Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 3, 763–779

Traces of Besov Spaces Revisited

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Abstract. For the trace of Besov spaces $B^s_{p,q}$ onto a hyperplane, the borderline case with $s = \frac{n}{p} - (n-1)$ and $0 < p < 1$ is analysed and a new dependence on the sum-exponent q is found. Through examples the restriction operator defined for s down to $\frac{1}{p}$, and valued in L_p, is shown to be distinctly different and, moreover, unsuitable for elliptic boundary problems. All boundedness properties (both new and previously known) are found to be easy consequences of a simple mixed-norm estimate, which also yields continuity with respect to the normal coordinate. The surjectivity for the classical borderline $s = \frac{1}{p}$ $(1 \le p < \infty)$ is given a simpler proof for all $q \in]0,1]$, using only basic functional analysis. The new borderline results are based on corresponding convergence criteria for series with spectral conditions.

Keywords: *Distributional trace operator, borderline cases, mixed-norm estimate, convergence criteria, elliptic boundary problems*

AMS subject classification: 46 E 35, 46 F 10

1. Introduction

This note concerns the (distributional) trace operator γ_0 that restricts to the hyperplane $\Gamma = \{x_n = 0\}$ in $\mathbb{R}^n$ for $n \ge 2$,

$$\gamma_0 : \ f(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_{n-1}, 0). \tag{1.1}$$

The title should indicate both that there remains unexplored borderlines in the L_p-theory of γ_0 and that the existing literature do not reveal the full efficacy of the Fourier-analytic proof methods.

The main purpose is to describe the borderline cases for $0 < p < 1$. See the below Theorem 1.2 concerning $s = \frac{n}{p} - n + 1$, where it is shown that the smallest Besov space containing $\gamma_0(B^s_{p,q})$ has its integral-exponent equal to $\max(p, q)$, hence depending on both the integral-exponent and the sum-exponent of the domain. This result seems to be hitherto undescribed.

Secondly, Theorem 1.2 is proved in a mere two lines, deriving from the Paley-Wiener-Schwartz theorem and the Nikolskiĭ-Plancherel-Polya inequality a basic mixed-norm, in fact $L_p(\mathbb{R}^{n-1}; L_{\infty,x_n})$, estimate. In addition, all the known boundedness results are recovered equally easily from the same calculation. The ensuing *unified* treatment is in

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ISSN 0232-2064 / $ 2.50

contrast with the existing literature, which has various page-long arguments both for the generic cases ($s > \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+$) and the classical borderline $s = \frac{1}{p}$ ($1 \le p \le \infty$). The present paper should also be interesting for this reason.

Thirdly, another perspective on γ_0 is also gained from the mixed-norm estimate, for this yields (since the value $x_n = 0$ has no special significance) that all the treated spaces $B^s_{p,q}$ are contained in $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$ and that γ_0 is a restriction of the natural trace on the latter space. This property has not been given much attention in the Besov space literature (J. Peetre's report [18] seems to be the only example), although in practice γ_0 has been defined space by space by means of a limiting procedure. Evidently, this raises the question whether $\gamma_0 u$ is consistently defined when u belongs to both $C(\mathbb{R}^n)$ and $B^1_{1,1}(\mathbb{R}^n)$ or to another intersection of two spaces. However, the consistency is always assured by the below embedding into $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$.

Finally, the surjectivity of $\gamma_0 : B^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1})$ for $1 \le p < \infty$ and $0 < q \le 1$ is given a new proof by an easy extension of the Closed Range Theorem to quasi-Banach spaces.

For precision's sake it should be mentioned that γ_0 in the beginning of the analysis refers to a *working definition* of the trace as

$$\gamma_0 u = \sum (\check{\Phi}_k * u)|_{x_n=0},$$

whereby

$$u = \sum \mathcal{F}^{-1}(\Phi_k \hat{u})$$

is a Littlewood-Paley decomposition (cf. Section 3 below). Consistency and independence of the Φ_k are obtained post-festum (cf. (1.3) and Theorem 1.4 below).

As a point of departure, the generic properties of γ_0 are recalled:

Theorem 1.1 (see [12, 20]). *When applied to the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ with $0 < p, q \le \infty$, the trace γ_0 is continuous*

$$\gamma_0 : B^s_{p,q}(\mathbb{R}^n) \to B^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \tag{1.2}$$

for $s > \frac{1}{p}$ if $p \ge 1$, and for $s > \frac{n}{p} - n + 1$ if $p < 1$. Moreover, γ_0 has a right inverse K, which is a bounded operator from $B^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ to $B^s_{p,q}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$.

It is known, but proved explicitly here that, on the one hand, γ_0 in (1.2) is a restriction of the distributional trace, that is of

$$f(0) \quad \text{defined for } f \in C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1})). \tag{1.3}$$

(This is also denoted by $\gamma_0 f$ in the rest of the introduction.) On the other hand, the restriction of γ_0 to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ extends by continuity (cf. [5, 6, 13, 22]) to an operator

$$T : B^s_{p,q}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1}) \qquad \text{for } s > \tfrac{1}{p}, 0 < p < 1. \tag{1.4}$$

It should be emphasised that T is rather different from γ_0 when $s < \frac{n}{p} - n + 1 = \frac{1}{p} + (n-1)(\frac{1}{p} - 1)$ (whereby γ_0 acts only on the intersection of $B^s_{p,q}$ and $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$, cf. (1.3)). Their incompatibility may be exemplified by tensorising some $\varphi \in C_0^\infty(\mathbb{R})$ equal to 1 near $x_n = 0$ with the delta measure δ_0 in $\mathbb{R}^{n-1}$, for

$$\gamma_0\big(\delta_0(x') \otimes \varphi(x_n)\big) = \delta_0(x') \tag{1.5}$$

whereas

$$T\big(\delta_0(x') \otimes \varphi(x_n)\big) = 0. \tag{1.6}$$

Here (1.5) is clear by (1.3), since $a\delta_0$ depends continuously on the scalar a.

