

Paley-Wiener-Type Theorem for a Class of Integral Transforms Arising from a Singular Dirac System

A. I. Zayed and V. K. Tuan

Abstract. A characterization of weighted $L_2(I)$ spaces in terms of their images under various integral transformations is derived, where I is a finite interval. The class of integral transformations considered is related to certain singular Dirac systems on a half line.

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1. Introduction

The Paley-Wiener theorem [5] gives a characterization of the space $L_2[-\sigma, \sigma]$ in terms of its image under the Fourier transformation by showing that a function $f(x)$ is in $L_2[-\sigma, \sigma]$ if and only if its Fourier transform $\hat{f}(\omega)$ can be continued analytically to the whole complex plane as an entire function of exponential type at most σ whose restriction to the real axis belongs to $L_2(\mathbb{R})$. This characterization uses complex-variable techniques and as a result it does not lend itself very naturally to other more complicated integral transformations.

Alternative approaches using real analysis techniques have been developed in the last few years to characterize the images of spaces of the form $L_2[I, d\rho]$, for some measure $d\rho$ and an interval I (finite or infinite), under various integral transformations, such as the Mellin [11], Hankel [10], Y [9], and Airy transforms [12].

One of the first and simplest of such results was discovered by H. Bang [2]. It can be rephrased as follows: let $f \in C^\infty(\mathbb{R})$ be such that all its derivatives belong to $L_2(\mathbb{R})$. Then

$$\lim_{n \rightarrow \infty} \|f^{(n)}\|_2^{\frac{1}{n}} = \sup \{|y| : y \in \text{supp } \hat{f}\}.$$

In particular, a function f is the Fourier transform of a square integrable function vanishing outside the interval $[-\sigma, \sigma]$ if and only if

$$\lim_{n \rightarrow \infty} \|f^{(n)}\|_2^{\frac{1}{n}} \leq \sigma.$$

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In a recent paper [13], we proposed a unified approach to handle a large class of integral transforms that includes the Fourier sine and cosine transforms, the Hankel, the Weber, the Kontorovich-Lebedev and the Jacobi transforms. This class of integral transforms arises from two types of singular Sturm-Liouville problems: singular on a half line and singular on the whole line. The main idea of this approach goes as follows:

Let L be a differential operator, with a continuous spectrum Ω_1 , and $\phi(x, \lambda)$ be an eigenfunction with corresponding eigenvalue λ : $L\phi = -\lambda\phi$. In addition, suppose that $T : L_2(\Omega_1; d\rho_1) \rightarrow L_2(\Omega_2; d\rho_2)$,

$$f(x) = (TF)(x) = \int_{\Omega_1} F(\lambda)\phi(x, \lambda) d\rho_1(\lambda)$$

is a unitary transformation, i.e.

$$\int_{\Omega_2} |f(x)|^2 d\rho_2(x) = \int_{\Omega_1} |F(\lambda)|^2 d\rho_1(\lambda).$$

In that case, if $\lambda^n F(\lambda) \in L_2(\Omega_1; d\rho_1)$ for any $n \in \mathbb{N}_0$, we have

$$L^n f(x) = \int_{\Omega_1} (-\lambda)^n F(\lambda)\phi(x, \lambda) d\rho_1(\lambda)$$

and

$$\int_{\Omega_2} |L^n f(x)|^2 d\rho_2(x) = \int_{\Omega_1} |\lambda^n F(\lambda)|^2 d\rho_1(\lambda). \tag{1}$$

Raising both sides of (1) to the power $\frac{1}{2n}$ and taking the limit as $n \rightarrow \infty$ we get [8]

$$\lim_{n \rightarrow \infty} \|L^n f\|_{L_2(\Omega_2; d\rho_2)}^{\frac{1}{2n}} = \sup_{\lambda \in \text{supp } F} |\lambda| \tag{2}$$

where $\text{supp } F$ denotes the support of F , the smallest closed set, outside which the function F vanishes almost everywhere. From (2) it is obvious that if

$$\lim_{n \rightarrow \infty} \|L^n f\|_{L_2(\Omega_2; d\rho_2)}^{\frac{1}{2n}} < \infty, \tag{3}$$

then F has a compact support. Hence, formula (2) plays a decisive role in studying integral transforms of functions with compact supports. It can be shown that under some "extra conditions" on f inequality (3) gives the necessary and sufficient condition for a function f to be a T -transform of a function $F \in L_2(\Omega_1; d\rho_1)$ with compact support. Formula (3) yields Bang's formula when $\Omega_1 = (-\infty, \infty)$, $\phi(x, \lambda) = e^{ix\lambda}$, $d\rho_1(\lambda) = d\lambda$ and $L = \frac{d}{dx}$.

In this paper we extend this technique to study integral transforms arising from singular Dirac systems of differential equations [4]. The image of the space of functions with compact supports under those transforms is fully described. As examples, the Hartley transform and some transforms with the Airy and the Bessel functions as kernels are considered.

2. Singular Dirac system

Consider the one-dimensional singular Dirac system on the half line

$$\left. \begin{aligned} y_2' + p_{11}(x)y_1 + p_{12}(x)y_2 &= \lambda y_1 \\ -y_1' + p_{21}(x)y_1 + p_{22}(x)y_2 &= \lambda y_2 \end{aligned} \right\} \quad (0 < x < \infty) \tag{4}$$

where p_{ij} ($i, j = 1, 2$) are real-valued functions and λ is a parameter. Using matrix notation, we can write system (4) in the form

$$BY' + P(x)Y = \lambda Y$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}, \quad Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

In the case when $p_{12} = p_{21} = 0, p_{11}(x) = V(x) + m, p_{22}(x) = V(x) - m$, V is a potential function, and m is the mass of a particle, system (4) is called a *one-dimensional stationary Dirac system* in relativistic quantum theory.

