The Fundamental Solution of a Modified Oseen Problem

R. Farwig, A. Novotný and M. Pokorný

Abstract. We study asymptotic properties of the fundamental solution to an Oseen-type system coming from fluid mechanics. We show that the solution has similar anisotropic structure near infinity as the fundamental solution to the (classical) Oseen problem. We also study integral operators with kernels representing the second gradient of the fundamental solution.

Keywords: *Systems of partial differential equations, fundamental solutions, Oseen problem, Fourier transform, Fourier multipliers*

AMS subject classification: 35 A 08, 35 Q 35, 42 B 15

0. Introduction

This paper is devoted to the study of the Oseen-type problem

$$\left. \begin{aligned} -\Delta u + \mu \frac{\partial^2 u}{\partial x_1^2} + \beta \frac{\partial u}{\partial x_1} + \nabla \pi = f \\ \nabla \cdot u = 0 \end{aligned} \right\}, \tag{0.1}$$

$0 \le \mu < 1$, called here the *modified Oseen problem*. This problem appears e.g. in the study of asymptotic properties of solutions to the system of partial differential equations describing the steady flow of certain classes of incompressible viscoelastic fluids past an obstacle. In this case, the velocity v and the pressure p satisfy

$$\left. \begin{aligned} -\eta \Delta v + \varrho(v \cdot \nabla)v + \nabla \pi = \varrho f + \nabla \cdot \big[F(\nabla v, T) - \lambda \varrho((v \cdot \nabla)v) \otimes v \\ + \lambda \varrho f \otimes v + \lambda p (\nabla v)^T \big] \\ \nabla \cdot v = 0 \\ p + \lambda(v \cdot \nabla)p = \pi \\ T + \lambda(v \cdot \nabla)T + G(\nabla v, T) = 2\eta D(v) \end{aligned} \right\} \tag{0.2}$$

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where

> $F(\cdot, \cdot), G(\cdot, \cdot)$ are bilinear functions
> T is the extra stress tensor
> D is the symmetric part of the velocity gradient
> f is a given function (external force)
> ϱ is the constant density
> λ, η are positive constants.

We assume system (0.2) to hold in an exterior domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ and consider the boundary conditions

$$\left. \begin{array}{ll} v = 0 & \text{on } \partial\Omega \\ v \to v_\infty = \beta e_1 & \text{as } |x| \to \infty \end{array} \right\}. \tag{0.3}$$

We may construct a solution to system (0.2) - (0.3) by means of the following procedure. Define the linear operator

$$\mathcal{M} : (w, s) \to (u, \pi)$$

where

$$\left. \begin{array}{l} -\eta \Delta u + \lambda \varrho \beta^2 \dfrac{\partial^2 u}{\partial x_1^2} + \varrho \beta \dfrac{\partial u}{\partial x_1} + \nabla \pi = \varrho f + \nabla \cdot \Big[F(\nabla w, T) \\ \qquad\qquad - \lambda \varrho ((w \cdot \nabla) w) \otimes w - \varrho w \otimes w \\ \qquad\qquad - \lambda \varrho \beta \Big(\dfrac{\partial w}{\partial x_1} \otimes w + ((w \cdot \nabla) w) \otimes e_1 \Big) \\ \qquad\qquad + \lambda \varrho f \otimes (w + \beta e_1) + \lambda p (\nabla w)^T \Big] \\ \qquad\qquad\qquad \nabla \cdot u = 0 \\ \qquad\qquad p + \lambda ((w + v_\infty) \cdot \nabla) p = s \\ T + \lambda ((w + v_\infty) \cdot \nabla) T + G(\nabla w, T) = 2\eta D(w) \end{array} \right\} \tag{0.4}$$

and u satisfies the boundary conditions

$$\left. \begin{array}{ll} u \to 0 & \text{as } |x| \to \infty \\ u = -\beta e_1 & \text{on } \partial\Omega \end{array} \right\}. \tag{0.5}$$

Let $(\bar{u}, \bar{\pi})$ be a fixed point of $\mathcal{M}$. Then (v, p), with $v = \bar{u} + v_\infty$ and p solution to $(0.4)_3$ with $s = \bar{\pi}$ and $w = \bar{u}$, solves the original problem (0.2) - (0.3). We observe that the elliptic part of problem (0.4) is exactly system (0.1), i.e. the modified Oseen problem.

Let us note two peculiarities.

Instead of only constructing solutions to problem (0.2) - (0.3), we also want to study their asymptotic properties near infinity. Otherwise, we could put the term $\lambda \varrho \beta^2 \frac{\partial^2}{\partial x_1^2}$

to the right-hand side of $(0.4)_1$ and consider the (classical) Oseen problem instead of the modified one. But in this case we could not get weighted estimates of the solution due to certain loss in the estimates of convolutions with kernels representing the second gradient of the fundamental Oseen tensor (cf. [1, 6] or [3]). Therefore, the aim of this paper is to consider the asymptotic structure of the fundamental solution to the modified Oseen problem (0.1). The application of this analysis can be found in [6] or [4, 5].

Secondly, at a first glance it seems to be possible to change variables and modify the vector v in such a way that we would get the (classical) Oseen problem. This is not true; the condition $\nabla \cdot v = 0$ would be replaced by another condition and we could not treat the problem using the results on the (classical) Oseen problem.

Many properties of solutions to the (classical) Oseen problem are connected with the properties of the fundamental solution to the Oseen problem. The same holds also in the case of the modified version (cf. [6]). We are looking for $(\mathcal{O}^\mu, e)$ such that [1]

$$
\left.
\begin{aligned}
\left[-\Delta + \mu \frac{\partial^2}{\partial x_1^2} + \lambda \frac{\partial}{\partial x_1} \right] \mathcal{O}_{ij}^\mu(x; \lambda) + \frac{\partial}{\partial x_i} e_j(x) &= \delta_{ij}\delta \\
\frac{\partial \mathcal{O}_{ij}^\mu(x; \lambda)}{\partial x_i} &= 0
\end{aligned}
\right\}
\tag{0.6}
$$

where δ_{ij} denotes the Kronecker delta, δ is the Dirac δ-distribution, $0 \le \mu < 1$, and system (0.6) holds in the sense of distributions.

