Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 3, 677–693

On the Method of Approximation by Families of Linear Polynomial Operators

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Abstract. It is shown that best approximation by trigonometric polynomials is achieved in average by families of linear polynomial operators in the L_p-metric for all p, $0 < p \le \infty$. This is compared with approximation by Fourier means and interpolation means which is restricted to $1 \le p \le \infty$ and $p = \infty$, respectively.

Keywords: *Families of linear polynomial operators, best approximation by trigonometric polynomials, Fourier and interpolation means*

AMS subject classification: 42A10, 42A15

0. Introduction

In the present paper we deal with the method of trigonometric approximation of 2π-periodic functions by families of linear polynomial operators introduced in [6 - 8] for L_p-spaces with $0 < p \le 1$. It was proved that approximation by families in such spaces enables to achieve the best order of approximation. We will show that the same result is also valid in L_p-spaces with $1 < p \le +\infty$, that is, the method turns out to be an universal method of approximation for all L_p-spaces with $0 < p \le +\infty$ in difference to the methods of approximation by the Fourier means that make sense only for $1 \le p \le +\infty$ as well as to the methods of approximation by the interpolation means which is restricted to continuous functions.

We will study approximative properties of operators $\{\mathcal{L}_{n;\lambda}\}_{\lambda\in\mathbb{R},n\in\mathbb{N}_0}$ given by

$$\mathcal{L}_{n;\lambda}(f;x) = \tfrac{1}{2N+1} \sum_{k=0}^{2N} f\left(\frac{2\pi k}{2N+1} + \lambda\right) W_n\left(x - \frac{2\pi k}{2N+1} - \lambda\right)$$

where W_n is a kernel of Vallée-Poussin type. The main result is Theorem 4.2 where it is shown that

$$\left(\int_0^{2\pi} \|f - \mathcal{L}_{n;\lambda}(f)\|_p^p \, d\lambda\right)^{\frac{1}{p}} \le C_p E_n(f)_p$$

for all $f \in L_p$ $\quad(0 < p \le +\infty)$. Here $E_n(f)_p$ denotes the best L_p-approximation by trigonometric polynomials of order at most n. It means that the best order of

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ISSN 0232-2064 / \$ 2.50

approximation is achieved in average. For fixed λ (e.g. $\lambda = 0$) this is true for $p = +\infty$ only. Furthermore, Theorem 4.3 deals with the case of randomly chosen parameters. Then the best order of approximation can be realized with a priori given probability.

The paper is organized as follows. In Section 1 we introduce notations and give some definitions we will use henceforth. In Section 2 we study properties of trigonometric kernels. For the convenience of the reader we collect some well-known facts of the theory of approximation by linear operators in Section 3. In Section 4 we introduce families of linear polynomial operators and study their approximative properties. Here we also compare various methods of approximation mentioned above.

1. Preliminaries

1. We will deal with 2π-periodic functions in L_p, where $0 < p \le +\infty$ (as usual, $L_\infty = C$ with a standard norm), that sometimes will depend not only on the main variable x, but also on a parameter λ. By $\|\cdot\|_p$ or $\|\cdot\|_{p;x}$ we denote the p-norm (quasi-norm, if $0 < p < 1$) on x. For the p-(quasi-)norm on the parameter λ we use the symbol $\|\cdot\|_{p;\lambda}$. The notation $\|\cdot\|_{\mathbf{p}}$ will be used for $\big\| \; \|\cdot\|_{p;x} \big\|_{p;\lambda}$, that is, for the p-(quasi-) norm on x and λ together. For the sake of simplicity, we will use the notation "norm" for all $0 < p \le +\infty$.

2. As usual, the norm of a linear operator $\mathcal{L}$ in L_p $(0 < p \le +\infty)$ is given by

$$\|\mathcal{L}\|_{(p)} = \sup_{\|f\|_p = 1} \|\mathcal{L}f\|_p.$$

An operator $\mathcal{L}$ is bounded, if its norm is finite. By $\mathcal{S}_\lambda$ we denote the translation operator defined by

$$\mathcal{S}_\lambda f(x) = f(x + \lambda).$$

Clearly, its norm is equal to 1 for all $0 < p \le +\infty$.

3. Apart from single operators we will also consider one-parametric families of linear operators $\{\mathcal{L}_\lambda\}_{\lambda \in \mathbb{R}}$ in L_p $(0 < p \le +\infty)$ that are 2π-periodic on λ. We define the averaged norm of such family by

$$\begin{aligned} \|\{\mathcal{L}_\lambda\}\|_{(p)} &= (2\pi)^{-\frac{1}{p}} \sup_{\|f\|_p = 1} \|\mathcal{L}_\lambda f(x)\|_{\mathbf{p}} \\ &= \sup_{\|f\|_p = 1} \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} \|\mathcal{L}_\lambda f(x)\|_p^p \, d\lambda \right)^{\frac{1}{p}} & \text{if } 0 < p < +\infty \\ \max_\lambda \|\mathcal{L}_\lambda f(x)\|_\infty & \text{if } p = +\infty. \end{cases} \end{aligned}$$

In analogy to the case of operators we will say that a family $\{\mathcal{L}_\lambda\}$ is *bounded* in L_p, if its averaged norm is finite. The family of translation operators $\{\mathcal{S}_\lambda\}$ is an example of a family that is bounded for all $0 < p \le +\infty$.

4. By the symbol $f * g$ we denote the convolution of 2π-periodic L_1-functions f and g, that is

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + h) g(h) \, dh = \frac{1}{2\pi} \int_0^{2\pi} f(h) g(h - x) \, dh. \tag{1.1}$$

For 2π-periodic functions f and g that are defined on $\mathbb{R}$ we introduce its discrete counterpart

$$(f * g)_{(N)}(x) = \tfrac{1}{2N+1} \sum_{k=0}^{2N} f(t_N^k) g(t_N^k - x) \tag{1.2}$$

that we will call N-*discrete convolution* of f and g. In (1.2) $N \in \mathbb{N}_0$ and $t_N^k = \frac{2\pi k}{2N+1}$ $(k = 0, 1, \ldots, 2N)$ are points of uniform partition of $[0, 2\pi)$.

5. As usual, the best trigonometric approximation of a function f in L_p $(0 < p \leq +\infty)$ of order $n \in \mathbb{N}_0$ is given by

$$E_n(f)_p = \inf_{T \in \mathcal{T}_n} \|f - T\|_p$$

where $\mathcal{T}_n$ is the space of real-valued trigonometric polynomials of order at most n.