Under the assumption $p_{12} = p_{21}$ we can transform system (4) into the canonical form

$$BZ' + Q(x)Z = \lambda Z \tag{5}$$

where

$$\begin{aligned} Y &= H(x)Z \\ H(x) &= \begin{pmatrix} \cos \phi(x) & -\sin \phi(x) \\ \sin \phi(x) & \cos \phi(x) \end{pmatrix} \\ Q(x) &= H^{-1}(x)B \frac{d}{dx} H(x) + H^{-1}(x)P(x)H(x). \end{aligned}$$

In addition, if we choose

$$\phi(x) = \begin{cases} \frac{1}{2} \tan^{-1} \left(\frac{2p_{12}(x)}{p_{11}(x) - p_{22}(x)} \right) & \text{if } p_{11} \neq p_{22} \\ \frac{\pi}{4} & \text{if } p_{11} = p_{22} \end{cases},$$

then Q takes the simpler form $Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$ for some continuous functions q_1 and q_2 . Thus (5) takes the canonical form

$$\left. \begin{aligned} y_2' + q_1(x)y_1 &= \lambda y_1 \\ y_1' - q_2(x)y_2 &= -\lambda y_2 \end{aligned} \right\} \quad (0 < x < \infty).$$

In the sequel we will be dealing mainly with the equivalent system

$$\left. \begin{aligned} y_1' - q_2(x)y_2 &= \lambda y_2 \\ y_2' + q_1(x)y_1 &= -\lambda y_1 \end{aligned} \right\} \quad (0 < x < \infty) \tag{6}$$

where q_1 and q_2 are continuous on $[0, \infty)$. Let

$$\underline{\Phi}(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \quad \text{and} \quad \underline{\Theta}(x, \lambda) = \begin{pmatrix} \theta_1(x, \lambda) \\ \theta_2(x, \lambda) \end{pmatrix} \tag{7}$$

be the solutions of system (6) that satisfies

$$\left. \begin{matrix} \phi_1(0, \lambda) = -\cos \alpha \\ \phi_2(0, \lambda) = \sin \alpha \end{matrix} \right\} \quad \text{and} \quad \left. \begin{matrix} \theta_1(0, \lambda) = \sin \alpha \\ \theta_2(0, \lambda) = \cos \alpha \end{matrix} \right\}. \tag{8}$$

Let

$$\underline{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

be a vector-valued function such that $\underline{f} \in L_2(\mathbb{R}^+)$, i.e.

$$\|\underline{f}\|_{L_2(\mathbb{R}^+)}^2 = \int_0^\infty |\underline{f}(x)|^2 dx = \int_0^\infty (|f_1(x)|^2 + |f_2(x)|^2) dx < \infty.$$

It is known [4] that for any non-real λ there exists a function $m(\lambda)$, analytic in the upper and lower half-planes that are not necessarily analytic continuation of each other so that

$$\underline{\Theta}(x, \lambda) + m(\lambda)\underline{\Phi}(x, \lambda),$$

as a vector-valued function of x , is in $L_2(\mathbb{R}^+)$ for non-real λ . Moreover,

$$\frac{1}{\pi} \lim_{\delta \rightarrow 0^+} \int_\mu^\lambda \Im m(u + i\delta) du = \rho(\lambda) - \rho(\mu)$$

where ρ is a monotone increasing function (see [3]). If $\underline{f} \in L_2(\mathbb{R}^+)$, then

$$F(\lambda) = \int_0^\infty \underline{f}^T(x)\underline{\Phi}(x) dx = \int_0^\infty (f_1(x)\phi_1(x, \lambda) + f_2(x)\phi_2(x, \lambda)) dx,$$

the generalized Fourier transform of \underline{f} , belongs to $L_2(\mathbb{R}, d\rho)$ and the Parseval equality

$$\int_0^\infty |\underline{f}(x)|^2 dx = \int_{-\infty}^\infty |F(\lambda)|^2 d\rho(\lambda) \tag{9}$$

holds. In addition, if the integrals

$$\int_{-\infty}^\infty F(\lambda)\phi_1(x, \lambda) d\rho(\lambda) \quad \text{and} \quad \int_{-\infty}^\infty F(\lambda)\phi_2(x, \lambda) d\rho(\lambda)$$

converge absolutely and uniformly in x in each finite interval, then

$$\left. \begin{matrix} f_1(x) = \int_{-\infty}^\infty F(\lambda)\phi_1(x, \lambda) d\rho(\lambda) \\ f_2(x) = \int_{-\infty}^\infty F(\lambda)\phi_2(x, \lambda) d\rho(\lambda) \end{matrix} \right\}. \tag{10}$$

We also have the estimate

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda^2 + 1} < \infty. \tag{11}$$

Let us denote by L the operator

$$L = B \frac{d}{dx} - Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} - \begin{pmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{pmatrix}$$

so that system (6) takes the form

$$LY = \lambda Y.$$

Hence $\underline{\Phi}(x, \lambda)$ is a solution of the system

$$L\underline{\Phi} = \lambda \underline{\Phi}, \tag{12}$$

and

$$A^T \underline{\Phi}(0, \lambda) = 0 \tag{13}$$

where

$$A = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \quad \text{and} \quad \underline{\Phi}(0, \lambda) = \begin{pmatrix} \phi_1(0, \lambda) \\ \phi_2(0, \lambda) \end{pmatrix}.$$

We shall call a vector-valued function \underline{f} the $\underline{\Phi}$ -transform of F if

$$\underline{f}(x) = \int_{-\infty}^{\infty} F(\lambda) \underline{\Phi}(x, \lambda) d\rho(\lambda) \tag{14}$$

where $\underline{\Phi}$ satisfies (12) and (13). We shall assume that Q is infinitely differentiable, such that the spectrum of the system (12) is either \mathbb{R} or \mathbb{R}^+ .

3. Paley-Wiener-type theorem

The following lemma describes the image of the $\underline{\Phi}$ -transform of a function F such that $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for all $n \in \mathbb{N}_0$.

Lemma 1. *Let the function F be such that $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for all $n \in \mathbb{N}_0$. A vector-valued function \underline{f} is the $\underline{\Phi}$ -transform of F as given by (14) if and only if the following conditions are satisfied:*

- 1) \underline{f} is infinitely differentiable on \mathbb{R}^+ .
- 2) $L^n \underline{f} \in L_2(\mathbb{R}^+)$ for all n .
- 3) $\lim_{x \rightarrow 0} A^T \cdot (L^n \underline{f})(x) = 0$.
- 4) $\lim_{x \rightarrow \infty} (L^n \underline{f})(x) = 0$.