Throughout the paper we use the standard notation $L^p(\mathbb{R}^N)$ for Lebesgue spaces, $\mathcal{D}(\mathbb{R}^N)$ for the set of smooth functions with compact support and $\mathcal{S}$ for the Schwartz class. By $\mathcal{F}(f) = \hat{f}$ we denote the Fourier transform of the function f, either in the sense of $L^2(\mathbb{R}^N)$ or in the sense of $\mathcal{S}'$.

1. Fundamental solution to the Oseen problem

Before starting to study problem (0.6), let us recall several properties of the fundamental solution to the (classical) Oseen problem, i.e. problem (0.6) with $\mu = 0$. We shall write only $\mathcal{O}$ instead of $\mathcal{O}^0$ in this case. As is well known (see, e.g., [2]),

$$
e = \nabla \mathcal{E}(x)
$$

with $\mathcal{E}(x)$ the fundamental solution to the Laplace equation. The tensor $\mathcal{O}$ satisfies the homogeneity property

$$
\mathcal{O}(x, \lambda) = \lambda^{N-2} \mathcal{O}(\lambda x; 1)
$$

and therefore it is sufficient to study only the case $\lambda = 1$.

For small $|x|$,

$$
\mathcal{O}(x; 1) = \mathcal{S}(x) + \mathcal{R}(x)
$$

---

[1] We use the standard summation convention.

where $\mathcal{S}$ is the fundamental Stokes tensor (see, e.g., [2]) and in three dimensions [2)]

$$D^\alpha \mathcal{R}(x) \sim |x|^{-|\alpha|} \qquad \text{for } |x| \to 0$$

for $|\alpha| \geq 0$, while in two dimensions

$$\left. \begin{array}{l} \mathcal{R}(x) \sim 1 \\ D^\alpha \mathcal{R}(x) \sim \begin{cases} \ln|x| & \text{if } |\alpha| = 1 \\ |x|^{-|\alpha|+1} & \text{if } |\alpha| \geq 2 \end{cases} \end{array} \right\} \qquad \text{for } |x| \to 0.$$

Recall that

$$\mathcal{S}_{ij}(x) = \begin{cases} \dfrac{1}{4\pi}\left(\delta_{ij}\log\dfrac{1}{|x|} + \dfrac{x_i x_j}{|x|^2}\right) & \text{if } N = 2 \\ \dfrac{1}{8\pi}\left(\dfrac{\delta_{ij}}{|x|} + \dfrac{x_i x_j}{|x|^3}\right) & \text{if } N = 3. \end{cases}$$

Therefore the second derivatives of $\mathcal{S}$ represent Calderón-Zygmund singular integral kernels, i.e. there exists $C = C(p, N)$ such that

$$T : \mathcal{D}(\mathbb{R}^N) \mapsto \mathcal{D}'(\mathbb{R}^N), \qquad (Tf)(x) = \int_{\mathbb{R}^N} \frac{\partial \mathcal{S}_{kl}(x-y)}{\partial x_i} \frac{\partial f(y)}{\partial y_j} \, dy,$$

satisfies

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}$$

for all $f \in \mathcal{D}(\mathbb{R}^N)$ and $1 < p < \infty$. Thus the operator T can be continuously extended onto $L^p(\mathbb{R}^N)$. Moreover, it can be shown that

$$(Tf)(x) = \text{v.p.} \int_{\mathbb{R}^N} \frac{\partial^2 \mathcal{S}_{kl}(x'-y)}{\partial x_i \partial x_j} f(y) \, dy + c_{ijkl} f(x).$$

Define $s(x) = |x| - x_1$. For $|x| \to \infty$ we have in three space dimensions

$$\mathcal{O}(x; 1) \sim |x|^{-1}(1 + s(x))^{-1}$$

$$D^\alpha \mathcal{O}(x; 1) \sim |x|^{-1-\frac{|\alpha|}{2}}(1 + s(x))^{-1-\frac{|\alpha|}{2}} \quad (|\alpha| \geq 1).$$

The derivatives with respect to the first variable decay faster; more precisely,

$$D^\alpha \mathcal{O}(x; 1) \sim |x|^{-1-\alpha_1 - \frac{|\alpha|-\alpha_1}{2}}(1 + s(x))^{-1-\frac{|\alpha|-\alpha_1}{2}} \qquad (|\alpha| \geq 1).$$

[2)] Throughout this paper, $f \sim g$ for $|x| \to 0$ means that there exists $\mathcal{U}_\varepsilon(0)$, a neighborhood of 0, and $C_1, C_2 \in \mathbb{R}$ such that $C_1|f(x)| \leq g(x) \leq C_2|f(x)|$ for $x \in \mathcal{U}_\varepsilon(0)$; analogously for $|x| \to \infty$.

In two space dimensions the structure is even more complicated. Here we have for $|x| \to \infty$ (recall that $\mathcal{O}_{ij}(x; 1) = \mathcal{O}_{ji}(x; 1)$)

$$\mathcal{O}_{11}(x; 1) \sim |x|^{-\frac{1}{2}}(1 + s(x))^{-\frac{1}{2}}$$

$$\mathcal{O}_{ij}(x; 1) \sim |x|^{-1} \quad ((i,j) \neq (1,1))$$

$$\frac{\partial \mathcal{O}_{11}(x; 1)}{\partial x_2} \sim |x|^{-1}(1 + s(x))^{-1}$$

$$\frac{\partial \mathcal{O}_{11}(x; 1)}{\partial x_1}, \frac{\partial \mathcal{O}_{12}(x; 1)}{\partial x_2} \sim |x|^{-\frac{3}{2}}(1 + s(x))^{-\frac{1}{2}}$$

$$\frac{\partial \mathcal{O}_{12}(x; 1)}{\partial x_1}, \frac{\partial \mathcal{O}_{22}(x; 1)}{\partial x_i} \sim |x|^{-2} \quad (i = 1, 2)$$

$$\frac{\partial^2 \mathcal{O}_{11}(x; 1)}{\partial x_2^2} \sim |x|^{-\frac{3}{2}}(1 + s(x))^{-\frac{3}{2}}$$

$$\frac{\partial^2 \mathcal{O}_{ij}(x; 1)}{\partial x_k \partial x_l} \sim |x|^{-2}(1 + s(x))^{-1} \quad ((i,j,k,l) \neq (1,1,2,2)).$$

By analogy with the three-dimensional case the derivatives with respect to the first variable decay faster. Especially,

$$\frac{\partial^2 \mathcal{O}_{ij}(x)}{\partial x_1^2} \sim |x|^{-\frac{5}{2}}(1 + s(x))^{-\frac{1}{2}} \qquad (i, j = 1, 2).$$

Our aim is to show similar properties also for the modified problem (0.6).