6. The Fourier transform of a function f that belongs to L_1 on $\mathbb{R}$ is defined by

$$\hat{f}(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-i\xi x} d_\xi.$$

2. Trigonometric kernels and their properties

Throughout the paper we will study trigonometric polynomials W_n $(n \in \mathbb{N}_0)$ of the special type

$$\left.\begin{aligned} &W_0(x) = 1 \\ &W_n(x) = \sum_{k \in \mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx} \quad (n \in \mathbb{N}) \end{aligned}\right\} \tag{2.1}$$

that are usually called *kernels*. Here we suppose that ψ is an even function defined on $\mathbb{R}$, such that

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 + \delta \end{cases} \tag{2.1$'$}$$

where δ is some positive number.

Lemma 2.1.

1. *The function* W_n $(n \in \mathbb{N}_0)$ *is a real-valued even trigonometric polynomial of order at most* $N = [(1+\delta)n]$.

2. *For any* T *in* $\mathcal{T}_n$ $(n \in \mathbb{N}_0)$ *we have*

$$(T * W_n)(x) = T(x) \tag{2.2}$$

$$(T * W_n)_{(N)}(x) = T(x) \tag{2.3}$$

for all $x \in \mathbb{R}$.

Proof. Part 1 follows immediately from the definition of kernels. It is clear because of linearity that it is sufficient to prove (2.2) and (2.3) for harmonics e^{imx} with $|m| \le n$. We get

$$\begin{aligned}(e^{im\,\cdot} * W_n)(x) &= \tfrac{1}{2\pi}\int_0^{2\pi} e^{imh} W_n(x-h)\,dh \\ &= \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx}\left(\tfrac{1}{2\pi}\int_0^{2\pi} e^{imh}e^{-ikh}dh\right) \\ &= \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big)\delta_{mk}e^{ikx} \\ &= e^{imx}.\end{aligned}$$

Now we recall that for $s \in \mathbb{Z}$

$$\delta(s;N) = \sum_{k=0}^{2N} \exp\left(\frac{2\pi i s k}{2N+1}\right) = \begin{cases} 2N+1 & \text{if } s \equiv 0(\mathrm{mod}(2N+1)) \\ 0 & \text{otherwise.}\end{cases} \tag{2.4}$$

Using (1.2) and (2.1) we have for $|m| \le n$

$$\begin{aligned}(e^{im\,\cdot} * W_n)_{(N)}(x) &= \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx}\,\tfrac{1}{2N+1}\sum_{\nu=0}^{2N}\exp(imt_N^\nu)\,\exp(-ikt_N^\nu) \\ &= \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx}\,\tfrac{1}{2N+1}\sum_{\nu=0}^{2N}\exp\left(\frac{2\pi i(m-k)}{2N+1}\nu\right) \\ &= \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx}\,\tfrac{1}{2N+1}\,\delta(m-k;N).\end{aligned} \tag{2.5}$$

Since $\psi\left(\frac{k}{n}\right) = 0$ for $|k| > N$, the summation in (2.5) is for $|k| \le N$, and as $|m| \le n$, we have $|m-k| \le |m| + |k| \le n + N \le 2N < 2N+1$. Therefore, $m - k \equiv 0(\mathrm{mod}(2N+1))$ if and only if $m = k$, and now (2.4) implies

$$(e^{im\,\cdot} * W_n)_{(N)}(x) = \psi\Big(\frac{m}{n}\Big)e^{imx} = e^{imx}.$$

The proof is complete ∎

For $n \in \mathbb{N}_0$, $N = [(1+\delta)n]$, $g \in L_p$ $(0 < p \le +\infty)$ let us define

$$J_{N;p}(g) = \begin{cases} (2\pi)^{-\frac{1}{p}}\|g\|_p & \text{if } 0 < p \le 1 \\ \frac{1}{2N+1}\sum_{k=0}^{2N}\left(\frac{1}{\tau}\int_{t_N^k}^{t_N^{k+1}} |g(h)|^p dh\right)^{\frac{1}{p}} & \text{if } 1 \le p + \infty \\ \frac{1}{2N+1}\sum_{k=0}^{2N} \max\limits_{h\in[t_N^k, t_N^{k+1}]} |g(h)| & \text{if } p = +\infty \end{cases}$$

where $\tau = \frac{2\pi}{2N+1}$. We will often denote $J_{N;p}(W_n)$ by $J_{n;p}$.

Lemma 2.2. *Let $0 < p \le 1$. If ψ defined by $(2.1)'$ is continuous and $\hat{\psi}$ belongs to $L_p(\mathbb{R})$, then*

$$\left.\begin{aligned} &J_{0;p} = 1 \\ &J_{n;p} \le (2\pi)^{\frac{1}{2}-\frac{1}{p}} \|\hat{\psi}\|_{L_p(\mathbb{R})} n^{1-\frac{1}{p}} \quad (n \in \mathbb{N}) \end{aligned}\right\}. \tag{2.6}$$

Proof. To prove this lemma we need the Poisson summation formula

$$W_n(x) = \sum_{k\in\mathbb{Z}} \psi\Big(\frac{k}{n}\Big) e^{ikx} = \sqrt{2\pi}\, n \sum_{k\in\mathbb{Z}} \hat{\psi}(n(x+2\pi k)). \tag{2.7}$$

It is proved [11: p. 252/Corollary 2.6] that, for φ belonging to the Schwartz space S of rapidly decreasing test functions, (2.7) is valid.

If $\hat{\psi} \in L_p(\mathbb{R})$ for $0 < p < 1$, then $\hat{\psi} \in L_1(\mathbb{R})$ and, hence, it is enough to prove (2.7) for $p = 1$. It is known [13: p. 22/Section 1.4.1] that for any $\varepsilon > 0$ there is a function $\varphi \in S$ with compact support such that

$$\|\hat{\psi} - \hat{\varphi}\|_{L_1(\mathbb{R})} < \frac{\varepsilon}{2\sqrt{2\pi}(N+1)}. \tag{2.8}$$

We denote by I_ψ and J_ψ the left- and right-hand sides of (2.7), respectively. Then we get

$$\begin{aligned} \|J_\psi - J_\varphi\|_1 &\le \sqrt{2\pi}\, n \sum_{k\in\mathbb{Z}} \big\|\hat{\psi}(n(x+2\pi\nu)) - \hat{\varphi}(n(x+2\pi\nu))\big\|_1 \\ &= \sqrt{2\pi}\, n \big\|\hat{\psi}(nx) - \hat{\varphi}(nx)\big\|_{L_1(\mathbb{R})} \\ &= \sqrt{2\pi}\, \|\hat{\psi} - \hat{\varphi}\|_{L_1(\mathbb{R})}. \end{aligned} \tag{2.9}$$