Proof . *Necessity:* Let $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for all $n \in \mathbb{N}_0$. It is easy to see that $\lambda^n F(\lambda) \in L_1(\mathbb{R}, d\rho)$. Indeed, applying formula (11) and the Cauchy-Schwartz inequality, we get

$$\left(\int_{-\infty}^{\infty} |\lambda^n F(\lambda)| d\rho(\lambda) \right)^2 \leq \left(\int_{-\infty}^{\infty} (\lambda^{2n+2} + \lambda^{2n}) |F(\lambda)|^2 d\rho(\lambda) \right) \left(\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda^2 + 1} \right) < \infty.$$

1) Since Q is infinitely differentiable, so are $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ with respect to the variable x . It also follows as in the Sturm-Liouville case [7] that $\frac{\partial^n \phi_1(x, \lambda)}{\partial x^n}$ and $\frac{\partial^n \phi_2(x, \lambda)}{\partial x^n}$ are of order λ^n as $\lambda \rightarrow \infty$. Therefore,

$$\underline{f}^{(n)}(x) = \int_{-\infty}^{\infty} F(\lambda) \frac{\partial^n \underline{\Phi}(x, \lambda)}{\partial x^n} d\rho(\lambda)$$

exists for all n , where

$$\underline{f}^{(n)}(x) = \begin{pmatrix} f_1^{(n)}(x) \\ f_2^{(n)}(x) \end{pmatrix}.$$

2) By applying the differential operator L to (14) n times, we have

$$(L^n \underline{f})(x) = \int_{-\infty}^{\infty} F(\lambda) L^n \underline{\Phi}(x, \lambda) d\rho(\lambda) = \int_{-\infty}^{\infty} \lambda^n F(\lambda) \underline{\Phi}(x, \lambda) d\rho(\lambda),$$

and since $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ we have $L^n \underline{f} \in L_2(\mathbb{R}^+)$.

3) By taking the matrix product with A^T , we get

$$\begin{aligned} \lim_{x \rightarrow 0^+} A^T \cdot (L^n \underline{f})(x) &= \lim_{x \rightarrow 0^+} \int_{-\infty}^{\infty} \lambda^n F(\lambda) (A^T \cdot \underline{\Phi}(x, \lambda)) d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \lambda^n F(\lambda) (A^T \cdot \underline{\Phi}(0, \lambda)) d\rho(\lambda) \\ &= 0. \end{aligned}$$

Taking the limit inside the integral is justified by the Lebesgue dominated convergence theorem.

4) Let

$$\underline{\tilde{f}}(x) = B \underline{\bar{f}}(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{f}_1(x) \\ \bar{f}_2(x) \end{pmatrix} = \begin{pmatrix} -\bar{f}_2(x) \\ \bar{f}_1(x) \end{pmatrix}.$$

We have

$$\begin{aligned} &\int_0^N \underline{\tilde{f}}^T(x) \cdot L \underline{f}(x) dx \\ &= \int_0^N \left\{ \bar{f}_2(x) [f_2'(x) + q_1(x) f_1(x)] + \bar{f}_1(x) [f_1'(x) - q_2(x) f_2(x)] \right\} dx \\ &= [|f_1(x)|^2 + |f_2(x)|^2] \Big|_0^N \\ &\quad + \int_0^N \left\{ [-\bar{f}_1'(x) + \bar{f}_2(x) q_1(x)] f_1(x) + [-\bar{f}_2'(x) - \bar{f}_1(x) q_2(x)] f_2(x) \right\} dx \\ &= [|f_1(x)|^2 + |f_2(x)|^2] \Big|_0^N + \int_0^N \underline{f}^T(x) \cdot L \underline{\tilde{f}}(x) dx. \end{aligned}$$

Hence

$$\begin{aligned} &|f_1(N)|^2 + |f_2(N)|^2 \\ &= |f_1(0)|^2 + |f_2(0)|^2 + \int_0^N \underline{\tilde{f}}^T(x) \cdot L \underline{f}(x) dx - \int_0^N \underline{f}^T(x) \cdot L \underline{\tilde{f}}(x) dx. \end{aligned} \tag{15}$$

Since \underline{f} and $L\underline{f}$ are in $L_2(\mathbb{R}^+)$, it follows that $\tilde{\underline{f}}$ and $L\tilde{\underline{f}}$ are also in $L_2(\mathbb{R}^+)$. Therefore, the right-hand side of (15) approaches a limit as $N \rightarrow \infty$; hence so is the left-hand side of (15). But because f_1 and f_2 belong to $L_2(\mathbb{R}^+)$, the limit must be zero, i.e.

$$\lim_{N \rightarrow \infty} f_1(N) = \lim_{N \rightarrow \infty} f_2(N) = 0.$$

Similarly one can show that $\lim_{x \rightarrow \infty} (L^n \underline{f})(x) = 0$ for all n .

Sufficiency: Let \underline{f} satisfy conditions 1 - 4 of the lemma. Since $\underline{f} \in L_2(\mathbb{R}^+)$ (condition 1 with $n = 0$), there exists $F \in L_2(\mathbb{R}, d\rho)$ such that

$$F(\lambda) = \int_0^\infty \underline{f}^T(x) \cdot \underline{\Phi}(x, \lambda) dx.$$

Then

$$\lambda^n F(\lambda) = \int_0^\infty \lambda^n \underline{f}^T(x) \cdot \underline{\Phi}(x, \lambda) dx = \int_0^\infty \underline{f}^T(x) \cdot L^n \underline{\Phi}(x, \lambda) dx.$$

We show by induction on n that

$$\lambda^n F(\lambda) = \int_0^\infty (L^n \underline{f})^T(x) \cdot \underline{\Phi}(x, \lambda) dx.$$

For $n = 1$,

$$\begin{aligned} \lambda F(\lambda) &= \int_0^\infty \underline{f}^T(x) \cdot L\underline{\Phi}(x, \lambda) dx \\ &= \int_0^\infty \left\{ -f_1(x) [\phi_2'(x, \lambda) + q_1(x)\phi_1(x, \lambda)] \right. \\ &\quad \left. + f_2(x) [\phi_1'(x, \lambda) - q_2(x)\phi_2(x, \lambda)] \right\} dx \\ &= \left[-f_1(x)\phi_2(x, \lambda) + f_2(x)\phi_1(x, \lambda) \right] \Big|_0^\infty \\ &\quad + \int_0^\infty \left\{ [-\phi_1(x, \lambda)f_2'(x) + \phi_2(x, \lambda)f_1'(x)] \right. \\ &\quad \left. - [\phi_1(x, \lambda)f_1(x)q_1(x) + \phi_2(x, \lambda)f_2(x)q_2(x)] \right\} dx \\ &= \left[-f_1(x)\phi_2(x, \lambda) + f_2(x)\phi_1(x, \lambda) \right] \Big|_0^\infty \\ &\quad + \int_0^\infty \left\{ [-f_2'(x) - f_1(x)q_1(x)]\phi_1(x, \lambda) \right. \\ &\quad \left. + [f_1'(x) - f_2(x)q_2(x)]\phi_2(x, \lambda) \right\} dx \\ &= \left[-f_1(x)\phi_2(x, \lambda) + f_2(x)\phi_1(x, \lambda) \right] \Big|_0^\infty \\ &\quad + \int_0^\infty [(L\underline{f})(x)]^T \cdot \underline{\Phi}(x, \lambda) dx. \end{aligned}$$