2. Fundamental solution to the modified Oseen problem

We shall search $\mathcal{O}^\mu$ in the form

$$\mathcal{O}^\mu(x; \lambda) = \mathcal{O}(x; \lambda) + E^\mu(x; \lambda) \tag{2.1}$$

where $\mathcal{O}$ is the (classical) fundamental Oseen tensor and

$$\left[\Delta - \mu \frac{\partial^2}{\partial x_1^2} - \lambda \frac{\partial}{\partial x_1}\right] E_{ij}^\mu = \mu \frac{\partial^2 \mathcal{O}_{ij}}{\partial x_1^2} \tag{2.2}$$

in the sense of distributions. As in the Stokes and also classical Oseen problem the fundamental pressure e is given by $e_j(x) = \frac{\partial \mathcal{E}(x)}{\partial x_j}$. Therefore it is sufficient to study E^μ, the solution to (2.2). We further need

$$\frac{\partial E_{ij}^\mu}{\partial x_i} = 0 \tag{2.3}$$

in the sense of distributions, which will be proved in Lemma 2.2 below.

First we consider the fundamental solution to (2.2), i.e.

$$\left[\Delta - \mu \frac{\partial^2}{\partial x_1^2} - \lambda \frac{\partial}{\partial x_1}\right]E^* = \delta \tag{2.4}$$

in the sense of distributions. For these reasons define

$$X_i = \frac{x_i}{\sqrt{1 - \mu \delta_{i1}}} \qquad \text{and} \qquad \overline{\lambda} = \frac{\lambda}{\sqrt{1 - \mu}}. \tag{2.5}$$

Lemma 2.1. *The fundamental solution to* (2.4) *is given by*

$$E^*(x; \lambda) = \begin{cases} \frac{-1}{2\pi\sqrt{1-\mu}} K_0\left(\frac{\overline{\lambda}R}{2}\right)e^{\overline{\lambda}X_1/2} & \text{if } N = 2 \\ \frac{-1}{4\pi\sqrt{1-\mu}} \frac{e^{-\overline{\lambda}s(X)/2}}{R} & \text{if } N = 3. \end{cases} \tag{2.6}$$

Here X and $\overline{\lambda}$ are defined by (2.5), *$R = |X|$, $s(X) = R - X_1$, and K_0 denotes a modified Bessel function of the second kind satisfying*

$$K_0(z) = \begin{cases} -\ln z + O(1) & \text{for } |z| \to 0 \\ \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 + O\left(\frac{1}{|z|}\right)\right) & \text{for } |z| \to \infty. \end{cases}$$

Proof. Using (2.5) we transform (2.4) into

$$\left(\Delta - \overline{\lambda}\frac{\partial}{\partial X_1}\right)\overline{E}(X; \overline{\lambda}) = \delta$$

where

$$\overline{E}(X; \overline{\lambda}) = \sqrt{1 - \mu}\, E^*(x(X); \lambda), \quad \Delta = \Delta_X, \quad \delta = \delta(X).$$

This problem has the well-known fundamental solution

$$\begin{cases} -\frac{1}{2\pi} K_0\left(\frac{\overline{\lambda}R}{2}\right)^{\overline{\lambda}X_1/2} & \text{if } N = 2 \\ -\frac{1}{4\pi} \frac{e^{-\overline{\lambda}s(X)/2}}{R} & \text{if } N = 3 \end{cases}$$

(see [2]) ∎

Remark 2.1. By (2.6),

$$\overline{E}^*(X; \overline{\lambda}) \sim \begin{cases} |\overline{\lambda}X|^{-\frac{1}{2}}e^{-\overline{\lambda}s(X)/2} & \text{if } N = 2 \\ |X|^{-1}e^{-\overline{\lambda}s(X)/2} & \text{if } N = 3 \end{cases} \qquad (\text{as } R = |X| \to \infty).$$

Using the change of variables (2.5) we see that

$$\sqrt{1 - \mu}R \le |x| \le R$$
$$\frac{\sqrt{1-\mu}}{2}s(x) \le s(X) \le 2s(x) \qquad \text{if } x_1 \ge 0 \tag{2.7}$$
$$\tfrac{1}{2}s(x) \le s(X) \le \frac{2}{\sqrt{1-\mu}}s(x) \qquad \text{if } x_1 \le 0$$

with $s(X) = R - X_1$. Thus for the study of the asymptotic behaviour, it is not necessary to distinguish between the behaviour in the variables x and X.

From (2.7) and Lemma 2.1, setting $r = |x|$, we see:

For $N = 2$,

$$E^* \in L^p(\mathbb{R}^2) \qquad \text{for } p \in (3, \infty)$$
$$\partial_1 E^* \in L^p(\mathbb{R}^2) \qquad \text{for } p \in (1, 2)$$

since

$$E^* \sim \begin{cases} r^{-\frac{1}{2}} e^{-s(x)} & (r \to \infty) \\ \ln r & (r \to 0) \end{cases}, \quad \partial_1 E^* \sim r^{-\frac{3}{2}} e^{-s(x)} \ (r \to \infty), \quad \nabla E^* \sim r^{-1} \ (r \to 0).$$

For $N = 3$,

$$E^* \in L^p(\mathbb{R}^3) \qquad \text{for } p \in (2, 3)$$
$$\partial_1 E^* \in L^p(\mathbb{R}^3) \qquad \text{for } p \in (1, \tfrac{3}{2})$$

since

$$E^* \sim \begin{cases} r^{-1} e^{-s} & (r \to \infty) \\ r^{-1} & (r \to 0) \end{cases}, \quad \partial_1 E^* \sim r^{-2} e^{-s} \ (r \to \infty), \quad \nabla E^* \sim r^{-2} \ (r \to 0).$$