Furthermore,

$$\begin{aligned} \|I_\psi - I_\varphi\|_1 &\le \sum_{k\in\mathbb{Z}} \Big|\psi\Big(\frac{k}{n}\Big) - \varphi\Big(\frac{k}{n}\Big)\Big|\, \|e^{ikx}\|_1 \\ &\le 2\pi(2N+1)\, \|\psi - \varphi\|_{C(\mathbb{R})} \\ &\le \sqrt{2\pi}(2N+1)\, \|\hat{\psi} - \hat{\varphi}\|_{L_1(\mathbb{R})}. \end{aligned} \tag{2.10}$$

Recalling that $I_\varphi = J_\varphi$, we get from (2.8) - (2.10)

$$\|I_\psi - J_\psi\|_1 \le \|I_\psi - I_\varphi\|_1 + \|J_\psi - J_\varphi\|_1 \le 2\sqrt{2\pi}(N+1)\, \|\hat{\psi} - \hat{\varphi}\|_{L_1(\mathbb{R})} < \varepsilon$$

that completes the proof of (2.7).

Now we prove (2.6). We have from (2.7) (using $0 < p < 1$)

$$\begin{aligned} \|W_n\|_p^p &\le (2\pi)^{\frac{p}{2}} n^p \sum_{k\in\mathbb{Z}} \|\hat{\psi}(n(x+2\pi k))\|_p^p \\ &= (2\pi)^{\frac{p}{2}} n^p \|\hat{\psi}(nx)\|_{L_p(\mathbb{R})}^p \\ &= (2\pi)^{\frac{p}{2}} n^{p-1} \|\hat{\psi}\|_{L_p(\mathbb{R})}^p. \end{aligned}$$

The proof is finished ∎

Lemma 2.3. *Let $1 \le p \le +\infty$. If ψ defined by $(2.1)'$ is continuous and $\hat{\psi}$ belongs to $L_1(\mathbb{R})$, then*

$$\left.\begin{aligned} &J_{0;p} = 1 \\ &J_{n;p} \le J_{n;\infty} \le (\pi+1)(2\pi)^{-\frac{1}{2}} \|\hat{\psi}\|_{L_1(\mathbb{R})} \quad (n \ge 1) \end{aligned}\right\}. \tag{2.11}$$

Proof. The first part of inequality (2.11) is obvious. There exist points $\xi_N^k \in [t_N^k, t_N^{k+1}]$ $(k = 0, 1, \ldots, 2N)$ such that

$$J_{n;\infty} = \tfrac{1}{2N+1} \sum_{k=0}^{2N} \max_{h \in [t_N^k, t_N^{k+1}]} |W_n(h)| = \tfrac{1}{2N+1} \sum_{k=0}^{2N} |W_n(\xi_N^k)|. \tag{2.12}$$

We consider the function

$$F(\lambda) = \tfrac{1}{2N+1} \sum_{k=0}^{2N} |W_n(t_N^k + \lambda)|.$$

Clearly $\left(\tau = t_N^1 = \frac{2\pi}{2N+1}\right)$,

$$\tfrac{1}{\tau} \int_0^\tau F(\lambda)\, d\lambda = \tfrac{1}{2\pi} \sum_{k=0}^{2N} \int_{t_N^k}^{t_N^{k+1}} |W_n(h)|\, dh = \tfrac{1}{2\pi} \|W_n\|_1. \tag{2.13}$$

Hence, there is a number $\lambda^* \in [0, \tau]$ such that

$$F(\lambda^*) = \tfrac{1}{2\pi} \|W_n\|_1. \tag{2.14}$$

By using Bernstein's inequality that is valid with constant 1 (see [1]), we get from (2.12) and (2.14)

$$\begin{aligned} J_{n;\infty} &\le \tfrac{1}{2N+1} \sum_{k=0}^{2N} \left| W_n(t_N^k + \lambda^*) - W_n(\xi_N^k) \right| + \tfrac{1}{2\pi} \|W_n\|_1 \\ &\le \tfrac{1}{2N+1} \sum_{k=0}^{2N} \int_{t_N^k}^{t_N^{k+1}} |W_n'(h)|\, dh + \tfrac{1}{2\pi} \|W_n\|_1 \\ &\le \tfrac{1}{2N+1} \|W_n'\|_1 + \tfrac{1}{2\pi} \|W_n\|_1 \\ &\le \left(\tfrac{N}{2N+1} + \tfrac{1}{2\pi}\right) \|W_n\|_1 \\ &\le \tfrac{1}{2}\left(1 + \tfrac{1}{\pi}\right) \|W_n\|_1 \\ &= (\pi + 1) J_{n;1}. \end{aligned} \tag{2.15}$$

Applying (2.6) with $p = 1$ we obtain (2.11) ∎

Remark 2.2. In this section we considered kernels that guarantee "a preserving" of a trigonometric polynomial in the sense of part 2 of Lemma 2.1. As is well-known, the Dirichlet kernel that corresponds to the characteristic function of $[-1, 1]$ as well as the Vallée-Poussin kernel that corresponds to ψ with extralinear connection on the segments

$[-1-\delta, -1]$ and $[1, 1+\delta]$ have this property. We also mention that in approximation theory there are many other kernels like kernels of Fejér, Favard, Jackson, Rogozinski, Bochner-Riesz that do not preserve a polynomial. Their properties are described in many papers and monographs (see, for example, [12]).

The idea to present kernels in the form (2.1) as well as the idea to apply Fourier transform methods to study their properties was developed in [3, 9] and many other papers. For example, an approach based on the use of Nikolskii's inequality was elaborated in [9]. The idea to use Poisson's summation formula appeared in [3].

Some sufficient conditions on a function ψ that guarantee the validity of Poisson's summation formula (2.7) were obtained in terms of smooth characteristics of ψ or in terms of its Fourier transform (see, for example, [11]). The condition $\hat{\psi} \in L_1$ seems to be new. We want to emphasize that the proof of (2.7) with this condition was based on properties of the spaces L_p^Ω [13: p. 24].