Since $\underline{f}(x)$ and $\underline{\Phi}(x, \lambda)$ satisfy the same initial condition (13) it follows that

$$f_1(0)\phi_2(0, \lambda) - f_2(0)\phi_1(0, \lambda) = 0.$$

As $x \rightarrow \infty$, from condition 4 ($n = 0$) it follows that $\lim_{x \rightarrow \infty} \underline{f}(x) = 0$, and because $\underline{\Phi}(x, \lambda)$ is bounded in x for each fixed λ we have

$$\lim_{x \rightarrow \infty} [f_1(x)\phi_2(x, \lambda) - f_2(x)\phi_1(x, \lambda)] = 0.$$

Hence,

$$\lambda F(\lambda) = \int_0^\infty [(L\underline{f})(x)]^T \cdot \underline{\Phi}(x, \lambda) dx.$$

As $L\underline{f} \in L_2(\mathbb{R}^+)$, we have $\lambda F(\lambda) \in L_2(\mathbb{R}, d\rho)$.

Now assume that

$$\lambda^n F(\lambda) = \int_0^\infty [(L^n \underline{f})(x)]^T \cdot \underline{\Phi}(x, \lambda) dx.$$

Then

$$\begin{aligned} \lambda^{n+1} F(\lambda) &= \int_0^\infty [(L^n \underline{f})(x)]^T \cdot L\underline{\Phi}(x, \lambda) dx \\ &= [-g_1(x)\phi_2(x, \lambda) + g_2(x)\phi_1(x, \lambda)] \Big|_0^\infty + \int_0^\infty [(L\underline{g})(x)]^T \cdot \underline{\Phi}(x, \lambda) dx \end{aligned}$$

where

$$\underline{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = (L^n \underline{f})(x).$$

For $\underline{g}(x)$ and $\underline{\Phi}(x, \lambda)$ satisfy the same initial condition (13), then by conditions 3 and 4 we have

$$[-g_1(x)\phi_2(x, \lambda) + g_2(x)\phi_1(x, \lambda)] \Big|_0^\infty = 0.$$

Thus,

$$\lambda^{n+1} F(\lambda) = \int_0^\infty [(L^{n+1} \underline{f})(x)]^T \cdot \underline{\Phi}(x, \lambda) dx$$

and since $L^{n+1} \underline{f} \in L_2(\mathbb{R}^+)$, it follows that $\lambda^{n+1} F(\lambda) \in L_2(\mathbb{R}, d\rho)$ ■

Lemma 2. Let \underline{f} be the $\underline{\Phi}$ -transform of a function F as given by (14). Let $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for $n \in \mathbb{N}_0$. Then

$$\lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} = \sup_{\lambda \in \text{supp } F} |\lambda|.$$

Proof. From the relation

$$(L^n \underline{f})(x) = \int_{-\infty}^\infty \lambda^n F(\lambda) \underline{\Phi}(x, \lambda) d\rho(\lambda)$$

and the Parseval equation (9) we have

$$\|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^2 = \int_{-\infty}^\infty \lambda^{2n} |F(\lambda)|^2 d\rho(\lambda).$$

First, let F have a compact support: $\sup_{\lambda \in \text{supp } F} |\lambda| = \delta < \infty$. Then

$$\int_{-\infty}^{\infty} \lambda^{2n} |F(\lambda)|^2 d\rho(\lambda) = \int_{-\delta}^{\delta} \lambda^{2n} |F(\lambda)|^2 d\rho(\lambda) \leq \delta^{2n} \int_{-\delta}^{\delta} |F(\lambda)|^2 d\rho(\lambda).$$

Hence,

$$\limsup_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} \leq \delta \limsup_{n \rightarrow \infty} \left\{ \int_{-\delta}^{\delta} |F(\lambda)|^2 d\rho(\lambda) \right\}^{\frac{1}{2n}} = \delta.$$

On the other hand, because δ is the supremum of the support of F , for any ε with $0 < \varepsilon < \delta$ we have

$$\int_{\delta-\varepsilon < |\lambda| < \delta} |F(\lambda)|^2 d\rho(\lambda) > 0.$$

Therefore,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} &\geq \liminf_{n \rightarrow \infty} \left\{ \int_{\delta-\varepsilon < |\lambda| < \delta} \lambda^{2n} |F(\lambda)|^2 d\rho(\lambda) \right\}^{\frac{1}{2n}} \\ &\geq (\delta - \varepsilon) \liminf_{n \rightarrow \infty} \left\{ \int_{\delta-\varepsilon < |\lambda| < \delta} |F(\lambda)|^2 d\rho(\lambda) \right\}^{\frac{1}{2n}} \\ &= \delta - \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} = \delta.$$

Now let F have an unbounded support. Then for any N large enough

$$\int_{|\lambda| > N} |F(\lambda)|^2 d\rho(\lambda) > 0.$$

Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} &\geq \lim_{n \rightarrow \infty} \left\{ \int_{|\lambda| > N} \lambda^{2n} |F(\lambda)|^2 d\rho(\lambda) \right\}^{\frac{1}{2n}} \\ &\geq N \lim_{n \rightarrow \infty} \left\{ \int_{|\lambda| > N} |F(\lambda)|^2 d\rho(\lambda) \right\}^{\frac{1}{2n}} \\ &= N. \end{aligned}$$

Because N is arbitrary, we obtain $\lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} = \infty$ ■

Now we are ready to state and prove the following Paley-Wiener-type theorem.