Lemma 2.2. *The solution $E^\mu = (E_{i,j}^\mu)_{i,j=1}^N$ to (2.2) can be expressed in the form*

$$E^\mu(x; \lambda) = \mu \int_{\mathbb{R}^N} \frac{\partial}{\partial x_1} E^*(x - y; \lambda) \frac{\partial}{\partial y_1} \mathcal{O}(y; \lambda) \, dy. \tag{2.8}$$

This convolution can be understood in the usual notion of L^p-spaces yielding for every $\lambda > 0$

$$E^\mu \in \begin{cases} L^p(\mathbb{R}^2) & \text{for } p \in (1, \infty) \\ L^p(\mathbb{R}^3) & \text{for } p \in (1, 3). \end{cases}$$

Furthermore, $E_j^\mu = (E_{ij}^\mu)_{i=1}^N$ satisfies in the sense of Schwartz distributions

$$\operatorname{div} E_j^\mu = 0 \qquad \text{for } j = 1, \ldots, N.$$

Proof. The integrability properties of $\partial_1 E^*$ and $\partial_1 \mathcal{O}$ and Young's inequality imply that $\partial_1 E^* * \partial_1 \mathcal{O} \in L^p(\mathbb{R}^N)$ with $p \in (1, \infty)$ when $N = 2$ and $p \in (1, 3)$ when $N = 3$. In particular, E^μ can be considered as a distribution in the Schwartz' class $\mathcal{S}'$.

Let $(\varphi_k) \subset C_0^\infty(\mathbb{R}^N)$ be defined by $\varphi_k(x) = \varphi_1(\frac{x}{k})$ where $\varphi_1 = 1$ in the ball $B_1 \subset \mathbb{R}^N$ of radius 1 with center 0 and $\varphi_1 = 0$ in B_2^c. Then elementary L^q-estimates prove that in the sense of $\mathcal{S}'$

$$\partial_1 E^* * \partial_1(\varphi_k \mathcal{O}) \to \partial_1 E^* * \partial_1 \mathcal{O} \qquad \text{as } k \to \infty.$$

Furthermore, defining the operator $A = \Delta - \mu \partial_1^2 - \lambda \partial_1$,

$$A\big(\partial_1 E^* * \partial_1(\varphi_k \mathcal{O})\big) = \partial_1(AE^*) * \partial_1(\varphi_k \mathcal{O})$$
$$= (AE^*) * \partial_1^2(\varphi_k \mathcal{O})$$
$$= \delta * \partial_1^2(\varphi_k \mathcal{O})$$
$$= \partial_1^2(\varphi_k \mathcal{O})$$
$$\to \partial_1^2 \mathcal{O}$$

in the sense of $\mathcal{S}'$. Consequently, $A(\partial_1 E^* * \partial_1 \mathcal{O}) = \partial_1^2 \mathcal{O}$ in $\mathcal{S}'$ proving that $E^\mu(\cdot; \lambda)$ given by (2.8) is a solution of (2.2).

To prove div $E_j^\mu = 0$ we use Fourier transform in $\mathcal{S}'$ to conclude from (2.2) that

$$\left(-|\xi|^2 + \mu\xi_1^2 - i\lambda\xi_1 \right)\widehat{E_{kj}^\mu} = -\mu\xi_1^2 \widehat{\mathcal{O}_{kj}}.$$

Since div $\mathcal{O}_j = 0$ or, equivalently, $\xi \cdot \widehat{\mathcal{O}_j} = 0$, we get

$$\left(-|\xi|^2 + \mu\xi_1^2 - i\lambda\xi_1 \right)\mathcal{F}(\operatorname{div} E_j^\mu) = 0.$$

Consequently, $\operatorname{supp}\mathcal{F}(\operatorname{div} E_j^\mu) = \{0\}$ implying that div E_j^μ is a polynomial on $\mathbb{R}^N$. However, since $E_{kj}^\mu \in L^p(\mathbb{R}^N)$ for some $p > 1$,

$$\int_{B_1(x)} (1 - |x - y|^2)\operatorname{div} E_j^\mu(y)\,dy = -2\int_{B_1(x)} (x - y) \cdot E_j^\mu(y)\,dy \to 0$$

as $|x| \to \infty$ by Lebesgue's theorem on dominated convergence. Thus the polynomial div E_j^μ vanishes identically in $\mathbb{R}^N$ ∎

3. Asymptotic properties of the fundamental solution

We start to study the asymptotic behaviour of E^μ near infinity. For notational convenience, we define
$$\eta_b^a(x) = (1 + |x|)^a (1 + s(x))^b.$$

We have (again, as for the classical Oseen problem, we put $\lambda = 1$)

Lemma 3.1. *Let $N = 3$. Then, for $k \geq 0$, $\varepsilon > 0$ arbitrarily small and $|x| \geq R \gg 1$,*

$$|\nabla^k E_{ij}^\mu(x; 1)| \leq C|x|^{-\frac{4+k-\varepsilon}{2}}(1 + s(x))^{-\frac{2+k}{2}}. \tag{3.1}$$

Proof. We divide the convolution integral (2.8) into three parts:

$$E_{ij}^\mu(x; 1) = \mu\left(\int_{B_1(0)} + \int_{B_1(x)} + \int_{\mathbb{R}^3 \setminus B_1(0) \setminus B_1(x)} \right) \frac{\partial E^*(x - y; 1)}{\partial x_1} \frac{\partial O_{ij}(y; 1)}{\partial y_1}\,dy$$

$$\equiv I_1(x) + I_2(x) + I_3(x).$$

Obviously,
$$|I_1(x)| + |I_2(x)| \leq C\eta_{-1}^{-2}(x)$$

for $|x|$ sufficiently large. For I_3 it is enough to estimate the convolution

$$|I_3(x)| \leq C\int_{\mathbb{R}^3} (1 + |x - y|)^{-2}e^{-s(x-y)}\eta_{-1}^{-2}(y)\,dy$$

which can be assumed as $(\eta_{-m}^{-2} * \eta_{-1}^{-2})(x)$ for m arbitrarily large. We shortly sketch how to estimate such convolutions (for more details see, e.g., [1, 3] or [6]). Let $x' = (x_2, \ldots, x_N)$. We distinguish the following four situations:

    a) $|x| \leq R_0$

    b) $x_1 > 0$, $|x'| < \sqrt{x_1}$, $|x| > R_0$

    c) $x_1 > 0$, $|x'| = \frac{1}{2}|x|^{\frac{1}{2}+\sigma}$ $(\sigma \in [0, \frac{1}{2}], |x| > R_0)$

    d) $x_1 > 0$, $|x'| > \frac{|x|}{2}$, $|x| > R_0$ or $x_1 < 0$, $|x| > R_0$.