3. Approximation by linear polynomial operators

In this section we discuss some aspects of the classical and wide-spread method of approximation by linear polynomial operators. We will also show that this method turns out to be unapplicable in the case $0 < p < 1$. Usually, they deal with Fourier means and interpolation means that are given by

$$\mathcal{F}_n(f) = f * W_n \tag{3.1}$$
$$\mathcal{I}_n(f) = (f * W_n)_{(N)} \tag{3.2}$$

respectively. In (3.1) and (3.2) $N = [(1+\delta)n]$. W_n is defined by (2.1) and (2.1)$'$, and the convolution and its discrete counterpart are given by (1.1) and (1.2), respectively. The Fourier means are linear operators mapping L_p $(1 \le p \le +\infty)$ into $\mathcal{T}_N$. The interpolation means that are correctly defined on C and are linear mappings from C into $\mathcal{T}_N$.

In this paper we consider only one of the approximation problems connected with the constructions (3.1) and (3.2), namely, we are interested in conditions implying the validity of the inequality ($\mathcal{L}_n$ is $\mathcal{F}_n$ or $\mathcal{I}_n$)

$$\|f - \mathcal{L}_n(f)\|_p \le C_1 E_n(f)_p \tag{3.3}$$

where the positive constant C_1 does not depend on f and n. Inequality (3.3) has a long history. As is known, if W_n is the Vallée-Poussin kernel, (3.3) is valid for all $1 \le p \le +\infty$ in the case of Fourier means and for $p = +\infty$ in the case of interpolation means. In the case of the Dirichlet kernel inequality (3.3) fails for both types of operators, if $p = +\infty$, and additionally for the type (3.1), if $p = 1$. However, as it was proved by M. Riesz, (3.3) holds in this case, if $1 < p < +\infty$. For more details and other references we refer, for instance, to [4: Chapter 4].

Now let us consider the general case.

Definition 3.1. A sequence of linear operators $(\mathcal{L}_n)_{n\in\mathbb{N}_0}$ mapping $L_p \quad (1 \leq p \leq +\infty)$ into $\mathcal{T}_N \quad (N = [(1+\delta)n], \delta \geq 0)$ is of *Vallée-Poussin type* in L_p, if

1. $\mathcal{L}_n(T) = T$ for any $T \in \mathcal{T}_n \quad (n \in \mathbb{N}_0)$

2. $\|\mathcal{L}_n\|_{(p)} \leq C_2$, where the positive constant C_2 does not depend on n.

Lemma 3.1. *If $(\mathcal{L}_n)_{n\in\mathbb{N}_0}$ is of Vallée-Poussin type in L_p for some $1 \leq p \leq +\infty$, then* (3.3) *is valid.*

Proof. For an arbitrary $\varepsilon > 0$ we consider T in $\mathcal{T}_n$ such that

$$\|f - T\|_p \leq E_n(f)_p + \varepsilon.$$

Now we get

$$\begin{aligned}\|f - \mathcal{L}_n(f)\|_p &= \|f - T + \mathcal{L}_n(T) - \mathcal{L}_n(f)\|_p \\ &\leq \|f - T\|_p + \|\mathcal{L}_n(f - T)\|_p \\ &\leq (1 + \|\mathcal{L}_n\|_{(p)})\,(E_n(f)_p + \varepsilon) \\ &\leq C_1(E_n(f)_p + \varepsilon)\end{aligned}$$

that implies (3.3) ∎

At first we consider Fourier means.

Theorem 3.1. *If ψ defined by* $(2.1)'$ *is continuous and $\hat{\psi}$ belongs to $L_1(\mathbb{R})$, then the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ given by* (3.1) *is of Vallée-Poussin type in L_p for any $1 \leq p \leq +\infty$ and* (3.3) *holds.*

Proof. As it follows from part 2 of Lemma 2.1, $\mathcal{F}_n$ preserves any polynomial T in $\mathcal{T}_n$. On the basis of Lemma 2.2 and the generalized Minkovski inequality we get

$$\begin{aligned}\|\mathcal{F}_n\|_{(p)} &= \sup_{\|f\|_p=1} \|\mathcal{F}_n f\|_p \\ &\leq \tfrac{1}{2\pi} \sup_{\|f\|_p=1} \int_0^{2\pi} \|f(x+h)\|_p |W_n(h)|\, dh \\ &\leq \tfrac{1}{2\pi}\, \|W_n\|_1 \\ &< C_2\end{aligned}$$

that completes the proof ∎

Theorem 3.2. *If ψ defined by* $(2.1)'$ *is continuous and $\hat{\psi}$ belongs to $L_1(\mathbb{R})$, the sequence $(\mathcal{I}_n)_{n\in\mathbb{N}_0}$ given by* (3.2) *is of Vallée-Poussin type in L_∞ and* (3.3) *holds for $p = +\infty$.*

Proof. As it follows from part 2 of Lemma 2.1, $\mathcal{I}_n$ preserves any polynomial T in

$\mathcal{T}_n$. On the basis of Lemma 2.3 we get

$$\begin{aligned}\|\mathcal{I}_n\|_{(\infty)} &\le \tfrac{1}{2N+1} \sup_{\|f\|_\infty=1} \max_{x\in[0,2\pi)} \sum_{k=o}^{2N} |f(t_N^k)|\,|W_n(x-t_N^k)| \\ &\le \tfrac{1}{2N+1} \sum_{k=o}^{2N} \max_{x\in[0,\tau]} |W_n(x-t_N^k)| \\ &= J_{n;\infty} \\ &\le C_2\end{aligned}$$

that completes the proof ∎

Remark 3.1. It follows from Lemmas 2.2, 2.3 and the proofs of Theorems 3.1, 3.2 that in the case of Fourier means (3.3) is valid for all $1 \le p \le +\infty$ with the constant

$$C_1 = 1 + \tfrac{1}{\sqrt{2\pi}}\,\|\hat{\psi}\|_{L_1(\mathbb{R})}$$

and in the case of interpolation means for $p = +\infty$ with the constant

$$C_1 = 1 + \tfrac{\pi+1}{\sqrt{2\pi}}\,\|\hat{\psi}\|_{L_1(\mathbb{R})}.$$

In Lemma 3.1 and Theorems 3.1, 3.2 we followed the classical scheme that has many applications in approximation theory. In the next section we will show how this scheme works in the case of families of linear polynomial operators.

We notice that the requirement for operators to be bounded is quite natural, because of its necessity for the validity of (3.3). Indeed, for $0 < p \le +\infty$ inequality (3.3) implies ($\tilde{p} = \min(1,p)$)

$$\|\mathcal{L}_n f\|_p^{\tilde{p}} \le \|f\|_p^{\hat{p}} + \|f - \mathcal{L}_n(f)\|_p^{\tilde{p}} \le \|f\|_p^{\tilde{p}} + C_1^{\tilde{p}}\,E_n(f)_p^{\tilde{p}} \le (1 + C_1^{\tilde{p}})\,\|f\|_p^{\tilde{p}}$$

for all $f \in L_p$.