The result in (1.6) is connected to the fact that the co-domain L_p is not continuously embedded into $\mathcal{D}'$ when $p < 1$; this fact is elementary, for when $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\int \eta = 1$, then $k^n \eta(k\cdot)$ tends to δ_0 in $\mathcal{D}'$ and to 0 in L_p for $k \to \infty$ because

$$\|k^n \eta(k\cdot)|L_p\| = \|\eta|L_p\| k^{n(1-\frac{1}{p})} \to 0 \qquad \text{for each } p < 1. \tag{1.7}$$

With a similar $\eta \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\psi_k(x) = k^{n-1}\eta(kx')\varphi(x_n)$,

$$\gamma_0 \psi_k = k^{n-1}\eta(kx') \to \delta_0 \qquad \text{in } \mathcal{D}' \tag{1.8}$$

whereas

$$T\psi_k = k^{n-1}\eta(kx') \to 0 \qquad \text{in } L_p, \tag{1.9}$$

so the sequence (ψ_k) is treated rather differently by γ_0 and T (in fact (1.6) can be proved thus, cf. Remark 8.1 below). These phenomena also depend on the *domain* chosen in (1.4). Indeed, γ_0 in (1.3) is for $p < 1$ continuous $B^{\frac{n}{p}-n+1}_{p,q} \to \mathcal{D}'$ only if $q \le 1$ (and a fortiori not at all for $s < \frac{n}{p} - n + 1$) by [16: Lemma 2.8] or [14: Lemma 2.5.2]; however, the counterexample there does not contradict (1.4), cf. Remark 8.2. (Similarly, for $s = \frac{1}{p}$ and $q > 1$, hence for $s < \frac{1}{p}$, it was shown too that γ_0 is never continuous from $B^s_{p,q}$, *regardless* of the co-domain.) Moreover, the severe shortcomings of T in connection with elliptic boundary problems for $s \le \frac{n}{p} - n + 1$ are reviewed in Remark 8.3 below.

Altogether T discards so much information that it is inconsistent with the distribution trace γ_0, seemingly to the extent that it is inappropriate, for the usual applications, to maintain $s = \frac{1}{p}$ as the borderline when $p < 1$.

In view of the above, it is natural to analyse $s = \frac{n}{p} - n + 1$ when $p < 1$. The main point is that $q \le p \le 1$ and $p < q \le 1$ constitute two rather different cases:

Theorem 1.2. *For $0 < p < 1$ the operator γ_0 is continuous*

$$\gamma_0 : B^{\frac{n}{p}-n+1}_{p,q}(\mathbb{R}^n) \to B^{(n-1)(\frac{1}{p}-1)}_{p,\infty}(\mathbb{R}^{n-1}) \qquad \textit{if } q \le p < 1, \tag{1.10}$$

whereas it is bounded

$$\gamma_0 : B^{\frac{n}{p}-n+1}_{p,q}(\mathbb{R}^n) \to B^{(n-1)(\frac{1}{q}-1)}_{q,\infty}(\mathbb{R}^{n-1}) \qquad \textit{when } p < q \le 1. \tag{1.11}$$

Furthermore, q is the smallest possible integral-exponent for the co-domain in (1.11), *for even $B^t_{r,\infty}$ can only receive when $r \geq q$.*

This shows that the smallest Besov space one may use as a co-domain of γ_0 is $B^{(n-1)(\frac{1}{r}-1)}_{r,\infty}$ with $r = \max(p,q)$ when $s = \frac{n}{p} - n + 1$ and $0 < p < 1$; in addition, neither (1.10) nor (1.11) is a surjection (hence the range is not a Besov space, cf. Remark 1.5 below). Altogether this makes a noteworthy contrast with Theorem 1.1.

To elucidate Theorem 1.2, one can observe that the above-mentioned operator T is a continuous surjection (see [5: Theorem 5.1] and [22: 4.4.3]),

$$T : B^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1}) \qquad \text{for } 0 < q \leq p < 1. \tag{1.12}$$

Here the condition $q \leq p$ is known to be necessary, and formally a distinction between the same cases appear in Theorem 1.2, too. This seems surprising and unnoticed hitherto, and a fortiori the theorem is a novelty (cf. Remark 1.5 below).

As an interpretation of (1.1), note that it follows from (1.10) when combined with a Sobolev embedding. In fact, given (1.10), then

$$B^{\frac{n}{p}-n+1}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\frac{n}{q}-n+1}_{q,q}(\mathbb{R}^n) \xrightarrow{\gamma_0} B^{(n-1)(\frac{1}{q}-1)}_{q,\infty}(\mathbb{R}^{n-1}), \tag{1.13}$$

and since q is the optimal integral-exponent on the right-hand side of (1.11) (cf. Section 7 below), this is the *only* way to apply γ_0 when $p < q \leq 1$. Moreover, in both (1.10) and (1.11) one can take $L_1(\mathbb{R}^{n-1})$ as the receiving space, for by a Sobolev embedding into $B^1_{1,1}(\mathbb{R}^n)$ the question is reduced to a case (viz. $p = 1$) of the following

Theorem 1.3. *Let $1 \leq p \leq \infty$ and $0 < q \leq 1$. Then γ_0 in* (1.3) *is bounded*

$$\gamma_0 : B^{\frac{1}{p}}_{p,q}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1}). \tag{1.14}$$

Moreover, (1.14) *is a surjection if $1 \leq p < \infty$ and $0 < q \leq 1$.*

Earlier Burenkov, Gol'dman and Peetre [2, 10, 18] proved surjectivity for $q = 1$ (the latter two even for anisotropic spaces), but the first to consider this borderline were seemingly Agmon and Hörmander [1] (cf. their note), who covered $p = 2$. However, the borderline itself was found in 1951 by Nikolskiĭ [17]. Using atomic decompositions, Frazier and Jawerth [5] proved the surjectivity for $0 < q \leq 1$. An alternative argument is given below by means of a short application of the Closed Range Theorem (extended to quasi-Banach spaces); it should be interesting because of the simplicity.

Theorems 1.1 − 1.3 are proved and re-proved here, for they may actually all be obtained by combining general principles with a single, mixed-norm estimate; in its turn, this estimate follows straightforwardly from the Paley-Wiener-Schwartz theorem and the Nikolskiĭ-Plancherel-Polya inequality (see Section 4 below). Besides being a unified proof, it is also simple compared to those, e.g., in [3, 5, 21].

The mixed-norm estimate actually shows $\mathcal{S}'$-convergence of the series entering the working definition of $\gamma_0 u$ in (3.1) below. In Theorem 1.3 this is a consequence of L_p's completeness, and for the generic cases in Theorem 1.1 it follows from known convergence criteria for series with spectral conditions, summed up in part (ii) of Theorem 3.1 below.

Furthermore, a small reflection about this estimate yields

Theorem 1.4. *Let $s \geq \frac{1}{p} + (n-1)(\frac{1}{p}-1)_+$, and suppose $q \leq 1$ holds in the case of equality. Then there is an inclusion*

$$B^s_{p,q}(\mathbb{R}^n) \subset C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1})), \tag{1.15}$$

and the working definition of γ_0 equals the restriction to $B^s_{p,q}(\mathbb{R}^n)$ of the natural trace on the space $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$.

For the two cases in Theorem 1.2 it is also noteworthy that they stem from an analogous destinction in part (iii) of Theorem 3.1 below. However, part (iii) of the latter theorem is actually a generalisation of the criteria to the borderline $s = \frac{n}{p} - n$, and the necessity of the splitting into two cases is shown in Proposition 3.2. Hence this paper also contributes to the convergence criteria in general Besov spaces.

