solution of the form $y(\xi) = b_0\xi + b_1$ ($b_0, b_1 \in \mathbb{C}$). Substituting y into (37) we obtain $b_0 = -i$ and $b_1 = -1$. Now with the help of the substitution $f_0(\xi) = y(\xi) + z(\xi)$ we obtain the Bernoulli equation

$$z' + 2iyz = -iz^2 \tag{38}$$

for z which by the substitution $z_1 = \frac{1}{z}$ can be transformed into a linear equation, and this way we obtain the general solution

$$z(\xi) = \left(i \int e^{(-\xi^2 + 2i\xi)} \, d\xi + b_2\right)^{-1} \cdot e^{(-\xi^2 + 2i\xi)}$$

of equation (38) where $b_2 \in \mathbb{C}$ is an arbitrary constant. Thus,

$$f_0(\xi) = -i\xi - 1 + \frac{e^{(-\xi^2 + 2i\xi)}}{i \int e^{(-\xi^2 + 2i\xi)} \, d\xi + b_2}.$$

Finally, we obtain $e^{i \int f_0(\xi) \, d\xi} \in \ker\left(\Delta + \xi(2i - \xi)I\right)$.

7. Factorization of the Schrödinger operator

In [3] (see also [4]) it was noticed that the Schrödinger operator can be factorized as

$$-\Delta - vI = (D + M^{\vec{b}})(D - M^{\vec{b}}) \tag{39}$$

where the vector $\vec{b} = \operatorname{grad} \phi$ is a solution of the equation

$$D\vec{b} - |\vec{b}|^2 = v. \tag{40}$$

Equality (39) can be verified by a direct calculation. Note that when $v = 0$, equation (40) can be reduced to a linear one (see [10]).

For the solutions of the Schrödinger equation

$$\Delta u + vu = 0 \tag{41}$$

there was obtained [3] the representation

$$u(x) = e^{\phi(x)}\left(T[e^{-\phi(x)}\vec{v}(x)] + \Phi(x)\right) \tag{42}$$

where $\vec{v} \in \ker(D + M^{\vec{b}})$ and $\Phi \in \ker D$. Unfortunately, not for any functions $\vec{v}$ and Φ from the indicated kernels the expression on the right-hand side of (42) gives us a solution of equation (41). The function u on the left-hand side of (42) is scalar. Consequently, we have an additional restrictive condition

$$\text{Vec}\left(T[e^{-\phi(x)}\vec{v}(x)] + \Phi(x)\right) = 0$$

for $\vec{v}$ and Φ which complicates the things.

Let us notice that the problem of factorization of the Schrödinger operator is equivalent to finding a particular solution of the Schrödinger equation (41). In fact, let $\vec{b} = \text{grad}\,\phi$ be a vector satisfying equation (40). First, let us represent it as $\vec{b} = \frac{\text{grad}\,u}{u}$ where $u = e^{\phi}$. Then applying D to $\vec{b}$ we obtain

$$D\vec{b} = D\left[\frac{\text{grad}\,u}{u}\right] = u^{-2}|\text{grad}\,u|^2 - u^{-1}\Delta u = |\vec{b}|^2 - u^{-1}\Delta u.$$

Using (40) we obtain that u must be a solution of equation (41), and vice versa, if u is a solution of equation (41), then $\vec{b} = \frac{\text{grad}\,u}{u}$ is a solution of equation (40). Thus, the results of Section 6 may be used to factorize the Schrödinger operator and to obtain representation (42) for its null-solutions.

8. Fundamental solutions

A distribution U will be called *particular fundamental solution* of the operator $D - f_0(\xi(x))I$ if

$$(D - f_0(\xi(x)))U(x) = \delta(\xi(x)). \tag{43}$$

We will see that the technique described in the previous sections in some cases allows us to construct particular fundamental solutions and, consequently, to obtain the integral representation of the general solution of equation (25) depending only on ξ.