Theorem. A vector-valued function \underline{f} is the $\underline{\Phi}$ -transform (14) of a function $F \in L_2(\mathbb{R}, d\rho)$ with compact support if and only if \underline{f} satisfies conditions 1 – 4 of Lemma 1 and

$$\lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} < \infty. \tag{16}$$

Proof. Let $F \in L_2(\mathbb{R}, d\rho)$ have a compact support. Then $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for all $n \in \mathbb{N}_0$. Consequently, by Lemma 1, \underline{f} satisfies conditions 1 - 4 of Lemma 1, and by Lemma 2

$$\lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} = \sup_{\lambda \in \text{supp } F} |\lambda| < \infty.$$

Conversely, let \underline{f} satisfy conditions 1 - 4 of Lemma 1 and (16). By Lemma 1, \underline{f} is the $\underline{\Phi}$ -transform (14) of a function F such that $\lambda^n F(\lambda) \in L_2(\mathbb{R}, d\rho)$ for all $n \in \mathbb{N}_0$. By Lemma 2 and equation (16) we have

$$\sup_{\lambda \in \text{supp } F} |\lambda| = \lim_{n \rightarrow \infty} \|L^n \underline{f}\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} < \infty.$$

Hence, F has a compact support ■

4. Examples

As examples, the Hartley and some transforms with the Airy and the Bessel functions as kernels are derived, and the Paley-Wiener theorems are obtained for these transforms.

Example 1 (Fourier and Hartley transform). Consider the boundary-value problem (6) - (8) in which $q_1 = 0 = q_2$. It is easy to see that

$$\underline{\Phi}(x, \lambda) = \begin{pmatrix} -\cos(\lambda x + \alpha) \\ \sin(\lambda x + \alpha) \end{pmatrix} \quad \text{and} \quad \underline{\Theta}(x, \lambda) = \begin{pmatrix} \sin(\lambda x + \alpha) \\ \cos(\lambda x + \alpha) \end{pmatrix}$$

and $m(\lambda) = i$, since

$$\underline{\Theta}(x, \lambda) + m(\lambda)\underline{\Phi}(x, \lambda) = \begin{pmatrix} -ie^{i(\lambda x + \alpha)} \\ e^{i(\lambda x + \alpha)} \end{pmatrix} \in L_2(\mathbb{R}^+)$$

for $\Im \lambda > 0$. Thus $d\rho(\lambda) = \frac{1}{\pi} d\lambda$,

$$F(\lambda) = \int_0^\infty \left\{ -\cos(\lambda x + \alpha)f_1(x) + \sin(\lambda x + \alpha)f_2(x) \right\} dx$$

and

$$\left. \begin{aligned} f_1(x) &= -\frac{1}{\pi} \int_{-\infty}^\infty \cos(\lambda x + \alpha) F(\lambda) d\lambda \\ f_2(x) &= \frac{1}{\pi} \int_{-\infty}^\infty \sin(\lambda x + \alpha) F(\lambda) d\lambda \end{aligned} \right\}. \tag{17}$$

The Paley-Wiener-type theorem for the transforms (17) takes the following form.

Corollary 1. *A vector-valued function \underline{f} is the $\underline{\Phi}$ -transform (17) of a function $F \in L_2(\mathbb{R})$ with compact support if and only if \underline{f} satisfies the following conditions:*

- 1) \underline{f} is infinitely differentiable on \mathbb{R}^+ .
- 2) $\left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{smallmatrix}\right)^n \underline{f}(x) \in L_2(\mathbb{R}^+)$ for all n .
- 3) $\lim_{x \rightarrow 0} (\sin \alpha, \cos \alpha) \cdot \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{smallmatrix}\right)^n \underline{f}(x) = 0$.
- 4) $\lim_{x \rightarrow \infty} \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{smallmatrix}\right)^n \underline{f}(x) = 0$.
- 5) $\lim_{n \rightarrow \infty} \left\| \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ -\frac{d}{dx} & 0 \end{smallmatrix}\right)^n \underline{f}(x) \right\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} < \infty$.

Let $\alpha = 0$ and

$$2f(x) = -f_1(|x|) + i \operatorname{sign} x f_2(|x|) \quad (x \in \mathbb{R}).$$

Then we obtain the pair of Fourier transforms

$$\left. \begin{aligned} F(\lambda) &= \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) dx \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda) d\lambda \end{aligned} \right\}.$$

If we put $\alpha = 0$ and

$$2f(x) = -f_1(|x|) + \operatorname{sign} x f_2(|x|) \quad (x \in \mathbb{R}),$$

then we arrive at the pair of Hartley transforms

$$\left. \begin{aligned} F(\lambda) &= \int_{-\infty}^{\infty} (\cos \lambda x + \sin \lambda x) f(x) dx \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cos \lambda x + \sin \lambda x) F(\lambda) d\lambda \end{aligned} \right\}.$$

Example 2 (Airy transform). Consider system (6) - (8) with $q_1 = 0, q_2(x) = -x$ and $\alpha = 0$:

$$\left. \begin{aligned} y_1' + xy_2 &= \lambda y_2 \\ y_2' &= -\lambda y_1 \end{aligned} \right\} \quad (0 < x < \infty). \tag{18}$$

Eliminating y_1 from this system yields

$$y_2'' - (\lambda x - \lambda^2)y_2 = 0.$$

Setting $t + \tilde{\lambda} = \lambda^{\frac{4}{3}} - \lambda^{\frac{1}{3}}x$ we obtain

$$\frac{d^2 y_2}{dt^2} + (t + \tilde{\lambda})y_2 = 0 \quad \text{and} \quad y_1 = \lambda^{-\frac{2}{3}} \frac{dy_2}{dt}.$$

The general solution of the first equation is

$$y_2(t) = A(t + \tilde{\lambda})^{\frac{1}{2}} J_{\frac{1}{3}}(X) + B(t + \tilde{\lambda})^{\frac{1}{2}} Y_{\frac{1}{3}}(X)$$

where

$$X = \frac{2}{3}(t + \tilde{\lambda})^{\frac{3}{2}} = \frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}$$

and J_μ and Y_μ are the Bessel functions of the first and second kinds, respectively. Hence

$$y_2(x) = \lambda^{\frac{1}{6}} \left\{ A(\lambda - x)^{\frac{1}{2}} J_{\frac{1}{3}}(X) + B(\lambda - x)^{\frac{1}{2}} Y_{\frac{1}{3}}(X) \right\}$$

and

$$y_1(x) = \lambda^{-\frac{5}{6}} \left\{ A\lambda^{\frac{1}{2}}(\lambda - x) J'_{\frac{1}{3}}(X) + \frac{A}{2\sqrt{\lambda - x}} J_{\frac{1}{3}}(X) \right. \\ \left. + B\lambda^{\frac{1}{2}}(\lambda - x) Y'_{\frac{1}{3}}(X) + \frac{B}{2\sqrt{\lambda - x}} Y_{\frac{1}{3}}(X) \right\}.$$