The most restrictive case is c). Here we divide $\mathbb{R}^N$ into 16 subdomains, as shown in Figure 1 below. Here $R = |x|$ and $\nu = \sigma + \frac{1}{2}$.

Figure 1

    Now we calculate the convolutions over each subdomain Ω_i $(i = 0, 1, \ldots, 15)$. After some lengthy computations we get exponents $e_i, f_i \in \mathbb{R}$ such that

$$\int_{\Omega_i} \eta_{-d}^{-c}(x - y)\eta_{-b}^{-a}(y)\, dy \leq C\eta_{-f_i}^{-e_i}(x).$$

Now, taking $e \leq \min\{e_i\}_{i=0}^{15}$ and $f \leq \min\{f_i\}_{i=0}^{15}$, we obtain the desired inequality $(\eta_{-d}^{-c} * \eta_{-b}^{-a})(x) \leq C\eta_{-f}^{-e}(x)$. The situation is slightly more complicated by the presence of some logarithmic terms which can be absorbed by taking $e < \min\{e_i\}_{i=0}^{15}$ and/or $f < \min\{f_i\}_{i=0}^{15}$. The calculations are quite technical and the results are summarized in [1] when $N = 3$ and in Tables 1 - 4 in [3] or [6] for $N = 2, 3$; the results for $N > 3$ can be read from the calculations performed in the above references. Thus we get

$$|I_3(x)| \leq C\eta_{-1}^{-2+\varepsilon}(x)$$

where the ε-loss is caused by logarithmic factors. Continuing with the first derivatives we easily observe that

$$\left|\frac{\partial}{\partial x_i} I_1(x)\right| + \left|\frac{\partial}{\partial x_i} I_2(x)\right| \leq C\eta_{-\frac{3}{2}}^{-\frac{5}{2}}(x).$$

The term $\frac{\partial}{\partial x_i} I_3(x)$ needs more care. In subdomains $\Omega_0, \Omega_2, \Omega_6, \Omega_8$ and Ω_{11} (and in the corresponding situations in $\Omega_4, \Omega_9, \Omega_{10}, \Omega_{14}$ and Ω_{15}) we simply take the derivative of I_3, otherwise we first use the change of variables $\mathbf{z} = x - y$ and then calculate the derivative. Using Tables 1 - 2 from [3] or [6] we again obtain

$$\left|\frac{\partial}{\partial x_i} I_3(x)\right| \leq C\eta_{-\frac{3}{2}}^{-\frac{5}{2}+\varepsilon}(x).$$

Analogously we proceed for higher derivatives ■

Similarly, in two space dimensions:

Lemma 3.2. *Let $N = 2$. Then, for $k \geq 0$, $\varepsilon > 0$ arbitrarily small and $|x| \geq R \gg 1$,*

$$|\nabla^k E_{ij}^\mu(x; 1)| \leq C|x|^{-\frac{3+k-\varepsilon}{2}}(1 + s(x))^{-\frac{1+k}{2}}. \tag{3.2}$$

Proof. It is analogous to the three-dimensional case ■

Remark 3.1. Obviously, E^μ and ∇E^μ decay faster than $\mathcal{O}$ or $\nabla\mathcal{O}$, respectively. Only for $\nabla^2 E^\mu$, due to a certain ε-loss, $\frac{\partial^2 \mathcal{O}}{\partial x_1^2}$ decays faster near the axis x_1. Nevertheless, it can be shown that the logarithmic factor does not appear and that $\frac{\partial^2 E^\mu}{\partial x_1^2}$ decays as fast as $\frac{\partial^2 \mathcal{O}}{\partial x_1^2}$. Note also that in applications we usually do not distinguish the asymptotic behaviour of particular components of $\nabla^2 \mathcal{O}$.

The investigation of E^μ near zero is more delicate. We wish to verify not only that $\nabla^2 E^\mu \sim |x|^{-N}$ for $|x| \to 0$, but also that the singular part of $\nabla^2 E^\mu$ represents, similarly as $\nabla^2 \mathcal{S}$, a kernel of a bounded operator from $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$.

For simplicity we omit in the following the parameter $\lambda = 1$ in (2.8). Introduce two cut-off functions $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\varphi = \begin{cases} 1 & \text{on } B_2 \\ 0 & \text{on } B_3^c \end{cases} \quad \text{and} \quad \psi = \begin{cases} 1 & \text{on } B_4 \\ 0 & \text{on } B_5^c. \end{cases}$$

Then

$$E^\mu = \mu\partial_1 E^* * \partial_1 \mathcal{O} = I_1 + I_2 + I_3$$

where

$$I_1(x) = \mu \int \partial_1(\psi E^*)(x - y)\partial_1(\varphi\mathcal{O})(y)\, dy$$

is the most crucial term,

$$I_2 = \mu\partial_1((1 - \psi)E^*) * \partial_1(\varphi\mathcal{O})$$

vanishes in B_1, and

$$I_3 = \mu \partial_1 E^* * \partial_1((1 - \varphi)\mathcal{O})$$

is easily seen to be bounded together with all its derivatives in B_1. Obviously,

$$J = I_1 + I_2 = \mu \partial_1 E^* * \partial_1(\varphi \mathcal{O})$$

satisfies

$$(\Delta - \mu \partial_1^2 - \partial_1)J = \mu \partial_1^2(\varphi \mathcal{O}) \qquad \text{in } \mathscr{S}'$$

(cf. Lemma 2.2). Since $I_2 = 0$ in B_1, we conclude that $I_1 \in L^p(B_1) \cap C^\infty(B_1 \backslash \{0\})$ where $p \in (1, \infty)$ for $N = 2$ and $p \in (1, 3)$ for $N = 3$.