Let $\xi = ax_1 + bx_2 + cx_3 + d$ with $a, b, c, d \in \mathbb{R}$ and let $C = \sqrt{a^2 + b^2 + c^2}$. The distribution U will be constructed in the form $U = Q^+ h^+$ where the idempotent Q^+ will be slightly modified compared with the previous sections and the choice of the scalar function h^+ will be discussed below. We will look for the vector $\text{grad}\,\mu$ in the form

$$\text{grad}\,\mu = \xi\psi(\xi)\,\text{grad}\,\xi$$

($\mu = \int \xi\psi(\xi)\,d\xi$). The eikonal equation (27) gives us the relation

$$\psi^2(\xi) = -\frac{1}{\gamma^2(\xi)} = -\frac{1}{C^2}. \tag{44}$$

Let us require $D[Q^+] = A\delta(\xi)$ where $A \in \mathbb{C}$ is constant. We have $D[\psi(\xi)\text{grad}\,\xi] = 2A\delta(\xi)$ which is equivalent to the equation $-C^2\psi'(\xi) = 2A\delta(\xi)$. From the last equality we obtain

$$\psi(\xi) = -\frac{A}{C^2}\text{sign}\,\xi \tag{45}$$

where by definition

$$\operatorname{sign} \xi = \begin{cases} 1 & \text{for } \xi \geq 0 \\ -1 & \text{for } \xi < 0. \end{cases}$$

Comparing (45) with (44) we obtain $A = iC$ and $\psi(\xi) = -\frac{i}{C}\operatorname{sign}\xi$. Thus,

$$Q^+ = \frac{1}{2}\left(1 - \frac{i\operatorname{sign}\xi}{C}\operatorname{grad}\xi\right).$$

Now, let us consider the expression

$$\operatorname{grad} h^+(\xi)Q^+ = \frac{\partial h^+(\xi)}{\partial \xi}\operatorname{grad}\xi Q^+$$

which must be equal to $f_0(\xi)h^+(\xi)Q^+$ (see the proof of Theorem 1). Note that

$$\operatorname{grad}\xi \cdot Q^+ = \frac{1}{2}(\operatorname{grad}\xi + iC\operatorname{sign}\xi)$$

$$= iC\operatorname{sign}\xi\frac{1}{2}\left(1 - \frac{i}{C}\operatorname{sign}\xi\operatorname{grad}\xi\right)$$

$$= iC\operatorname{sign}\xi \cdot Q^+.$$

Then we obtain for h^+ the equation

$$iC\operatorname{sign}\xi\,\frac{\partial h^+(\xi)}{\partial \xi} - f_0(\xi)h^+(\xi) = 0$$

the solution of which is the function $h^+(\xi) = Be^{-\frac{i}{C}\int \operatorname{sign}\xi f_0(\xi)\,d\xi}$. The antiderivative F_0 of the function f_0 we fix by the condition $F_0(0) = 0$. Then the constant B we find from the requirement $h^+(\xi)DQ^+ = iCh^+(\xi)\delta(\xi) = \delta(\xi)$. Consequently, $B = \frac{1}{iC}$. Finally, we obtain that the distribution

$$U(\xi) = \frac{1}{2iC}\left(1 - \frac{i\operatorname{sign}\xi}{C}(ai_1 + bi_2 + ci_3)\right)e^{-\frac{i}{C}\operatorname{sign}\xi F_0(\xi)}$$

is a solution of equation (43).

9. Solutions of the Dirac equation in "traditional" form

As it was shown in Section 2 the Dirac operators $\mathbb{D}^{ps}_{\omega,m}, \mathbb{D}^{sc}_{\omega,m}, \mathbb{D}^{el}_{\omega,m}$ with pseudoscalar, scalar and electric potential, respectively, can be rewritten in biquaternionic form with the aid of the maps $\mathcal{A}$ and $\mathcal{A}^{-1}$. Using some projection operators (Section 3), the corresponding biquaternionic operators $R^{ps}_\alpha, R^{sc}_\alpha, R^{el}_\alpha$ can be reduced to an operator of the form $D - f_0 I$. In subsequent sections we obtained some null-solutions of this operator. Now, in order to write down the solutions for $\mathbb{D}^{ps}_{\omega,m}, \mathbb{D}^{sc}_{\omega,m}, \mathbb{D}^{el}_{\omega,m}$ we have to apply the inverse procedure. We consider here only the case of pseudoscalar potential – the two other cases may be considered by analogy.