Since $X = \frac{2}{3}\lambda^2$ at $x = 0$ we have

$$y_2(0) = A\lambda^{\frac{2}{3}} J_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) + B\lambda^{\frac{2}{3}} Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) = 0$$

and

$$y_1(0) = \frac{A}{\lambda^{\frac{5}{6}}} \left\{ \lambda^{\frac{3}{2}} J'_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) + \frac{1}{2\sqrt{\lambda}} J_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) \right\} \\ + \frac{B}{\lambda^{\frac{5}{6}}} \left\{ \lambda^{\frac{3}{2}} Y'_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) + \frac{1}{2\sqrt{\lambda}} Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda^2\right) \right\} = 1.$$

Solving this system for A and B , we obtain ϕ_1 and ϕ_2 . The explicit solution for ϕ_1 and ϕ_2 can be found in [7: pp. 92 - 93] and therefore the kernels of the integrals in (10) can be expressed in terms of the Bessel functions J_μ and Y_μ .

However, since the argument of the Bessel functions,

$$X = \frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}$$

may be complex or real depending on whether x is greater or less than λ , and hence the integrals have to be split into different regions with different kinds of Bessel functions as kernels, it is more convenient to use the Airy functions. In so doing, we can express f_1 and f_2 as single integrals extending over the whole real line (see (23) below).

So, let us make the change of variable $\lambda x - \lambda^2 = \lambda^{\frac{2}{3}} z$ to obtain the Airy differential equation

$$\frac{d^2 y_2}{dz^2} - z y_2 = 0$$

whose general solution is

$$y_2 = a \text{Ai}(z) + b \text{Bi}(z).$$

Here Ai and Bi are the Airy functions [1]. Hence,

$$y_2(x, \lambda) = a \operatorname{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + b \operatorname{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}). \quad (19)$$

Since $y_1 = -\lambda^{-1}y_2'$ we have

$$y_1(x, \lambda) = -a\lambda^{-\frac{2}{3}}\operatorname{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - b\lambda^{-\frac{2}{3}}\operatorname{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}). \quad (20)$$

Thus, (18) and (19) are the general solution of system (18). The solution $\underline{\Phi}(x, \lambda)$ satisfies the initial condition

$$\left. \begin{aligned} \phi_1(0, \lambda) &= -1 \\ \phi_2(0, \lambda) &= 0 \end{aligned} \right\}.$$

Thus

$$\left. \begin{aligned} a \operatorname{Ai}(-\lambda^{\frac{4}{3}}) + b \operatorname{Bi}(-\lambda^{\frac{4}{3}}) &= 0 \\ a \lambda^{-\frac{2}{3}} \operatorname{Ai}'(-\lambda^{\frac{4}{3}}) + b \lambda^{-\frac{2}{3}} \operatorname{Bi}'(-\lambda^{\frac{4}{3}}) &= 1 \end{aligned} \right\}.$$

Using the Wronskian equality for the Airy functions [1]

$$W(\operatorname{Ai}(x), \operatorname{Bi}(x)) := \operatorname{Ai}(x) \operatorname{Bi}'(x) - \operatorname{Ai}'(x) \operatorname{Bi}(x) = \frac{1}{\pi}$$

we get

$$\left. \begin{aligned} a &= -\pi \lambda^{\frac{2}{3}} \operatorname{Bi}(-\lambda^{\frac{4}{3}}) \\ b &= \pi \lambda^{\frac{2}{3}} \operatorname{Ai}(-\lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

Thus,

$$\left. \begin{aligned} \phi_1(x, \lambda) &= \pi \operatorname{Bi}(-\lambda^{\frac{4}{3}}) \operatorname{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \pi \operatorname{Ai}(-\lambda^{\frac{4}{3}}) \operatorname{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \phi_2(x, \lambda) &= -\pi \lambda^{\frac{2}{3}} \operatorname{Bi}(-\lambda^{\frac{4}{3}}) \operatorname{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \pi \lambda^{\frac{2}{3}} \operatorname{Ai}(-\lambda^{\frac{4}{3}}) \operatorname{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

Similarly, the solution $\underline{\Theta}(x, \lambda)$ satisfies the initial condition

$$\left. \begin{aligned} \theta_1(0, \lambda) &= 0 \\ \theta_2(0, \lambda) &= 1 \end{aligned} \right\},$$

that yields

$$\left. \begin{aligned} a \operatorname{Ai}(-\lambda^{\frac{4}{3}}) + b \operatorname{Bi}(-\lambda^{\frac{4}{3}}) &= 1 \\ a \lambda^{-\frac{2}{3}} \operatorname{Ai}'(-\lambda^{\frac{4}{3}}) + b \lambda^{-\frac{2}{3}} \operatorname{Bi}'(-\lambda^{\frac{4}{3}}) &= 0 \end{aligned} \right\}.$$

Solving this system we get

$$\left. \begin{aligned} a &= \pi \operatorname{Bi}'(-\lambda^{\frac{4}{3}}) \\ b &= -\pi \operatorname{Ai}'(-\lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

Thus

$$\left. \begin{aligned} \theta_1(x, \lambda) &= -\pi \lambda^{-\frac{2}{3}} \operatorname{Bi}'(-\lambda^{\frac{4}{3}}) \operatorname{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \pi \lambda^{-\frac{2}{3}} \operatorname{Ai}'(-\lambda^{\frac{4}{3}}) \operatorname{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \theta_2(x, \lambda) &= \pi \operatorname{Bi}'(-\lambda^{\frac{4}{3}}) \operatorname{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \pi \operatorname{Ai}'(-\lambda^{\frac{4}{3}}) \operatorname{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