To establish the singular behaviour of I_1 near 0 we estimate its Fourier transform

$$\hat{I}_1 = (2\pi)^{\frac{N}{2}} \mu \mathcal{F}(\partial_1(\psi E^*)) \cdot \mathcal{F}(\partial_1(\psi \mathcal{O})) = -(2\pi)^{\frac{N}{2}} \mu \xi_1^2 \widehat{\psi E^*} \cdot \widehat{\varphi \mathcal{O}}$$

and the ξ-derivative $D_\xi^\alpha \hat{I}_1$. Note that this formula for $\hat{I}_1$ is justified by [7: Theorem 30.4]. Since $x^\alpha \partial_1(\psi E^*), x^\alpha \partial_1(\varphi \mathcal{O}) \in L^1(\mathbb{R}^N)$ for all $\alpha \in \mathbb{N}_0^N$, the functions $D_\xi^\alpha \mathcal{F}(\partial_1(\psi E^*))$ and $D_\xi^\alpha \mathcal{F}(\partial_1(\varphi \mathcal{O}))$ are continuous on $\mathbb{R}^N$ vanishing for $|\xi| \to \infty$, the same result being true when the partial derivative ∂_1 is omitted. Thus for all $\alpha \in \mathbb{N}_0^N$

$$D_\xi^\alpha \hat{I}_1(\xi) = O(|\xi|^{2-|\alpha|}) \qquad \text{for } |\xi| \to 0. \tag{3.3}$$

For large $|\xi|$ we use the representation (see [7: Theorem 30.4])

$$\widehat{\psi E^*}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{R^N} \hat{\psi}(\xi - \lambda) \widehat{E^*}(\lambda) \, d\lambda$$

where $\hat{\psi} \in \mathcal{S}$ and

$$|\widehat{E^*}(\lambda)| \leq \frac{c}{|\lambda|^2 + |\lambda_1|} \leq \frac{c}{|\lambda_1|^{\frac{2}{3}} |\lambda'|^{\frac{2}{3}}} \qquad (\lambda = (\lambda_1, \lambda')).$$

Consequently,

$$|\widehat{\psi E^*}(\xi)| \leq \frac{c_M}{|\xi|^M} \int_{|\lambda| \leq \frac{|\xi|}{2}} \frac{1}{|\lambda_1|^{\frac{2}{3}} |\lambda'|^{\frac{2}{3}}} \, d\lambda + \frac{c_M}{|\xi|^2} \int_{|\lambda| > \frac{|\xi|}{2}} \frac{1}{(1 + |\xi - \lambda|)^M} \, d\lambda$$

$$\leq c(|\xi|^{-M+N+-\frac{4}{3}} + |\xi|^{-2})$$

for $M \in \mathbb{N}$ arbitrarily large. Hence, dealing analogously with $\widehat{\varphi \mathcal{O}}$,

$$|\widehat{\psi E^*}(\xi)| + |\widehat{\varphi \mathcal{O}}(\xi)| = O(|\xi|^{-2}) \qquad \text{as } |\xi| \to \infty.$$

For the partial derivative

$$D_\xi^\alpha \widehat{\psi E^*}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} (-1)^{|\alpha|} D_\lambda^\alpha \hat{\psi}(\xi - \lambda) \widehat{E^*}(\lambda) \, d\lambda$$

$$= \int_{|\lambda| \leq \frac{|\xi|}{2}} \ldots + \int_{|\lambda| \geq \frac{|\xi|}{2}} \ldots$$

we use integration by parts $|\alpha|$-times in the second integral. Here the first integral on $|\lambda| \leq \frac{|\xi|}{2}$ and $|\alpha|$-many boundary integrals on $|\lambda| = \frac{|\xi|}{2}$ can be estimated by $c_M |\xi|^{-M}$ for M arbitrarily large. Finally, for the remaining integral,

$$\left| \int_{|\lambda| \geq \frac{|\xi|}{2}} \hat{\psi}(\xi - \lambda) D_\lambda^\alpha \widehat{E^*}(\lambda) \, d\lambda \right| \leq c_M \int_{|\lambda| \geq \frac{|\xi|}{2}} \frac{1}{(1 + |\xi - \lambda|)^M} |\xi|^{-2-|\alpha|} \, d\lambda$$

$$\leq c|\xi|^{-2-|\alpha|}.$$

Thus we proved that for all $\alpha \in \mathbb{N}_0^n$

$$|D_\xi^\alpha \widehat{\psi E^*}(\xi)| + |D_\xi^\alpha \widehat{\varphi \mathcal{O}}(\xi)| = O(|\xi|^{-2-|\alpha|}) \qquad \text{as } |\xi| \to \infty. \tag{3.4}$$

Summarizing (3.3) and (3.4) we obtain

$$|D_\xi^\alpha \hat{I}_1(\xi)| \leq c_\alpha \frac{|\xi|^{2-|\alpha|}}{(1 + |\xi|)^4} \qquad \text{for } \xi \neq 0, \, \alpha \in \mathbb{N}_0^N. \tag{3.5}$$

We shall now reconstruct the asymptotic properties of $I_1(x)$ using the following lemma.

Lemma 3.3. *Let $\mathcal{F}(G) \in C^{m+N-1}(\mathbb{R}^N \setminus \{0\})$ $(N \geq 2)$ be such that*

$$A := \sup_{\substack{|\alpha| \leq m+N-1 \\ \xi \in \mathbb{R}^N \setminus \{0\}}} (1 + |\xi|)^4 |\xi|^{|\alpha|-2} |D^\alpha \mathcal{F}(G)(\xi)| < \infty.$$

Then $G \in C^m(\mathbb{R}^N \setminus \{0\})$ and there exists $c_{m,N} > 0$ such that for all $0 \neq x \in B_1$

$$|G(x)| \leq \begin{cases} c_{m,N} A |\ln |x|| & \text{if } N = 2 \\ c_{m,N} A |x|^{2-N} & \text{if } N \geq 3 \end{cases}$$

$$|D^\beta G(x)| \leq c_{m,N} A |x|^{2-N-|\beta|} \quad \text{if } N \geq 2, 0 < |\beta| \leq m.$$