Equality (9) gives us the equivalence

$$q \in \ker \mathbb{D}^{ps}_{\omega,m}(\Omega) \iff u := \mathcal{A}[q] \in \ker R^{ps}_\alpha(\tilde{\Omega}).$$

First, let $\omega^2 \neq m^2$. Then from (13) we have

$$u \in \ker R^{ps}_\alpha(\tilde{\Omega}) \iff u = P^+ u^+ + P^- u^-$$

where $P^\pm$ are defined by (12), u^+ and u^- are solutions of equations (14) and (15), respectively.

Now, let in (4) $g = g(\xi(x))$ where $\xi \in \Xi(\Omega)$. Note that in this case $\tilde{\xi} \in \Xi(\tilde{\Omega})$ where as above Ω may coincide with the whole $\mathbb{R}^3$. Using Theorem 1 we obtain that the function

$$u^+ = Q^+ h_1^+(\tilde{\xi}) A^+ + Q^- h_1^-(\tilde{\xi}) A^-$$

is a solution of equation (14) in $\tilde{\Omega}$ where

$$Q^\pm = \frac{1}{2}\left(1 \pm \frac{i\,\mathrm{grad}\,\tilde{\xi}}{|\mathrm{grad}\,\tilde{\xi}|}\right) \quad \text{and} \quad h_1^\pm(\tilde{\xi}) = \exp\left(\pm i \int \frac{ig(\tilde{\xi}) - \sqrt{\omega^2 - m^2}}{\gamma(\tilde{\xi})}\, d\tilde{\xi}\right)$$

and $A^\pm$ are arbitrary constant complex quaternions. By analogy,

$$u^- = Q^+ h_2^+(\tilde{\xi}) B^+ + Q^- h_2^-(\tilde{\xi}) B^-$$

is a solution of equation (15) in $\tilde{\Omega}$ where

$$h_2^\pm(\tilde{\xi}) = \exp\left(\pm i \int \frac{ig(\tilde{\xi}) + \sqrt{\omega^2 - m^2}}{\gamma(\tilde{\xi})}\, d\tilde{\xi}\right)$$

and $B^\pm$ are arbitrary constant complex quaternions. Thus,

$$u = P^+\left(Q^+ h_1^+(\tilde{\xi}) A^+ + Q^- h_1^-(\tilde{\xi}) A^-\right) + P^-\left(Q^+ h_2^+(\tilde{\xi}) B^+ + Q^- h_2^-(\tilde{\xi}) B^-\right)$$
$$\in \ker R^{ps}_\alpha(\tilde{\Omega}). \tag{46}$$

To obtain the corresponding solution q of equation (4) $(q = \mathcal{A}^{-1}[u])$ we have to rewrite equation (46) in component-wise form and then apply the map $\mathcal{A}^{-1}$. Let us introduce the notations

$$u^{++} = P^+ Q^+ h_1^+(\tilde{\xi}) A^+$$
$$u^{+-} = P^+ Q^- h_1^-(\tilde{\xi}) A^-$$
$$u^{-+} = P^- Q^+ h_2^+(\tilde{\xi}) B^+$$
$$u^{--} = P^- Q^- h_2^-(\tilde{\xi}) B^-.$$

Each of the four functions belongs to $\ker R^{ps}_\alpha(\tilde{\Omega})$. Thus, each of their images

$$q^{++} = \mathcal{A}^{-1}[u^{++}]$$
$$q^{+-} = \mathcal{A}^{-1}[u^{+-}]$$
$$q^{-+} = \mathcal{A}^{-1}[u^{-+}]$$
$$q^{--} = \mathcal{A}^{-1}[u^{--}]$$

satisfies equation (4). We will consider in detail the function u^{++} and the corresponding solution q^{++} of equation (4). For the three other solutions q^{+-}, q^{-+}, q^{--} we will write down only the final result.