In contrast to the case $1 \le p \le +\infty$ the following statement holds.

Lemma 3.2. *For any $m \in \mathbb{N}_0$ and $0 < p < 1$ there are no non-trivial linear bounded operators mapping L_p into $\mathcal{T}_m \subset L_p$.*

Proof. On the contrary, if $\mathcal{L}$ is such an operator, by virtue of Nikolski's inequality [10: p. 145] we get for any f in L_p

$$\|\mathcal{L}f\|_1 \le c(m+1)^{\frac{1}{p}-1}\|\mathcal{L}f\|_p \le c(m+1)^{\frac{1}{p}-1}\,\|\mathcal{L}\|_p\|f\|_p$$

and $\mathcal{L}$ is a bounded operator from L_p into L_1, that contradicts to the fact that for $0 < p < 1$ there are no non-trivial linear bounded operators mapping L_p into any Banach space X (see, for instance, [5: p. 37]) ∎

4. Approximation by families of linear polynomial operators

It will be shown in this section that bounded families of linear polynomial operators exist in all L_p-spaces with $0 < p \le +\infty$ in contrast to single operators existing for $1 \le p \le +\infty$ only. We will also see that families have approximative properties that are similar to the properties of operators in a certain sense. We apply the scheme described in the previous section and we give its realization by constructing appropriate families of linear polynomial operators.

We consider a family of linear operators $\{\mathcal{L}_{n;\lambda}\}_{\lambda\in\mathbb{R}}$ mapping L_p $(0 < p \le +\infty)$ into $\mathcal{T}_N$, where, as usual, $N = [(1+\delta)n]$ for some $\delta \ge 0$. We suppose that $\mathcal{L}_{n;\lambda} = \mathcal{L}_{n;\lambda+2\pi}$ for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, and for each $f \in L_p$ the function $\mathcal{L}_{n;\lambda}(f;x)$ of variables λ and x belongs to L_p on the two-dimensional torus $\mathbb{T}^2$.

Definition 4.1. A family $\{\mathcal{L}_{n;\lambda}\}$ is of *Vallée-Poussin type* in L_p $(0 < p \le +\infty)$ if

1. $\mathcal{L}_{n;\lambda}(T) = T$ for any $T \in \mathcal{T}_n$ $(n \in \mathbb{N}_0)$ and $\lambda \in \mathbb{R}$

2. $\|\{\mathcal{L}_{n;\lambda}\}\|_{(p)} \le C_3$, where the positive constant C_3 does not depend on n.

Theorem 4.1. *If $\{\mathcal{L}_{n;\lambda}\}$ is of Vallée-Poussin type in L_p $(0 < p \le +\infty)$, then*

$$(2\pi)^{-\frac{1}{p}}\|f - \mathcal{L}_{n;\lambda}(f)\|_{\mathbf{p}} \le C\,E_n(f)_p \qquad (n \in \mathbb{N}_0) \tag{4.1}$$

where the positive constant C does not depend on f and n.

Proof. We put $\tilde{p} = \min\{1, p\}$. For an arbitrary $\varepsilon > 0$ we consider $T \in \mathcal{T}_n$ such that

$$\|f - T\|_p \le E_n(f)_p + \varepsilon.$$

Then, for any real λ,

$$\begin{aligned}\|f - \mathcal{L}_{n;\lambda}(f)\|_p^{\tilde{p}} &\le \|f - T + \mathcal{L}_{n;\lambda}(T) - \mathcal{L}_{n;\lambda}(f)\|_p^{\tilde{p}} \\ &\le \|f - T\|_p^{\tilde{p}} + \|\mathcal{L}_{n;\lambda}(f - T)\|_p^{\tilde{p}}.\end{aligned} \tag{4.2}$$

Taking the p-norm with respect to λ if $1 \le p \le +\infty$, and and integrating with respect to λ from 0 to 2π if $0 < p < 1$, on both sides of inequality (4.2), we obtain

$$\begin{aligned}(2\pi)^{-\frac{\tilde{p}}{p}}\|f - \mathcal{L}_{n;\lambda}(f)\|_{\mathbf{p}}^{\tilde{p}} &\le (2\pi)^{-\frac{\tilde{p}}{p}}\|f - T\|_{\mathbf{p}}^{\tilde{p}} + (2\pi)^{-\frac{\tilde{p}}{p}}\|\mathcal{L}_{n;\lambda}(f - T)\|_{\mathbf{p}}^{\tilde{p}} \\ &\le \|f - T\|_p^{\tilde{p}} + \|\{\mathcal{L}_{n;\lambda}\}\|_{(p)}^{\tilde{p}}\|f - T\|_p^{\tilde{p}} \\ &\le \left(1 + \|\{\mathcal{L}_{n;\lambda}\}\|_{(p)}^{\tilde{p}}\right)\left(E_n(f)_p + \varepsilon\right)^{\tilde{p}}\end{aligned}$$

that implies (4.1) ∎

Remark 4.1. The proof of Theorem 4.1 enables us to estimate the constant C. More precisely, (4.1) is valid with

$$C = \Big(1 + \sup_n \|\{\mathcal{L}_{n;\lambda}\}\|_{(p)}^{\tilde{p}}\big)^{\frac{1}{\tilde{p}}} = \sup_n C(n). \tag{4.3}$$

For computational needs, when n is a priori given, one can use the constant $C(n)$.

Let now δ be a non-negative real number, W_n $(n \in \mathbb{N}_0)$ be given by (2.1) and (2.1)$'$, $N = [(1+\delta)n]$. Henceforth, we deal with the family

$$\mathcal{L}_{n;\lambda}(\cdot) = S_{-\lambda}(S_\lambda(\cdot) * W_n)_{(N)}, \tag{4.4}$$

that is,

$$\mathcal{L}_{n;\lambda}(f;x) = \tfrac{1}{2N+1} \sum_{k=0}^{2N} f(t_N^k + \lambda) W_n(x - t_N^k - \lambda). \tag{4.5}$$

It follows immediately from (4.5) that if a 2π-periodic function f is defined almost everywhere on $[0, 2\pi]$, then for almost every λ all numbers $f(t_N^k + \lambda)$ $(k = 0, 1, \ldots, 2N)$ are defined and the function $\mathcal{L}_{n;\lambda}(f;x)$ of the variable x is a trigonometric polynomial of order at most N. Moreover, if the functions f_1 and f_2 coincide almost everywhere, then $\mathcal{L}_{n;\lambda}(f_1;x) = \mathcal{L}_{n;\lambda}(f_2;x)$ $(x \in \mathbb{R})$ for almost every λ. These remarks show that the families (4.4) are correctly defined for all L_p-spaces with $0 < p \leq +\infty$. As we see, the introduction of the parameter, that has the sense of translation of knots, enables us to extend the construction of interpolation, that make sense only for $p = +\infty$, to the general case $0 < p \leq +\infty$.