Furthermore, for $x \in B_1^c$, $N \geq 2$ and $0 \leq |\beta| \leq m$,

$$|D^\beta G(x)| \leq c_{m,N} A |x|^{1-N-|\beta|}.$$

Proof. Fix $\Phi \in C_0^\infty(\mathbb{R}^N)$ such that $\Phi = 1$ in $B_{\frac{1}{2}}$ and $\Phi = 0$ in B_1^c. For $\beta \in \mathbb{N}_0^N$ and $\lambda > 0$ we define $v_{\lambda,\beta}$ and $w_{\lambda,\beta}$ by

$$\mathcal{F}(v_{\lambda,\beta})(\xi) = (i\xi)^\beta \widehat{G}(\xi) \Phi\left(\frac{\xi}{\lambda}\right)$$

$$\mathcal{F}(w_{\lambda,\beta})(\xi) = (i\xi)^\beta \widehat{G}(\xi) \left(1 - \Phi\left(\frac{\xi}{\lambda}\right)\right)$$

yielding

$$\mathcal{F}(v_{\lambda,\beta}) + \mathcal{F}(w_{\lambda,\beta}) = (i\xi)^\beta \widehat{G}(\xi) = \mathcal{F}(D^\beta G)(\xi).$$

Moreover, $\mathcal{F}(v_{\lambda,\beta}) \in L^1(\mathbb{R}^N)$ and therefore

$$|v_{\lambda,\beta}(x)| \leq cA \int_{|\xi| \leq \lambda} |\xi|^{|\beta|+2}(1+|\xi|)^{-4}\,d\xi \leq cA \int_0^\lambda \frac{r^{|\beta|+N+1}}{(1+r)^4}\,dr.$$

For $\lambda < 1$ we get $|v_{\lambda,\beta}(x)| \leq cA\lambda^{|\beta|+N+2}$, while for $\lambda > 1$ we distinguish between the cases $\beta = 0, N = 2$ and $|\beta| + N > 2$. Obviously,

$$|v_{\lambda,0}(x)| \leq cA(1 + \ln \lambda) \qquad \text{if } N = 2$$
$$|v_{\lambda,\beta}(x)| \leq cA\lambda^{|\beta|+N-2} \qquad \text{if } |\beta| + N > 2.$$

Now choose $p \in \mathbb{N}_0^N$ such that $|p| = |\beta| + N - 1$. We will show that $x^p w_{\lambda,\beta}(x)$ is a continuous function tending to 0 as $|x| \to \infty$, satisfying

$$|x^p w_{\lambda,\beta}(x)| \leq cA.$$

The function $\mathcal{F}(w_{\lambda,\beta})$ has its support in $B^c_{\lambda/2}$. Evidently, $D^p \mathcal{F}(w_{\lambda,\beta}) \in L^1(\mathbb{R}^N)$ since it is a linear combination of

$$\xi^{\beta-q} D^\alpha \widehat{G}(\xi) D^r \left(1 - \Phi\left(\frac{\xi}{\lambda}\right)\right)$$

where $\alpha + q + r = p$ and $q \leq \beta$. If $r \neq 0$, its support lies between two spheres with diameters $\frac{\lambda}{2}$ and λ. Therefore

$$c\lambda^{-|r|} \int_{\frac{\lambda}{2} < |\xi| < \lambda} |\xi|^{|\beta|-|q|}|D^\alpha \widehat{G}(\xi)|\,d\xi$$
$$\leq cA\lambda^{-|r|} \int_{\frac{\lambda}{2} < |\xi| < \lambda} |\xi|^{|\beta|-|q|+2-|\alpha|}(1+|\xi|)^{-4}\,d\xi$$
$$\leq cA\lambda^{-|p|+|\beta|+N+2}(1+\lambda)^{-4}.$$

If $r = 0$, then the support of $\xi^{\beta-q} D^\alpha \widehat{G}(\xi)\left(1 - \Phi\left(\frac{\xi}{\lambda}\right)\right)$ lies in $B^c_{\lambda/2}$ and

$$cA \int_{|\xi| \geq \frac{\lambda}{2}} |\xi|^{|\beta|-|q|+2-|\alpha|}(1+|\xi|)^{-4}\,d\xi \leq cA \int_{\frac{\lambda}{2}}^\infty r^{|\beta|-|p|+N+1}(1+r)^{-4}\,dr$$
$$\leq cA(1+\lambda)^{-1}.$$

Summarizing the different estimates of $x^p w_{\lambda,\beta}(x)$ we obtain

$$|x|^{|p|}|w_{\lambda,\beta}(x)| \leq cA\lambda^{-|p|+|\beta|+N-1}(1+\lambda)^{-1}.$$

Setting $\lambda = \frac{1}{|x|}$ we get the result $\blacksquare$

In particular, we have reproved that

$$I_1(x) \sim \begin{cases} \ln |x| & \text{if } N = 2 \\ \frac{1}{|x|} & \text{if } N = 3 \end{cases} \qquad (|x| \to 0).$$

Moreover, we have obtained that for $0 < |\beta| \leq 2$

$$D^\beta I_1(x) \sim \frac{1}{|x|^{|\beta|+N-2}} \qquad \text{in } B_1.$$

Therefore the integral operator

$$\int_{\mathbb{R}^N} \frac{\partial I_1}{\partial x_i}(x - y) f(y) \, dy$$

is well defined on $C_0^\infty(\mathbb{R}^N)$ and can be (eventually) extended due to a density argument onto some $L^q(\mathbb{R}^N)$. Let us consider the integral operator

$$Tf(x) = \int_{\mathbb{R}^N} I_1(x - y) \frac{\partial^2 f(y)}{\partial y_i \partial y_j} \, dy$$

for $f \in C_0^\infty(\mathbb{R}^N)$. Evidently,

$$\mathcal{F}(Tf)(\xi) = (2\pi)^{\frac{N}{2}} \hat{I}_1(\xi)(-\xi_i \xi_j) \hat{f}(\xi)$$

where the multiplication is to be understood in the sense of $\mathcal{S}'$. Nevertheless, thanks to the properties of $\hat{I}_1(\xi)$ we easily see that

$$\mathcal{F}(Tf)(\xi) = -(2\pi)^{\frac{N}{2}} \xi_i \xi_j \hat{I}_1(\xi) \hat{f}(\xi) = m(\xi) \hat{f}(\xi)$$

where $m(\xi) \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \setminus \{0\})$ and

$$|D^\alpha m(\xi)| \leq \frac{C}{|\xi|^{|\alpha|}} \qquad \text{for all } \xi \neq 0, \alpha \in \mathbb{N}_0^n.$$

Therefore, $m(\xi)$ is an L^p-multiplier, $1 < p < \infty$. Thus we get the following lemma.