First, let us rewrite u^{++} in the component-wise form

$$u^{++} = \frac{ih_1^+(\tilde{\xi})}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\tilde{\xi})} \Big\{ \big(-id_0\gamma(\tilde{\xi}) + d_1\partial_1\tilde{\xi} + d_2\partial_2\tilde{\xi} + d_3\partial_3\tilde{\xi} \big)i_0$$

$$+ \big(id_1\gamma(\tilde{\xi}) + d_0\partial_1\tilde{\xi} - d_3\partial_2\tilde{\xi} + d_2\partial_3\tilde{\xi} \big)i_1$$

$$+ \big(id_2\gamma(\tilde{\xi}) + d_3\partial_1\tilde{\xi} + d_0\partial_2\tilde{\xi} - d_1\partial_3\tilde{\xi} \big)i_2$$

$$+ \big(id_3\gamma(\tilde{\xi}) - d_2\partial_1\tilde{\xi} + d_1\partial_2\tilde{\xi} + d_0\partial_3\tilde{\xi} \big)i_3 \Big\}$$

where

$$d_0 = A_0^+\sqrt{\omega^2 - m^2} + i\omega A_1^+ + mA_2^+$$
$$d_1 = -A_1^+\sqrt{\omega^2 - m^2} + i\omega A_0^+ - mA_3^+$$
$$d_2 = -A_2^+\sqrt{\omega^2 - m^2} + i\omega A_3^+ + mA_0^+$$
$$d_3 = -A_3^+\sqrt{\omega^2 - m^2} - i\omega A_2^+ + mA_1^+.$$

Let us notice that although the constants $A_0^+, A_1^+, A_2^+, A_3^+$ are independent, the introduced constants d_0, d_1, d_2, d_3 due to the action of projection operators are not already independent. A simple calculation shows that

$$\left.\begin{array}{l} (\omega + m)(-d_0 + id_3) = -\sqrt{\omega^2 - m^2}(d_2 - id_1) \\ (\omega + m)(d_2 + id_1) = -\sqrt{\omega^2 - m^2}(d_0 + id_3) \end{array}\right\}. \tag{47}$$

Applying the transform $\mathcal{A}^{-1}$ to the function u^{++} we obtain the solution

$$q^{++} = \frac{h_1^+(\xi)}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\xi)}$$

$$\times \begin{pmatrix} (d_2 + id_1)\gamma(\xi) + (d_0 - id_3)\partial_1\xi - (d_3 + id_0)\partial_2\xi - (d_2 + id_1)\partial_3\xi \\ (-d_0 + id_3)\gamma(\xi) - (d_2 + id_1)\partial_1\xi + (d_1 - id_2)\partial_2\xi - (d_0 - id_3)\partial_3\xi \\ (d_0 + id_3)\gamma(\xi) - (d_2 - id_1)\partial_1\xi + (d_1 + id_2)\partial_2\xi - (d_0 + id_3)\partial_3\xi \\ (d_2 - id_1)\gamma(\xi) - (d_0 + id_3)\partial_1\xi + (d_3 - id_0)\partial_2\xi + (d_2 - id_1)\partial_3\xi \end{pmatrix}$$

of equation (4) in Ω. Finally, let us introduce the notations

$$a_1^+ = d_2 + id_1$$
$$a_2^+ = -d_0 + id_3.$$

Then taking into account relations (47) we rewrite the obtained solution in the form

$$q^{++} = \frac{h_1^+(\xi)}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\xi)} \begin{pmatrix} q_0^{++} \\ q_1^{++} \\ -\frac{(\omega+m)}{\sqrt{\omega^2-m^2}}q_0^{++} \\ -\frac{(\omega+m)}{\sqrt{\omega^2-m^2}}q_1^{++} \end{pmatrix}$$

where

$$q_0^{++} = a_1^+ \gamma(\xi) - a_2^+ \partial_1 \xi + i a_2^+ \partial_2 \xi - a_1^+ \partial_3 \xi$$
$$q_1^{++} = a_2^+ \gamma(\xi) - a_1^+ \partial_1 \xi - i a_1^+ \partial_2 \xi + a_2^+ \partial_3 \xi.$$