Theorem 4.2. *Let $0 < p \leq +\infty$. If ψ defined by (2.1)$'$ is a continuous and $\hat{\psi}$ belongs to $L_{\tilde{p}}(\mathbb{R})$ where $\tilde{p} = \min\{1, p\}$, then the family $\{\mathcal{L}_{n;\lambda}\}$ given by (4.4) is of Vallée-Poussin type in L_p and (4.1) holds.*

Proof. Using part 2 of Lemma 2.1 we notice that

$$\mathcal{L}_{n;\lambda}(T) = S_{-\lambda}\big((S_\lambda(T) * W_n)_{(N)}\big) = S_{-\lambda}(S_\lambda(T)) = T$$

for $T \in \mathcal{T}_n$ $(n \in \mathbb{N}_0)$ and $\lambda \in \mathbb{R}$. Now we check the boundedness of $\{\mathcal{L}_{n;\lambda}\}$. Let first $p = +\infty$. On the basis of Lemma 2.3 we get

$$\begin{aligned}
\|\{\mathcal{L}_{n;\lambda}\}\|_{(\infty)} &\leq \tfrac{1}{2N+1} \sup_{\|f\|_\infty = 1} \max_{x,\lambda} \sum_{k=0}^{2N} |f(t_N^k + \lambda)|\,|W_n(x - t_N^k - \lambda)| \\
&\leq \tfrac{1}{2N+1} \sum_{k=0}^{2N} \max_{h \in [0,\tau]} |W_n(h - t_N^k)| \\
&= J_{n;\infty} \\
&\leq C_3.
\end{aligned}$$

In the case $0 < p \leq 1$ we have for $f \in L_p$ and $n \in \mathbb{N}_0$

$$\begin{aligned}
\|\mathcal{L}_{n;\lambda}(f)\|_{\mathbf{p}}^p &\leq (2N+1)^{-p} \sum_{k=0}^{2N} \int_0^{2\pi} |f(t_N^k + \lambda)|^p \left(\int_0^{2\pi} |W_n(x - t_N^k - \lambda)|^p dx \right) d\lambda \\
&= (2N+1)^{1-p} \|f\|_p^p \, \|W_n\|_p^p.
\end{aligned} \tag{4.6}$$

By Lemma 2.2 we obtain from (4.6) for $n \in \mathbb{N}$

$$\|\{\mathcal{L}_{n;\lambda}\}\|_{(p)} \leq (2\pi)^{-\frac{1}{p}} (2N+1)^{\frac{1}{p}-1} \|W_n\|_p = (2N+1)^{\frac{1}{p}-1} J_{n;p} \leq C_3.$$

Finally, we consider the case $1 < p < +\infty$. For almost every λ from $[0, \tau]$ (we recall that $\tau = \frac{2\pi}{2N+1}$) we have

$$\begin{aligned} J(\lambda) &= \|\mathcal{L}_{n;\lambda}(f;x)\|_{p;x}^p \\ &\leq (2N+1)^{-p} \int_0^{2\pi} \left(\sum_{k=0}^{2N} |f(t_N^k + \lambda)| \, |W_n(x - t_N^k - \lambda)| \right)^p dx \\ &= (2N+1)^{-p} \sum_{m=0}^{2N} \int_{t_N^m}^{t_N^{m+1}} \left(\sum_{k=0}^{2N} |f(t_N^k + \lambda)| \, |W_n(x - t_N^k - \lambda)| \right)^p dx \\ &= (2N+1)^{-p} \sum_{m=0}^{2N} \int_0^{\tau} \left(\sum_{k=0}^{2N} |f(t_N^k + \lambda)| \, |W_n(x - t_N^{k-m} - \lambda)| \right)^p dx \\ &= (2N+1)^{-p} \sum_{m=0}^{2N} \int_0^{\tau} \left(\sum_{k=0}^{2N} |f(t_N^{k+m} + \lambda)| \, |W_n(x - t_N^k - \lambda)| \right)^p dx. \end{aligned} \tag{4.7}$$

We consider the space of vector-functions

$$\bar{a}(x) = (a_0(x), \ldots, a_{2N}(x))$$

with components in $L_p[0, \tau]$ that is equipped with the norm

$$\|\bar{a}(x)\|_{\{p\}} = \left(\sum_{m=0}^{2N} \int_0^{\tau} |a_m(x)|^p dx \right)^{\frac{1}{p}}.$$

We consider also the vector-functions

$$\bar{a}^{(k)}(x) = \left(a_0^{(k)}(x), \ldots, a_{2N}^{(k)}(x)\right) \qquad (k = 0, 1, \ldots, 2N)$$

where

$$a_m^{(k)}(x) = |f(t_N^{k+m} + \lambda)| \, |W_n(x - t_N^k - \lambda)| \qquad (m = 0, \ldots, 2N).$$

From (4.7) we have

$$\begin{aligned} J(\lambda) &\leq (2N+1)^{-p} \left\| \sum_{k=0}^{2N} \bar{a}^{(k)}(x) \right\|_{\{p\}}^p \\ &\leq (2N+1)^{-p} \left(\sum_{k=0}^{2N} \|\bar{a}^{(k)}(x)\|_{\{p\}} \right)^p \\ &= (2N+1)^{-p} \left(\sum_{k=0}^{2N} \left(\sum_{m=0}^{2N} |f(t_N^{k+m} + \lambda)|^p \int_0^{\tau} |W_n(x - t_N^k - \lambda)|^p dx \right)^{\frac{1}{p}} \right)^p \end{aligned} \tag{4.8}$$

$$= (2N+1)^{-p} \left(\sum_{k=0}^{2N} \left(\int_0^{\tau} |W_n(x - t_N^k - \lambda)|^p dx \right)^{\frac{1}{p}} \left(\sum_{m=0}^{2N} |f(t_N^{k+m} + \lambda)|^p \right)^{\frac{1}{p}} \right)^p$$

$$= (2N+1)^{-p} \sum_{m=0}^{2N} |f(t_N^m + \lambda)|^p \left(\sum_{k=0}^{2N} \left(\int_{t_N^k}^{t_N^{k+1}} |W_n(h+\lambda)|^p dh \right)^{\frac{1}{p}} \right)^p$$

$$= \tau \sum_{m=0}^{2N} |f(t_N^m + \lambda)|^p \big(J_{N;p}(W_n(\cdot + \lambda)) \big)^p$$

$$\leq \tau \sum_{m=0}^{2N} |f(t_N^m + \lambda)|^p (J^*_{n;p})^p$$

where

$$J^*_{n;p} = \max_{\lambda \in [0,\tau]} J_{N;p}(W_n(\cdot + \lambda)). \tag{4.9}$$