Remark 1.5. In a subsequent joint work [7], inspired by the present article, especially Theorems 1.2 and 1.4, the traces of all admissible Besov and Triebel-Lizorkin spaces were determined. In particular, the exact ranges in (1.10) and (1.11) were found to be the approximation space $A^{(n-1)(\frac{1}{p}-1)}_{p,q}$ in both cases. So although $r = \max(p,q)$ is the smallest possible integral-exponent when the co-domain is stipulated to be a Besov space (as in Theorem 1.2 ff. and throughout this paper), the situation is different if the scale of $A^s_{p,q}$ spaces is adopted.

Acknowledgement. In the early stages I benefitted from discussions with Prof. H. Triebel, who also kindly provided [18].

2. Preliminaries

For the general notions in distribution theory standard notation is used, similarly to [11]; $C(\mathbb{R}, X)$ denotes the vector space of continuous functions from $\mathbb{R}$ to X, and if X is a Banach space, $C_{\mathrm{b}}(\mathbb{R}, X)$ stands for the sup-normed space of continuous bounded functions.

For the Besov spaces $B^s_{p,q}$ the conventions of [23] are adopted, so the norm is defined from a Littlewood-Paley decomposition $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$, where the $\Phi_j(\xi)$ vanish unless $\frac{11}{20}2^j \leq |\xi| \leq \frac{13}{10}2^j$ when $j > 0$. This may, moreover, be obtained by letting $\Phi_0 = \Psi_0$ and $\Phi_j = \Psi_j - \Psi_{j-1}$ when $\Psi_j(\xi) = \Psi(2^{-j}|\xi|)$ for some real C^∞ function $\Psi(t)$ on $\mathbb{R}$ vanishing for $t > \frac{13}{10}$ and equalling 1 for $t < \frac{11}{10}$; in this case $\Psi_j = \Phi_0 + \ldots + \Phi_j$. Then $B^s_{p,q}$ is defined to consist of the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\|u|B^s_{p,q}\| = \left(\sum_{k=0}^{\infty} 2^{skq} \|\mathcal{F}^{-1}(\Phi_k \hat{u})|L_p\|^q\right)^{\frac{1}{q}} < \infty. \tag{2.1}$$

On $\mathbb{R}^{n-1}$ a partition of unity $1 = \sum \Phi'_j$ with $\Phi'_j(\xi') = \Phi_j(\xi', 0)$ is used.

Equivalently, a partition may be used in which each function is a product of n factors, each depending on a single coordinate ξ_j of ξ. This is folklore, but for precision the following easy construction and Lemma 2.1 below are given. Let $\Phi_k^{(1)}$ and $\Psi_k^{(1)}$ denote the functions obtained in the manner above for $n = 1$. Then

$$\tilde\Psi_k(\xi) := \Psi_k^{(1)}(\xi_1)\cdots\Psi_k^{(1)}(\xi_n) \tag{2.2}$$

equals 1 in $B_\infty(0, \frac{11}{10}2^k)$, the max-norm ball of radius $\frac{11}{10}2^k$, centred at the origin; $\operatorname{supp}\tilde\Psi_k$ lies in $B_\infty(0, \frac{13}{10}2^k)$. Now insertion of $\Psi_k^{(1)} = \Psi_{k-1}^{(1)} + \Phi_k^{(1)}$ gives, for $k \ge 1$,

$$\tilde\Psi_k(\xi) = \tilde\Psi_{k-1}(\xi) + \sum_{\emptyset\ne J\subset\{1,\dots,n\}} \Theta_{J,k}(\xi) \tag{2.3}$$

whereby

$$\Theta_{J,k}(\xi) = \prod_{j\in J}\Phi_k^{(1)}(\xi_j)\prod_{j\notin J}\Psi_{k-1}^{(1)}(\xi_j). \tag{2.4}$$

Letting $\Theta_{J,0} = \tilde\Psi_0$, this yields a smooth partition of unity since for $\xi \in \mathbb{R}^n$

$$1 = \sum_{k=0}^{\infty}\sum_J \Theta_{J,k}(\xi). \tag{2.5}$$

When $k \ge 1$, then evidently

$$\operatorname{supp}\Theta_{J,k} \subset B_\infty(0, \tfrac{13}{10}2^k) \setminus B_\infty(0, \tfrac{11}{10}2^{k-1}). \tag{2.6}$$

Observe also the tensor product structure of the function $\Theta_{J,k}$ and that $\Theta_{J,k}(\xi) = \Theta_{J,1}(2^{-(k-1)}\xi)$ for $k \ge 1$.

Finally, the next lemma may be proved in the usual way by means of part (iv) in Theorem 3.1 below, using also that independently of k there are (1 or) $2^n - 1$ terms in the sum over J.

Lemma 2.1. *For every $s \in \mathbb{R}$ and $p, q \in \,]0,\infty]$ the Besov space $B^s_{p,q}(\mathbb{R}^n)$ coincides with the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm*

$$\|u|B^s_{p,q}\|^\Theta = \Big(\sum_{k=0}^{\infty}\sum_J 2^{skq}\|\mathcal{F}^{-1}(\Theta_{J,k}\hat u)|L_p\|^q\Big)^{\frac1q} \tag{2.7}$$

is finite. Moreover, $\|\cdot|B^s_{p,q}\|^\Theta$ is an equivalent quasi-norm for $B^s_{p,q}$.

3. Definition of the trace

3.1 The working definition. When dealing with $\gamma_0 u$ it is convenient to take a Littlewood-Paley partition of unity, say $1 = \sum_{j=0}^{\infty} \Phi_j$, and let

$$\gamma_0 u = \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\Phi_j \mathcal{F} u)\big|_{x_n=0} \tag{3.1}$$

for those $u \in \mathcal{S}'(\mathbb{R}^n)$ for which the sum converges in $\mathcal{D}'(\mathbb{R}^{n-1})$: by the Paley-Wiener-Schwartz theorem each summand $\mathcal{F}^{-1}(\Phi_j \mathcal{F} u)$ is an entire analytic function for which restriction to $x_n = 0$ makes sense. However, the limit in (3.1) might depend on the Φ_j, but in Proposition 5.1 below, this is shown not to be the case for the spaces treated here. (The procedure in (3.1) was used to define the trace in [12], but without justification or relation to other trace notions.)

The usefulness of (3.1) depends on the availability of easy-to-apply results for the convergence of a series $\sum_{j=0}^{\infty} u_j$. While for a general Banach space X a finite norm series, $\sum_{j=0}^{\infty} \|u_j|X\| < \infty$, is such a criterion, $B^s_{p,q}$ has a variant with $\ell^s_q(L_p)$-norms without the troublesome $\mathcal{F}^{-1}\Phi_j\mathcal{F}$ acting on u_j.