To verify that q^{++} is a solution of equation (4) when $g = g(\xi)$ with $\xi \in \Xi(\Omega)$, one can simply substitute it into the equation using its explicit form:

$$\left.\begin{array}{l} \partial_1 q_3 - i\partial_2 q_3 + \partial_3 q_2 - g q_2 + i(\omega + m)q_0 = 0 \\ \partial_1 q_2 + i\partial_2 q_2 - \partial_3 q_3 - g q_3 + i(\omega + m)q_1 = 0 \\ -\partial_1 q_1 + i\partial_2 q_1 - \partial_3 q_0 + g q_0 - i(\omega - m)q_2 = 0 \\ -\partial_1 q_0 - i\partial_2 q_0 + \partial_3 q_1 + g q_1 - i(\omega - m)q_3 = 0. \end{array}\right\}$$

By analogy, we obtain the other three solutions:

$$q^{+-} = \frac{h_1^-(\xi)}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\xi)} \begin{pmatrix} q_0^{+-} \\ q_1^{+-} \\ -\frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_0^{+-} \\ -\frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_1^{+-} \end{pmatrix} \tag{49}$$

where

$$q_0^{+-} = a_1^- \gamma(\xi) + a_2^- \partial_1 \xi - i a_2^- \partial_2 \xi + a_1^- \partial_3 \xi$$
$$q_1^{+-} = a_2^- \gamma(\xi) + a_1^- \partial_1 \xi + i a_1^- \partial_2 \xi - a_2^- \partial_3 \xi,$$

$$q^{-+} = \frac{h_2^+(\xi)}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\xi)} \begin{pmatrix} q_0^{-+} \\ q_1^{-+} \\ \frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_0^{-+} \\ \frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_1^{-+} \end{pmatrix} \tag{50}$$

where

$$q_0^{-+} = b_1^+ \gamma(\xi) - b_2^+ \partial_1 \xi + i b_2^+ \partial_2 \xi - b_1^+ \partial_3 \xi$$
$$q_1^{-+} = b_2^+ \gamma(\xi) - b_1^+ \partial_1 \xi - i b_1^+ \partial_2 \xi + b_2^+ \partial_3 \xi,$$

$$q^{--} = \frac{h_2^-(\xi)}{4\sqrt{\omega^2 - m^2} \cdot \gamma(\xi)} \begin{pmatrix} q_0^{--} \\ q_1^{--} \\ \frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_0^{--} \\ \frac{(\omega+m)}{\sqrt{\omega^2-m^2}} q_1^{--} \end{pmatrix} \tag{51}$$

where

$$q_0^{--} = b_1^- \gamma(\xi) + b_2^- \partial_1 \xi - i b_2^- \partial_2 \xi + b_1^- \partial_3 \xi$$
$$q_1^{--} = b_2^- \gamma(\xi) + b_1^- \partial_1 \xi + i b_1^- \partial_2 \xi - b_2^- \partial_3 \xi.$$

Thus, the following statement is proved.

Theorem 3. *The functions defined by (48) - (51), where $a_1^\pm, a_2^\pm, b_1^\pm, b_2^\pm \in \mathbb{C}$ are arbitrary (independent) constants, belong to $\ker \mathbb{D}_{\omega,m}^{ps}(\Omega)$ when $g = g(\xi)$, $\xi \in \Xi(\Omega)$ and $\omega^2 \neq m^2$.*

Now, let us consider the case $\omega^2 = m^2$. We have that if $v \in \ker(D - i\tilde{g}I)$, then $u := v\alpha \in \ker R_\alpha^{ps}$ (see Section 3). That is, the function

$$u = Q^+ h^+(\tilde{\xi})a^+\alpha + Q^- h^-(\tilde{\xi})a^-\alpha \qquad \text{where} \quad h^\pm(\tilde{\xi}) = e^{\mp \int \frac{g(\tilde{\xi})}{\gamma(\tilde{\xi})} d\tilde{\xi}}$$

and $a^\pm$ are arbitrary constant complex quaternions, is a solution of equation (17). Denote