Noticing that the function J is τ-periodic, we have from (4.8)

$$\begin{aligned} \|\{\mathcal{L}_{n;\lambda}\}\|_{(p)} &= (2\pi)^{-\frac{1}{p}} \sup_{\|f\|_p = 1} \|\mathcal{L}_{n;\lambda}(f;x)\|_p \\ &= \sup_{\|f\|_p=1} \left(\tfrac{1}{2\pi} \int_0^{2\pi} J(\lambda)\, d\lambda \right)^{\frac{1}{p}} \\ &= \sup_{\|f\|_p=1} \left(\tfrac{1}{\tau} \int_0^{\tau} J(\lambda)\, d\lambda \right)^{\frac{1}{p}} \\ &\leq \sup_{\|f\|_p=1} \left(\sum_{m=0}^{2N} \int_0^{\tau} |f(t_N^m + \lambda)|^p d\lambda \right)^{\frac{1}{p}} J^*_{n;p} \\ &= J^*_{n;p}. \end{aligned} \tag{4.10}$$

Applying (2.15) and Lemma 2.2 for $p = 1$ we get from (4.10)

$$\begin{aligned} \|\{\mathcal{L}_{n;\lambda}\|_{(p)} &\leq \max_{\lambda \in [0,\tau]} J_{N;\infty}(W_n(\cdot + \lambda)) \\ &\leq \max_{\lambda \in [0,\tau]} \tfrac{\pi+1}{2\pi} \|W_n(\cdot + \lambda)\|_1 \\ &= (\pi + 1) J_{n;1} \\ &\leq C_3. \end{aligned}$$

Theorem 4.2 is completely proved ∎

Now we make some conclusions. We have considered the family $\{\mathcal{L}_{n;\lambda}\}$ given by formula (4.4). Comparing it with formula (3.2) we get

$$\mathcal{L}_{n;\lambda} = S_{-\lambda} \circ \mathcal{I}_n \circ S_{\lambda} \tag{4.11}$$

that establishes a connection between $\{\mathcal{L}_{n;\lambda}\}$ and $\mathcal{I}_n$. However, as it was mentioned above the introduction of the parameter λ enables us to consider $\{\mathcal{L}_{n;\lambda}\}$ in L_p-spaces for all $0 < p \le +\infty$ in contrast to the interpolation means that make sense for $p = +\infty$ only.

Integrating formula (4.5) with respect to the parameter λ we obtain a connection between $\{\mathcal{L}_{n;\lambda}\}$ and the Fourier means given by (3.1) in the case $1 \le p \le +\infty$, that can be symbolically represented in the form

$$\tfrac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{n;\lambda}\, d\lambda = \mathcal{F}_n. \tag{4.12}$$

In the present section we have considered the value

$$e_n(f)_p = e_n(f)_{p;\psi} = \begin{cases} \left(\tfrac{1}{2\pi}\int_0^{2\pi} \|f-\mathcal{L}_{n;\lambda}(f)\|_p^p d\lambda\right)^{\frac{1}{p}} & \text{if } 0 < p \le +\infty \\ \max_\lambda \|f-\mathcal{L}_{n;\lambda}(f)\|_\infty & \text{if } p = +\infty. \end{cases}$$

It can be called an *averaged approximation* by $\{\mathcal{L}_{n;\lambda}\}$. We have proved that if $0 < p \le +\infty$, ψ is continuous and $\hat{\psi}$ belongs to $L_{\tilde{p}}(\mathbb{R})$, where $\tilde{p} = \min\{1, p\}$, then

$$e_n(f)_p \le C_n(p;\psi)\, E_n(f)_p \tag{4.13}$$

where the constants $C_n(p;\psi)$ are uniformly bounded on n. As it follows from the proofs of Theorems 4.1, 4.2 and from Lemmas 2.2, 2.3, they can be estimated in one of the following ways:

$$C_n(p;\psi) \le \begin{cases} \left(1+(2N+1)^{1-p} J^p_{n;p}\right)^{\frac{1}{p}} & \text{if } 0 < p \le 1 \\ 1 + J^*_{n;p} & \text{if } 1 \le p < +\infty \\ 1 + J_{n;+\infty} & \text{if } p = +\infty \end{cases} \tag{4.14}$$

$$C_0(p;\psi) \le 2^{\frac{1}{p}}$$
$$C_n(p;\psi) \le \begin{cases} \left(1+(2\pi)^{\frac{p}{2}-1}\|\hat{\psi}\|^p_{L_p(\mathbb{R})}(2(1+\delta)+\tfrac{1}{n})^{1-p}\right)^{\frac{1}{p}} & \text{if } 0 < p \le 1 \\ 1+(\pi+1)(2\pi)^{-\frac{1}{2}}\|\hat{\psi}\|_{L_1(\mathbb{R})} & \text{if } 1 < p \le +\infty \end{cases} \tag{4.15}$$

where $n \in \mathbb{N}$. If we replace the value $\left(2(1+\delta)+n^{-1}\right)$ by $3+2\delta$, we get a universal constant $C(p;\psi)$ that bounds all $C_n(p;\psi)$.