For the reader's sake, these criteria for series with spectral conditions are recalled with [23: Theorems 3.6 and 3.7] in parts (ii) and (iv) of the theorem below, together with supplements on the borderline cases for $s = \max(0, \frac{n}{p} - n)$ in parts (i) and (iii).

Theorem 3.1. *Let a series $\sum_{j=0}^{\infty} u_j$ of distributions u_j in $\mathcal{S}'(\mathbb{R}^n)$ be given together with numbers $s \in \mathbb{R}$ and $p, q \in\,]0, \infty]$, and consider then*

$$B = \Big(\sum_{j=0}^{\infty} 2^{sjq} \|u_j|L_p\|^q \Big)^{\frac{1}{q}} \tag{3.2}$$

as a constant in $[0, \infty]$ (with sup-norm over j if $q = \infty$). Then the following assertion is valid:

(i) *If $s = 0$, $1 \le p \le \infty$ and $q \le 1$, then $B < \infty$ implies that $\sum u_j$ converges in $L_p(\mathbb{R}^n)$ to a sum u for which $\|u|L_p\| \le B$ holds.*

In addition, suppose that for some $A > 0$ the spectral condition

$$\operatorname{supp} \mathcal{F} u_j \subset \{\xi \mid |\xi| \le A 2^j\} \tag{3.3}$$

is satisfied by each u_j $(j \ge 0)$. Then one has:

(ii) *If $s > \max(0, \frac{n}{p} - n)$, then $B < \infty$ implies convergence of $\sum u_j$ in $\mathcal{S}'(\mathbb{R}^n)$ to a limit u in $B^s_{p,q}(\mathbb{R}^n)$ for which $\|u|B^s_{p,q}\| \le cB$ holds for some constant $c = c(n, s, p, q)$.*

(iii) *If $s = \frac{n}{p} - n$, $p \in\,]0, 1[$ and $q \in\,]0, 1]$, then $B < \infty$ implies convergence of $\sum u_j$ in $L_1(\mathbb{R}^n)$ to a limit u in L_1 for which $\|u|L_1\| \le cB$ holds for some constant $c = c(n, p, q)$.*

Moreover, there is then a constant $c = c(n,p,q)$ such that u belongs to $B^{\frac{n}{p}-n}_{p,\infty}$ or $B^{\frac{n}{q}-n}_{q,\infty}$ and satisfies the estimate

$$\|u|B^{\frac{n}{p}-n}_{p,\infty}\| \le cB \qquad \text{when } q \le p < 1 \tag{3.4}$$

$$\|u|B^{\frac{n}{q}-n}_{q,\infty}\| \le cB \qquad \text{when } p < q \le 1, \tag{3.5}$$

respectively.

(iv) *Furthermore, if the stronger condition*

$$\operatorname{supp}\mathcal{F}u_j \subset \big\{\xi \mid A^{-1}2^j \le |\xi| \le A2^j\big\} \tag{3.6}$$

holds for $j > 0$, then assertion (ii) *holds for all $s \in \mathbb{R}$.*

Proof. The completeness of L_p easily gives assertion (i) (cf. [15: Proposition 2.5]). The L_1-part of assertion (iii) may be reduced to assertion (i) by means of the Nikolskiĭ-Plancherel-Polya inequality (cf. [15: Proposition 2.6]; modulo typos there: L^p should have been L_1 and the corresponding estimate $\|u|L_1\| \le cB$). This gives the existence of u, and since

$$\mathcal{F}^{-1}(\Phi_j\hat{u}) = \sum_{k=j-h}^{\infty} \mathcal{F}^{-1}(\Phi_j\hat{u}_k)$$

for some fixed $h \in \mathbb{Z}$, we may for $q \le p$ use $\ell_q \hookrightarrow \ell_p$ to get

$$\begin{aligned}\|\mathcal{F}^{-1}(\Phi_j\hat{u})|L_p\| &\le \Bigg(\sum_{k=j-h}^{\infty} \|\check{\Phi}_j * u_k|L_p\|^p\Bigg)^{\frac{1}{p}} \\ &\le c\Bigg(\sum_{k=j-h}^{\infty} 2^{k(\frac{n}{p}-n)p}\|\check{\Phi}_j|L_p\|^p\|u_k|L_p\|^p\Bigg)^{\frac{1}{p}} \\ &\le c\max\big(\|\check{\Phi}_0|L_p\|, \|\check{\Phi}_1|L_p\|\big)2^{j(n-\frac{n}{p})}B.\end{aligned} \tag{3.7}$$

Therefore u is in $B^s_{p,\infty}$ for $s = \frac{n}{p} - n$ with the required estimate. For $p < q$ the Nikolskiĭ-Plancherel-Polya inequality applied to B reduces the question to the case with $p = q$ ∎

It was also shown in [15: Example 2.4] that in both the assertions (i) and (iii) of Theorem 3.1 the restriction $q \le 1$ is optimal; for $q > 1$ there exists series diverging in $\mathcal{D}'(\mathbb{R}^n)$ for which the associated B is finite.

In addition to this, the receiving spaces in assertion (iii) must have sum-exponents equal to infinity (see [7: Theorem 6], where this is derived from trace estimates) and the integral-exponents cannot be smaller than p and q, respectively:

Proposition 3.2. *If for some $t \in \mathbb{R}$ and $r > 0$ there exists $c \in]0,\infty[$ such that every $u \in \mathcal{S}(\mathbb{R}^n)$ satisfies*

$$\|u|B^t_{r,\infty}\| \le c\Bigg(\sum_{j=0}^{\infty} 2^{j(\frac{n}{p}-n)q}\|u_j|L_p\|^q\Bigg)^{\frac{1}{q}} \tag{3.8}$$

whenever $u = \sum u_j$ is a decomposition satisfying (3.3)*, then $r \geq q$.*

Consequently, for $p < q \leq 1$ in part (iii) of Theorem 3.1*, the receiving space $B^{\frac{n}{q}-n}_{q,\infty}$ is optimal with respect to the integral-exponent.*

Proof. The latter statement follows from the former, for on the one hand $B^{\frac{n}{q}-n}_{q,\infty} \hookrightarrow B^{\frac{n}{r}-n}_{r,\infty}$ for $r \geq q$, and if, on the other hand, $B^{\frac{n}{r}-n}_{r,\infty}$ receives with an estimate for some $r < q$, then (3.8) holds. In particular, it does so when $u = \sum u_j$ is a decomposition of a Schwartz function, so the contradicting conclusion $r \geq q$ follows.