$$u^+ = Q^+ h^+(\tilde{\xi})a^+\alpha$$
$$u^- = Q^- h^-(\tilde{\xi})a^-\alpha.$$

Then applying the transformation $\mathcal{A}^{-1}$ to u^+ and u^- we obtain the corresponding solutions $q^+ = \mathcal{A}^{-1}[u^+]$ and $q^- = \mathcal{A}^{-1}[u^-]$ of equation (4):

$$q^+ = \frac{h^+(\xi)}{2\gamma(\xi)} \times$$

$$\begin{pmatrix} (\omega - m)((-a_0^+ + ia_3^+)\gamma(\xi) - (a_2^+ - ia_1^+)\partial_1\xi + i(a_2^+ - ia_1^+)\partial_2\xi + (a_0^+ - ia_3^+)\partial_3\xi) \\ (\omega - m)((a_2^+ - ia_1^+)\gamma(\xi) + (a_0^+ - ia_3^+)\partial_1\xi + i(a_0^+ - ia_3^+)\partial_2\xi - (a_2^+ - ia_1^+)\partial_3\xi) \\ (\omega + m)((a_2^+ + ia_1^+)\gamma(\xi) - (a_0^+ + ia_3^+)\partial_1\xi + i(a_0^+ + ia_3^+)\partial_2\xi - (a_2^+ + ia_1^+)\partial_3\xi) \\ (\omega + m)((a_0^+ + ia_3^+)\gamma(\xi) - (a_2^+ + ia_1^+)\partial_1\xi - i(a_2^+ + ia_1^+)\partial_2\xi + (a_0^+ + ia_3^+)\partial_3\xi) \end{pmatrix}$$

and

$$q^- = \frac{h^-(\xi)}{2\gamma(\xi)} \times$$

$$\begin{pmatrix} (\omega - m)((-a_0^- + ia_3^-)\gamma(\xi) + (a_2^- - ia_1^-)\partial_1\xi - i(a_2^- - ia_1^-)\partial_2\xi - (a_0^- - ia_3^-)\partial_3\xi) \\ (\omega - m)((a_2^- - ia_1^-)\gamma(\xi) - (a_0^- - ia_3^-)\partial_1\xi - i(a_0^- - ia_3^-)\partial_2\xi + (a_2^- - ia_1^-)\partial_3\xi) \\ (\omega + m)((a_2^- + ia_1^-)\gamma(\xi) + (a_0^- + ia_3^-)\partial_1\xi - i(a_0^- + ia_3^-)\partial_2\xi + (a_2^- + ia_1^-)\partial_3\xi) \\ (\omega + m)((a_0^- + ia_3^-)\gamma(\xi) + (a_2^- + ia_1^-)\partial_1\xi + i(a_2^- + ia_1^-)\partial_2\xi - (a_0^- + ia_3^-)\partial_3\xi) \end{pmatrix}$$

Note that for both cases $\omega = m$ or $\omega = -m$ we obtain two-component solutions. When $\omega = m$, the two first components of each solution are zero. Similarly, when $\omega = -m$, the last two components are zero. Thus, the following statement is true.

Theorem 4. *The functions q^+ and q^-, where $a_k^\pm \in \mathbb{C}$ ($k = 0, \ldots, 3$) are arbitrary (independent) constants, belong to $\ker \mathbb{D}_{\omega,m}^{ps}(\Omega)$ when $g = g(\xi)$, $\xi \in \Xi(\Omega)$ and $\omega^2 = m^2$.*

Concluding Remark. A new method for obtaining solutions of the Dirac equation and of the Schrödinger equation is proposed. The method permits to construct exact solutions for any potential which depends on a combination ξ of the variables x_1, x_2, x_3 from a quite ample class which we called Ξ-class. At least, we obtain solutions for any

potential depending on the linear combination $\xi = ax_1 + bx_2 + cx_3 + d$. Moreover, for the linear combination we are able to construct a fundamental solution for the Dirac operator. The main tool of the method are some special biquaternionic projection operators which are constructed after having solved an eikonal equation corresponding to the combination ξ. Let us notice that the use of biquaternionic form of the Dirac operator and the methods of biquaternionic analysis allowed us to make all essential calculations quite transparent. Perhaps the most laborious part is the return of the solutions obtained in quaternionic form into the "traditional" form, but after all this is only arithmetic.

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