We have

$$\begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) &= \left[-\lambda^{-\frac{2}{3}}\text{Bi}'(-\lambda^{\frac{4}{3}}) + m(\lambda)\text{Bi}(-\lambda^{\frac{4}{3}}) \right] \pi\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ &\quad + \left[\lambda^{-\frac{2}{3}}\text{Ai}'(-\lambda^{\frac{4}{3}}) - m(\lambda)\text{Ai}(-\lambda^{\frac{4}{3}}) \right] \pi\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) &= \left[\text{Bi}'(-\lambda^{\frac{4}{3}}) - m(\lambda)\lambda^{\frac{2}{3}}\text{Bi}(-\lambda^{\frac{4}{3}}) \right] \pi\text{Ai}(\lambda^{-\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ &\quad + \left[-\text{Ai}'(-\lambda^{\frac{4}{3}}) + m(\lambda)\lambda^{\frac{2}{3}}\text{Ai}(-\lambda^{\frac{4}{3}}) \right] \pi\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}). \end{aligned}$$

If we take

$$m(\lambda) = \lambda^{-\frac{2}{3}} \frac{\text{Ai}'(-\lambda^{\frac{4}{3}}) + i\text{Bi}'(-\lambda^{\frac{4}{3}})}{\text{Ai}(-\lambda^{\frac{4}{3}}) + i\text{Bi}(-\lambda^{\frac{4}{3}})},$$

then

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) &= -\lambda^{-\frac{2}{3}} \frac{\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + i\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\text{Ai}(-\lambda^{\frac{4}{3}}) + i\text{Bi}(-\lambda^{\frac{4}{3}})} \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) &= \frac{\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + i\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\text{Ai}(-\lambda^{\frac{4}{3}}) + i\text{Bi}(-\lambda^{\frac{4}{3}})} \end{aligned} \right\}.$$

The relations between the Airy functions and the Hankel functions $H_\nu^{(2)}$ [1]

$$\left. \begin{aligned} H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}z^{\frac{3}{2}}\right) &= e^{-i\frac{\pi}{6}}\sqrt{\frac{3}{z}}(\text{Ai}(-z) + i\text{Bi}(-z)) \\ H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}z^{\frac{3}{2}}\right) &= e^{-i\frac{\pi}{6}}\frac{\sqrt{3}}{z}(\text{Ai}'(-z) + i\text{Bi}'(-z)) \end{aligned} \right\} \tag{21}$$

yield

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) &= -\frac{(\lambda - x)H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right)}{\lambda H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}\lambda^2\right)} \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) &= \frac{(\lambda - x)^{\frac{1}{2}}H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right)}{\lambda^{\frac{1}{2}}H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}\lambda^2\right)} \end{aligned} \right\}.$$

Using the asymptotic expansion for the Hankel function [1]

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}e^{-i(z - \frac{\nu}{2}\pi - \frac{\pi}{4})}\left(1 + O\left(\frac{1}{z}\right)\right)$$

as $z \rightarrow \infty$ ($-2\pi < \arg z < \pi$), it is clear that

$$\left. \begin{aligned} (\lambda - x)H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right) \\ (\lambda - x)^{\frac{1}{2}}H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right) \end{aligned} \right\} \in L_2(\mathbb{R}^+). \tag{22}$$

for any non-real λ with $\Im \lambda > 0$. Hence,

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) \end{aligned} \right\} \in L_2(\mathbb{R}^+).$$

We have

$$m(\lambda) = \frac{\text{Ai}(-\lambda^{\frac{4}{3}})\text{Ai}'(-\lambda^{\frac{4}{3}}) + \text{Bi}(-\lambda^{\frac{4}{3}})\text{Bi}'(-\lambda^{\frac{4}{3}}) + iW\left(\text{Ai}(-\lambda^{\frac{4}{3}}), \text{Bi}(-\lambda^{\frac{4}{3}})\right)}{\lambda^{\frac{2}{3}}\left(\text{Ai}^2(-\lambda^{\frac{4}{3}}) + \text{Bi}^2(-\lambda^{\frac{4}{3}})\right)}.$$

Thus

$$d\rho(\lambda) = \frac{1}{\pi} \Im m(\lambda) d\lambda = \frac{1}{\pi\lambda^{\frac{2}{3}}\left(\text{Ai}^2(-\lambda^{\frac{4}{3}}) + \text{Bi}^2(-\lambda^{\frac{4}{3}})\right)} d\lambda.$$

We arrive at the triple of integral transforms

$$F(\lambda) = \pi \int_0^\infty \left\{ \left[\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \right] f_1(x) + \lambda^{\frac{2}{3}} \left[-\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \right] f_2(x) \right\} dx$$

and

$$\left. \begin{aligned} f_1(x) &= \int_{-\infty}^\infty \frac{\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\lambda^{\frac{2}{3}}\left[\text{Ai}^2(-\lambda^{\frac{4}{3}}) + \text{Bi}^2(-\lambda^{\frac{4}{3}})\right]} F(\lambda) d\lambda \\ f_2(x) &= \int_{-\infty}^\infty \frac{-\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\text{Ai}^2(-\lambda^{\frac{4}{3}}) + \text{Bi}^2(-\lambda^{\frac{4}{3}})} F(\lambda) d\lambda \end{aligned} \right\}. \tag{23}$$

Theorem 1 holds for the transforms (23) if

$$A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{pmatrix}.$$

Corollary 2. *A vector-valued function \underline{f} is the $\underline{\Phi}$ -transform (23) of a function $F \in L_2(\mathbb{R}, d\rho)$ with compact support if and only if \underline{f} satisfies the following conditions:*

- 1) \underline{f} is infinitely differentiable on \mathbb{R}^+ .
- 2) $\left(\frac{0}{\frac{d}{dx}} - \frac{d}{x}\right)^n \underline{f}(x) \in L_2(\mathbb{R}^+)$ for all n .
- 3) $\lim_{x \rightarrow 0} (0, 1) \cdot \left(\frac{0}{\frac{d}{dx}} - \frac{d}{x}\right)^n \underline{f}(x) = 0$.
- 4) $\lim_{x \rightarrow \infty} \left(\frac{0}{\frac{d}{dx}} - \frac{d}{x}\right)^n \underline{f}(x) = 0$.
- 5) $\lim_{n \rightarrow \infty} \left\| \left(\frac{0}{\frac{d}{dx}} - \frac{d}{x}\right)^n \underline{f}(x) \right\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} < \infty$.