When (3.8) holds, one may for arbitrary fixed points $x_j \in \mathbb{R}^n$ define

$$\omega_N = \sum_{k=1}^{N} \check{\Psi}_k(x - x_k). \tag{3.9}$$

Independently of the choice of x_j, the right-hand side of (3.8) equals $cN^{\frac{1}{q}} \|\check{\Psi}_0 | L_p\|$, and it is well known that $x_1, x_2, \ldots$ may be chosen such that

$$\|\omega_N | B^t_{r,\infty}\| \geq c(r)\, N^{\frac{1}{r}}; \tag{3.10}$$

so in view of (3.8) the inequality $r \geq q$ must hold. For completeness' sake it is remarked that (3.10) may be seen thus: clearly, the fact $\Psi_k \equiv 1$ on $\operatorname{supp} \Phi_0$ yields

$$\|\omega_N | B^t_{r,\infty}\| \geq \|\check{\Phi}_0 * \omega_N | L_r\| = \left\| \sum_{k=1}^{N} \check{\Phi}_0(\cdot - x_k) \middle| L_r \right\|. \tag{3.11}$$

Moreover, $\Phi_0(\xi) = \Phi_0(-\xi) \geq 0$, so $\check{\Phi}_0$ is real-valued with $\check{\Phi}_0(0) > 0$, hence some $\delta > 0$ fulfils $\check{\Phi}_0(x) > \frac{1}{2}\check{\Phi}_0(0) > 0$ for $|x| < \delta$. There is also $R > \delta$ such that $|\check{\Phi}_0(x)| < \frac{1}{2N}\check{\Phi}_0(0)$ for $|x| > R$, so if $x_j = 3jR(1, 0, \ldots, 0)$,

$$\|\omega_N | B^t_{r,\infty}\| \geq \frac{1}{2}\left(\sum_{k=1}^{N} \int_{B(x_k,\delta)} |\check{\Phi}_0(x - x_k)|^r \, dx \right)^{\frac{1}{r}} = c(r, \Phi_0, \delta)\, N^{\frac{1}{r}}. \tag{3.12}$$

Indeed,

$$|\check{\Phi}_0 * \omega_N| \geq \check{\Phi}_0(\cdot - x_j) - \tfrac{N-1}{2N}\check{\Phi}_0(0) \geq \tfrac{1}{2}\check{\Phi}_0(\cdot - x_j)$$

holds on the ball $B(x_j, \delta)$ because $|x_k - x| > R$ does so for $k \neq j$. This shows (3.10) ∎

Remark 3.3. In assertions (ii) and (iv) of Theorem 3.1, the series $u = \sum u_j$ converges in $B^s_{p,q}$ if $q < \infty$ and in $B^{s-\varepsilon}_{p,1}$ for $\varepsilon > 0$ if $q = \infty$. This is a well-known easy consequence of the completeness and the norm estimate in the theorem.

Remark 3.4. The spectral conditions in (3.3) are robust under restriction: when $x = (x', x'')$ is a splitting of the variables and x'' is kept fixed, then

$$\operatorname{supp} \mathcal{F}_{x' \to \xi'} u_j(\cdot, x'') \subset \{\xi' \mid |\xi'| \leq A2^j\} \tag{3.13}$$

by the Paley-Wiener-Schwartz theorem, for $u_j(\cdot, x'')$ is still an analytic function satisfying the relevant estimates in $\operatorname{Re} z'$ and $\operatorname{Im} z'$. By the same argument, (3.6) goes over into (3.13) for $u_j(\cdot, x'')$.

3.2 The distribution trace. A rather general definition of the trace is obtained as $r_0 f := f(0)$ on the subspace

$$C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1})) \subset \mathcal{D}'(\mathbb{R}^n). \tag{3.14}$$

For the spaces considered in this note, the working definition in (3.1) actually amounts to a restriction of r_0. This is proved in Proposition 5.1 below by means of the injection in (3.14), so this folklore is explicated (in lack of a reference):

Proposition 3.5. *Let $f \in C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$, whereby $\mathcal{D}'(\mathbb{R}^{n-1})$ has the w^*-topology. Then*

$$\langle \Lambda_f, \varphi \rangle = \int_{\mathbb{R}} \langle f(t), \varphi(\cdot, t) \rangle \, dt \qquad (\varphi \in C_0^\infty(\mathbb{R}^n)) \tag{3.15}$$

defines an injection of $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$ into $\mathcal{D}'(\mathbb{R}^n)$.

Proof. When $\varphi \in C_0^\infty$ is supported by the rectangle $K = [-k, k]^n$, bilinearity and the Banach-Steinhaus theorem for $C_0^\infty([-k, k]^{n-1})$ give continuity of the map $t \mapsto \langle f(t), \varphi(\cdot, t) \rangle$ and, for constants $c_k, N_k < \infty$, the bound

$$\Big| \int_{-k}^{k} \langle f(t), \varphi(\cdot, t) \rangle \, dt \Big| \le 2kc_k \sup \big\{ D^\alpha \varphi(x) \mid x \in K, |\alpha| \le N_k \big\}, \tag{3.16}$$

while φ of the form $\psi(x')\chi(t)$ yields the injectivity of $f \mapsto \Lambda_f$ ∎

While it is meaningful, for every subspace X of $\mathcal{D}'(\mathbb{R}^n)$, to ask whether

$$X \subset C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1})), \tag{3.17}$$

it is for arbitrary $u \in \mathcal{D}'(\mathbb{R}^n)$ meaningless to ask whether the dependence on x_n is continuous. Despite this peculiarity, the estimates yielding boundedness of γ_0 in (3.1) do also give inclusions like (3.17) for the domains of γ_0 (cf. Proposition 5.1).

Remark 3.6. On $X = C_{\mathrm{b}}(\mathbb{R}^n)$, where the inclusion in (3.17) is clear, it follows that (3.1) converges to the continuous function obtained from the operation in (1.1) as expected. Indeed, since $\Psi_k = \Phi_0 + \ldots + \Phi_k$ gives an approximative identity, viz. $\mathcal{F}^{-1}\Psi_k$, for the convolution algebra X,

$$u(0) = \lim_{k \to \infty} \check{\Psi}_k * u(\cdot, 0) = \gamma_0 u. \tag{3.18}$$

Remark 3.7. Considering $\rho_0 : H^1(\overline{\mathbb{R}}_+) \to \mathbb{C}$ given by $\rho_0 u = u(0)$, the restriction $\rho_0|_{C_0^\infty}$ extends by continuity to the zero-operator $L_2 \to \mathbb{C}$. This exemplifies that when a restriction of an operator is extended by continuity between *another* pair of spaces, the resulting map may be very different from the original one. A less obvious example is $\gamma_0|_{\mathcal{S}}$ extended as T in (1.4) (cf. (1.6) – (1.8)).

Remark 3.8. To avoid phenomena as those in Remark 3.7, the approach of this paper is first of all to define r_0 as the distributional trace on $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$; for this reason Proposition 3.5 is included. Secondly, boundedness of $\gamma_0 \colon X \to Y$ is obtained together with the identity $\gamma_0 = r_0|_X$ without extension by continuity.