Lemma 3.4. *Let $0 < |\alpha| \leq 2$. Then we have for $|x| \leq 1$*

$$|E_{ij}^\mu(x; 1)| \leq \begin{cases} C \ln |x| & \text{if } N = 2 \\ C|x|^{-1} & \text{if } N = 3 \end{cases} \qquad \text{and} \qquad |D^\alpha E_{ij}^\mu(x; 1)| \leq C|x|^{-|\alpha|-N+2}.$$

Moreover,

$$E_{ij}^\mu(x; 1) = I_1(x) + I_2(x)$$

where $|D^\alpha I_2(x)| \leq C$ for $|x| \leq 1$, $|\alpha| = 2$, and $I_1(x)$, representing the singular part of the second gradient of $E_{ij}^\mu(x; 1)$, has the property

$$\mathcal{F}\left(\int_{\mathbb{R}^N} I_1(\cdot - y) D^\alpha f(y) \, dy \right)(\xi) = m(\xi) \hat{f}(\xi) \qquad (|\alpha| = 2)$$

where $m(\xi)$ represents an L^p-Fourier multiplier, $1 < p < \infty$. Therefore, if $|\alpha| = 2$, the integral operator T,

$$Tf(x) = \int_{\mathbb{R}^N} I_1(x - y) D^\alpha f(y) \, dy,$$

maps $C_0^\infty(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ $(1 < p < \infty)$ and

$$\|Tf\|_{p,\mathbb{R}^N} \le C\|f\|_{p,\mathbb{R}^N}$$
$$\|Tf\|_{p,(g),\mathbb{R}^N} \le C\|f\|_{p,(g),\mathbb{R}^N}$$

for all weights g from the Muckenhoupt class A_p.

Remark 3.2. The non-negative weight g belongs to the Muckenhoupt class A_r $(1 \le r < +\infty)$ if there is a constant C such that

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q g(x)\, dx\right)\left(\frac{1}{|Q|} \int_Q g(x)^{-\frac{1}{r-1}}\, dx\right)^{r-1}\right] \le C < \infty$$

if $r \in (1, \infty)$ and

$$\sup_Q \frac{1}{|Q|} \int_Q g(x) \, dx \le Cg(x_0) \qquad \forall\, x_0 \in \mathbb{R}^N$$

if $r = 1$. In the first case, the supremum is taken over all cubes Q in $\mathbb{R}^N$, in the second case only over those cubes which contain x_0; $|Q|$ denotes the Lebesgue measure of Q. The constant does not depend on x_0.

Finally, we study how

$$E_{ij}^\mu(x; \lambda) = \mu \int_{\mathbb{R}^N} \frac{\partial E^*(x - y; \lambda)}{\partial x_1} \frac{\partial \mathcal{O}_{ij}(y; \lambda)}{\partial y_1} \, dy$$

behaves for $\lambda \ne 1$. Recall that we usually assume $\lambda \ll 1$. Since

$$\mathcal{O}_{ij}(x; \lambda) = \lambda^{N-2} \mathcal{O}_{ij}(\lambda x; 1)$$
$$E^*(x; \lambda) = \lambda^{N-2} E^*(\lambda x; 1)$$

we get

$$E_{ij}^\mu(x; \lambda) = \lambda^{N-2} E_{ij}^\mu(\lambda x; 1). \tag{3.6}$$

We can now summarize the results.

Theorem 3.1. *Let $N = 2, 3$ and $0 \le \mu < 1$. Then the solution to (0.6) can be written in the form*

$$\mathcal{O}^\mu(x; \lambda) = \mathcal{O}(x; \lambda) + E^\mu(x; \lambda)$$
$$e(x) = \nabla \mathcal{E}(x);$$

here $\mathcal{E}(x)$ denotes the fundamental solution to the Laplace equation, $\mathcal{O}(x; \lambda)$ is the fundamental Oseen tensor and $E^\mu(x; \lambda)$ has the following properties:

a) *For $|x| \geq R \gg 1$, $|\alpha| \geq 0$ and $\varepsilon > 0$ arbitrarily small*

$$|D^\alpha E_{ij}^\mu(x;1)| \leq C|x|^{-\frac{N+1+|\alpha|-\varepsilon}{2}}(1+s(x))^{-\frac{N-1+|\alpha|}{2}}.$$

b) *For $|x| \leq 1$ and $|\alpha| = 1, 2$*

$$|E_{ij}^\mu(x;1)| \leq \begin{cases} C\ln|x| & \text{if } N = 2 \\ C|x|^{-1} & \text{if } N = 3 \end{cases}$$

$$|D^\alpha E_{ij}^\mu(x;1)| \leq C|x|^{2-N-|\alpha|} \quad \text{if } N = 2, 3.$$

Moreover, $E_{ij}^\mu(x;1) = I_1(x) + I_2(x)$, where $|D^\alpha I_2(x)| \leq C$ for all $|\alpha| \leq 2$ and $|x| \leq 1$, and $D^\alpha I_1(x), |\alpha| = 2$, representing the singular part of the second gradient of $E^\mu(x;1)$, defines an integral operator

$$Tf(x) = \int_{\mathbb{R}^N} I_1(x-y)D^\alpha f(y)\, dy \qquad (|\alpha| = 2)$$

which maps $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ and $\overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{p,(g)}}$ into $L^p(\mathbb{R}^N; g)$ for $1 < p < \infty$, $g \in A_p$ (Muckenhoupt class). Moreover,

$$E^\mu(x;\lambda) = \lambda^{N-2}E^\mu(\lambda x; 1).$$

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