Example 3 (Second Airy transform). Consider the same system (18), but with $\alpha = \frac{\pi}{2}$. The solution $\underline{\Phi}(x, \lambda)$ satisfying the initial condition

$$\left. \begin{aligned} \phi_1(0, \lambda) &= 0 \\ \phi_2(0, \lambda) &= 1 \end{aligned} \right\}$$

is

$$\left. \begin{aligned} \phi_1(x, \lambda) &= -\pi\lambda^{-\frac{2}{3}}\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \pi\lambda^{-\frac{2}{3}}\text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \phi_2(x, \lambda) &= \pi\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \pi\text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

Similarly, the solution $\underline{\Theta}(x, \lambda)$ satisfying the initial condition

$$\left. \begin{aligned} \theta_1(0, \lambda) &= 1 \\ \theta_2(0, \lambda) &= 0 \end{aligned} \right\}$$

is

$$\left. \begin{aligned} \theta_1(x, \lambda) &= -\pi\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \pi\text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \theta_2(x, \lambda) &= \pi\lambda^{\frac{2}{3}}\text{Bi}(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \pi\lambda^{\frac{2}{3}}\text{Ai}(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

We have

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) &= \left[-\text{Bi}(-\lambda^{\frac{4}{3}}) - m(\lambda)\lambda^{-\frac{2}{3}}\text{Bi}'(-\lambda^{\frac{4}{3}}) \right] \pi\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ &\quad + \left[\text{Ai}(-\lambda^{\frac{4}{3}}) + m(\lambda)\lambda^{-\frac{2}{3}}\text{Ai}'(-\lambda^{\frac{4}{3}}) \right] \pi\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) &= \left[\lambda^{\frac{2}{3}}\text{Bi}(-\lambda^{\frac{4}{3}}) + m(\lambda)\text{Bi}'(-\lambda^{\frac{4}{3}}) \right] \pi\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \\ &\quad + \left[-\lambda^{\frac{2}{3}}\text{Ai}(-\lambda^{\frac{4}{3}}) - m(\lambda)\text{Ai}'(-\lambda^{\frac{4}{3}}) \right] \pi\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \end{aligned} \right\}.$$

If we take

$$m(\lambda) = -\frac{\lambda^{\frac{2}{3}} \left(\text{Ai}(-\lambda^{\frac{4}{3}}) + i\text{Bi}(-\lambda^{\frac{4}{3}}) \right)}{\text{Ai}'(-\lambda^{\frac{4}{3}}) + i\text{Bi}'(-\lambda^{\frac{4}{3}})},$$

then the relations between the Airy and the Hankel functions (21) yield

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) &= \frac{(\lambda - x)H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right)}{\lambda H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}\lambda^2\right)} \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) &= -\frac{(\lambda - x)^{\frac{1}{2}}H_{-\frac{1}{3}}^{(2)}\left(\frac{2}{3}\lambda^{\frac{1}{2}}(\lambda - x)^{\frac{3}{2}}\right)}{\lambda^{\frac{1}{2}}H_{\frac{2}{3}}^{(2)}\left(\frac{2}{3}\lambda^2\right)} \end{aligned} \right\}.$$

From (22) it is clear that

$$\left. \begin{aligned} \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) \\ \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) \end{aligned} \right\} \in L_2(\mathbb{R}^+).$$

We have

$$m(\lambda) = -\lambda^{\frac{2}{3}} \frac{\text{Ai}(-\lambda^{\frac{4}{3}})\text{Ai}'(-\lambda^{\frac{4}{3}}) + \text{Bi}(-\lambda^{\frac{4}{3}})\text{Bi}'(-\lambda^{\frac{4}{3}}) - iW\left(\text{Ai}(-\lambda^{\frac{4}{3}}), \text{Bi}(-\lambda^{\frac{4}{3}})\right)}{\text{Ai}'^2(-\lambda^{\frac{4}{3}}) + \text{Bi}'^2(-\lambda^{\frac{4}{3}})}.$$

Thus

$$d\rho(\lambda) = \frac{1}{\pi} \Im m(\lambda) d\lambda = \frac{\lambda^{\frac{2}{3}}}{\pi \left(\text{Ai}'^2(-\lambda^{\frac{4}{3}}) + \text{Bi}'^2(-\lambda^{\frac{4}{3}})\right)} d\lambda.$$

We arrive at the triple of integral transforms

$$F(\lambda) = \pi \int_0^\infty \left\{ \lambda^{-\frac{2}{3}} \left[-\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \right] f_1(x) + \left[\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) \right] f_2(x) \right\} dx$$

and

$$\left. \begin{aligned} f_1(x) &= \int_{-\infty}^\infty \frac{-\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) + \text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}'(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\text{Ai}'^2(-\lambda^{\frac{4}{3}}) + \text{Bi}'^2(-\lambda^{\frac{4}{3}})} F(\lambda) d\lambda \\ f_2(x) &= \int_{-\infty}^\infty \frac{\text{Bi}'(-\lambda^{\frac{4}{3}})\text{Ai}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}}) - \text{Ai}'(-\lambda^{\frac{4}{3}})\text{Bi}(\lambda^{\frac{1}{3}}x - \lambda^{\frac{4}{3}})}{\text{Ai}'^2(-\lambda^{\frac{4}{3}}) + \text{Bi}'^2(-\lambda^{\frac{4}{3}})} F(\lambda) \lambda^{\frac{2}{3}} d\lambda \end{aligned} \right\}. \quad (24)$$

Theorem 1 holds for transforms (24) if

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{pmatrix}.$$

Corollary 3. *A vector-valued function \underline{f} is the Φ -transform (24) of a function $F \in L_2(\mathbb{R}, d\rho)$ with compact support if and only if \underline{f} satisfies the following conditions:*

- 1) \underline{f} is infinitely differentiable on \mathbb{R}^+ .
- 2) $\left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{smallmatrix}\right)^n \underline{f}(x) \in L_2(\mathbb{R}^+)$ for all n .
- 3) $\lim_{x \rightarrow 0} (1, 0) \cdot \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{smallmatrix}\right)^n \underline{f}(x) = 0$.
- 4) $\lim_{x \rightarrow \infty} \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{smallmatrix}\right)^n \underline{f}(x) = 0$.
- 5) $\lim_{n \rightarrow \infty} \left\| \left(\begin{smallmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & x \end{smallmatrix}\right)^n \underline{f}(x) \right\|_{L_2(\mathbb{R}^+)}^{\frac{1}{n}} < \infty$.

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