4. Boundedness

To obtain the continuity properties, observe that since $\mathcal{F}^{-1}(\Phi_j\hat{u})$ has spectrum in the ball $B(0,R2^j)$ for $R=\frac{13}{10}$, it follows from Remark 3.4 by freezing x' that $\mathcal{F}^{-1}(\Phi_j\hat{u})(x',\cdot)$ has spectrum in $[-R2^j,R2^j]$, hence by the Nikol'skiĭ-Plancherel-Polya inequality that

$$\big\|\mathcal{F}^{-1}(\Phi_j\hat{u})(x',\cdot)\big|L_\infty(\mathbb{R})\big\| \le c(R2^j)^{\frac{1}{p}}\big\|\mathcal{F}^{-1}(\Phi_j\hat{u})(x',\cdot)\big|L_p(\mathbb{R})\big\|, \tag{4.1}$$

when the latter is applied in the x_n-variable only. Integration with respect to x' then gives the basic L_p-L_∞ estimate

$$\Big\|\sup_{x_n\in\mathbb{R}}\big|\mathcal{F}^{-1}(\Phi_j\hat{u})(\cdot,x_n)\big|\,\Big|L_p(\mathbb{R}^{n-1})\Big\| \le c2^{\frac{j}{p}}\big\|\mathcal{F}^{-1}(\Phi_j\hat{u})\big|L_p(\mathbb{R}^n)\big\|, \tag{4.2}$$

and taking in particular $x_n=0$,

$$\big\|\mathcal{F}^{-1}(\Phi_j\hat{u})(\cdot,0)\big|L_p(\mathbb{R}^{n-1})\big\| \le c2^{\frac{j}{p}}\big\|\mathcal{F}^{-1}(\Phi_j\hat{u})\big|L_p(\mathbb{R}^n)\big\|. \tag{4.3}$$

The boundedness in Theorems 1.1 – 1.3 now follows by Theorem 3.1 and Remark 3.4.

For example, that $u\in B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$ means that the right-hand side of (4.3) is in ℓ_1, so $\sum_{j=0}^\infty \mathcal{F}^{-1}(\Phi_j\mathcal{F}u)\big|_{x_n=0}$ converges in L_p (because of its convergent norm series); hence also in $\mathcal{D}'(\mathbb{R}^{n-1})$ when $1\le p\le\infty$. So, with the limit denoted $\gamma_0 u$ according to the working definition of γ_0,

$$\|\gamma_0 u|L_p\| \le \sum_{j=0}^\infty \|\mathcal{F}^{-1}(\Phi_j\hat{u})(\cdot,0)|L_p\| \le c\|u|B^{\frac{1}{p}}_{p,1}\|. \tag{4.4}$$

For $B^{\frac{1}{p}}_{p,q}$ with $0<q<1$ part (i) of Theorem 3.1 applies.

When $s=\frac{n}{p}-n+1$ for $p<1$, then (4.3) may be multiplied by $2^{j(s-\frac{1}{p})}$ and the ℓ_q-norm of both sides calculated. By Remark 3.4 – this time applied with the freezing $x_n=0$ – and (iii) of Theorem 3.1, the properties in (1.10) – (1.11) are obtained. Observe here that the assumption on s is equivalent to

$$s-\tfrac{1}{p}=(n-1)(\tfrac{1}{p}-1) \tag{4.5}$$

which is required when part (iii) of Theorem 3.1 is applied to the co-domain $B^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$. In the same way (4.3) and part (ii) of Theorem 3.1 may be used to show the boundedness in Theorem 1.1.

Following [21: 2.7.2], the right inverse K of γ_0 may be taken as

$$Kv=\sum_{j=0}^\infty \psi(2^j x_n)\mathcal{F}^{-1}(\Phi'_j\hat{v})(x') \tag{4.6}$$

when $\psi \in \mathcal{S}(\mathbb{R})$ has $\operatorname{supp} \mathcal{F}\psi \subset [-1,1]$ and $\psi(0) = 1$. Indeed, letting $v_j = \mathcal{F}^{-1}\Phi'_j \mathcal{F} v$,

$$\operatorname{supp} \mathcal{F}(\psi(2^j \cdot) v_j) \subset \left\{ \xi \in \mathbb{R}^n \mid 2^j \le |\xi| \le 3 \cdot 2^j \right\} \tag{4.7}$$

$$\|\psi(2^j \cdot) v_j | L_p(\mathbb{R}^n)\| = 2^{-\frac{j}{p}} \|\psi | L_p(\mathbb{R})\| \, \|v_j | L_p(\mathbb{R}^{n-1})\|, \tag{4.8}$$

so part (iv) of Theorem 3.1 gives that Kv is well defined and that, for $s \in \mathbb{R}$,

$$\big\| Kv \big| B^s_{p,q}(\mathbb{R}^n) \big\| \le c \big\| v \big| B^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \big\|. \tag{4.9}$$

Moreover, for $s > \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+$ the already shown continuity of γ_0 gives

$$\gamma_0 K v = \sum \gamma_0 \big(\psi(2^j x_n) v_j(x') \big) = \sum \psi(0) v_j = v. \tag{4.10}$$

This reproves the claims on K in Theorem 1.1.

Remark 4.1. The spaces $B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$ with $1 \le p \le \infty$ are maximal among those under consideration, for when $s > \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+$,

$$B^s_{p,q} \hookrightarrow B^{\frac{1}{r}}_{r,1} \qquad \text{for } r = \max(1,p) \tag{4.11}$$

and this also holds when $s = \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+$ and $q \le 1$.

5. Continuity in the x_n-variable

In view of Remark 4.1, the proof of Theorem 1.4 need only be conducted for the $B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$ spaces with $1 \le p \le \infty$. Clearly, $x_n = 0$ does not play a special role, for the mixed norm estimate in (4.2) 'absorbs' any value equally well: obviously,

$$\sup_{x_n \in \mathbb{R}} \big\| \mathcal{F}^{-1}(\Phi_j \hat{u})(\cdot, x_n) \big| L_p(\mathbb{R}^{n-1}) \big\| \le c 2^{\frac{j}{p}} \big\| \mathcal{F}^{-1}(\Phi_j \hat{u}) \big| L_p(\mathbb{R}^n) \big\| \tag{5.1}$$

follows in the same way as (4.3). This means that the function series

$$t \mapsto \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\Phi_j \hat{u}) \big|_{x_n = t} \tag{5.2.}$$

converges in the Banach space $C_{\mathrm{b}}(\mathbb{R}, L_p(\mathbb{R}^{n-1}))$, say, with the limit denoted by $f_u(t)$. So for every $u \in B^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$ with $1 \le p \le \infty$,

$$f_u \in C_{\mathrm{b}}(\mathbb{R}, L_p(\mathbb{R}^{n-1})) \hookrightarrow \mathcal{D}'(\mathbb{R}^n) \tag{5.3}$$

and $f_u(0) = \gamma_0 u$ by the working definition of γ_0. By (3.5), the injection in (5.3) is well defined and continuous; in fact

$$\begin{aligned} |\langle f, \varphi \rangle| &\le \int \|f(t) | L_p\| \, \|\varphi(\cdot, t) | L_{p'}\| \, dt \\ &\le (\operatorname{diam} \operatorname{supp} \varphi)^{\frac{1}{p}} \|f | C_{\mathrm{b}}(\mathbb{R}, L_p)\| \, \|\varphi | L_{p'}(\mathbb{R}^n)\| \end{aligned} \tag{5.4}$$

for every test function φ, when $p + p' = pp'$. However, since the series of C^∞-functions in (5.2) converges to the given u in $\mathcal{S}'(\mathbb{R}^n)$, hence in $\mathcal{D}'(\mathbb{R}^n)$, it follows from (5.3) – (5.4) that $u = f_u$. This proves