Now we compare the approximative properties of Fourier means, interpolation means and families of linear polynomial operators (in the cases, when it makes sense). Clearly,

$$\|f-\mathcal{I}_n(f)\|_\infty \le e_n(f)_\infty \qquad (f \in C, n \in \mathbb{N}_0). \tag{4.16}$$

As it follows from Remark 4.1, the right-hand side of (4.16) can be estimated by $1 + \|\{\mathcal{L}_{n;\lambda}\}\|_{(\infty)}$. However,

$$\begin{aligned} \|\mathcal{I}_n\|_{(\infty)} &\le \|\{\mathcal{L}_{n;\lambda}\}\|_{(\infty)} \\ &= \sup_{\|f\|_\infty=1} \max_\lambda \|S_{-\lambda}\circ\mathcal{I}_n\circ S_\lambda(f)\|_\infty \\ &= \sup_{\|f\|_\infty=1} \max_\lambda \|\mathcal{I}_n\circ S_\lambda(f)\|_\infty \\ &\le \sup_{\|f\|_\infty=1} \max_\lambda \|\mathcal{I}_n\|_{(\infty)}\|S_\lambda(f)\|_\infty \\ &= \|\mathcal{I}_n\|_{(\infty)}, \end{aligned}$$

and (4.16) gives the same estimate for $\|f - \mathcal{I}_n(f)\|_\infty$ as we have obtained in the proof of Lemma 3.1. Hence, the substitution $\mathcal{I}_n$ by $\{\mathcal{L}_{n;\lambda}\}$ does not lead to essential increasing of the approximation error.

In the case of Fourier means we have the same situation. Let now $1 \le p \le +\infty$. Using Minkovski's generalized inequality, Hölder's inequality, and relation (4.12) we obtain for $n \in \mathbb{N}_0$

$$\begin{aligned}\|\mathcal{F}_n\|_{(p)} &= \sup_{\|f\|_p=1} \|\mathcal{F}_n(f)\|_p \\ &\le \tfrac{1}{2\pi} \sup_{\|f\|_p=1} \int_0^{2\pi} \|\mathcal{L}_{n;\lambda}(f)\|_p d\lambda \\ &\le \tfrac{1}{2\pi} \left(\int_0^{2\pi} d\lambda\right)^{\frac{1}{q}} \sup_{\|f\|_p=1} \|\mathcal{L}_{n;\lambda}(f)\|_p \\ &= \|\mathcal{L}_{n;\lambda}(f)\|_{(p)}.\end{aligned}$$

By the same arguments we get

$$\|f - \mathcal{F}_n(f)\|_p \le e_n(f)_p \qquad (f \in L_p, n \in \mathbb{N}_0). \tag{4.17}$$

Thus, Theorem 3.1 is an immediate consequence of Theorem 4.2.

As we have seen the approximation by families of linear polynomial operators gives the best order of approximation in average on the parameter λ. Now we will show that the same outcome can be obtained with an a priori defined probability error, if the parameters λ are randomly chosen. More exactly, the following theorem holds. As usual, we put $N = [(1+\delta)n]$ and $\tau = \frac{2\pi}{2N+1}$. The constants $C_n = C_n(p;\psi)$ have the same meaning as in inequality (4.13). By $P(A)$ we denote the probability of an event A. It will follow from below that we practically use the geometrical concept of probability.

Theorem 4.3 *Let $0 < p < +\infty$, ψ defined by $(2.1)'$ be continuous and $\hat{\psi}$ belong to $L_{\tilde{p}}(\mathbb{R})$, where $\tilde{p} = \min\{1, p\}$. Let also $0 < \sigma < 1$, $\gamma > 1$, $k \in \mathbb{N}$ satisfy $\gamma^{-pk} \le \sigma$ and $\xi_j \ \ (j = 0, 1, \ldots, 2N)$ be independent random variables uniformly distributed on $[0, \tau]$. Then for $f \in L_p$ and $n \in \mathbb{N}_0$*

$$P\left\{\min_{j=1,\ldots,k} \|f - \mathcal{L}_{n;\xi_j}(f)\|_p \le \gamma C_n E_n(f)_p\right\} \ge 1 - \sigma. \tag{4.18}$$

Proof. First we notice that if ξ is a uniformly distributed on $[0, \tau]$ random value, then

$$P\{\xi \in e\} = \tfrac{1}{\tau}\mu\{\lambda \in e\} \tag{4.19}$$

for each set $e \subset [0, \tau]$ measurable in the Lebesgue sense. As it follows from formula (4.5), the function $\|f - \mathcal{L}_{n;\lambda}(f)\|_p$ is τ-periodic on λ, and therefore

$$\begin{aligned}\tfrac{1}{2\pi}\int_0^{2\pi} \|f - \mathcal{L}_{n;\lambda}(f)\|_p^p d\lambda &= \tfrac{1}{2\pi}\sum_{k=0}^{2N} \int_{t_N^k}^{t_N^{k+1}} \|f - \mathcal{L}_{n;\lambda}(f)\|_p^p d\lambda \\ &= \tfrac{1}{2\pi}\sum_{k=0}^{2N} \int_0^{\tau} \|f - \mathcal{L}_{n;\lambda - t_N^k}(f)\|_p^p d\lambda \\ &= \tfrac{1}{\tau}\int_0^{\tau} \|f - \mathcal{L}_{n;\lambda}(f)\|_p^p d\lambda.\end{aligned} \tag{4.20}$$

Taking into account that the random values ξ_j are independent and have the same distribution law we get using (5.2), (5.3), (4.13) and Chebyshev's inequality

$$
\begin{aligned}
P\Big\{\min_{j=1,\dots,k}\|f-\mathcal{L}_{n;\xi_j}(f)\|_p \le \gamma\, C_n E_n(f)_p\Big\}
&= 1 - P\left\{\bigcap_{j=1}^{k}\{\|f-\mathcal{L}_{n;\xi_j}(f)\|_p > \gamma\, C_n E_n(f)_p\}\right\}\\
&= 1 - \prod_{j=1}^{k} P\{\|f-\mathcal{L}_{n;\xi_j}(f)\|_p > \gamma\, C_n E_n(f)_p\}\\
&= 1 - \Big(P\{\|f-\mathcal{L}_{n;\xi_j}(f)\|_p > \gamma\, C_n E_n(f)_p\}\Big)^k\\
&= 1 - \left(\frac{1}{\tau}\mu\Big\{\lambda\in[0,\tau]\Big|\, \|f-\mathcal{L}_{n;\lambda}(f)\|_p^p > (\gamma\, C_n E_n(f))_p^p\Big\}\right)^k\\
&\ge 1 - \left(\frac{1}{\tau}\big(\gamma\, C_n E_n(f)_p\big)^{-p}\int_0^\tau \|f-\mathcal{L}_{n;\lambda}(f)\|_p^p d\lambda\right)^k\\
&\ge 1 - \Big(\big(\gamma\, C_n E_n(f)_p\big)^{-p}\big(C_n E_n(f)\big)^p\Big)^k\\
&= 1-\gamma^{-pk}\\
&\ge 1-\sigma.
\end{aligned}
$$

The proof is complete ∎

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Received 09.08.1999