Proposition 5.1. *Let $u \in B^{\frac{1}{p}}_{p,q}(\mathbb{R}^n)$ for some $p \in [1,\infty]$ and $q \le 1$. Then the function f_u given by (5.2) – (5.3) defines a distribution Λ_{f_u}, by Proposition (3.5), that coincides with u; that is, $\Lambda_{f_u} = u$.*

Thereby (3.17) has been verified for the result in Theorem 1.3, so the distribution trace $u(0)$ is defined for every $u \in B^{\frac{1}{p}}_{p,1}$; viewing u as an element of $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^n))$ gives $u(0) = f_u(0) = \gamma_0 u$ as desired. In particular, $\gamma_0 u$ in (3.1) is independent of the choice of partition of unity.

6. Surjectivity

Since γ_0 in (1.14) has dense range, it is for $q = 1$ surjective precisely when its adjoint γ_0^* has a bounded inverse from $\operatorname{ran}(\gamma_0^*)$ to L_p^* (see, e.g., [19: Theorem 4.15]).

For $1 \le p < \infty$ and $q = 1$ the adjoint is bounded, when $p + p' = pp'$,

$$\gamma_0^* : L_{p'}(\mathbb{R}^{n-1}) \to B^{\frac{1}{p'}-1}_{p',\infty}(\mathbb{R}^n) \tag{6.1}$$

(cf. [23] for the dual space) and $\gamma_0^* u = u \otimes \delta_0$ for $u \in L_{p'}$ since for $\varphi \in \mathcal{S}$

$$\langle \gamma_0^* u, \varphi \rangle = \langle u, \varphi(\cdot, 0) \rangle = \langle u \otimes \delta_0, \varphi \rangle. \tag{6.2}$$

It remains to be shown, with primes omitted for simplicity, that

$$\|u|L_p\| \le c\|u \otimes \delta_0 | B^{\frac{1}{p}-1}_{p,\infty}\| =: c\, B(u) \tag{6.3}$$

for all $u \in L_p(\mathbb{R}^{n-1})$ whenever $p \in]1,\infty]$. Using Lemma 2.1 we have a partition of unity $1 = \sum_{k=0}^{\infty} \sum_{J \ne \emptyset} \Theta_{J,k}$, where each $\Theta_{J,k}$ is a product:

$$\begin{gathered} \Theta_{J,k}(\xi) = \eta_{J,k}(\xi')\, \theta_{J,k}(\xi_n) \\ \eta_{J,k}(\xi') = \eta_J(2^{-k}\xi'), \quad \theta_{J,k}(\xi_n) = \theta_J(2^{-k}\xi_n) \quad (k > 0). \end{gathered} \tag{6.4}$$

By (6.1), the corresponding $B^{\frac{1}{p}-1}_{p,\infty}$-norm with supremum over (J,k) gives

$$\|\check{\theta}_J|L_p\|\,\|\check{\eta}_{J,k} * u|L_p\| = 2^{j(\frac{1}{p}-1)}\big\|\mathcal{F}^{-1}(\Theta_{J,k}\mathcal{F}(u \otimes \delta_0))|L_p\big\| \le B(u) < \infty. \tag{6.5}$$

Since $\eta_J(0) \ne 0$ for some J, we can take J such that

$$\check{\eta}_{J,k} * u \to a \cdot u \qquad \text{in } \mathcal{D}' \text{ for } k \to \infty \tag{6.6}$$

if $a := \int \check{\eta}_J \ne 0$. The w*-compactness of the balls in L_p together with (6.5) – (6.6) show that (6.3) holds with c equal to $(a\|\check{\theta}_J|L_p\|)^{-1}$. From the Besov spaces' point of view the surjectivity is proved in a natural way above; essentially it is known from the technical report [18].

For $q \le 1$ the dual of $B^{\frac{1}{p}}_{p,q}$ is independent of q, because $(B^{\frac{1}{p}}_{p,q})^* = B^{-\frac{1}{p}}_{p',\infty}$ then (cf. [21: 2.11.2]). Therefore the adjoint remains equal to (6.1) for $q < 1$, so it suffices to show that the Closed Range Theorem is valid when the domain is a quasi-Banach space. Observe first, for precision, that $B^s_{p,q}$ is an F-space in Rudin's terminology [19] when $d(u,v) := \|u - v|B^s_{p,q}\|^\lambda$ and $\lambda = \min(1,p,q)$. Hence continuity and boundedness are equivalent for operators between these quasi-Banach spaces [19].

Moreover, defining the operator norm in the usual way, $B(X,Y)$ becomes a quasi-Banach space; $\|S+T\| \le c(\|S\| + \|T\|)$ holds with the same constant as it does for $\|\cdot|Y\|$. In particular, X^* is always a Banach space. As usual each $T \in B(X,Y)$ has an adjoint $T^* \in B(Y^*,X^*)$.

Proposition 6.1. *Let X be a quasi-Banach space such that $\|\cdot|X\|^\lambda$ is subadditive for some $\lambda \in]0,1]$, let Y be a Banach space and $T\colon X \to Y$ be a bounded linear operator. When $\overline{T(X)} = Y$, then boundedness of T^{*-1} from $T^*(Y^*)$ to Y^* implies that T is surjective, i.e. $T(X) = Y$.*

Proof. Since $\ker T^* \subset T(X)^\perp = \{0\}$, the inverse is well defined; by assumption there is a constant $c < \infty$ such that

$$\|y^*|Y^*\| \le c\,\|T^*y^*|X^*\| \qquad \text{for all } y^* \in Y^*. \tag{6.7}$$

This inequality implies that T is open. Indeed, if X is a Banach space, this is the content of [19: Lemma 4.13]. When only Y is assumed to be a Banach space, the reduction from part (b) to (a) in the proof of [19: Lemma 4.13] carries over verbatim (since the Hahn-Banach theorem is only used for Y), and in the proof of (a) the sequence (ε_n) should be picked in ℓ_λ such that $\sum_{n=1}^\infty \varepsilon_n^\lambda < 1 - \|y_1|Y\|^\lambda$. Then the sequences (x_n) and (y_n) defined there satisfy

$$\sum_{n=1}^\infty \|x_n|X\|^\lambda \le \|x_1|X\|^\lambda + \sum_{n=1}^\infty \varepsilon_n^\lambda < \|y_1|Y\|^\lambda + (1 - \|y_1|Y|^\lambda) = 1; \tag{6.8}$$

hence $x = \sum x_n$ converges in X and has $\|x|X\| < 1$ as desired. Thus (6.7) implies that T is an open mapping, but as such it's necessarily surjective ∎

Altogether this shows that $L_p(\mathbb{R}^{n-1})$ is the image of $B^{\frac{1}{p}}_{p,q}(\mathbb{R}^n)$ under γ_0 for every $q \le 1$ when $1 \le p < \infty$.

Remark 6.2. It is known that every quasi-Banach space X has an equivalent quasi-norm such that $\|\cdot|X\|^\lambda$ is sub-additive for some $\lambda \in]0,1]$. In view of this, Proposition 6.1 holds for all quasi-Banach spaces.

7. The borderline for $0 < p < 1$

Since the boundedness in Theorem 1.2 is proved in Section 4 above, it remains to show the claim on the integral-exponents. That it is necessary for $p < q \le 1$ in Theorem 1.2 to let $B^{(n-1)(\frac{1}{q}-1)}_{q,\infty}(\mathbb{R}^{n-1})$ receive $\gamma_0 u$ follows because the inequality $r \ge q$ is implied by the estimate

$$\|\gamma_0 u|B^t_{r,\infty}\| \le c\|u|B^{\frac{n}{p}-n+1}_{p,q}\|. \tag{7.1}$$

To show this implication, it suffices to extend the ω_N in the proof of Proposition 3.2 by taking some $\eta \in \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}\eta \subset]1,2[$ and $\check{\eta}(0) = 1$ and set

$$E\omega_N(x) = \sum_{k=1}^{N} \check{\eta}(2^k x_n)\check{\Psi}_k(x' - x'_k). \tag{7.2}$$

Using (7.1) and part (ii) of Theorem 3.1 to estimate the Besov norm of $E\omega_N$, it is easily seen that

$$\big\|\omega_N\big|B^t_{r,\infty}(\mathbb{R}^{n-1})\big\| = \big\|\gamma_0 E\omega_N\big|B^t_{r,\infty}\big\| \le \big\|E\omega_N\big|B^{\frac{n}{p}-n+1}_{p,q}(\mathbb{R}^n)\big\| \le cN^{\frac{1}{q}}. \tag{7.3}$$

Because of (3.10) the inequality $r \ge q$ holds.

8. Final remarks

Some of the claims made after (1.4) in the introduction shall now be explained for the reader's sake. While the first two remarks concern the difference between γ_0 and T, the third observation about the boundary problems might be of general interest.

Remark 8.1. To show (1.6), note first that in addition to (1.9),

$$\psi_k := 2^{k(n-1)}\eta(2^k\cdot)\varphi \to \delta \otimes \varphi$$

in $B^s_{p,1}$ for $k \to \infty$ when $\frac{1}{p} < s < \frac{n}{p} - n$ (which entails $p < 1 - \frac{1}{n}$), at least if $\hat{\eta} = 1$ in a ball around $\xi' = 0$. Indeed, by Remark 3.3,

$$\delta_0 = \eta + \sum_{k=1}^{\infty}\Big(2^{k(n-1)}\eta(2^k\cdot) - 2^{(k-1)(n-1)}\eta(2^{k-1}\cdot)\Big)$$

converges in $B^s_{p,1}(\mathbb{R}^{n-1})$ while $\cdot \otimes \varphi$ maps continuously into $B^s_{p,1}(\mathbb{R}^n)$ by [9]. Hence $\psi_k \to \delta_0 \otimes \varphi$ there, and $T\psi_k \to 0$ as shown in (1.9); i.e. (1.6) holds.

Remark 8.2. For $s = \frac{n}{p} - n + 1$ it is useful to consider

$$v_k(x) = \frac{1}{k}\sum_{l=k+1}^{2k} 2^{l(n-1)} f(2^l x')g(2^l x_n) \tag{8.1}$$

for Schwartz functions f and g with their spectra in balls of radius $\frac{1}{2}$ such that $\int f = 1$ and $g(0) = 1$. As shown in [16: Lemma 2.8], $\gamma_0 v_k \to \delta_0$ in $\mathcal{D}'$ while $v_k \to 0$ in $B^{\frac{n}{p}-n+1}_{p,q}$ if $q > 1$, so that γ_0 is only continuous from $B^{\frac{n}{p}-n+1}_{p,q}$ if $q \leq 1$.

However,

$$T v_k = \tfrac{1}{k} \sum_{l=k+1}^{2k} 2^{l(n-1)} f(2^l \cdot)$$

since $v_k \in \mathcal{S}$, and $\|T v_k | L_p\|$ is $\mathcal{O}(k^{(n-1)(1-\frac{1}{p})})$ and so tends to 0 for $k \to \infty$; that is, already at the borderline γ_0 and T behave differently.

Remark 8.3. For Ω equal to the unit ball in $\mathbb{R}^n$ $(n \geq 3)$, Franke and Runst [8: Section 6.5] proved that $B^{\frac{n}{p}-n+1}_{p,\infty}(\overline{\Omega})$ contains an infinite-dimensional solution space for the problem

$$\left.\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega \\ T u &= 0 \qquad \text{on } S^{n-1} \end{aligned}\right\}. \tag{8.2}$$

In fact, for each boundary point $z \in S^{n-1}$ they showed that $\Phi(x-z) - \frac{1}{n-2} z \cdot \operatorname{grad} \Phi(x-z)$, where $\Phi(x) = c|x|^{2-n}$ is the fundamental solution of $-\Delta$, belongs to this space and solves problem (8.2).

Moreover, in [14] it was proved that the Boutet de Monvel calculus of pseudo-differential boundary operators (for elliptic problems) extends nicely to spaces with $p < 1$. However, for trace operators and $P_\Omega + G$ that precisely *have* class $r \in \mathbb{Z}$, it was proved that $s \geq \frac{n}{p} - n + r$ is *necessary* for continuity from $B^s_{p,q}$ to $\mathcal{D}'$ when $p < 1$.

Taken together, these facts show that not only the usual Fredholm properties but also the continuity of solution operators for elliptic problems would break down for $p < 1$ unless $s = \frac{n}{p} - n + r$ is taken as the borderline for operators of class r. (For the Dirichlét realisation of Δ, the latter fact was also shown by Chang, Krantz and Stein [4].)

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Received 13.09.1999